DIFFERENTIAL FORMS IN THE MODEL THEORY OF DIFFERENTIAL FIELDS

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ABSTRACT. Fields of characteristic zero with several commuting derivations can be treated as fields equipped with a *space* of derivations that is closed under the Lie bracket. The existentially closed instances of such structures can then be given a coordinate-free characterization in terms of differential forms. The main tool for doing this is a generalization of the Frobenius Theorem of differential geometry.

CONTENTS

Date: March 17, 2003.

2000 *Mathematics Subject Classification.* Primary 03C60; Secondary 03C10, 12H05, 12L12, 13Nxx.

Much of this work was done at McMaster University, Hamilton, Ontario, Canada.

0. Introduction

Basic definitions, examples and aims. As it has usually been understood, a **differential field** is a field K of characteristic zero equipped with a **derivation**, that is, an additive endomorphism D taking an arbitrary product $x \cdot y$ to the linear combination $Dx \cdot y + Dy \cdot x$. (Equivalently, the map $a \mapsto a + Da \cdot X + (X^2)$ from K to $K[X]/(X^2)$ is a ring-homomorphism.) In this paper, we allow the differential field to have *several* derivations, with a restriction. Say the derivations are D_0, $\dots$, D_{m-1}. The **(Lie) bracket** $[D_i, D_j]$ is the derivation $D_iD_j - D_jD_i$. We shall require these brackets to be K-linear combinations of the original derivations D_k.

The differential field $(K, D_0, \dots, D_{m-1})$ determines the pair (K, E), where E is the linear span over K of the D_i. The space E is a *Lie ring*, but is not a Lie algebra over K unless its dimension over K is zero. Every element of E is definable in $(K, D_0, \dots, D_{m-1})$ by a term with parameters from K; so the pair (K, E) determines the definable sets of the original differential field. However, possibly the dimension of E is strictly less than m. In this case, the differential field $(K, D_0, \dots, D_{m-1})$ itself carries additional information, namely that in some differential field in which this structure embeds, the named derivations D_i span a space whose dimension *is* m.

Examples 0.1. (0) K is an algebraic extension of $\mathbb{Q}(X^0, \dots, X^{m-1})$, and the D_i are the 'partial' derivations $\partial/\partial X^i$. Here, $[D_i, D_j] = 0$.

(1) K is the field of germs of meromorphic functions at a point of a manifold, and the D_i are the vector-fields determined by the coordinate-functions on a neighborhood of the point. Here again, the derivations commute.

(2) In the last example, suppose the manifold is a complex Lie group, and the point is the identity. The Lie algebra $\mathfrak{g}$ of the group is a vector space over $\mathbb{C}$ whose members act as derivations of K. So let E be the space $K \otimes_{\mathbb{C}} \mathfrak{g}$ of K-linear combinations of elements of $\mathfrak{g}$. The bracket is defined on E and is $\mathbb{C}$-bilinear, but not K-bilinear. It will turn out that E has a commuting basis, although $\mathfrak{g}$ itself need not have.

In the class of differential fields in this broader sense, axioms picking out the *existentially closed* structures can be developed from earlier work ([McG] and [Y]). Instead of doing this, the present paper takes a novel approach in terms of *differential forms*. The result is a succinct characterization of the existentially closed differential fields (and one which has an evident translation into first-order axioms). The proof of this result relies on the algebraic fact underlying the Frobenius Theorem of differential geometry.

Context: the model theory of fields. A differential field can be understood as a certain kind of structure in the language of (unital) rings with unary function-symbols for the derivations. For any positive integer m, the differential fields with m derivations are just the models of a certain first-order theory, here called DF^m. This theory is *inductive*; that is, the union of an (increasing) chain of models is a model. Therefore existentially closed differential fields do exist. Over such a differential field, by definition, any (finite) system of differential-polynomial equations and inequations with a solution in some extension has a solution from the original field. (See [H, § 8.1–2].)

The existentially closed differential fields with m derivations turn out to be just the models of a certain first-order theory, here called DCF^m (for 'differentially closed field with m independent derivations'). In particular, DCF^m is *model-complete* and is the *model-companion* of DF^m, which is therefore *companionable.* (See [H, § 8.3].) Abraham Robinson provides a contrasting example in [R] by showing that the theory of algebraically closed fields with proper algebraically closed subfield is not model-complete (in the obvious signature), but becomes complete when a characteristic is specified. (The 'problem of Tarski' thus solved is to axiomatize the theory of $(\mathbb{C}, A)$, where A is the field of algebraic numbers.) The theory of fields with distinguished algebraically closed subfield is therefore not companionable. (This slightly generalizes Angus Macintyre's observation in [HML, ch. A.4, § 3.3].) Indeed, reflexion shows that the existentially closed models of the last theory are the algebraically closed fields *of transcendence-degree one* over an algebraically closed subfield. (Robinson's example shows the failure of 'local modularity' in algebraically closed fields: see [P96, ch. 2, p. 77, example 1.8].)

When $m > 1$, then the theory DF^m fails to have the *amalgamation property*, since two extensions of a differential field might not embed in a third. The obstruction was noted above: The m derivations need not be linearly independent *on the original field*. That is, the restrictions of the derivations to the original field need not be linearly independent (even over that field). The derivations may become independent when extended to larger fields, but they may do so incompatibly—that is, their brackets may be different linear combinations of themselves on each of the larger fields. However, there is a sentence α in DCF^m (given below in Lemma 2.6) saying that the m derivations are independent. Then $\mathrm{DF}^m \cup \{\alpha\}$ has the amalgamation property, so that DCF^m is the *model-completion* of this larger theory. (See [H, exercise 8.4.9].) Similarly, the theory of fields with an automorphism has a model-companion—call it T. (See [Mac]—or [ChHr], where T is called ACFA.) This theory T is the model-completion of the theory of *algebraically closed* fields with an automorphism.

Another way to augment DF^m is with the axiom

$$(\sigma) \qquad \forall x \bigwedge_{i<j<m} \sum_{k<m} a^k_{i\,j} \cdot D_k x = [D_i, D_j]x,$$

which *specifies* the linear relations that hold amongst the m derivations and their brackets. The language must accordingly be augmented with the constant-symbols $a^k_{i\,j}$. Then even the universal part of $\mathrm{DF}^m \cup \{\sigma\}$ has the amalgamation property; therefore, $\mathrm{DCF}^m \cup \{\sigma\}$ admits elimination of quantifiers (by [H, Theorem 8.4.1], for example).

In [McG], Tracey McGrail gives a model-companion for the theory of fields with m *commuting* derivations. (She also shows that the model-companion is complete and ω-stable, and she characterizes forking.) Yoav Yaffe's independent work in [Y] can be understood as follows. Let $(K, D_0, \dots, D_{m-1})$ be a particular model of DF^m. In the axiom σ above, suppose the coefficients $a^k_{i\,j}$ are understood to be appropriate members of K. Then Yaffe gives a model-companion for the theory

$$\mathrm{DF}^m \cup \{\sigma\} \cup \mathrm{diag}(K, D_0, \dots, D_{m-1})$$

(and shows that the model-companion is complete and ω-stable). Presumably, Yaffe's methods could be adjusted to show that the bare theory DF^m is companionable. However, McGrail's results yield the same fact, because in any model of DF^m, the span of the derivations D_i has a basis of commuting derivations. (If the derivations D_i happen not to be independent, then one will want to move to a larger field on which they *are* independent before choosing the commuting basis. Also, the Lie algebra generated by the D_i *over their constant-field* will not in general have a commuting basis. Yaffe alludes to these facts in [Y, Remark 0.8]; however, they do not prevent the special case of commuting derivations from yielding companionability in the general case.)

The argument. A derivation on K need not have co-domain K. For the definition to make sense, it is enough that the co-domain be a vector-space over K. We have two such spaces at hand, namely the span E of the derivations D_i, and the dual space E^*. We also have the pairing $(D,\alpha) \mapsto D\alpha : E \times E^* \to K$. Hence we have a derivation $\mathrm{d} : K \to E^*$ given by $D(\mathrm{d}\,x) = Dx$. If $\dim_K E = m$, then E^* has a basis $(\mathrm{d}\,t^i : i < m)$ for some t^i in K. This basis is dual to a basis $(\partial_i : i < m)$ of E, and the latter basis commutes. Thus we can dispense with the original basis of E, which may not have commuted. Even if it did commute, its dual may not have consisted of images under d. But having such a dual is useful, since we can now write d as the map $x \mapsto \sum_{i<m} \mathrm{d}\,t^i \cdot \partial_i x$, combining the several derivations ∂_i into one.

A derivation on K has a unique extension to the algebraic closure of K. In particular, the ∂_i will continue to commute when extended to this algebraic closure, as their brackets will still be zero. So we may assume that K is algebraically closed.

We want to understand systems of polynomial equations over K whose variables are some symbols X^j and their formal derivatives with respect to the ∂_i. By introducing new variables, we need only consider a system consisting of various equations

$$f = 0 \quad \text{and} \quad \partial_i X^k = g_i^k,$$

where the f and the g_i^k are from $K[X^0, \dots, X^{n-1}]$. Suppose that this system has a solution $\mathbf{a}$. More precisely, this means that there is a differential field $(L', \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1})$ of which $(K, \partial_0, \dots, \partial_{m-1})$ is a substructure, and where $K(\mathbf{a}) \subseteq L'$, such that

$$f(\mathbf{a}) = 0 \quad \text{and} \quad \tilde{\partial}_i a^k = g_i^k(\mathbf{a})$$

in each case. After various manipulations—which may include lengthening $\mathbf{a}$, introducing additional polynomials g_i^k, and re-indexing—we can write down a sentence

$$(*) \qquad \bigwedge_{k<r} \bigwedge_{i<m} \partial_i a^k = g_i^k(\mathbf{a})$$

that is true in $(L', \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1})$, where $\{a^k : k < r\}$ is algebraically independent over K. Moreover, we may assume that the truth of $(*)$ is sufficient to ensure that $\mathbf{a}$ is a solution of the original system. Let us refer to $(L', \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1})$ as a **generic solution** to the system $(*)$. A **solution in** K to $(*)$ is a K-rational specialization $\mathbf{b}$ of $\mathbf{a}$ such that

$$\bigwedge_{k<r} \bigwedge_{i<m} \partial_i b^k = g_i^k(\mathbf{b}).$$

Then (K, E) is existentially closed just in case every system $(*)$ with a generic solution has a solution in K.

We want to understand just when generic solutions exist. We can proceed in two ways. One way is to work directly with the equations $\partial_i a^k = g_i^k(\mathbf{a})$. The other way is to rewrite $(*)$ in terms of the derivation d.

To take the first approach, let $L = K(\mathbf{a})$. We may assume that $\mathbf{a}$ is literally

$$(X^0, \dots, X^{s-1}, a^s, \dots, a^{n-1}),$$

where $r \leqslant s < n$, and $(X^\ell : \ell < s)$ is a transcendence-basis of L over K. If we formally differentiate the equations in $(*)$, then, since the derivations should commute, we get a new system, namely

$$(\dagger) \qquad \bigwedge_{k<r} \bigwedge_{i<j<m} \partial_j(g_i^k(\mathbf{a})) = \partial_i(g_j^k(\mathbf{a})).$$

Each member of each equation is an affine combination over L of derivatives $\partial_k X^\ell$ such that $\ell < s$. We can also replace $\partial_k X^\ell$ with $g_k^\ell(\mathbf{a})$ if $\ell < r$. So we can consider $(\dagger)$ as a linear system over L in the tuple $(\partial_k X^\ell : k < m \wedge r \leqslant \ell < s)$ of variables. The core result of this paper is the following.

Theorem A. *The differential system $(*)$ has a generic solution if and only if the linear system $(\dagger)$ is consistent.*

Since consistency of $(\dagger)$ can be checked in K, axioms for DCF^m can be written down.

If $m = 1$, then $(\dagger)$ is empty. So DCF^1 is axiomatized by the statements that all systems $(*)$ have solutions. The axioms given in [PP] can be understood as saying just this.

For arbitrary m, it may still be that the system $(\dagger)$ is trivially consistent. This is the Frobenius integrability condition, and it holds just in case there is a generic solution to $(*)$ as described above in which $L' = L$. However, there are systems $(*)$ with generic solutions in which the field L' necessarily has infinite transcendence-degree over K:

Example 0.2. The following is given in [JRR] and mentioned in [McG, Remark 3.2.5]. Let K be the field $\mathbb{Q}(Z_\sigma : \sigma \in {}^2\omega)$, on which commuting derivations ∂_0 and ∂_1 are defined in an obvious way, so that $\partial_0 Z_{(i,j)} = Z_{(i+1,j)}$, and $\partial_1 Z_{(i,j)} = Z_{(i,j+1)}$. The system

$$\partial_0 X^0 = Z_{(0,0)} \wedge \partial_1 X^0 = X^1$$

in the variables X^0 and X^1 has a generic solution. For, let L' be $K(X^i : i \in \omega)$, and define $\tilde{\partial}_0 X^i = Z_{(0,i)}$ and $\tilde{\partial}_1 X^i = X^{i+1}$. So $\tilde{\partial}_0$ and $\tilde{\partial}_1$ respectively send elements down and to the right in the following table.

X^0	X^1	X^2	$\dots$
$Z_{(0,0)}$	$Z_{(0,1)}$	$Z_{(0,2)}$	$\dots$
$Z_{(1,0)}$	$Z_{(1,1)}$	$Z_{(1,2)}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$

In any generic solution, the set $\{\tilde{\partial}_1^i X^0 : i \in \omega\}$ must be algebraically independent over K, since each derivative $\tilde{\partial}_0 \tilde{\partial}_1^i X^0$ is just $\tilde{\partial}_1^i \tilde{\partial}_0 X^0$, that is, $Z_{(0,i)}$.

Proving Theorem A will involve replacing the several derivations ∂_i with the single derivation d. In $(*)$, multiply each equation by $\mathrm{d}\, t^i$, and combine like terms to get

$$(\ddagger) \qquad \bigwedge_{k<r} \mathrm{d}\, X^k = \sum_{i<m} \mathrm{d}\, t^i \cdot g_i^k(\mathbf{a}).$$

Let $\Omega^1_{L/E}$ be the vector-space over L spanned by the elements $\mathrm{d}\, t^i$ of E^* and the new symbols $\mathrm{d}\, X^\ell$ (where $\ell < s$). The system $(\ddagger)$ determines the subspace W of $\Omega^1_{L/E}$ spanned by the elements $\mathrm{d}\, X^k - \sum_{i<m} \mathrm{d}\, t^i \cdot g_i^k(\mathbf{a})$, where $k < r$.

The space $\Omega^1_{L/E}$ also has a coordinate-free definition. It is the dual of the space $\mathrm{Der}(L/E)$ of derivations $D : L \to L$ such that $D|_K \in L \otimes_K E$. Then d extends to a derivation from L to $\Omega^1_{L/E}$.

Suppose $(K, \partial_0, \dots, \partial_{m-1})$ is a substructure of the differential field $(L', \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1})$, and let $\tilde{E}$ be the space spanned over L' by the $\tilde{\partial}_i$. If $L \subseteq L'$, then we have a linear map

$$(\Phi) \qquad \mathrm{d}\, x \longmapsto \sum_{i<m} \mathrm{d}\, t^i \cdot \tilde{\partial}_i x : \Omega^1_{L/E} \longrightarrow \tilde{E}^*.$$

The kernel of this map includes W just in case $(L', \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1})$ is a generic solution of $(*)$. Let us say that an arbitrary subspace of $\Omega^1_{L/E}$ is **integrable** if it is included in the kernel of such a linear map.

More generally, let $\Omega^p_{L/E}$ be the space of alternating multi-linear maps from $\mathrm{Der}(L/E)^p$ to L, and let $\Omega_{L/E}$ be the direct sum of these spaces. Then $\Omega_{L/E}$ can be equipped with the *wedge-product*, making it a (non-commutative) algebra over L. Also d extends to a group-endomorphism of $\Omega_{L/E}$ that interacts with the wedge-product in a derivation-like way and is such that $\mathrm{d}^2 = 0$.

There is a trivial case, $\Omega_{K/E}$. Here $\Omega^1_{K/E}$ is just E^*, and this, or rather $E^* \otimes_K L$, embeds in $\Omega^1_{L/E}$. (The embedding takes α to the map $D \mapsto (D|_K)\alpha$.) More generally, $\Omega_{K/E} \otimes_K L$ embeds in $\Omega_{L/E}$.

A subspace W of $\Omega^1_{L/E}$ generates both an ordinary ideal $\mathcal{I}(W)$ and a differential ideal $\mathcal{D}(W)$ of $\Omega_{L/E}$. Theorem A can now be expressed as follows.

Theorem B. *A differential system of the form (∗) has a generic solution if and only if*

$$\mathcal{D}(W) \cap (\Omega_{K/E} \otimes_K L) = \{0\}$$

—that is, $\mathcal{D}(W)$ *and* $\Omega_{K/E} \otimes_K L$ *are* linearly disjoint*—, where* W *is the corresponding subspace of* $\Omega^1_{L/E}$.

The hard direction of Theorem A seems to be easier to prove when expressed in these terms.

For an arbitrary subspace W of $\Omega^1_{L/E}$, it may be that $\mathcal{D}(W)$ and $\Omega_{K/E} \otimes_K L$ are linearly disjoint, but W is not integrable. For, the map Φ induces a homomorphism from $(\Omega_{L/E}, \mathrm{d})$ to $(\Omega_{L'/\tilde{E}}, \mathrm{d})$, and integrability of W would mean $\mathcal{D}(W)$ was included in the kernel of this homomorphism; but it need not be, even if it is linearly disjoint from $\Omega_{L/E} \otimes_K L$, as the following shows:

Example 0.3. Let $m = 4$, let $L = K(X^0, X^1)$, and let W be spanned by the form

$$-\mathrm{d}\,X^0 \cdot X^1 + \mathrm{d}\,X^1 \cdot X^0 + (\mathrm{d}\,t^0 \cdot t^1 + \mathrm{d}\,t^2 \cdot t^3) \cdot 2.$$

Call this form θ. Then $\mathcal{D}(W)$ is $\mathcal{I}(\theta, \mathrm{d}\,\theta)$. But $\mathrm{d}\,\theta$ is

$$(\mathrm{d}\,X^0 \wedge \mathrm{d}\,X^1 - \mathrm{d}\,t^0 \wedge \mathrm{d}\,t^1 - \mathrm{d}\,t^2 \wedge \mathrm{d}\,t^3) \cdot 2.$$

Then $\mathcal{D}(W)$ is linearly disjoint from $\Omega_{K/E} \otimes_K L$. If W were integrable, then the map Φ witnessing this would have to send $\mathrm{d}\,X^0 \wedge \mathrm{d}\,X^1$ to $\mathrm{d}\,t^0 \wedge \mathrm{d}\,t^1 + \mathrm{d}\,t^2 \wedge \mathrm{d}\,t^3$; but the map would also have to send wedge-products to wedge-products, and $\mathrm{d}\,t^0 \wedge \mathrm{d}\,t^1 + \mathrm{d}\,t^2 \wedge \mathrm{d}\,t^3$ is not a wedge-product.

References and notation. This paper makes rigorous and proves the preceding claims. As noted, [H, ch. 8] collects the basic facts about model-companions. The conversion of a differential system to a system of differential forms is described also in [ChCh]. This reference is given in [Sh, ch. 1, § 5, p. 59], and indeed a talk by Richard Sharpe in Ankara in 1999 led ultimately to the discovery of the results given here.

Some notational conventions of this paper have already been used in this introduction: The ordered monoid of natural numbers is ω, and the partially ordered monoid of functions from $\{0,\dots,m-1\}$ to ω is ${}^m\omega$. The same n-tuple may be written $(a^0,\dots,a^{n-1})$ or $\mathbf{a}$. Capital letters X denote independent transcendentals (or free variables); small letters x denote specializations of these (or bound variables). Symbols for relations are generally verbs; so '$i < m$' is a clause, interchangeable with '$m > i$'.

1. The theories of differential fields

All **rings** in this paper have units and are of characteristic zero. The definition of a derivation given in § 0 is meaningful for maps from a commutative ring R into a right R-module. The derivations from R into itself compose the left R-module $\mathrm{Der}(R)$, which is closed under the bracket. As defined in § 0, the theory DF^m of differential fields with m derivations is the theory of structures $(K, D_0, \dots, D_{m-1})$ such that:

- K is an integral domain of characteristic zero;
- each D_i is a derivation from K to K;
- each bracket $[D_i, D_j]$ is a K-linear combination of the derivations D_k;
- every non-zero element of K has a multiplicative inverse.

The last axiom is $\forall\exists$, and the first three are universal. Indeed, the third axiom is that each linear system

$$(\S)\qquad \bigwedge_{y\in K}\sum_{k<m} X^k_{i\,j}\cdot D_k y = [D_i, D_j]y$$

in the tuple $(X^k_{i\,j} : k < m)$ of unknowns is soluble. This just means that every sub-system of $m+1$ equations is soluble, which in turn means that row-reduction in the corresponding matrices does not yield absurdity—that is,

$$\Big(\bigwedge_{k<m}\sum_{\ell\leqslant m} D_k y^\ell\cdot z_\ell = 0\Big) \implies \sum_{\ell\leqslant m}[D_i, D_j]y^\ell\cdot z_\ell = 0$$

for all y^ℓ and z_ℓ in K, for each i and j such that $i < j < m$.

So DF^m is an $\forall\exists$ theory; equivalently, it is inductive. The universal part $T_\forall$ of any theory T is the theory of substructures of models of T. If $(R, D_0, \dots, D_{m-1})$ is a model of $\mathrm{DF}^m_\forall$, then R is an integral domain, and the derivations $D_k : R \to R$ are such that

$$b_{i\,j}[D_i, D_j] = \sum_{k<m} c^k_{i\,j}\cdot D_k$$

for some $b_{i\,j}$ and $c^k_{i\,j}$ in R, where $b_{i\,j} \neq 0$, for all i and j such that $i < j < m$. If K is the quotient-field of R, then there are unique elements $\tilde{D}_i$ of $\operatorname{Der}(K)$ such that $\tilde{D}_i|_R = D_i$; these are given by the usual quotient-rule. Then $(K, \tilde{D}_0, \dots, \tilde{D}_{m-1}) \models \mathrm{DF}^m$.

2. Extensions

Extensions of derivations. Let K be a field, with algebraic closure K^{a}. Suppose $D \in \operatorname{Der}(K)$. It is known that D has a unique **extension** to K^{a}; that is, there is a unique derivation $\tilde{D}$ on K^{a} such that $\tilde{D}|_K = D$. In fact, $\tilde{D} \in \operatorname{Der}(K^{\mathrm{a}})$. Moreover, if $f^0, \dots, f^{n-1} \in K(X^0, \dots, X^{n-1})$, then D has a unique extension to $K(X^0, \dots, X^{n-1})$ taking each X^j to f^j; this extension is in $\operatorname{Der}(K(X^0, \dots, X^{n-1}))$. The most general statement of these facts that is useful to us is as follows.

Fact 2.1. *Let L/K be a field-extension with transcendence-basis $(X^j : k < \mu)$. Then for each j less than μ, the zero-derivation on K has a unique extension $\partial/\partial X^j$ to L taking each X^k to δ^k_j (the Kronecker delta); in fact, $\partial/\partial X^j \in \operatorname{Der}(L)$.*

Let also E be a vector-space over K, and suppose δ is a derivation from K to E^*. Then δ has a unique extension to L taking each X^j to 0. This extension can be denoted by $f \mapsto f^\delta$ and has co-domain $E^* \otimes_K L$.

Suppose now $\tilde{E}^*$ is a space over L that includes $E^* \otimes_K L$. Then the map

$$\tilde{\delta} \longmapsto (\tilde{\delta}(X^j) : j < \mu)$$

is a one-to-one correspondence between the set of extensions of δ to L with co-domain $\tilde{E}^*$, and $(\tilde{E}^*)^\mu$. The inverse of this correspondence takes $(a^j : j < \mu)$ to the well-defined derivation

$$f \longmapsto \sum_{j<\mu} a^j \cdot \frac{\partial}{\partial X^j} f + f^\delta.$$

Proof. The claims follows from two observations:

(0) The claims hold when $L = K(X^j : j < \mu)$.

(1) The derivation δ extends uniquely to $K(a)$—where it has co-domain $E^* \otimes_K K(a)$—when $a \in K^{\mathrm{a}} \setminus K$.

Observation (0) is a special case of [J, Theorem 8.39], and Observation (1) is [J, Proposition 8.17]. □

We shall need the following to justify the definition of the space $\operatorname{Der}(L/E)$ mentioned in § 0.

Lemma 2.2. *Let L/K be a field-extension. If $D \in \operatorname{Der}(L)$, then $D|_K \in L \otimes_K \operatorname{Der}(K)$.*

Proof. Let $(X^j : j < \mu)$ be a transcendence-basis of K over its prime field $\mathbb{Q}$. The derivation $D|_K$ is an extension to K of the zero-derivation on $\mathbb{Q}$. By Fact 2.1, we have

$$Df = \sum_{j<\mu} DX^j \cdot \frac{\partial}{\partial X^j} f$$

whenever $f \in K$. Since $DX^j \in L$, we have that $D|_K$ is an L-linear combination of the derivations $\partial/\partial X^j$ in $\operatorname{Der}(K)$. □

Extensions of differential fields. An **extension** of a differential field is another differential field of which the first is a substructure. So, an extension of the differential field $(K, D_0, \dots, D_{m-1})$ is a model $(L, \tilde{D}_0, \dots, \tilde{D}_{m-1})$ of the theory

$$\mathrm{DF}^m \cup \operatorname{diag}(K, D_0, \dots, D_{m-1}),$$

where 'diag' stands for the (Robinson or basic) diagram. Strictly, L must literally include K; in this case, $\tilde{D}_i$ extends D_i.

The following is a direct consequence of [P02, Lemma 1.5(b)].

Fact 2.3. DF^1 *has the amalgamation-property, that is, any two extensions of a model have a common extension.*

Lemma 2.4. *Any differential field $(K, D_0 \dots, D_{m-1})$ has a unique extension whose underlying field is K^{a}.*

Proof. Each linear system (§) in § 1 is satisfied by some $(a^k_{i\,j} : k < m)$ in K. By Fact 2.1, each D_k has a unique extension $\tilde{D}_k$ in $\operatorname{Der}(K^{\mathrm{a}})$. Then $\sum_{k<m} a^k_{i\,j} \cdot \tilde{D}_k$ and $[\tilde{D}_i, \tilde{D}_j]$ agree on K, hence on K^{a}. □

The theory DF^m fails to have the amalgamation-property if $1 < m$, because of examples like the following.

Example 2.5. Let the derivations D_0 and D_1 be zero on K. Extend them to $K(X^0, X^1)$ by defining $D_i X^j$ to be δ^j_i (the Kronecker delta). Extend them to $K(Y^0, Y^1)$ by defining

$$D_0 Y^j = \delta^j_0, \quad D_1 Y^0 = Y^0, \quad D_1 Y^1 = 1.$$

Then the solution-set to the equation

$$Z^0 \cdot D_0 + Z^1 \cdot D_1 = [D_0, D_1]$$

is either $\{(0,0)\}$ or $\{(1,0)\}$, depending on whether the D_i are on $K(X^0, X^1)$ or $K(Y^0, Y^1)$. So these extensions of (K, D_0, D_1) have no common extension.

The problem in the example of course is that $\{D_0, D_1\}$ is not linearly independent on K (that is, is not linearly independent when the D_i are the derivations on K).

Lemma 2.6. *Suppose $(K, D_0, \dots, D_{m-1})$ is a differential field. The D_i are linearly independent elements of $Der(K)$ if and only if $(K, D_0, \dots, D_{m-1})$ satisfies the sentence*

$$(\alpha) \qquad \exists(x^0, \dots, x^{m-1}) \ \det((D_i x^j)^{j<m}_{i<m}) \neq 0.$$

Then $\mathrm{DF}^m \cup \{\alpha\}$ *is consistent and has the amalgamation-property.*

Proof. The first part is clear. The theory $\mathrm{DF}^m \cup \{\alpha\}$ has the model given as (0) of the Examples 0.1, since in that case the matrix $(D_i X^j)^j_i$ is the identity. In any model $(K, D_0, \dots, D_{m-1})$ of $\mathrm{DF}^m \cup \{\alpha\}$, each system (§) in § 1 has a *unique* solution $(a^k_{i\,j} : k < m)$ in K. Say this model has extensions with underlying fields L_0 and L_1. By Fact 2.3, these extensions are substructures of $(L_0 L_1, \tilde{D}_0, \dots, \tilde{D}_{m-1})$, where each $(L_0 L_1, \tilde{D}_i)$ is a differential field. But then $\sum_{i<m} a^k_{i\,j} \cdot \tilde{D}_k$ and $[\tilde{D}_i, \tilde{D}_j]$ agree on a transcendence-basis of $L_0 L_1$, hence on $L_0 L_1$. □

3. The algebra of differential forms

Throughout this section, K is a field, and E is a vector space over K. We shall recall and build on some standard definitions. Some of the notions are used in geometry in case E is a tangent-space to a manifold. So, any differential-geometry book—possibly [S]—might serve as a reference. In our ultimate case of interest, E is a space of derivations.

Vector-spaces. First though, we assume no extra structure on the space E, but we do suppose that $\dim_K E$ is finite. Associated with E are the spaces $\mathrm{A}^p(E)$ of alternating p-multilinear maps $E^p \to K$. Let us call these maps the **p-forms** on E (although in geometry, a form is a certain *family* of such maps). In particular, $\mathrm{A}^0(E)$ is just K, and $\mathrm{A}^1(E)$ is E^*. The other spaces can be seen as duals as well. For example, by definition, $\mathrm{A}^2(E)$ comprises the *anti-symmetric* bilinear maps from $E \times E$ to K. Such a map induces a linear functional on the K-vector space with basis $E \times E$. Take the quotient of this space by the common kernel of all of the induced functionals. The result is the space denoted $E \wedge E$ or $\bigwedge^2(E)$, and $\mathrm{A}^2(E)$ is its dual. An element (D_0, D_1) of $E \times E$ has the image $D_0 \wedge D_1$ in $E \wedge E$.

We shall work with the pairings

$$((D_0,\dots,D_{m-1}),\theta)\mapsto(D_0,\dots,D_{m-1})\theta : E^p\times \mathrm{A}^p(E)\to K,$$

denoted thus by juxtaposition. We shall treat E and $\mathrm{A}^p(E)$ as left and right K-modules, respectively, so that there is no confusion when the elements of E are also derivations of K.

Suppose W is a subspace of E^*. Then the **kernel** of W—the common kernel of the elements of W—is a subspace $\ker W$ of E.

Fact 3.1. *The map $W\mapsto\ker W$ is an inclusion-reversing bijection between the sets of subspaces of E^* and E respectively. In particular, the co-dimension of W in E^* is the dimension of $\ker W$. Therefore $E^*/W\cong(\ker W)^*$.*

Proof. Say W has basis $(v^i : i<\ell)$, extending to the basis $(v^i : i<m)$ of E^*, which is dual to $(v_i : i<m)$. Then $(v_i : \ell\leqslant i<m)$ is a basis of $\ker W$, so $(v^i : \ell\leqslant i<m)$ can be understood as a basis both of $(\ker W)^*$ and of E^*/W. □

For each pair (p,q) of non-negative integers, there is a map

$$\wedge : \mathrm{A}^p(E)\times\mathrm{A}^q(E)\longrightarrow\mathrm{A}^{p+q}(E),$$

the exterior or **wedge-product**, a generic way to convert a pair (α,β) of forms into another form, $\alpha\wedge\beta$; this last form can be given by

$$(D_i : i<p+q)\alpha\wedge\beta=\sum_{\sigma\in\mathrm{sh}(p,q)}(D_{\sigma(i)} : i<p)\alpha\cdot(D_{\sigma(i)} : p\leqslant i<p+q)\beta\cdot\mathrm{sgn}(\sigma),$$

where $\mathrm{sh}(p,q)$ comprises the (p,q)-shuffle-permutations (that is, those permutations σ of $p+q$ such that $\sigma(i)<\sigma(j)$ when $i<j<p$ or $p\leqslant i<j$).

Fact 3.2. *The wedge-product is bilinear and associative and satisfies*

$$\beta\wedge\alpha=\alpha\wedge\beta\cdot(-1)^{pq}$$

when $\alpha\in\mathrm{A}^p(E)$ and $\beta\in\mathrm{A}^q(E)$; in particular, $x\wedge\alpha=\alpha\wedge x=\alpha\cdot x$ when $x\in K$.

The following gives an alternative definition of the spaces of forms.

Fact 3.3. *Suppose $(\theta^i : i<m)$ is a basis of E^*. Then the indexed set*

$$(\theta^{\sigma(0)}\wedge\dots\wedge\theta^{\sigma(p-1)} : \sigma(0)<\dots<\sigma(p-1)<m)$$

is a basis of $\mathrm{A}^p(E)$. So this space is trivial if $p>m$.

Letting $\mathrm{A}(E)$ be the direct sum of the spaces $\mathrm{A}^p(E)$, we have a **graded K-algebra**,

$$(\mathrm{A}(E), \{\mathrm{A}^p(E) : p \in \omega\}, +, -, \wedge, \{\cdot x : x \in K\}, 0, 1)$$

(which could be treated as a one-sorted structure).

Spaces of derivations. A consequence of the following was mentioned in the § 0.

Lemma 3.4. *Suppose E is a vector-space over the field K. Let D range over E, and let x range over K. Then the rule $D(\mathrm{d}\,x) = h(D)x$ determines a one-to-one correspondence between*

- *linear transformations $h : E \to \mathrm{Der}(K)$, and*
- *derivations $\mathrm{d} : K \to E^*$.*

If E is finite-dimensional, then h is injective just in case the range of d spans E^*.

Proof. The rule evidently gives a one-to-one correspondence between:

- maps h from E to the vector-space of maps $K \to K$, and
- maps d from K to the vector-space of maps $E \to K$.

From this correspondence, we claim two consequences:—that the map h is linear if and only if each map $\mathrm{d}\,x$ is linear,—and also that the map d is a derivation if and only if each map $h(D)$ is a derivation. These claims follow from consideration of the following diagrams:

$$\begin{array}{ccccc} h\left(\sum y^i \cdot D_i\right) x & = & \left(\sum y^i \cdot h(D_i)\right) x & = & \sum y^i \cdot (h(D_i)x) \\ \| & & & & \| \\ \left(\sum y^i \cdot D_i\right)(\mathrm{d}\,x) & & = & & \sum y^i \cdot (D_i(\mathrm{d}\,x)) \end{array}$$

$$\begin{array}{ccccc} D(\mathrm{d}(x+y)) & = & D(\mathrm{d}\,x + \mathrm{d}\,y) & = & D(\mathrm{d}\,x) + D(\mathrm{d}\,y) \\ \| & & & & \| \\ h(D)(x+y) & & = & & h(D)x + h(D)y \end{array}$$

$$\begin{array}{ccccc} D(\mathrm{d}(x \cdot y)) & = & D(\mathrm{d}\,x \cdot y + \mathrm{d}\,y \cdot x) & = & D(\mathrm{d}\,x) \cdot y + D(\mathrm{d}\,y) \cdot x \\ \| & & & & \| \\ h(D)(x \cdot y) & & = & & h(D)x \cdot y + h(D)y \cdot x \end{array}$$

The vertical equalities hold by the rule $D(\mathrm{d}\,x) = h(D)x$. In the top rows, the second equalities hold simply because $h(D)$ and $\mathrm{d}\,x$ are elements of vector-spaces of maps. Hence, of the remaining equalities in each diagram, one holds if and only if the other does. In the first diagram, this means that linearity of h and of $\mathrm{d}\,x$ are equivalent. The other two diagrams show that d and $h(D)$ are alike derivations, or not.

Finally, if h and d are corresponding maps as in the statement of the Lemma, then the following are equivalent:

- h is injective;
- if $D \neq 0$, then $h(D)x \neq 0$ for some x;
- if $D \neq 0$, then $D(\mathrm{d}\,x) \neq 0$ for some x;
- the span of the range of d has trivial kernel.

If $\dim_K E < \omega$, then by Fact 3.1, the last clause means $\{\mathrm{d}\,x : x \in K\}$ spans E^*. □

Suppose now that $(K, D_0, \dots, D_{m-1})$ is a differential field. Let E be the span over K of the D_i. As noted in § 0, the pair (K, E) determines the definable sets of $(K, D_0, \dots, D_{m-1})$. We may now refer to this pair also as a **differential field**. (Strictly speaking, the pair corresponds to an equivalence-class of differential fields, namely, the class comprising those models $(K, D'_0, \dots, D'_{m-1})$ of DF^m such that the D'_i span E; and the pair doesn't tell us what m is, unless it is known that the original D_i are independent, so that $m = \dim E$.)

By the Lemma, we have a derivation $\mathrm{d} : K \to E$, corresponding to the inclusion of E in $\mathrm{Der}(K)$, and so given by $D(\mathrm{d}\,x) = Dx$. Since E is closed under the bracket, this derivation d is just the first of the additive maps $\mathrm{d} : \mathrm{A}^p(E) \to \mathrm{A}^{p+1}(E)$ given by

$$(D_i : i < p+1)(\mathrm{d}\,\alpha) = \sum_{\sigma \in \mathrm{sh}(1,p)} D_{\sigma(0)}((D_{\sigma(i+1)} : i < p)\alpha) \cdot \mathrm{sgn}(\sigma) - \\ - \sum_{\sigma \in \mathrm{sh}(2,p-1)} ([D_{\sigma(0)}, D_{\sigma(1)}], D_{\sigma(2)}, \dots, D_{\sigma(p)})\alpha \cdot \mathrm{sgn}(\sigma).$$

Extending d additively, we get a group-endomorphism of $\mathrm{A}(E)$.

Fact 3.5. *If $\alpha \in \mathrm{A}^p(E)$ and $\beta \in \mathrm{A}^q(E)$, then*

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\,\alpha \wedge \beta + \mathrm{d}\,\beta \wedge \alpha \cdot (-1)^{pq}.$$

The following provides a further example of computation with forms.

Fact 3.6. *Suppose (K, E) is a differential field. Then $\mathrm{d}^2 = 0$ on $\mathrm{A}(E)$.*

Proof. Suppose $x, y \in K$. Then $(D_0, D_1)\mathrm{d}^2 x = D_0(D_1(\mathrm{d}\,x)) - D_1(D_0(\mathrm{d}\,x)) - [D_0, D_1]\mathrm{d}\,x = D_0 D_1 x - D_1 D_0 x - [D_0, D_1]x = 0$, so $\mathrm{d}^2 = 0$ on K. Also,

$$\mathrm{d}(\mathrm{d}\,x \cdot y) = \mathrm{d}^2\,x \wedge y + \mathrm{d}\,y \wedge \mathrm{d}\,x = \mathrm{d}\,y \wedge \mathrm{d}\,x,$$

so $\mathrm{d}^2(\mathrm{d}\,x \cdot y) = 0$. Since E^* is spanned by forms $\mathrm{d}\,x$, this means $\mathrm{d}^2 = 0$ on E^*. In general we have $\mathrm{d}^2(\alpha \wedge \beta) = \mathrm{d}^2\,\alpha \wedge \beta + \mathrm{d}^2\,\beta \wedge \alpha \cdot (-1)^{pq}$, so d^2 is zero on all forms, provided it is zero on 0- and 1-forms. □

We may now call d on $\mathrm{A}(E)$ a **derivation**. (Since $\mathrm{A}(E)$ is non-commutative, our original definition of 'derivation' does not apply.) A converse of our observations is included in the following.

Lemma 3.7. *Let E be a finite-dimensional subspace of* $\operatorname{Der} K$*, with corresponding derivation* $\mathrm{d} : K \to E^*$. *Then the following are equivalent.*

(0) (K, E) *is a differential field, that is, E is closed under the bracket.*
(1) *There is also an additive map* d *from E^* to $\mathrm{A}^2(E)$ such that $\mathrm{d}^2\,x = 0$ and $\mathrm{d}(\theta \cdot x) = \mathrm{d}\,\theta \cdot x + \mathrm{d}\,x \wedge \theta$ when $\theta \in E^*$ and $x \in K$.*

In either case, let t^i be chosen from K, and ∂_i from E, so that $(\mathrm{d}\,t^i : i < m)$ is a basis for E^* dual to $(\partial_i : i < m)$. Then

$$\mathrm{d}\,x = \sum_{i<m} \mathrm{d}\,t^i \cdot \partial_i x \tag{¶}$$

for all x in K. Also, the derivations ∂_i commute.

Proof. That (0) implies (1) is a consequence of Facts 3.5 and 3.6. For the converse, we compute

$$[\partial_i, \partial_j]t^k = \partial_i(\partial_j(\mathrm{d}\,t^k)) - \partial_j(\partial_i(\mathrm{d}\,t^k)) = \partial_i \delta^k_j - \partial_j \delta^k_i = 0.$$

This does not yet tell us that the derivation $[\partial_i, \partial_j]$ is zero, since we do not yet know that it is in E. We do have

$$\partial_j(\mathrm{d}\,x) = \partial_j x = \sum_{i<m} \partial_j(\mathrm{d}\,t^i) \cdot \partial_i x = \partial_j \sum_{i<m} \mathrm{d}\,t^i \cdot \partial_i x$$

for each j less than m, which by linearity means (¶) holds when $x \in K$. Since $\mathrm{d}^2\,t^j = 0$, we have

$$\begin{aligned}\mathrm{d}^2\,x = \mathrm{d} \sum_{j<m} \mathrm{d}\,t^j \cdot \partial_j x = \sum_{j<m} \mathrm{d}(\partial_j x) \wedge \mathrm{d}\,t^j = \\ = \sum_{j<m} \sum_{i<m} (\mathrm{d}\,t^i \cdot \partial_i \partial_j x) \wedge \mathrm{d}\,t^j = \sum_{i<j<m} \mathrm{d}\,t^i \wedge \mathrm{d}\,t^j \cdot [\partial_i, \partial_j]x.\end{aligned}$$

Since $\mathrm{d}^2 x$ is zero generally, and the forms $\mathrm{d}\,t^i \wedge \mathrm{d}\,t^j$ are linearly independent by Fact 3.3, we have $[\partial_i, \partial_j]x = 0$. Hence E is closed under the bracket. □

Let (K, E) be a differential field. A subset W of $\mathrm{A}(E)$ generates the following two subspaces.

- $\mathcal{I}(W)$ is the (two-sided) ideal generated by W.
- $\mathcal{D}(W)$ is the differential ideal generated by W, that is, $\mathcal{D}(W)$ is the smallest ideal of $\mathrm{A}(E)$ that includes W and is closed under application of d.

When W is a subspace of E^*, we shall work with the following subspaces of $\mathrm{A}^2(E)$.

- $E^* \wedge W$ is spanned by $\{\eta \wedge \theta : (\eta, \theta) \in E^* \times W\}$.
- $\mathrm{d}\,W$ is spanned by $\{\mathrm{d}\,\theta : \theta \in W\}$.

Lemma 3.8. *Suppose (K, E) is a differential field, and let W be a subspace of E^*. Then*

(0) $E^* \wedge W = \mathcal{I}(W) \cap \mathrm{A}^2(E)$;
(1) $\mathrm{d}\,W = \mathcal{D}(W) \cap \mathrm{A}^2(E)$;
(2) $\mathcal{D}(W) = \mathcal{I}(W \cup \mathrm{d}\,W)$; *and*
(3) $\mathcal{D}(W) \cap \mathrm{A}^p(E) = \mathcal{I}(\mathrm{d}\,W) \cap \mathrm{A}^p(E)$

when $p > 1$. Hence also

$$\mathcal{I}(W) = \mathcal{D}(W) \iff E^* \wedge W = \mathrm{d}\,W.$$

Proof. Equation (0) is clear. For (1), note that $\mathcal{D}(W) \cap \mathrm{A}^2(E)$ is the subspace of $\mathrm{A}^2(E)$ generated by $\mathrm{d}\,W$ and $E^* \wedge W$. Let E^* have a basis $(\mathrm{d}\,t^i : i < m)$ as in Lemma 3.7. Then $\mathrm{d}\,t^i \wedge \theta = \mathrm{d}(\theta \cdot t^i) - \mathrm{d}\,\theta \cdot t^i$, so this form is in $\mathrm{d}\,W$ if $\theta \in W$. But all elements of $E^* \wedge W$ are linear combinations of forms $\mathrm{d}\,t^i \wedge \theta$ such that $\theta \in W$. Hence $E^* \wedge W \subseteq \mathrm{d}\,W$, so (1) holds. Equation (2) holds because $\mathrm{d}^2 = 0$. Equation (3) is now a consequence of (2). The forward direction of the equivalence is by (0) and (1); the reverse, by (2). □

I am naming the following to indicate that the algebraic fact has been in play for a long time. I don't have a reference for the statement at this level of abstraction.

Lemma 3.9 (Frobenius)**.** *Suppose (K, E) is a differential field, and let W be a subspace of E^*. Then $\ker W$ is closed under the bracket if and only if $\mathcal{D}(W) = \mathcal{I}(W)$.*

Proof. From the inclusion of $\ker W$ in $\mathrm{Der}(K)$, we have a derivation $\mathrm{d} : K \to (\ker W)^*$ by Lemma 3.4. By Lemmas 3.7 and 3.8, we have to show that the corresponding map $\mathrm{d} : (\ker W)^* \to \mathrm{A}^2(\ker W)$ is well-defined just in case $\mathrm{d}\, W = E^* \wedge W$.

By Fact 3.1, we have an isomorphism $E^*/W \to (\ker W)^*$. For the same reasons, there is an isomorphism

$$\mathrm{A}^2(E)/(E^* \wedge W) \longrightarrow \mathrm{A}^2(\ker W).$$

The map $\mathrm{d} : (\ker W)^* \to \mathrm{A}^2(\ker W)$ exists just in case the corresponding map

$$\begin{aligned} E^*/W &\longrightarrow \mathrm{A}^2(E)/(E^* \wedge W) \\ \theta + W &\longmapsto \mathrm{d}\,\theta + E^* \wedge W. \end{aligned}$$

is well-defined; the latter map is well-defined just in case $\mathrm{d}\, W \subseteq E^* \wedge W$. □

Extensions. Now let (K, E) be a differential field, and let L be an extension of K. We want to understand extensions of (K, E) with underlying field L. By Lemma 2.2, we can define

$$\mathrm{Der}(L/E)$$

to be the pre-image of $L \otimes_K E$ under the restriction-map $D \mapsto D|_K : \mathrm{Der}(L) \to L \otimes_K \mathrm{Der}(K)$.

Lemma 3.10. $\dim_L \mathrm{Der}(L/E) = \dim_K E + \mathrm{tr.\ deg}(L/K)$.

Proof. The dimension of the kernel of the surjection $D \mapsto D|_K : \mathrm{Der}(L/E) \to L \otimes_K E$ is precisely the transcendence-degree of L over K, by Fact 2.1. □

Assume for now that L has finite transcendence-degree over K, so that $\mathrm{Der}(L/E)$ is finite-dimensional. We define

$$\Omega_{L/E}$$

to be the graded algebra $\mathrm{A}(\mathrm{Der}(L/E))$ equipped with the derivation d. The elements of the component spaces $\Omega^p_{L/E}$ can be called **differential p-forms**. The surjection of $\mathrm{Der}(L/E)$ onto $L \otimes_K E$ induces an injection of $E^* \otimes_K L$ into $\Omega^1_{L/E}$ that simply takes $\mathrm{d}\, x$ to $\mathrm{d}\, x$.

Suppose $\tilde{E} \subseteq \mathrm{Der}(L/E)$. For the pair $(L, \tilde{E})$ to be an extension of (K, E), there are two necessary conditions, which together are sufficient:

(0) $(L, \tilde{E})$ is a differential field;

(1) the composition $\tilde{E} \rightarrowtail \mathrm{Der}(L/E) \twoheadrightarrow L \otimes_K E$ is an isomorphism.

We can rewrite the latter condition in terms of the dual spaces. First, $\tilde{E}$ is precisely $\ker W$ for some subspace W of $\Omega^1_{L/E}$. So the conditions become:

(0) $\ker W$ is closed under the bracket;

(1) the composition $E^* \otimes_K L \rightarrowtail \Omega^1_{L/E} \twoheadrightarrow (\ker W)^*$ is an isomorphism.

We can rewrite Condition (0) using the Lemma 3.9. In Condition (1), we know by Fact 3.1 that the kernel of the second map is W. So the conditions are:

(0) $\mathrm{d}\, W = \Omega^1_{L/E} \wedge W$;

(1) the map $E^* \otimes_K L \to \Omega^1_{L/E}/W$ is an isomorphism.

The latter condition can be split in two:

(1) (a) $E^* \otimes_K L$ and W are **linearly disjoint** (have trivial intersection in $\Omega^1_{L/E}$);

(b) the co-dimension of W in $\Omega^1_{L/E}$ is $\dim_K E$.

Now suppose L has infinite transcendence-degree μ over K. We still have the derivation d from L to $\mathrm{Der}(L/E)^*$. But now we *define* $\Omega^1_{L/E}$ to be the span of the range of d. If $(X^j : j < \mu)$ is a transcendence-basis of L over K, then $(\mathrm{d}\, X^j : j < \mu)$ is a basis of $\Omega^1_{L/E}$, and we can define the other spaces $\Omega^p_{L/E}$ so that Fact 3.3 holds. Then the other relevant facts and lemmas remain true in this context, except that the map $W \mapsto \ker W$ in Fact 3.1 must be understood as taking subspaces of *finite* co-dimension in $\Omega^1_{L/E}$ to finite-dimensional subspaces of $\mathrm{Der}(L/E)$. In particular, we can repeat the proof of Lemma 3.9, and use Lemma 3.8, to get the following.

Theorem 3.11 (Frobenius). *Suppose (K, E) is a differential field, and L is an extension of K, and $\Omega^1_{L/E} = W \oplus (E^* \otimes_K L)$ for some subspace W of $\Omega^1_{L/E}$. Then $(L, \ker W)$ is a differential field extending (K, E) just in case $\mathrm{d}\, W = \Omega^1_{L/E} \wedge W$.*

We shall use this theorem to establish differential fields in which given differential systems have solutions. In this context, in the Introduction, we defined what it means for a subspace of $\Omega^1_{L/E}$ to be integrable. Now we can give a coordinate-free definition.

Lemma 3.12. *Let (K, E) be a differential field, and let W be a subspace of $\Omega^1_{L/E}$. Then W is integrable just in case (K, E) has an extension $(L', \tilde{E})$ such that $L \subseteq L'$, and $W \otimes_L L'$ is included in the kernel of the map*

$$\alpha \longmapsto (D \mapsto (D|_L)\alpha) : \Omega^1_{L/E} \longrightarrow \tilde{E}^*.$$

Proof. We just have to verify that the map on $\Omega^1_{L/E}$ described here is the same as the map Φ in § 0. The map here is well-defined, since the restriction of an element of $\tilde{E}$ to L is in $L' \otimes_K \mathrm{Der}(L/E)$ (since its further restriction to K is in E). Let t^i in K and ∂_i in E be as in Lemma 3.7. Each ∂_i extends uniquely to some $\tilde{\partial}_i$ in $\tilde{E}$, and these compose a basis as well. The conversion of α in $\Omega^1_{L/E}$ to $D \mapsto (D|_L)\alpha$ in $\tilde{E}^*$ can be described as the linear map taking $\mathrm{d}\, x$ to the linear map

$$\tilde{\partial}_j \longmapsto \tilde{\partial}_j x.$$

But $\tilde{\partial}_j x$ is just $\tilde{\partial}_j \sum_{i<m} \mathrm{d}\, t^i \cdot \tilde{\partial}_j x$. So the conversion of α is just the linear map Φ. □

4. Differential equations

We want to characterize the existentially closed differential fields. These are models $(K, D_0, \dots, D_{m-1})$ of DF^m (for some m) in which hold all **primitive** sentences that hold in an extension. A primitive sentence has the form

$$\exists \mathbf{x}\, \phi(\mathbf{x}),$$

where ϕ is a conjunction of literals, that is, atomic and negated atomic formulas. In the present case, the literals are equations and inequations of **terms** in the language of fields with m derivations and constants from K. We can identify two terms t_0 and t_1 if

$$\mathrm{DF}^m_\forall \cup \mathrm{diag}(K, D_0, \dots, D_{m-1}) \models \forall \mathbf{x}\, t_0(\mathbf{x}) = t_1(\mathbf{x}).$$

The resulting equivalence-classes of terms can be called **differential polynomials** over $(K, D_0, \dots, D_{m-1})$. We cannot turn the set of these differential polynomials into a model of $\mathrm{DF}^m_\forall$ without having the axiom σ given in § 0. We may have σ by fiat (as in [Y, § 1.3]). We may also have σ because the D_i are independent, as in the following. (This point is not trivial, since linear independence is not always preserved under passage to a larger scalar field: $\mathbb{C}$ has two dimensions as a real vector-space, but one as a complex.)

Lemma 4.1. *Let $(K, D_0, \dots, D_{m-1})$ be a differential field in which the D_i are linearly independent, and let T be the theory*

$$\mathrm{DF}^m_\forall \cup \mathrm{diag}(K, D_0, \dots, D_{m-1}).$$

Then there are $a^k_{i\,j}$ in K such that axiom σ is in T. Hence the differential polynomials over $(K, D_0, \dots, D_{m-1})$ compose a model of T. The reduct of this model to the language of rings is the polynomial-ring over K in the derivatives

$$D_0^{\sigma(0)} \cdots D_{m-1}^{\sigma(m-1)} v,$$

where $\sigma \in {}^m\omega$, and v ranges over the variables in the logical language.

Proof. Let E be the span of the D_i, and let ∂_i in E and t^i in K be as chosen in Lemma 3.7. Then $\partial_i t^j = \delta_i^j$. Hence $\{\tilde{\partial}_i : i < m\}$ is independent for all extensions $\tilde{\partial}_i$ of the ∂_i, and the same is true of the D_i. Hence there are $a_{i\,j}^k$ in K as desired. □

We may assume that the D_i *are* independent, because of the following.

Lemma 4.2. *In any existentially closed model of* DF^m*, the derivations* D_i *are linearly independent, and the underlying field is algebraically closed.*

Proof. By Lemmas 2.4 and 2.6, it is enough to show that any differential field $(K, D_0, \dots, D_{m-1})$ has an extension in which the D_i are linearly independent. Let E be the span over K of the D_i. We may assume that the tuple $(D_i : i < \ell)$ is a basis of E. By Lemma 3.7, the space E also has a basis $(\partial_i : i < \ell)$ of commuting derivations. Hence

$$D_i = \sum_{j<\ell} a_i^j \cdot \partial_j$$

for some a_i^j in K, for each i less than m. When $\ell \leqslant i < m$, define ∂_i to be the zero-derivation on K. We can extend all of the derivations ∂_i, *as commuting derivations,* to the field $K(X_\sigma : \sigma \in {}^m\omega)$ in an obvious way, so that $\partial_i X_\sigma = X_{\sigma+\mathbf{i}}$, where $\mathbf{i}$ is the characteristic function of $\{i\}$. On this field, define

$$\tilde{D}_i = \begin{cases} \sum_{j<\ell} a_i^j \cdot \partial_j, & \text{if } i < \ell; \\ \sum_{j<\ell} a_i^j \cdot \partial_j + \partial_i, & \text{if } \ell \leqslant i < m. \end{cases}$$

The result is an extension $(L, \tilde{D}_0, \dots, \tilde{D}_{m-1})$ of the original differential field in which the derivations are linearly independent. □

In short, the existentially closed models of DF^m are models of $\{\alpha\} \cup \mathrm{ACF}$, where α is as in Lemma 2.6 (and ACF is the theory of algebraically closed fields).

Theorem 4.3. *The theory* DF^m *has a model-companion,* DCF^m*. The completions of* DCF^m *are in one-to-one correspondence with the models of* $\mathrm{DF}^m_\forall \cup \{\sigma\}$ *generated by the constants* $a_{i\,j}^k$*. Such models are countably numerous, so* DCF^m *has countably many completions. Each completion of* DCF^m *is* ω*-stable.*

Proof. By [McG, Corollary 3.1.8], the theory of fields with m commuting derivations has a model-companion, m-DCF. The models of m-DCF are existentially closed models of DF^m. Conversely, if $(K, D_0, \dots, D_{m-1})$ is an existentially closed model of DF^m, then by Lemma 4.2, we can pick t^i and ∂_i as in Lemma 3.7. The derivations

∂_i are term-definable in $(K, D_0, \dots, D_{m-1})$ with parameters from K, and likewise the D_i are term-definable in $(K, \partial_0, \dots, \partial_{m-1})$. Moreover, an extension of one of the structures determines, by the same terms, an extension of the other. Hence $(K, \partial_0, \dots, \partial_{m-1})$ is a model of m-DCF.

Moreover, as a model-complete theory, m-DCF has $\forall\exists$ axioms. If $\forall \mathbf{x}\, \exists \mathbf{y}\, \phi$ is one of them, then in ϕ we can replace each instance of ∂_i with $\sum_{k<m} u_i^k D_k$, getting a formula $\hat{\phi}$; then the sentence

$$\forall \mathbf{x}\, \forall \mathbf{t}\, \forall \mathbf{u}\, \exists \mathbf{y}\, \exists z \Big(\bigwedge_{i<j<m} \sum_{k<m} u_i^k D_k t^j = \delta_i^j \to \\ \to \hat{\phi} \vee \bigvee_{i<j<m} \sum_{k<m} u_i^k D_k \Big(\sum_{\ell<m} u_j^\ell D_\ell z \Big) \neq \sum_{\ell<m} u_j^\ell D_\ell \Big(\sum_{k<m} u_i^k D_k z \Big) \Big)$$

holds in existentially closed models of DF^m. The set of all of these sentences—together with $\mathrm{DF}^m \cup \{\alpha\} \cup \mathrm{ACF}$—axiomatizes the class of existentially closed models of DF^m. Thus we have axioms for DCF^m.

As noted in § 0, the theory $\mathrm{DCF}^m \cup \{\sigma\}$ admits elimination of quantifiers. Let $(K, D_0, \dots, D_{m-1}; a_{i\,j}^k : i < j < m \wedge k < m)$ be a model of this theory, and let $\mathcal{M}$ be a substructure. Then the theory

$$(T) \qquad \mathrm{DCF}^m \cup \{\sigma\} \cup \operatorname{diag} \mathcal{M}$$

is complete. This is one of the theories LDCF_0 of [Y, § 5.1]. (For this to be literally true, M should have a sub*field* containing the constants $a_{i\,j}^k$; but the distinction is unimportant. See [Y, § 0.2].) Therefore the theory T is ω-stable, by [Y, Corollary 5.3]—or by [McG, Theorem 3.2.1], since we can rewrite types in terms of commuting derivations.

By taking the sentences of T that are in the signature of DF^m, we get the complete theory of $(K, D_0, \dots, D_{m-1})$. This theory must also be ω-stable. We may assume that $\mathcal{M}$ is *generated* by the constants $a_{i\,j}^k$. Hence the complete theory of $(K, D_0, \dots, D_{m-1})$ is determined, *modulo* DCF^m, by the type of the $a_{i\,j}^k$; but the ω-stability of DCF^m implies that there are only countably many such types. □

The remainder of this section and this paper is devoted to showing how to prove Theorem 4.3 using differential forms. Henceforth let $(K, D_0, \dots, D_{m-1})$ be a model of $\mathrm{DF}^m \cup \{\alpha\} \cup \mathrm{ACF}$. The span E of the derivations D_i has a commuting basis $(\partial_i : i < m)$ whose dual is $(\mathrm{d}\, t^i : i < m)$ for some t^i in K.

If $\exists \mathbf{x}\,\phi(\mathbf{x})$ is a primitive sentence over (K, E), then, by Lemma 4.1, we can write ϕ in the form

$$(\|) \qquad \bigwedge_f f(\partial_\sigma \mathbf{X} : \sigma \leqslant \tau) = 0 \wedge g(\partial_\sigma \mathbf{X} : \sigma \leqslant \tau) \neq 0,$$

where g and the f are ordinary polynomials, taken from $K[\partial_\sigma \mathbf{X} : \sigma \leqslant \tau]$ for some τ in ${}^m\omega$. So, ϕ is just a **differential system** in $\mathbf{X}$ over (K, E). The system ϕ is **consistent** just in case there is an extension $(L', \tilde{E})$ of (K, E) in which $\phi(\mathbf{a})$ holds for some tuple $\mathbf{a}$ from L'. Then $\mathbf{a}$ is a **solution** of ϕ.

We want to understand when ϕ is consistent. To do so, we seek to rewrite the sentence $\exists \mathbf{x}\,\phi(\mathbf{x})$ in a more tractable form. As noted in the Introduction, we shall ultimately (in Lemma 4.5) be able to write ϕ as a subspace W of $\Omega^1_{L/E}$ for some finitely generated extension L of K. There (and in Lemma 3.12) we defined what it means for W to be *integrable*. Let us say also that W is **eliminable** if it vanishes under some *place* of L over K onto K. Let us be clear about what this means:

A place of a field $K(\mathbf{a})$ over K is a (well-defined) map

$$f(\mathbf{a}) \longmapsto f(\mathbf{b}) : K(\mathbf{a}) \longrightarrow K(\mathbf{b}) \cup \{\infty\}.$$

(Correspondingly, $\mathbf{b}$ is a specialization of $\mathbf{a}$.) There is also a coordinate-free account of places. (See also [L, ch. 1].) A place of L over K corresponds to a *valuation-ring* $\mathfrak{O}$ of L over K. By definition, $\mathfrak{O}$ includes K and contains the reciprocal of every element of L that it does not contain. Therefore $\mathfrak{O}$ has a unique maximal ideal $\mathfrak{m}$, comprising the non-units of $\mathfrak{O}$; and distinct elements of K are incongruent *modulo* $\mathfrak{m}$. The corresponding place is the map $L \to \mathfrak{O}/\mathfrak{m} \cup \{\infty\}$ taking x to $x + \mathfrak{m}$ if $x \in \mathfrak{O}$, and otherwise to ∞. The *residue-field* $\mathfrak{O}/\mathfrak{m}$ can be treated as an extension of K.

For the ring $\mathfrak{O}$, we can define the $\mathfrak{O}$-module $\operatorname{Der}(\mathfrak{O}/E)$ just as we defined the L-vector-space $\operatorname{Der}(L/E)$. As a free module, $\operatorname{Der}(\mathfrak{O}/E)$ has a dual, $\Omega^1_{\mathfrak{O}/E}$, which is naturally a submodule of $\Omega^1_{L/E}$ into which d maps $\mathfrak{O}$.

Lemma 4.4. *Suppose (K, E) is a differential field, and suppose L is a finitely generated extension with valuation-ring $\mathfrak{O}$ over K. Let $\mathfrak{m}$ be the maximal ideal of $\mathfrak{O}$. If $\tilde{L}$ is the residue-field $\mathfrak{O}/\mathfrak{m}$, and $\tilde{x}$ is $x + \mathfrak{m}$ when $x \in \mathfrak{O}$, then the formula*

$$\mathrm{d}\,x \cdot y \longmapsto \mathrm{d}\,\tilde{x} \cdot \tilde{y}$$

determines a well-defined additive map of $\Omega^1_{\mathfrak{O}/E}$ onto $\Omega^1_{\tilde{L}/E}$.

Proof. Suppose that, for all $\tilde{D}$ in $\operatorname{Der}(\tilde{L}/E)$, there is D in $\operatorname{Der}(\mathfrak{O}/E)$ such that

$$(**) \qquad Dx + \mathfrak{m} = \tilde{D}\tilde{x}$$

for all x in $\mathfrak{O}$. Then, for all x and y in $\mathfrak{O}$, we have

$$\begin{aligned}\tilde{D}(\mathrm{d}\,\tilde{x}\cdot\tilde{y}) &= \tilde{D}\tilde{x}\cdot\tilde{y}\\ &= (Dx+\mathfrak{m})(y+\mathfrak{m})\\ &= Dx\cdot y+\mathfrak{m}\\ &= D(\mathrm{d}\,x\cdot y)+\mathfrak{m}.\end{aligned}$$

So, for the map $\mathrm{d}\,x\cdot y\mapsto \mathrm{d}\,\tilde{x}\cdot\tilde{y}$ to be well-defined, it is enough that, given $\tilde{D}$, we can find D. We have $\tilde{D}|_K\in\tilde{L}\otimes_K E$, which means

$$\tilde{D}|_K=\sum_{k<m}(x^k+\mathfrak{m})\cdot\partial_k$$

for some x^k in $\mathfrak{O}$; so we can define $D|_K$ to be $\sum_{k<m}x^k\cdot\partial_k$. Now we have to extend the definition to all of $\mathfrak{O}$. So, for each x in some transcendence-basis of $\mathfrak{O}$ over K, choose Dx from the coset $\tilde{D}(x+\mathfrak{m})$ of $\mathfrak{m}$. So $(**)$ holds for x in K, or in a transcendence-basis of $\mathfrak{O}$ over K; hence it holds for all x in $\mathfrak{O}$. □

We can now say precisely that a subspace W of $\Omega^1_{L/E}$ is *eliminable* just in case there is a valuation-ring $\mathfrak{O}$ of L over K such that $\mathfrak{O}/\mathfrak{m}=K$ and such that W has a basis in $\Omega^1_{\mathfrak{O}/E}$ that is sent to zero by the induced map into E^*.

Lemma 4.5. *The existentially closed models of* DF^m *are just the differential fields* $(K,D_0,\dots,D_{m-1})$ *such that:*

(0) *K is algebraically closed;*
(1) *the span E over K of the derivations D_i has dimension m;*
(2) *for any finitely generated extension L of K, every integrable subspace W of $\Omega^1_{L/E}$ is eliminable.*

The last condition can be weakened by requiring W to have, modulo $E^*\otimes_K L$, *a basis of the form* $(\mathrm{d}\,X^k : k<r)$.

Proof. By Lemma 4.2, it remains to check that Condition (2) is necessary and, with Conditions (0) and (1), sufficient.

Let $\phi(\mathbf{X})$ be a consistent differential system over (K,E), as (‖) above. Say $\mathbf{a}$ is a solution of ϕ in some extension of (K,E). We shall show that, for some other tuple $\mathbf{b}$ from this extension, there is a certain integrable subspace W of $\Omega^1_{L/E}$, where $L=K(\mathbf{a},\mathbf{b})$. If W is an arbitrary integrable subspace of $\Omega^1_{L/E}$, then we shall derive a consistent differential system ψ over (K,E), where ψ has a solution in K

if and only if W is integrable. If W has been derived from ϕ, then solubility in K of ψ will imply the same for ϕ.

First, by means of the 'Rabinowitsch trick' ([H, § 8.1]), we may assume that, in ϕ, the inequation $g(\partial_\sigma \mathbf{X} : \sigma \leqslant \tau) \neq 0$ is trivial (say, is $0 \neq 1$); otherwise, replace it with $y \cdot g(\partial_\sigma \mathbf{X} : \sigma \leqslant \tau) + 1 = 0$. Now we can write ϕ as

$$\bigwedge_f f(\mathbf{X}_\sigma : \sigma \leqslant \tau) = 0 \wedge \bigwedge_{i<m} \bigwedge_{\sigma+\mathbf{i} \leqslant \tau} \partial_i \mathbf{X}_\sigma = \mathbf{X}_{\sigma+\mathbf{i}},$$

where $\mathbf{i}$ is the characteristic function of $\{i\}$ on m, and where each equation $\partial_i \mathbf{X}_\sigma = \mathbf{X}_{\sigma+\mathbf{i}}$ stands for a conjunction of equations $\partial_i X^j_\sigma = X^j_{\sigma+\mathbf{i}}$. We shall want all (first) derivatives of variables in the f to appear in equations on the right. So let us introduce new variables $\mathbf{X}_{\sigma+\mathbf{i}}$, where $\sigma \leqslant \tau$, but $\sigma + \mathbf{i} \nleqslant \tau$. We now have another system equivalent to ϕ, namely

$$\bigwedge_f f(\mathbf{X}_\sigma : \sigma \leqslant \tau) = 0 \wedge \bigwedge_{\sigma \leqslant \tau} \bigwedge_{i<m} \partial_i \mathbf{X}_\sigma = \mathbf{X}_{\sigma+\mathbf{i}}.$$

(We said in the Introduction that this was the only kind of system we need consider.) Now we can understand each of the conjunctions $\bigwedge_{i<m} \partial_i X^j_\sigma = X^j_{\sigma+\mathbf{i}}$ as a single equation,

$$\sum_{i<m} \mathrm{d}\, t^i \cdot \partial_i X^j_\sigma = \sum_{i<m} \mathrm{d}\, t^i \cdot X^j_{\sigma+\mathbf{i}},$$

equivalently, $\mathrm{d}\, X^j_\sigma - \sum_{i<m} \mathrm{d}\, t^i \cdot X^j_{\sigma+\mathbf{i}} = 0$. We have assumed that $\mathbf{a}$ satisfies ϕ. We can consider $\mathbf{a}$ as the tuple of certain tuples $\mathbf{a}_\sigma$, where $\sigma \leqslant \tau$. Then of course $\partial_i \mathbf{a}_\sigma = \mathbf{a}_{\sigma+\mathbf{i}}$ (in the appropriate extension of (K, E)), if $\sigma + \mathbf{i} \leqslant \tau$. But now let $\mathbf{b}$ be the tuple of derivatives $\partial_i \mathbf{a}_\sigma$ such that $\sigma \leqslant \tau$, but $\sigma + \mathbf{i} \nleqslant \tau$. Then $(\mathbf{a}, \mathbf{b})$ is a solution of our latest systems. Let $\mathfrak{O}$ be the valuation-ring of $K(\mathbf{X}, \mathbf{Y})$ corresponding to the place $f(\mathbf{X}, \mathbf{Y}) \mapsto f(\mathbf{a}, \mathbf{b})$. Then $K[\mathbf{X}, \mathbf{Y}] \subseteq \mathfrak{O}$, so the forms $\mathrm{d}\, X^j_\sigma - \sum_{i<m} \mathrm{d}\, t^i \cdot X^j_{\sigma+\mathbf{i}}$ are in $\Omega^1_{\mathfrak{O}/E}$. By Lemma 4.4, the forms have well-defined images in $\Omega^1_{L/E}$, where $L = K(\mathbf{a}, \mathbf{b})$. These images span an integrable subspace W, which has a basis as in the weak form of Condition (2). (If we rewrite this basis in terms of ∂_i instead of d, then we shall have a system like $(*)$ in the Introduction.) If W is eliminable, then the appropriate place of L into K determines a solution of ϕ in K. This proves sufficiency.

Conversely, suppose W is an integrable subspace of $\Omega^1_{L/E}$, where L is some finitely generated extension $K(\mathbf{c})$ of K. Since it is integrable, W contains no non-trivial linear combination of the forms $\mathrm{d}\, t^i$ alone. There are some linearly independent components X^j of $\mathbf{c}$ such that, along with the $\mathrm{d}\, t^i$, the forms $\mathrm{d}\, X^j$ span $\Omega^1_{L/E}$.

Hence W has a basis consisting of some forms

$$\sum_j \mathrm{d}\,X^j \cdot g^j(\mathbf{c}) - \sum_{i<m} \mathrm{d}\,t^i \cdot f_i^j(\mathbf{c}),$$

where $f_i^j, g^j \in K[\mathbf{X}]$. Integrability of W implies consistency of the system consisting of the equations $\sum_j g_j \cdot \partial_i X^j = f_i^j$, along with the equations $h = 0$, where the h generate the ideal of polynomials in $K[\mathbf{X}]$ that are zero at $\mathbf{c}$. This system has a solution $\mathbf{d}$ in K if and only if W is eliminable by means of the place $f(\mathbf{c}) \mapsto f(\mathbf{d})$. □

To Theorem 3.11, we can give the following.

Corollary 4.6 (Frobenius)**.** *Let (K, E) be a differential field, let L be a finitely generated extension of K, and let W be a subspace of $\Omega^1_{L/E}$. If*

- *W and $E^* \otimes_K L$ are linearly disjoint, and*
- *$\mathrm{d}\,W = \Omega^1_{L/E} \wedge W$,*

then W is integrable.

Proof. The field L has a transcendence-basis $(X^k : k < n)$ over K. The images of the forms $\mathrm{d}\,X^k$ in $\Omega^1_{L/E}/(E^* \otimes_K L)$ compose a basis of this space. Under the stated conditions, W embeds in this space. So W has a basis

$$(\mathrm{d}\,X^\ell - \sum_{r \leqslant k<n} \mathrm{d}\,X^k \cdot a_k^\ell - \theta^\ell : \ell < r)$$

for some r less than n, where $a_k^\ell \in L$ and $\theta^\ell \in E^* \otimes_K L$. Let W' be the subspace of $\Omega^1_{L/E}$ spanned by

$$\{\mathrm{d}\,X^\ell - \sum_{r \leqslant k<n} \mathrm{d}\,X^k \cdot a_k^\ell - \theta^\ell : \ell < r\} \cup \{\mathrm{d}\,X^k : r \leqslant k < n\}.$$

Then $\Omega^1_{L/E}$ is the internal direct sum of W' and $E^* \otimes_K L$, and moreover, $\mathrm{d}\,W' = \Omega^1_{L/E} \wedge W'$, since $\mathrm{d}^2\,X^k = 0$. Hence $(L, \ker W')$ extends (K, E) by Theorem 3.11, so we are done by Lemma 3.12, since the kernel of the restriction-map $\Omega^1_{L/E} \to (\ker W')^*$, being W', includes W. □

The converse of this Corollary is false, as is shown by Example 0.2. The conditions can be adjusted to allow conversion; the result is the following.

Theorem 4.7. *Let (K, E) be a differential field, where E has basis $(\partial_i : i < m)$ whose dual is $(\mathrm{d}\,t^i : i < m)$ for some t^i in K. Let L be an extension of K with transcendence-basis $(X^k : k < n)$. Let W be a subspace of $\Omega^1_{L/E}$ with basis*

$$(\mathrm{d}\,X^\ell - \theta^\ell : \ell < r)$$

for some r less than n, where $\theta^\ell = \sum_{i<m} \mathrm{d}\, t^i \cdot u_i^\ell$ for some u_i^ℓ in L. Then the following are equivalent:

(0) *W is integrable.*

(1) *When the derivations ∂_i are extended to L so that*

$$\partial_i X^k = \begin{cases} u_i^k, & \text{if } k < r; \\ Y_i^k, & \text{if } r \leqslant k < n; \end{cases}$$

then the linear system

$$(\dagger\dagger) \qquad \bigwedge_{r \leqslant \ell < n} \bigwedge_{i<j<m} \partial_i u_j^\ell = \partial_j u_i^\ell$$

in the tuple $(Y_i^k : i < m \wedge r \leqslant k < n)$ of variables is consistent.

(2) *$\mathrm{Der}(L/E)$ has a subspace $\tilde{E}$, isomorphic to $L \otimes_K E$ under the restriction-map, such that $\tilde{E} \subseteq \ker W$ and $\tilde{E} \wedge \tilde{E} \subseteq \ker(\mathrm{d}\, W)$.*

(3) *The ideal $\mathcal{D}(W)$ and the subspace $\mathrm{A}(E) \otimes_K L$ of $\Omega_{L/E}$ are linearly disjoint.*

Proof. (0) $\Longrightarrow$ (1). Condition (0) is that there are commuting extensions $\tilde{\partial}_i$ of the ∂_i in $\mathrm{Der}(L')$, for some extension L' of L, such that $\tilde{\partial}_i X^\ell = u_i^\ell$ when $\ell < r$. Since the $\tilde{\partial}_i$ commute, we get the solution $(\tilde{\partial}_i X^k : i < m \wedge r \leqslant k < n)$ to $(\dagger\dagger)$.

(1) $\Longrightarrow$ (2). Suppose the linear system $(\dagger\dagger)$ is consistent. Then it has a solution $(u_i^k : i < m \wedge r \leqslant k < n)$ in L. Let $\tilde{\partial}_i$ be the extension of ∂_i in $\mathrm{Der}(L)$ such that $\tilde{\partial}_i X^k = u_i^k$ whenever $k < n$. Let $\tilde{E}$ be the span of the $\tilde{\partial}_i$. Since

$$\tilde{\partial}_i(\mathrm{d}\, X^\ell - \theta^\ell) = \tilde{\partial}_i X^\ell - u_i^\ell = 0,$$

we have $\tilde{E} \subseteq \ker W$. Since $\mathrm{d}(\mathrm{d}\, X^\ell - \theta^\ell) = -\,\mathrm{d}\,\theta^\ell = \sum_{i<m} \mathrm{d}\, t^i \wedge \mathrm{d}\, u_i^\ell$, we have

$$(\tilde{\partial}_i, \tilde{\partial}_j)\, \mathrm{d}\,\theta^\ell = \tilde{\partial}_i(\mathrm{d}\, u_j^\ell) - \tilde{\partial}_j(\mathrm{d}\, u_i^\ell) = \tilde{\partial}_i u_j^\ell - \tilde{\partial}_j u_i^\ell = 0,$$

so $\tilde{E} \wedge \tilde{E} \subseteq \ker(\mathrm{d}\, W)$.

(2) $\Longrightarrow$ (3). Condition (2) is that for each ∂_i there is an extension $\tilde{\partial}_i$ in $\mathrm{Der}(L/E)$ such that

$$\tilde{\partial}_i \alpha = 0 = (\tilde{\partial}_i, \tilde{\partial}_j)\beta$$

for all α in W and all β in $\mathrm{d}\, W$. But W and $\mathrm{d}\, W$ generate $\mathcal{D}(W)$ as an ordinary ideal (by Lemma 3.8). So Condition (2) means

$$(\tilde{\partial}_{i_0}, \dots, \tilde{\partial}_{i_{p-1}})\theta = 0$$

for all i_ℓ in m and all p-forms θ in $\mathcal{D}(W)$, for each positive integer p. By Fact 3.3, each nonzero element of $\mathrm{A}^p(E) \otimes_K L$ is nonzero at some element $(\tilde{\partial}_{i_0}, \dots, \tilde{\partial}_{i_{p-1}})$ of $\mathrm{Der}(L/E)$. Hence Condition (2) implies Condition (3).

(3) $\Longrightarrow$ (0). Suppose $\mathcal{D}(W)$ and $\mathrm{A}(E) \otimes_K L$ are linearly disjoint. If $\dim W = n$, then W is integrable by Theorem 3.11. So suppose $r < n$. We shall find θ^r in $E^* \otimes_K L(Y)$ so that

$$\mathcal{D}(W \cup \{\mathrm{d}\, X^r - \theta^r\})$$

is linearly disjoint from $\mathrm{A}(E) \otimes_K L(Y)$. If we can do this in general, then we can find a finitely generated extension L_1 of L and a subspace W_1 of $\Omega^1_{L_1/E}$ with basis

$$(\mathrm{d}\, X^\ell - \theta^\ell : \ell < n),$$

where $\theta^\ell \in E^* \otimes_K L_1$, such that $\mathcal{D}(W_1)$ and $\mathrm{A}(E) \otimes_K L_1$ are linearly disjoint. Hence there will be a chain

$$L \subseteq L_1 \subseteq L_2 \subseteq \dots,$$

with union L', and there will be a subspace of $\Omega^1_{L'/E}$ including $W \otimes_L L'$ to which Theorem 3.11 applies, showing that W is integrable.

Either we can choose θ^r to be 0, or not. Suppose not. Then $\mathrm{d}\, W$ must contain a form

$$\alpha \wedge \mathrm{d}\, X^r - \beta,$$

where α is a non-zero 1-form, and β is a non-zero element of $A^2(E) \otimes_K L$. Every form in $\Omega_{L/E}$ is a linear combination of wedge-products of forms $\mathrm{d}\, t^i$ and $\mathrm{d}\, X^k$. In a form in W though, $\mathrm{d}\, X^k$ never appears if $r \leqslant k < n$. Hence, if $\ell < k < n$, then $\mathrm{d}\, X^k \wedge \mathrm{d}\, X^\ell$ cannot appear in a form in $\mathrm{d}\, W$ unless $\ell < r$. Therefore α is a linear combination of forms $\mathrm{d}\, t^i$ and $\mathrm{d}\, X^\ell$ where $\ell < r$. By replacing $\mathrm{d}\, X^\ell$ with θ^ℓ, we may assume $\alpha \in E^* \otimes_K L$. We calculate

$$\alpha \wedge (\alpha \wedge \mathrm{d}\, X^r - \beta) = -\alpha \wedge \beta.$$

The right member is now in $\mathrm{A}^3(E) \otimes_K L$, and the left member is in $\mathcal{D}(W)$. Hence both members are zero. Therefore α is a factor of β, and we have

$$\alpha \wedge \mathrm{d}\, X^r - \beta = \alpha \wedge (\mathrm{d}\, X^r - \gamma)$$

for some γ in $E^* \otimes_K L$. So let θ^r be the element

$$\alpha \cdot Y + \gamma$$

of $E^* \otimes_K L(Y)$. We shall find an ordinary ideal of $\Omega_{L(Y)/E}$ that is linearly disjoint from $\mathrm{A}(E) \otimes_K L(Y)$, but that includes $\mathcal{D}(W \cup \{\mathrm{d}\, X^r - \theta^r\})$.

We have

$$\mathrm{d}(\mathrm{d}\, X^r - \theta^r) = -\,\mathrm{d}\, \theta^r = \alpha \wedge \mathrm{d}\, Y - (\mathrm{d}\, \alpha \cdot Y + \mathrm{d}\, \gamma).$$

By design, $\alpha\wedge(\mathrm{d}\,X^r-\theta^r)=\alpha\wedge(\mathrm{d}\,X^r-\gamma)$. Since the right member is the element $\alpha\wedge\mathrm{d}\,X^r-\beta$ of $\mathcal{D}(W)$, the left member and its derivative are also in $\mathcal{D}(W)$. Calculating this derivative, we get

$$\begin{aligned}\mathrm{d}(\alpha\wedge(\mathrm{d}\,X^r-\theta^r)) &= \mathrm{d}\,\alpha\wedge(\mathrm{d}\,X^r-\theta^r)+\alpha\wedge\mathrm{d}(\mathrm{d}\,X^r-\theta^r)\\ &= \mathrm{d}\,\alpha\wedge(\mathrm{d}\,X^r-\theta^r)-\alpha\wedge(\mathrm{d}\,\alpha\cdot Y+\mathrm{d}\,\gamma).\end{aligned}$$

Therefore $\alpha\wedge(\mathrm{d}\,\alpha\cdot Y+\mathrm{d}\,\gamma)$ is in the ideal I of $\Omega_{L/E}\otimes_L L(Y)$ generated by the 1-form $\mathrm{d}\,X^r-\theta^r$ and the forms in $W\cup\mathrm{d}\,W$. Hence

$$\mathrm{d}\,\alpha\cdot Y+\mathrm{d}\,\gamma=\alpha\wedge\zeta+\varepsilon,$$

where $\varepsilon\in I$ and $\zeta\in\Omega^1_{L/E}\otimes_L L(Y)$. Going back to the derivative of $\mathrm{d}\,X^r-\theta^r$, we have

$$\mathrm{d}(\mathrm{d}\,X^r-\theta^r)=\alpha\wedge(\mathrm{d}\,Y-\zeta)-\varepsilon.$$

This is in the ideal of $\Omega_{L(Y)/E}$ generated by $\mathrm{d}\,X^r-\theta^r$ and $\mathrm{d}\,Y-\zeta$ and the forms in $W\cup\mathrm{d}\,W$. This ideal is linearly disjoint from $\mathrm{A}(E)\otimes_K L(Y)$. So we have accomplished what we wanted, and W is integrable. □

By Lemma 4.5, Theorem 4.7 gives us an alternative axiomatization of DCF^m. As noted in § 0, the theory $\mathrm{DCF}^m\cup\{\sigma\}$ admits quantifier-elimination. Once one knows this, then one can say briefly that the completions of DCF^m are ω-stable because anti-chains are finite in the product-order of ω^m. The idea can be seen in [Y, § 0.3]; the argument itself can also be made with differential forms in the way sketched as follows.

Let $(L',\tilde{E})$ be an extension—generated by a tuple $\mathbf{a}$ of elements—of a differential field (K,E). Let $L=K(\mathbf{a})$. Of the kernel of the map Φ of § 0, let W be spanned by the elements that have the form $\mathrm{d}\,x-\theta$, where $\theta\in E^*\otimes_K L$. In the proof that (3) implies (0) in Theorem 4.7, we have a construction of an extension of (K,E) witnessing that W is integrable. We should modify that construction so that, if the form θ^r there *can* be zero, then we let it be $\sum_{i<m}\mathrm{d}\,t^i\cdot Y_i$ (that is, we let it be as generic as possible). Then the construction gives us an extension of (K,E) of which $(L',\tilde{E})$ can be seen as a specialization. To obtain the latter, we need only specialize the construction at finitely many steps, in a way to be described presently. This observation is equivalent to the ω-stability of the completions of DCF^m.

We need only consider the case where $\mathbf{a}$ has length 1 and hence has a single entry, a. If $\sigma\in{}^m\omega$, let a^σ be $\tilde{\partial}_0^{\sigma(0)}\cdots\tilde{\partial}_{m-1}^{\sigma(m-1)}a$, so that $L'=K(a^\sigma:\sigma\in{}^m\omega)$.

Write θ^σ for the form $\sum_{i<m} \mathrm{d}\, t^i \cdot \tilde{\partial}_i a^\sigma$ in $\tilde{E}^*$. Then $\tilde{E}$ is the kernel of the forms $\mathrm{d}\, a^\sigma - \theta^\sigma$ in $\Omega^1_{L'/E}$.

Now, θ^σ is just $\sum_{i<m} \mathrm{d}\, t^i \cdot a^{\sigma+\mathbf{i}}$; so, to choose θ^σ is to choose the immediate successors of a^σ. In the construction of $(L', \tilde{E})$, if $\sigma < \tau$, then a^σ can be chosen before a^τ, and θ^σ before θ^τ. If a^τ is algebraic over $K(a^\sigma : \sigma < \tau)$, then in fact each $\theta^{\tau+\upsilon}$ is *determined* by those θ^σ such that $\sigma < \tau + \upsilon$. Thus the construction need only be specialized in the choice of a^τ where τ is *minimal* such that a^τ is algebraic over $K(a^\sigma : \sigma < \tau)$. Such τ form an anti-chain in ${}^m\omega$ and are accordingly just finitely numerous.

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