

# de Sitter group as a symmetry for optical decoherence

S Başkal<sup>1</sup> and Y S Kim<sup>2</sup>

<sup>1</sup> Department of Physics, Middle East Technical University, 06531 Ankara, Turkey

<sup>2</sup> Department of Physics, University of Maryland, College Park, MD 20742, USA

E-mail: [baskal@newton.physics.metu.edu.tr](mailto:baskal@newton.physics.metu.edu.tr) and [yskim@physics.umd.edu](mailto:yskim@physics.umd.edu)

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## Abstract

Stokes parameters form a Minkowskian 4-vector under various optical transformations. As a consequence, the resulting two-by-two density matrix constitutes a representation of the Lorentz group. The associated Poincaré sphere is a geometric representation of the Lorentz group. Since the Lorentz group preserves the determinant of the density matrix, it cannot accommodate the decoherence process through the decaying off-diagonal elements of the density matrix, which yields to an increase in the value of the determinant. It is noted that the  $O(3, 2)$  de Sitter group contains two Lorentz subgroups. The change in the determinant in one Lorentz group can be compensated by the other. It is thus possible to describe the decoherence process as a symmetry transformation in the  $O(3, 2)$  space. It is shown also that these two coupled Lorentz groups can serve as a concrete example of Feynman's rest of the universe.

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## 1. Introduction

Traditionally the Poincaré sphere plays a central role in polarization optics [1]. It is also found to be useful to elucidate the properties of geometric phases [2], two-beam systems with partially coherent phase relations [3], symmetric scattering [4] and the decoherence in multiple scattering of light [5] as well as the decoherence in Bose Einstein condensates [6]. Apart from those, Thomas rotations are investigated in connection with the Bloch ball which is regarded as the Poincaré sphere model of the hyperbolic geometry [7].

Since the sphere is applicable to diverse branches of physics, and since its geometry is so appealing, the symmetry of the Poincaré sphere is a subject by its own right. It has three-dimensional rotational symmetries which are well known. What other symmetries does this sphere possess? This is one of the questions we would like to address in this paper.

Polarization optics can be formulated in terms of the two-by-two and four-by-four representations of the six-parameter Lorentz group. It was noted that the two-component Jones vector and the 4-component Stokes parameters are like the relativistic spinors and the Minkowskian 4-vectors, respectively [8, 9]. It is possible to identify the attenuator, rotator and phase shifter with appropriate transformation matrices of the Lorentz group. This formulation is not restricted to polarization optics. It can be applied to all two-beam systems with coherent or partially coherent phases.

If we use  $(t, z, x, y)$  as the Minkowskian 4-vector to which four-by-four Lorentz-transformation matrices are applicable, it is possible to write

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}, \quad (1)$$

with appropriate two-by-two transformation matrices applicable to both sides of this two-by-two representation of the 4-vector. These Lorentz transformations are unimodular transformations, keeping the determinant  $t^2 - z^2 - x^2 - y^2$  of the above matrix invariant.

If we write the Stokes parameters in this two-by-two form, the matrix becomes the density matrix. This density matrix can also be geometrically represented by the Poincaré sphere. Therefore, the symmetry of the Poincaré sphere is necessarily that of the Lorentz group [10]. In this Lorentzian regime, the determinant of the density matrix is an invariant quantity.

Unlike the Jones vectors, the Stokes parameters, the density matrix and the Poincaré sphere can deal with the lack of coherence between the two beams. The determinant of the density matrix vanishes when the two beams are completely coherent, and it increases as the beams lose coherence. The Lorentzian symmetry of the Poincaré sphere can describe the symmetry with a fixed value of the determinant, but it cannot describe the process in which the determinant changes its value. In other words, we cannot discuss the decoherence process within the framework of the Lorentz group [10].

Although there are other types of decoherences such as decoherence due to amplitude damping, in this paper we restrict our study to phase decoherence, since we are tempted to associate this damping problem with dissipation problems in physics [11]. The known mathematical method closest to group theoretical approaches is to introduce the concept semi-groups [12]. While semi-groups are quite promising in traditional dissipation problems, we choose to investigate the decoherence problem with a mathematical method which is already familiar to us.

Let us start with a pair of complex numbers  $a$  and  $b$ . From these numbers, we can construct the density matrix of the form

$$\rho = \begin{pmatrix} aa^* & ab^* e^{-\lambda t} \\ a^* b e^{-\lambda t} & bb^* \end{pmatrix}. \quad (2)$$

Indeed, the decay in the off-diagonal elements of this matrix plays fundamental role in decoherence processes [13, 14].

The determinant of this matrix is

$$aa^* bb^* (1 - e^{-2\lambda t}). \quad (3)$$

This density matrix enjoys the symmetry properties such as those of the  $X$  matrix given in equation (1), since the optical transformations applicable to the Stokes parameters are like Lorentz transformations. However, these determinant-preserving transformations cannot change the  $t$  variable.

When  $t = 0$ , the system is in a pure state, and the determinant is zero. As  $t$  increases, the value of the determinant in equation (3) increases from zero to  $aa^* bb^*$ , and consequently the system becomes decoherent.

The question is whether there is a symmetry group which will accommodate this transition process. We know the Lorentz group cannot, but this does not prevent us from looking for a larger symmetry group. The purpose of the present paper is to show that the de Sitter group  $O(3, 2)$  accommodates this decoherence process.

In section 2, we introduce the  $O(3, 2)$  de Sitter group and point out that it can act as two coupled  $O(3, 1)$  Lorentz groups. In section 3, we review the symmetries of the Stokes parameters and the density matrix. In section 4, we study the symmetries of the Poincaré sphere within the Lorentzian framework and discuss in detail what is possible and what is not possible. In section 5, it is shown that the  $O(3, 2)$  symmetry can provide a framework for the decoherence process. In section 6, we interpret the result of our paper in terms of Feynman's rest of the universe.

## 2. The $O(3, 2)$ de Sitter group as two coupled Lorentz groups

The de Sitter group is an extension of the Lorentz group applicable to a five-dimensional space consisting of three space coordinates  $(x, y, z)$  and two time coordinates  $(t, u)$ . It leaves the quadratic form

$$t^2 + u^2 - x^2 - y^2 - z^2 \quad (4)$$

invariant. This five-dimensional space admits three rotations around the three coordinate axes and for each of the two time coordinates there are two sets of boost transformations along the three coordinate axes. There is also one more rotation acting on the two time coordinates, all adding up to ten transformations. Thus, this ten-parameter group is the minimal extension of the  $O(3, 1)$  Lorentz group which is locally isomorphic to  $Sp(4, R)$  [15].

Spacetime structures with additional space or time variable(s) have been studied [16, 17], and its representations have been discussed in detail [18].

The generators  $M^{ab}$ , with  $M^{ab} = -M^{ba}$  satisfy the commutation relations:

$$[M^{ab}, M^{cd}] = i(g^{ad}M^{bc} - g^{ac}M^{bd} + g^{bc}M^{ad} - g^{bd}M^{ac}), \quad (5)$$

where  $g^{ab} = \text{diag}(-1, -1, 1, 1, 1)$ . From  $M^{ab}$ , the rotation generators  $J_i$  and the boost generators  $L_{ij}$  can respectively be read as:

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}, \quad L_{ij} = M^{ij+2} \quad (6)$$

where the late indices  $i, j, \dots$ , run from 1 to 3. They satisfy the commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, L_{jk}] = i\epsilon_{ijl}L_{lk} \quad [L_{ij}, L_{kl}] = -i\delta_{jl}\epsilon_{ikm}J_m. \quad (7)$$

Although, this may sound like a mathematical exercise remote from the physical reality, we would like to emphasize that the  $O(3, 2)$  de Sitter group is already a standard theoretical tool in optical sciences, specifically as a mathematical basis for two-mode squeezed states [19, 20], as well as in the theory of elementary particles together with the  $O(4, 1)$  group. As Paul A M Dirac noted in 1963, the  $O(3, 2)$  group is the fundamental symmetry group for two coupled harmonic oscillators [15]. This two-oscillator system often serves as a mathematical basis for soluble models such as the Lie model in quantum field theory [21] and the Bogoliubov transformations in superconductivity [22].

However, in this paper, we are interested in the fact that the  $O(3, 2)$  group contains two  $O(3, 1)$  Lorentz groups, where the two time variables are linearly combined through the one-parameter rotation group. We will consider them as two coupled Minkowskian spaces. Let us consider two Minkowskian 4-vectors  $(t, x, y, z)$  and  $(u, x, y, z)$ . In the first Minkowskian space,

$$(t^2 - x^2 - y^2 - z^2) \quad (8)$$

is invariant under Lorentz transformations while

$$(u^2 - x^2 - y^2 - z^2) \quad (9)$$

is invariant in the other Minkowskian space.

Let us introduce two notations  $T$  and  $U$  defined as

$$T = \sqrt{t^2 - x^2 - y^2 - z^2}, \quad U = \sqrt{u^2 - x^2 - y^2 - z^2}. \quad (10)$$

These two positive quantities are Lorentz-invariants in their respective Minkowskian spaces. However, they do not have to remain invariant in the five-dimensional de Sitter space. If we write the 5-vector as

$$(t, x, y, z, u), \quad (11)$$

it is possible to have a Lorentz frame in which  $x = y = z = 0$ . Then the above 5-vector becomes

$$(T, 0, 0, 0, U). \quad (12)$$

In this particular frame, the  $O(3, 2)$  group contains rotations which will allow us to write

$$\begin{pmatrix} T \\ U \end{pmatrix} = \begin{pmatrix} \cos \chi & -\sin \chi \\ \sin \chi & \cos \chi \end{pmatrix} \begin{pmatrix} Z \\ 0 \end{pmatrix}, \quad (13)$$

where

$$Z^2 = T^2 + U^2. \quad (14)$$

The variables  $T$  and  $U$  are invariant in their respective four-dimensional Minkowskian space; however the quantity invariant in the de Sitter space is  $Z$ . Indeed, we can say that these two Minkowskian spaces are coupled by equation (13) within the five-dimensional de Sitter space.

It has been shown in the literature that the Stokes parameters behave like Minkowskian 4-vectors [10]. Furthermore, they represent the density matrix for two optical beams. We therefore note that the quantities  $T$  and  $U$  correspond to the determinants of those density matrices measuring coherence of each system. Therefore, equation (14) tells a conservation of coherence in the total system defined in the de Sitter space.

The loss of coherence in one Lorentzian space will yield to a gain in the other space. We shall show that our symmetry model will constitute a concrete example of Feynman's rest of the universe. The first Lorentzian space is the world in which we make physical observations, and the second space belongs to the rest of the universe [23, 24].

It has been a question for many years whether time-irreversible systems such as dissipative and decoherent systems can be formulated as symmetry problems by introducing the rest of the universe clearly defined by Feynman. We shall see in the following sections whether this is possible for two-beam optical systems.

As for the values of  $T$  and  $U$ , we assume here that they are positive and can become as small as we wish, but do not vanish completely. This is a perfectly valid procedure in dealing with vanishing numbers in physics. However, there is a big difference in mathematics. It required a procedure called 'group contractions' [25]. We shall avoid in this section group contractions.

### 3. Stokes parameters as Minkowskian 4-vectors

Let us start with a plane wave propagating along the  $z$  direction. Then, it has polarizations along the  $x$  and  $y$  directions. We can then write the Jones vector as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A \exp\{i(kz - \omega t)\} \\ B \exp\{i(kz - \omega t)\} \end{pmatrix}. \quad (15)$$

Even though the Jones vector was developed originally for polarized light waves, the formalism can be extended to all two-beam systems such as interferometers [10].

If the two beams are mixed, we use the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad (16)$$

applicable to column vector of equation (15).

These two beams can go through two different optical path lengths, resulting in a phase difference. If the phase difference is  $\phi$ , the phase shift matrix is

$$P(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}. \quad (17)$$

When reflected from mirrors, or while going through beam splitters, there are intensity losses for both beams. The rate of loss is not the same for the beams. This results in the attenuation matrix of the form

$$\begin{pmatrix} e^{-\eta_1} & 0 \\ 0 & e^{-\eta_2} \end{pmatrix} = e^{-(\eta_1+\eta_2)/2} \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \quad (18)$$

with  $\eta = \eta_2 - \eta_1$ . This attenuator matrix tells us that the electric fields are attenuated at two different rates. The exponential factor  $e^{-(\eta_1+\eta_2)/2}$  reduces both components at the same rate and does not affect the degree of polarization. The effect of polarization is solely determined by the squeeze matrix

$$S(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}. \quad (19)$$

It was shown in [8, 10, 26] that repeated applications of the rotation matrices of the form of equation (16), shift matrices of the form of equation (17) and squeeze matrices of the form of equation (19) lead to a two-by-two representation of the six-parameter Lorentz group. The transformation matrix in general takes the form

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (20)$$

applicable to the column vector of equation (15), where all four elements are complex numbers with the condition that the determinant of the matrix be one. This matrix contains six free parameters. The above  $G$  matrix constitutes the two-by-two representation of the six-parameter Lorentz group, commonly called  $SL(2, c)$ .

Indeed, the two-component Jones vector provides the representation space for the two-by-two representation of the Lorentz group. However, the Jones vectors cannot describe whether the two beams are coherent. This is the reason why we have to resort to the coherency matrix

$$C = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \quad (21)$$

with

$$\begin{aligned} S_{11} &= \langle \psi_1^* \psi_1 \rangle, & S_{22} &= \langle \psi_2^* \psi_2 \rangle, \\ S_{12} &= \langle \psi_1^* \psi_2 \rangle, & S_{21} &= \langle \psi_2^* \psi_1 \rangle. \end{aligned} \quad (22)$$

This coherency matrix also serves as the density matrix [23].

Under the influence of the  $G$  transformation given in equation (20), this density matrix is transformed as

$$\begin{aligned} C' &= GCG^\dagger = \begin{pmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}. \end{aligned} \quad (23)$$

This leads to the four-by-four transformation

$$\begin{pmatrix} S'_{11} \\ S'_{22} \\ S'_{12} \\ S'_{21} \end{pmatrix} = \begin{pmatrix} \alpha^* \alpha & \gamma^* \beta & \gamma^* \alpha & \alpha^* \beta \\ \beta^* \gamma & \delta^* \delta & \delta^* \gamma & \beta^* \delta \\ \beta^* \alpha & \delta^* \alpha & \beta^* \beta & \delta^* \beta \\ \alpha^* \gamma & \gamma^* \gamma & \alpha^* \delta & \gamma^* \delta \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{22} \\ S_{12} \\ S_{21} \end{pmatrix}. \quad (24)$$

It is sometimes more convenient to use the following combinations of parameters:

$$\begin{aligned} S_0 &= \frac{S_{11} + S_{22}}{\sqrt{2}}, & S_1 &= \frac{S_{11} - S_{22}}{\sqrt{2}}, \\ S_2 &= \frac{S_{12} + S_{21}}{\sqrt{2}}, & S_3 &= \frac{S_{12} - S_{21}}{\sqrt{2}i}. \end{aligned} \quad (25)$$

These four parameters are called the Stokes parameters in the literature [27], usually in connection with polarized light waves. However, as was mentioned before, the Stokes parameters are useful to all two-beam systems. We can write the above expression as

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (S_{11} + S_{22}) \\ (S_{11} - S_{22}) \\ (S_{12} + S_{21}) \\ i(S_{21} - S_{12}) \end{pmatrix}. \quad (26)$$

Then the four-by-four matrix which transforms  $(S_{11}, S_{22}, S_{12}, S_{21})$  to  $(S_0, S_1, S_2, S_3)$  is

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -i & i \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{22} \\ S_{12} \\ S_{21} \end{pmatrix}. \quad (27)$$

This matrix enables us to construct the transformation matrix applicable to the Stokes parameters, widely known as the Mueller matrix. The transformation matrix applicable to the Stokes parameters of equation (25) can be derived from equation (24), and its form has been discussed in detail in [10, 8]. The above Stokes parameters form a Minkowskian 4-vector like  $(t, z, x, y)$ , and the transformation matrix applicable to the Stokes parameters represents a Lorentz transformation.

The four-by-four representation is like the Lorentz transformation matrix applicable to the spacetime Minkowskian vector  $(t, z, x, y)$  [10]. This allows us to study spacetime symmetries in terms of the Stokes parameters which are applicable to interferometers. Let us first see how the rotation matrix of equation (16) is translated into the four-by-four formalism. In this case,

$$\alpha = \delta = \cos(\theta/2), \quad \gamma = -\beta = \sin(\theta/2). \quad (28)$$

Thus, the corresponding four-by-four matrix takes the form

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (29)$$

Let us next see how the phase-shift matrix of equation (17) is translated into this four-dimensional space. For this two-by-two matrix,

$$\alpha = e^{-i\phi/2}, \quad \beta = \gamma = 0, \quad \delta = e^{i\phi/2}. \quad (30)$$

For these values, the four-by-four transformation matrix takes the form

$$P(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \sin \phi & \cos \phi \end{pmatrix}. \quad (31)$$

For the squeeze matrix of equation (19),

$$\alpha = e^{\eta/2}, \quad \beta = \gamma = 0, \quad \delta = e^{-\eta/2}. \quad (32)$$

As a consequence, its four-by-four equivalent is

$$S(\eta) = \begin{pmatrix} \cosh \eta & \sinh \eta & 0 & 0 \\ \sinh \eta & \cosh \eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (33)$$

If the above matrices are applied to the four-dimensional Minkowskian space of  $(t, z, x, y)$ , the above squeeze matrix will perform a Lorentz boost along the  $z$  or  $S_1$  axis with  $S_0$  as the time variable. The rotation matrix of equation (29) will perform a rotation around the  $y$  or  $S_3$  axis, while the phase shifter of equation (31) performs a rotation around the  $z$  or the  $S_1$  axis. Matrix multiplications with  $R(\theta)$  and  $P(\phi)$  lead to the three-parameter group of rotation matrices applicable to the three-dimensional space of  $(S_1, S_2, S_3)$ .

The phase shifter  $P(\phi)$  of equation (31) commutes with the squeeze matrix of equation (33), but the rotation matrix  $R(\theta)$  does not. This aspect of matrix algebra leads to many interesting mathematical identities which can be tested in laboratories. One of the interesting cases is that we can produce a rotation by performing three squeezes. This aspect is widely known as the Wigner rotation as discussed in the literature.

In this paper, we are interested in studying the time-dependent density matrix of the form

$$C(t) = \begin{pmatrix} S_{11} & S_{12} e^{-\lambda t} \\ S_{21} e^{-\lambda t} & S_{22} \end{pmatrix}. \quad (34)$$

This matrix can be translated into the Minkowskian 4-vector

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 e^{-\lambda t} \\ S_3 e^{-\lambda t} \end{pmatrix}. \quad (35)$$

As  $t$  increases, the third and fourth components of this Minkowskian 4-vector become smaller.

Lorentz transformations preserve the  $(\text{length})^2$  of the 4-vector which in the Minkowskian metric takes the form

$$S_0^2 - S_1^2 - (S_2^2 + S_3^2) e^{-2\lambda t}. \quad (36)$$

This is also the determinant of the density matrix  $D(t)$ . If this quantity increases as the time  $t$  increases, we cannot handle the problem within the framework of the Lorentz group [10].

One option is to assert that this is not a reversible problem and invent a mathematical tool other than group theory [12]. Another approach is to look for a larger group which contains the Lorentz group as a subgroup. This is precisely what we intend to do in this paper. In section 5, we shall introduce the  $O(3, 2)$  de Sitter group which contains two Lorentz groups. Before getting into the world of the  $O(3, 2)$  symmetry, let us study the geometry of the Poincaré sphere in the following section.

#### 4. Lorentz symmetries of the Poincaré sphere

The Poincaré sphere has a long history, and its spherical symmetry is well known [1]. The Lorentz group has the three-dimensional rotation group as its subgroup. Thus, the Lorentz symmetry of the Poincaré sphere includes the traditional rotational symmetry. Let us study in this section the symmetries associated with Lorentz boosts.

If we use the expressions of  $\psi_1$  and  $\psi_2$  given in equation (15), the density matrix  $C$  of equation (21) becomes

$$D(t) = \begin{pmatrix} A^2 & AB e^{(-\lambda t - i\phi)} \\ AB e^{(-\lambda t + i\phi)} & B^2 \end{pmatrix}. \quad (37)$$

Here  $\phi$  is the phase difference between  $\psi_1^* \psi_2$  and  $\psi_1 \psi_2^*$ . The  $\lambda t$  factor in the exponent describes the loss of coherence. We assume that the off-diagonal terms decrease exponentially in the time variable. The determinant of this density matrix is

$$(AB)^2 (1 - e^{-2\lambda t}). \quad (38)$$

This determinant is zero when  $t = 0$ , but increases to  $(AB)^2$  as  $t$  becomes larger.

The corresponding 4-vector is

$$\frac{1}{2} \begin{pmatrix} A^2 + B^2 \\ A^2 - B^2 \\ 2AB(\cos \phi) e^{-\lambda t} \\ 2AB(\sin \phi) e^{-\lambda t} \end{pmatrix}. \quad (39)$$

For a fixed value of  $t$ , the geometry of the Poincaré sphere is the geometry defined by the three parameters  $A$ ,  $B$  and  $\phi$ . This sphere consists of two spheres: one is the outer sphere whose radius is the time-like component of the above 4-vector

$$s = \frac{(A^2 + B^2)}{2}, \quad (40)$$

and the other is the inner sphere whose radius is the magnitude of the three-vector contained in the 4-vector of equation (39)

$$r = \frac{1}{2} \sqrt{(A^2 - B^2)^2 + 4(AB)^2 e^{-2\lambda t}}. \quad (41)$$

Then the quantity

$$s^2 - r^2 \quad (42)$$

is Lorentz-invariant, and is equal to the value of the determinant given in equation (38). The inner radius is equal to the outer radius when  $t = 0$ , and becomes  $(A^2 - B^2)/2$  as  $t$  becomes very large.

We can now introduce a spherical coordinate system with

$$\begin{aligned} r_z &= (A^2 - B^2)/2 = r(\cos \theta), \\ r_x &= AB(\cos \phi) e^{-\lambda t} = r(\sin \theta) \cos \phi, \\ r_y &= AB(\sin \phi) e^{-\lambda t} = r(\sin \theta) \sin \phi. \end{aligned} \quad (43)$$

Then the Lorentz symmetry allows rotations in this three-dimensional system. Now, with the appropriate rotation it is possible to bring 4-vector of equation (39) to

$$\begin{pmatrix} s \\ r \\ 0 \\ 0 \end{pmatrix}. \quad (44)$$

The rotations do not change the radii of the outer and inner spheres, and  $r$  and  $s$  remain invariant under the rotations.

However, the Lorentz symmetry allows the Lorentz boosts of the 4-vector of equation (44) along the  $-z$  direction. If we apply the inverse of the boost matrix of equation (33), then the 4-vector becomes

$$\begin{pmatrix} s(\cosh \eta) - r(\sinh \eta) \\ r(\cosh \eta) - s(\sinh \eta) \\ 0 \\ 0 \end{pmatrix}. \quad (45)$$

This transformation changes the outer and inner radii, but keeps  $(s^2 - r^2)$  invariant, as we can see from

$$[s(\cosh \eta) - r(\sinh \eta)]^2 - [r(\cosh \eta) - s(\sinh \eta)]^2 = s^2 - r^2. \quad (46)$$

It is now possible to choose the value of  $\eta$  such that

$$r(\cosh \eta) - s(\sinh \eta) = 0, \quad (47)$$

which leads to  $\tanh \eta = r/s$ . If this condition is met, then the 4-vector of equation (45) becomes

$$\begin{pmatrix} \sqrt{s^2 - r^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} AB\sqrt{1 - e^{-2\lambda t}} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (48)$$

Indeed, the Lorentz symmetry allows us to bring the Poincaré sphere to a one-number system. We are now tempted to change the value of  $(r^2 - s^2)$  in the above expression by changing the time variable  $t$ . This is precisely what is not allowed within the framework of the Lorentz group. We shall see whether this can be achieved when symmetry group is enlarged.

## 5. $O(3, 2)$ symmetry of the Poincaré sphere

In order to deal with the above problem, we introduce the  $O(3, 2)$  de Sitter space. As we emphasized in section 1, this group has already been exploited in optical sciences.

Let us consider a 5-vector  $(0, 0, 0, 0, m)$  in the de Sitter space, and a five-by-five rotation matrix acting on the two time coordinates

$$\begin{pmatrix} \cos \chi & 0 & 0 & 0 & \sin \chi \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\sin \chi & 0 & 0 & 0 & \cos \chi \end{pmatrix}. \quad (49)$$

This rotation matrix changes the 5-vector to

$$(m(\sin \chi), 0, 0, 0, m(\cos \chi)). \quad (50)$$

We have noted in section 2 that the de Sitter space contains two Minkowskian subspaces, with their respective invariants of equations (8) and (9), while the invariant quantity in this larger space is given in equation (4). If  $z = x = y = 0$ , we let this invariant quantity to be  $(t^2 + u^2) = m^2$ . Thus, in the Minkowskian space with the coordinate system  $(t, z, x, y)$ , the

invariant quantity is  $m^2 \sin^2 \chi$ , while  $m^2 \cos^2 \chi$  is the invariant quantity in the Minkowskian space with the coordinate system  $(u, z, x, y)$ , where now the four vectors in these spaces are

$$\begin{pmatrix} m(\sin \chi) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} m(\cos \chi) \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (51)$$

respectively.

Let us compare the first 4-vector of equation (51) with the 4-vector of equation (48). If we identify the parameter  $m(\sin \chi)$  in equation (51) with  $\sqrt{s^2 - r^2}$  of equation (48), we have

$$s^2 - r^2 = m^2 \sin^2 \chi. \quad (52)$$

This further allows us to identify  $m$  as  $AB$  in equation (48), and

$$(AB)^2 (\sin \chi)^2 = (AB)^2 (1 - e^{-2\lambda t}), \quad (53)$$

which leads to

$$\cos \chi = e^{-\lambda t}. \quad (54)$$

We concluded in section 4 that the  $t$  parameter cannot be changed in the Lorentzian regime. However, we have shown that this decoherence parameter can be identified with the angle variable  $\chi$  in the de Sitter space.

After changing the  $t$  variable, we can make inverse transformations to return to the 4-vector of the form given in equation (39). Indeed, it is gratifying to note that we now have the freedom of changing this time variable with a symmetry operation. In terms of this symmetry parameter, we can write the density matrix as

$$\rho(\chi) = \begin{pmatrix} A^2 & AB e^{-i\phi}(\cos \chi) \\ AB e^{i\phi}(\cos \chi) & B^2 \end{pmatrix}. \quad (55)$$

If  $\chi = 0$  and  $t = 0$ , the system is in a pure state. As  $t$  becomes large, the angle  $\chi$  approaches  $90^\circ$ . Therefore the de Sitter parameter  $\chi$  neatly takes care of the loss of coherence in the two-beam system.

## 6. Physical interpretation

In this paper, we introduced two separate Minkowskian spaces by insinuating the de Sitter space and consequently we have converted the decoherence problem into a symmetry problem. The first Minkowskian space was defined by the coordinate variables  $(t, z, x, y)$ , and the second one by  $(u, z, x, y)$ . When we discussed the Lorentzian symmetry of the Poincaré sphere we worked with the first Minkowskian space. Our analysis for the second Minkowskian space would be exactly the same, except that  $\sin \chi$  is replaced by  $\cos \chi$  as can be seen from equation (49). The density matrix in this second space can then be written as

$$\sigma(\chi) = \begin{pmatrix} A^2 & AB e^{-i\phi}(\sin \chi) \\ AB e^{i\phi}(\sin \chi) & B^2 \end{pmatrix}. \quad (56)$$

This density matrix gains coherence as the density matrix of equation (55) loses coherence. The determinants of these two density matrices are  $(AB)^2 \sin^2 \chi$  and  $(AB)^2 \cos^2 \chi$  respectively. The sum of these two determinants is  $(AB)^2$  and is independent of the angle variable  $\chi$ . Indeed, these two density matrices or the two Lorentzian subspaces are ‘coupled’ in a Pythagorean manner.

Next, in order to discuss the density matrix, let us go back to equation (55) and write the amplitudes  $A$  and  $B$  as

$$A = \sqrt{2r} \cos(\theta/2), \quad B = \sqrt{2r} \sin(\theta/2). \quad (57)$$

Then the density matrix takes the form

$$\rho(\chi) = 2r \begin{pmatrix} \cos^2(\theta/2) & [\sin(\theta/2) \cos \theta/2] e^{-i\phi} \cos \chi \\ [\sin(\theta/2) \cos \theta/2] e^{+i\phi} \cos \chi & \sin^2(\theta/2) \end{pmatrix}. \quad (58)$$

Since the density matrix is invariant within a given Poincaré regime, we can evaluate the above matrix for a convenient value of  $\theta$ . So we choose  $\theta = 90^\circ$ . If we impose the normalization condition  $\text{Tr}(\rho) = 1$ , the density matrix  $\rho(\chi)$  of equation (55) becomes

$$\rho(\chi) = \frac{1}{2} \begin{pmatrix} 1 & e^{-i\phi} \cos \chi \\ e^{i\phi} \cos \chi & 1 \end{pmatrix}. \quad (59)$$

This matrix can be diagonalized into the form

$$\frac{1}{2} \begin{pmatrix} 1 + \cos \chi & 0 \\ 0 & 1 - \cos \chi \end{pmatrix}. \quad (60)$$

Then the entropy can be calculated from the formula

$$S = -\text{Tr}(\rho \ln \rho), \quad (61)$$

and the result is

$$S = -\left(\frac{1 - \cos \chi}{2}\right) \ln\left(\frac{1 - \cos \chi}{2}\right) - \left(\frac{1 + \cos \chi}{2}\right) \ln\left(\frac{1 + \cos \chi}{2}\right). \quad (62)$$

This quantity becomes 0 when  $\chi = 0$  (fully coherent) and  $\ln 2$  when  $\chi = 90^\circ$  (fully incoherent). This is consistent with the prevailing definition of entropy for two optical waves.

The entropy for the second space is

$$S' = -\left(\frac{1 - \sin \chi}{2}\right) \ln\left(\frac{1 - \sin \chi}{2}\right) - \left(\frac{1 + \sin \chi}{2}\right) \ln\left(\frac{1 + \sin \chi}{2}\right). \quad (63)$$

The entropy  $S$  of the first space is monotonically increasing function of  $\chi$ , while that of the second space  $S'$  is a decreasing function. Thus, an increase in entropy in the first space leads to a decrease in the second space. Then we can ask whether the sum of these two entropies becomes independent of  $\chi$ , leading to an entropy conservation of the total system. The answer is No. However, this does not cause problems for us, because the second space is not necessarily a physical space. It could be meaningless to use the same definition of entropy for this space.

Even if we insist that the second space be a physical space, the increase of entropy is not a strange concept to us. On the other hand, we insist on a conservation of some physical quantity, we can use the sum of the determinants of the density matrices given in equations (55) and (56).

What is the meaning of this second space? In his book on statistical mechanics [23], Feynman makes the following statement about the density matrix. *When we solve a quantum-mechanical problem, what we really do is divide the universe into two parts—the system in which we are interested and the rest of the universe. We then usually act as if the system in which we are interested comprised the entire universe. To motivate the use of density matrices, let us see what happens when we include the part of the universe outside the system.*

Feynman did not specify whether the rest of the universe is observable or not. In either case, it is an interesting exercise to construct a model of the rest of the universe behaving like

a physical world. With this point in mind, Han *et al* studied two coupled harmonic oscillators in which one of the oscillators corresponds to the physical world and the other to the rest of the universe [24]. In this example, the rest of the universe is the same as the world in which we do physics. In thermal field theory [28], even though based on the same mathematics as that of the coupled oscillators, the rest of the universe is not physically identified, except that it causes thermal excitations of the oscillators in the physical world.

In the case of decoherence, the concept of thermal bath as the cause of decoherence was noted by Feynman and Vernon [29]. The decoherence effect in tunnelling processes was studied by Caldeira and Leggett in 1983 [30]. In their review paper, Leggett *et al* discuss two-state systems coupled to a dissipative system [31]. The two-level decoherence within the field-theoretic framework was studied in detail by Anastopoulos and Hu [32]. Recently, Shiokawa and Hu were able to apply this two-level decoherence to qubit systems [33].

While the concept of decoherence occupies one of the central places in the current development of physics, the decoherence effect in two-optical beams comes from the phase-randomizing process discussed by McAlister and Raymer [34], precisely in the form of the two-by-two matrix discussed in this paper. As for the decoherence in the rest of the universe introduced in this paper, the system becomes more coherent as the time variable increases. Although this ‘recoherence’ process was considered by Anglin and Zurek [35], it is premature to expect a two-state system to gain coherence in the real world. It is thus very safe to say that the second Minkowskian space introduced in this paper remains in Feynman’s rest of the universe.

However, this does not prevent us from constructing a physical system analogous to the decoherent system coupled to a recoherent system. It was noted by the present authors that para-axial lens systems constitute a very rich resource of symmetries of the Lorentz group [36]. Thus, it may be possible to construct a system of lenses which will illustrate the combination of decoherence and recoherence processes discussed in the present paper.

## 7. Concluding remarks

It has been widely believed that the decoherence problem could not be treated as a symmetry problem. In this paper, we have presented a different view, using an extra time-like dimension in the Lorentz group. The de Sitter group we used has been one of the standard tools in relativistic quantum mechanics [17] and elementary particle physics including one of the most recent models in string theory. Also, this group is not new in optical sciences. In 1963, Paul A M Dirac observed that the de Sitter group  $O(3, 2)$  serves as a symmetry group for coupled harmonic oscillators [15]. This group is the fundamental scientific language for two-mode squeezed states of light [19, 20]. We are thus not carrying the burden of introducing a new mathematical device in this paper.

As we noted in section 6, the  $O(3, 2)$  group can serve as an illustrative example of Feynman’s rest of the universe [23]. One Lorentz subgroup represents the system under examination, while the other appears as the rest of the universe. As Feynman noted, it is more satisfying to understand the entire system including the rest of the universe.

By mentioning his rest of the universe, Feynman introduced the concept of two entangled worlds. In view of the current trend in physics, it is worth studying physical examples of Feynman’s rest of the universe in connection with entangled systems. For instance, the universe consisting of two coupled oscillators serves an illustrative example [24]. In this context, we also note that Feshbach and Tikochinsky studied a dissipative oscillator using two coupled oscillators [37]. It would be interesting to observe further symmetries associated with dissipative systems.

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