# Dual metrics and non-generic supersymmetries for a class of Siklos spacetimes

D. Baleanu $^\ast$ 

Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Çankaya University, 06531 Ankara, Turkey <sup>†</sup>Institute of Space Sciences, P. O. Box, MG-23, R-76900, Magurele-Bucharest, Romania

S. Başkal<sup>‡</sup>

Department of Physics, University of Maryland, College Park, MD 20742, USA § Department of Physics. Middle East Technical University, 06531 Ankara, Turkey

The presence of Killing-Yano tensors implies the existence of non-generic supercharges in spinning point particle theories on curved backgrounds. Dual metrics are defined through their associated non-degenerate Killing tensors of valence two. Siklos spacetimes, which are the only non-trivial Einstein spaces conformal to non-flat pp-waves are investigated in regards to the existence of their corresponding Killing and Killing-Yano tensors. It is found that under some restrictions, pp-wave metrics and Siklos spacetimes admit dual metrics and non-generic supercharges. Possible significance of those dual spacetimes are discussed.

## I. INTRODUCTION

The most interesting feature of Killing-Yano (KY) tensors [1] is established in the context of pseudo-classical spinning point particles, with N=1 world line supersymmetry, as an object generating supercharges that depend on the background metric [2], [3], [4], [5], [6], [7]. More intriguingly, the separability of the Dirac equation in the Kerr geometry is traced back to the existence of KY tensor, in that background [8], [9]. On the other hand Killing tensors, in some cases can be considered the "square" of KY tensors. They give rise to the associated constant of motion and play an important role in the complete solution of the geodesic equation. Furthermore, it has been shown that there is a reciprocal relation between spaces admitting non-degenerate Killing tensors of valence two and the so-called "dual" spaces whose metrics are specified through those Killing tensors [10]. Such novel aspects promoted literature on Killing and KY tensors, which have long known to relativists [11], [12], [13], [14], [15]. Recently, the Killing spacetimes for Stäckel systems of three dimensional separable coordinates were investigated [16]. More recently, further generalizations of Killing tensors and their existence criteria were discussed [17]. Non-degenerate Killing tensors of valence two were investigated for a class of metrics describing pure radiative spacetimes [18]. Lax tensors [19], in some specific cases are shown to be Killing-Yano tensors of order three [20]. Static axisymmetric spacetimes [21] and the Euclidean Taub-NUT metric [6] were analyzed within the framework of pseudo-classical spinning particles, as well as the spacetime manifolds with constant curvature [22], [23].

The purpose of this paper is twofold. First, we believe that a systematical investigation of spacetimes admitting Killing-Yano tensor will strengthen the connection between the background geometry and the non-generic supersymmetries, especially in such cases when the background metrics receive significant physical interpretations. We also know that properly contracted KY tensors generate a constant of motion by defining a Killing tensor. Furthermore, those Killing tensors, as well as those that are solved through the defining equations describe dual spacetime, on the condition that they are non-degenerate. On the other hand, finding non-degenerate Killing tensors on a particular spacetime, is not an easy task, because the condition of non-degeneracy is imposed by hand and has no connection with symmetries of the equations. Therefore, providing examples of physical significance is a step further towards a better understanding of dual spacetimes.

Secondly, the spacetimes that we are going to investigate from the above point of view, have been of great interest since they were first introduced. The metric describing plane fronted waves with parallel rays (pp-wave) is very well-known [15]. On the other hand Siklos spacetimes are the only non-trivial Einstein spaces conformal to non-flat pp-waves [24]. In fact, the initial motivation to study such spacetimes was that they occurred naturally in N = 1 supergravity. The presence of a negative cosmological constant in the Siklos metric implies that the space is not asymptotically flat, a probable feature of how our real universe may be. It has been shown that a through analysis of

<sup>\*</sup>electronic address: dumitru@cankaya.edu.tr

<sup>&</sup>lt;sup>†</sup>On leave of absence

<sup>&</sup>lt;sup>‡</sup>electronic address: baskal@newton.physics.metu.edu.tr

<sup>&</sup>lt;sup>§</sup>On leave of absence

particle motion in Siklos spacetimes require rotating reference frames [25], [26]. Therefore, the importance of constants of motion in such spacetimes can hardly be overestimated. Because of those reasons we believe that a systematical investigation of Killing and Killing-Yano tensors of pp-waves and Siklos type spacetimes is worthwhile.

We will carry out our investigation according to the following strategy: We will solve the KY equation for valence two and three and classify the metrics. We will optimize between having a less restrictive metric and attaining the maximum number of non-vanishing KY components. We will exclude all flat background solutions. From those KY tensors, of valence two and three, it is straightforward to calculate the associated Killing tensors, and the dual metrics. However, those are not the only Killing tensors on the manifold. We will also solve the Killing equations. Since we are interested in dual spaces, we will look for Killing tensors having the same surviving components as that of the original metric.

In the following section, we will give the basic formulations. We will also discuss the number of independent equations to be solved for KY tensors. In the third section we will classify the pp-wave and the Siklos metric, so that they admit Killing and KY tensors of valence two and three. A class of Killing and KY tensor solutions will be written explicitly. Associated dual spacetimes will also be calculated. The last section will be devoted to concluding remarks.

## **II. KILLING AND KILLING-YANO TENSORS**

## A. Killing-Yano tensors and non-generic supercharges

A Killing-Yano tensor of valence n,  $f_{\nu_1\nu_2...\nu_n}$ , is an antisymmetric tensor fulfilling the following equations:

$$\mathcal{M}_{\lambda\,\nu_1\nu_2\cdots\nu_n} \equiv f_{\nu_1\nu_2\cdots(\nu_n;\lambda)} = 0, \tag{2.1}$$

where semicolon denotes the covariant derivative.

Since we shall be investigating the solutions of (2.1) for n = 2 and for n = 3, in four dimensions, it is worthwhile to discuss the number of independent equations. For n = 2, the number of independent equations can be found by analyzing the symmetries of  $\mathcal{M}_{\mu\nu\alpha}$ , in regards to its repeated and all-distinct indices: If all the indices are repeated then  $\mathcal{M}_{\alpha\alpha\alpha}$  is identically zero. If two indices are repeated then the symmetry is  $\mathcal{M}_{\alpha\nu\nu} = -\mathcal{M}_{\nu\alpha\nu} = -\mathcal{M}_{\nu\nu\alpha}$ , yielding 12 independent components. If all the indices are distinct then  $\mathcal{M}_{\mu\nu\alpha} = \mathcal{M}_{\nu\mu\alpha}$ , resulting in another 12 independent components. Therefore, there are totally 24 independent equations, to be solved for the six independent components of the Killing-Yano tensor of rank two.

Similar arguments are also valid for Killing-Yano tensors of valence three. If all the indices or three of the indices are the same then the equations are identically zero. If two are the same then  $\mathcal{M}_{\alpha\alpha\mu\nu} = -\mathcal{M}_{\alpha\alpha\nu\mu}$ , and there are 12 independent equations. If all are distinct then  $\mathcal{M}_{\alpha\beta\mu\nu} = \mathcal{M}_{\beta\alpha\mu\nu} = -\mathcal{M}_{\alpha\beta\nu\mu}$  and so there are 4 more equations, all sum up to 16 independent equations to be solved for four independent components.

The spinning particle model was constructed to be supersymmetric [3], therefore independent of the form of the metric there is always a conserved supercharge  $Q_0 = \Pi_{\mu}\psi^{\mu}$ . Here,  $\Pi_{\mu}$  is the covariant momenta and  $\psi^{\mu}$  are odd Grassmann variables. The existence of Killing-Yano of valence *n* are related to non-generic supersymmetries described by the following supercharge

$$Q_f = f_{\nu_1 \nu_2 \cdots \nu_r} \Pi^{\nu_1} \psi^{\nu_2} \cdots \psi^{\nu_r}$$
(2.2)

which is a superinvariant:  $\{Q_0, Q_f\} = 0$ . The Jacobi identities and (2.1) guarantee that it is also a constant of motion:  $\{Q_f, H\} = 0$ , with

$$H = \frac{1}{2}g^{\mu\nu}\Pi_{\mu}\Pi_{\nu} \tag{2.3}$$

and with the appropriate definitions of the brackets.

## B. Killing tensors and dual spacetimes

A Killing tensor of valence two is defined through the equation

$$K_{(\mu\nu;\alpha)} = 0. \tag{2.4}$$

It has been shown in detail in reference [10] that  $K^{\mu\nu}$  and  $g^{\mu\nu}$  are reciprocally the contravariant components of the Killing tensors with respect to each other.

If  $K^{\mu\nu}$  is non-degenerate, then through the relation

$$K^{\mu\alpha}k_{\alpha\nu} = \delta^{\mu}{}_{\nu}, \tag{2.5}$$

the second rank non-degenerate tensor  $k_{\mu\nu}$ , can be viewed as the metric on the "dual" space. The relation between the Christoffel symbols,  $\hat{\Gamma}^{\mu}{}_{\alpha\beta}$  of the dual and of the initial manifold can be expressed by writing  $\hat{\Gamma}^{\mu}{}_{\alpha\beta}$  in terms of the Killing tensor and taking (2.4) into account [18]:

$$\hat{\Gamma}^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\alpha\beta} - \mathcal{K}^{\mu\delta} K_{\alpha\beta;\delta}, \qquad (2.6)$$

where  $\mathcal{K}^{\mu\alpha}K_{\alpha\nu} = \delta^{\mu}{}_{\nu}$ .

The notion of geometric duality extends to that of phase space. The constant of motion  $K = \frac{1}{2}K^{\mu\nu}\Pi_{\mu}\Pi_{\nu}$ , generates symmetry transformations on the phase space linear in momentum:  $\{x^{\mu}, K\} = K^{\mu\nu}\Pi_{\nu}$ , and in view of (2.4) the Poisson brackets satisfy  $\{H, K\} = 0$ , where H is as in (2.3). Thus, in the phase space there is a reciprocal model with constant of motion H and the Hamiltonian K.

Killing-Yano tensors of any valence can be considered as the square root of the Killing tensors of valence two in the sense that, their appropriate contractions yield

$$K_{\mu\nu} = g^{\alpha\beta} f_{\mu\alpha} f_{\beta\nu} \tag{2.7}$$

or for valence three it can be written as

$$K_{\mu\nu} = g^{\alpha\delta} g^{\beta\gamma} f_{\mu\alpha\beta} f_{\gamma\delta\nu}.$$
 (2.8)

# **III. THE PP-WAVE METRIC AND SIKLOS SPACETIMES**

#### A. The pp-wave metric

The very well-known pp-wave metric, describes plane fronted waves with parallel rays, admit a non-expanding shear-free and twist-free null- geodesic congruence, can be expressed in the form [28]:

$$ds^{2} = 2dudv + dx^{2} + dy^{2} + h(x, y, u)du^{2}.$$
(3.1)

Here,  $x^{\mu} = (v, x, y, v)$ , x and y are spacial coordinates, u is the retarded time. The Killing vector  $l^{\mu} = \delta_1^{\mu}$  is at the same time tangent to the null geodesics, so the coordinate v can be considered as an affine parameter. Many of its characteristics have been exploited in the literature for various purposes [29], [30].

## 1. Subclasses admitting Killing-Yano tensors

Here, we shall investigate the Killing-Yano tensors of the pp-wave metric. We will search for KY tensors with maximum number of components, with minimum restrictions on the metric function. Here and thereafter, we have excluded all flat solutions. We shall spare the reader from all computational details, and give the results. Incidentally, it may be interesting to note that, we can have Killing-Yano tensors with no restrictions on the metric as in the following: A two-component KY tensor

$$f_{24} = c_1, \qquad f_{34} = c_2 \tag{3.2}$$

exist for any form of h(x, y, u). However, it is apparent that this tensor is trivial  $(f_{\mu\nu;\alpha} = 0)$ .

This metric admits a KY tensor with at most four non-zero components:

$$f_{12} = 0, \quad f_{13} = 0, \qquad f_{14} = c_1, f_{23} = c_2, \quad f_{24} = r(u), \quad f_{34} = s(u).$$
(3.3)

with the following restrictions on the metric function:

$$2r(u)_{,u} - c_1h_{,x} + c_2h_{,y} = 0,$$
  

$$2s(u)_{,u} - c_1h_{,y} - c_2h_{,x} = 0.$$
(3.4)

Analyzing the integrability conditions it is found that, for pp-wave metric there is no solution for non-zero  $f_{12}$  and/or  $f_{13}$  components, even if one nullifies some of or all of the other components.

A KY tensor of order three has three non-vanishing components:

$$f_{123} = 0, \quad f_{124} = c_1, \\ f_{134} = c_2, \quad f_{234} = q(u)$$
(3.5)

where q(u) and the metric function are subject to:

$$2q(u)_{,u} - c_2 h(x, y, u)_{,x} + c_1 h(x, y, u)_{,y} = 0.$$
(3.6)

Similar arguments as above also apply here about the vanishing of  $f_{123}$ .

### 2. Killing tensors

Recently, Killing tensors for pp-wave metric are presented within the framework of dual spacetimes, by solving (2.4) [18]. It is found that the associated dual metrics fall into a class of spacetimes describing parallel null one-planes given by Walker [31], [32]. It is possible to associate a massless particle, with its four-momentum vector to be the basis of the null one-plane. If  $\tilde{l}_{\mu}$  is the basis of the plane, then  $\tilde{l}_{\mu;\nu} = \alpha_{\nu}\tilde{l}_{\mu}$ , where  $\alpha_{\nu}$  is the recurrence vector of the plane. Here we shall not repeat the same calculations; instead we will give the Killing tensors that can be obtained from KY tensors. From (2.7) and (3.3) they become:

$$K_{14} = c_1^2, \qquad K_{22} = -c_2^2, \qquad K_{33} = -c_2^2$$

$$K_{24} = c_1 r(u) + c_2 s(u),$$

$$K_{34} = c_1 s(u) - c_2 r(u),$$

$$K_{44} = c_1^2 h(x, y, u) - r(u)^2 - s(u)^2.$$
(3.7)

Dual metric obtained from above is:

$$k_{14} = 1/c_1^2, \qquad k_{22} = -1/2c_2, \qquad k_{33} = -1/c_2^2,$$

$$k_{24} = (c_1r(u) + c_2s(u))/2c_1^2c_2,$$

$$k_{34} = (c_1s(u) - c_2r(u))/c_1^2c_2^2,$$

$$k_{44} = \{2c_2^2(r(u)^2 + s(u)^2 + c_1^2h(x, y, u) - 2(c_1s(u) - c_2r(u))^2 - c_2(c_1r(u) + c_2s(u))^2\}/2c_1^4c_2^2.$$
(3.8)

With  $c_1 = 1$ , and  $c_2 \neq 0$ , this metric also falls into a class of Walker's metric.

Killing tensors obtained from (2.8) and (3.5) are:

$$K_{14} = -2(c_2^2 + c_2^2), K_{22} = -2c_1^2, K_{23} = -2c_1c_2, K_{24} = -2c_2q(u), K_{33} - 2c_2^2, K_{34} = 2c_1q(u), K_{34} = 2c_1q(u), K_{44} = 2(q(u)^2 - (c_1^2 + c_2^2)h(x, y, u)) (3.9)$$

A straightforward calculation shows that, dual metric cannot be obtained from this tensor, because its contravariant components turn out to form a singular matrix.

Killing tensors obtained from (3.2) has only one component:

$$K_{44} = c_1^2 + c_2^2, (3.10)$$

and cannot be considered as a metric for the dual manifold, as (3.9) is.

## B. The Siklos metric

The Siklos metric is expressed as

$$ds^{2} = \frac{\beta^{2}}{x^{2}} \left[ 2 \, du \, dv + dx^{2} + dy^{2} + h(x, y, u) \, du^{2} \right]$$
(3.11)

where  $\beta = \sqrt{-3/\Lambda}$  and  $\Lambda$  is the negative cosmological constant. The presence of a negative cosmological constant implies that the spacetime is not asymptotically flat. It is demonstrated in [24], that they represent the only nontrivial Einstein spaces conformal to non-flat pp-waves. As in pp-wave metric here also u is the retarded time and the principal null vector can be expressed as  $l^{\mu} = \delta_4^{\mu}$ . Similar to that of the pp-wave metric the Siklos metric also has vanishing optical parameters.

# 1. Subclasses admitting Killing-Yano tensors

Thr Siklos metric admits a second order Killing-Yano tensor with four non-zero components:

$$f_{12} = 0, f_{13} = 0 f_{14} = (2r(u) - cy)/2x^3, f_{23} = -c/2x^2 f_{24} = r(u)_{,u}/x^2, f_{34} = (y r(u)_{,u} + s(u) + cv)/x^3$$
(3.12)

where the functions r(u) and s(u) are related to the metric function through the following differential equations:

$$(c_1y - 2r(u))h_{,x} - c_1xh_{,y} + 4xr(u)_{,uu} = 0, c_1xh_{,x} + (c_1y - 2r(u))h_{,y} - 2c_1h + 4yr(u)_{,uu} + 4s(u)_{,u} = 0,$$
(3.13)

with h depending on all of its arguments. A particular solution to these equations are found as:

$$\frac{r(u)_{,uu}}{r(u)} = h1(u), \qquad s(u) = c_2 \tag{3.14}$$

where  $c, c_1$  and  $c_2$  are arbitrary constants. Then, the metric function is found to be

$$h = h1(u)(x^2 + y^2). (3.15)$$

Here, also as in the case of the pp-wave metric, there is no solution for non-zero  $f_{12}$  and/or  $f_{13}$  components, due to the integrability conditions.

The solutions for the third rank Killing-Yano tensor has only three non-vanishing components:

$$f_{123} = 0, f_{124} = 1/x^3, f_{134} = (r(u) + y)/x^4, f_{234} = r(u)_{,u}/x^3. (3.16)$$

The metric function h(x, y, u) and r(u) are subject to the solutions of:

$$2x r(u)_{,uu} - r(u) h_{,x} - y h_{,x} + x h_{,y} = 0.$$
(3.17)

We are led to assume the form of the metric as:  $h = h_3(u)[h_1(x) + h_2(y)]$ . Then the metric function becomes

$$h(x, y, u) = h3(u)(x^2 - y^2),$$
(3.18)

where h3(u) can be found through

$$h3(u) = r(u)_{,uu}/r(u).$$
(3.19)

## 2. Subclasses admitting Killing tensors

Here, we will investigate the Killing tensors for the Siklos metric. We solved (2.4), by imposing the condition that the Killing tensor sustains the form of the metric. We have distinguished two cases in regards to the dependencies of the metric function h(x, y, u).

Case a) The function h(y, u) depends on y and u. The surviving components read:

$$K_{14} = K_{33} = c_1/x^2 + c_2/x^4,$$

$$K_{22} = c_1/x^2,$$

$$K_{44} = (c_1/x^2 + c_2/x^4) h(y, u) + c_3/x^4.$$
(3.20)

The dual metric corresponding to the above Killing tensor is of the following form:

$$k_{14} = k_{33} = \beta^4 / (c_1 x^2 + c_2),$$
  

$$k_{22} = \beta^4 / c_1 x^2,$$
  

$$k_{44} = \beta^4 ((c_1 x^2 + c_2) h(y, u) - c_3) / (c_1 x^2 + c_2)^2.$$
  
(3.21)

We observe that, when  $c_2 = c_3 = 0$ , we obtain the original metric, with h independent of x. Case **b**) Here the metric function h(x) is only a function of x. The solutions for the components of the Killing tensor are the same as above except for  $K_{44}$  which is:

$$K_{44} = \left(\frac{c_1}{x^2} + \frac{2c_2}{x^4}\right)h(x) + \frac{c_3}{x^4}.$$
(3.22)

For case b) the dual metric is

$$k_{44} = \frac{\beta^4 (c_1 x^2 h(x) - c_3)}{(c_1 x^2 + c_2)^2}.$$
(3.23)

Similarly, when  $c_2 = c_3 = 0$ , we obtain the original metric, with h(x).

From (3.12) the components of the Killing tensors are obtained as:

$$K_{14} = x^{8} (c y - 2r(u))^{2} / 4\beta^{2}, \qquad K_{22} = K_{33} = -c^{2} x^{8} / 4\beta^{2},$$

$$K_{24} = -x^{2} (c (y r(u)_{,u} + s(u) + c v) + x (c y - 2r(u)) r(u)_{,u}) / 2\beta^{2},$$

$$K_{34} = -x^{2} ((y r(u)_{u} + s + cv)) (c y - 2r) - cx r(u)_{u}) / 2\beta^{2},$$

$$K_{44} = (x^{2} (x^{10} (c y - 2r(u))^{2} h(x, y, u) - 4r(u)^{2}_{,u}) - 4(y r(u)_{,u} + s(u) + c v)^{2}) / 4\beta^{2} x^{4},$$
(3.24)

where r(u), s(u) and h(x, y, u) are as in (3.14) and (3.15). The dual metric for this Killing tensor is calculated, but it turned out to be somewhat cumbersome with a very remote chance of having any interpretation.

From (2.8) and (3.16), the Killing tensors are:

$$K_{14} = -2((y+r(u))^{2} + x^{2})/\beta^{4}x^{4},$$

$$K_{22} = -2/\beta^{2}x^{4},$$

$$K_{23} = -2(r(u) + y)/\beta^{2}x^{5},$$

$$K_{24} = -2r(u)_{,u}(r(u) + y)/\beta^{4}x^{3},$$

$$K_{33} = -2(r(u) + y)^{2}/\beta^{2}x^{6},$$

$$K_{34} = 2r(u)_{,u}/\beta^{4}x^{2},$$

$$K_{44} = -2\{\beta^{2}h(x, y, u)[(r(u) + y)^{2} + x^{2}] + x^{4}r(u)^{2}_{,u}\}/\beta^{4}x^{6}.$$
(3.25)

This tensor is also degenerate, as the other tensors obtained from third rank KY tensors.

# IV. CONCLUDING REMARKS

In this article we have solved the KY equations, and we have found valence two and valence three KY tensors, both for the pp-wave and the Siklos metric. In fact, the system of equations (2.1) are complicated, involving 24 coupled equations to be solved for six component KY tensors of valence two. Similarly, one has to solve for 16 coupled equations for four component KY tensors of valence three. In order to find analytical and non-trivial solutions we have looked for non-flat subspaces admitting the maximum number of non-zero KY components, with minimum restrictions on the metric functions. By analyzing the integrability conditions, we have found that there can be at most four and at most three non-zero components for valence two and valence three KY tensors, respectively. We have explicitly given the relations between the metric functions and the functions appearing in the KY tensors and thereby we have classified the pp-wave and the Siklos metric admitting KY tensors.

As to our knowledge a valence three KY tensor for a curved spacetime have not been exemplified yet. Therefore, we have attained further non-generic supercharges as well as the usual ones obtained from valence two KY tensors. Moreover, valence three KY tensors can be considered as a particular kind of Lax tensors on the manifold. We also note that KY tensors with maximum number of non-zero components are all non-trivial.

In general, for an arbitrary metric, one cannot predict in advance that the Killing tensor equations admit nondegenerate and non-trivial solutions, because there is not a well defined technique to solve this problem. For this purpose we have analyzed in detail equation (2.4) and looked for non-degenerate Killing tensors that are of the same form as that of the initial Siklos metric. Furthermore, we have obtained second rank Killing tensors from valence two and three KY tensors. We have noted that, the Killing tensors obtained by solving the Killing equation are quite different, from those obtained by contracting the KY tensors. It is well known that, irrespective of how they are obtained, Killing tensors determine the first integrals, and play an important role in the solution of the geodesic equation. We have found that the Killing tensors obtained from the second rank KY tensors give non-degenerate and non-trivial Killing tensors, which can be considered as dual metrics. On the other hand, we have observed that Killing tensors obtained from valence three KY tensors yielded degenerate Killing tensors that are not suitable for metric generation. We have investigated the dual metrics for possible physical interpretations and we have found that the dual metric obtained from the pp-wave Killing tensor falls into a more general class of metrics describing parallel null one-planes.

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