# ON THE EXISTENCE OF A FACTORIZED UNBOUNDED OPERATOR BETWEEN FRÉCHET SPACES

#### ERSIN KIZGUT AND MURAT YURDAKUL

ABSTRACT. For locally convex spaces X and Y, the continuous linear map  $T: X \to Y$  is called bounded if there is a zero neighborhood U of X such that T(U) is bounded in Y. Our main result is that the existence of an unbounded operator T between Fréchet spaces E and F which factors through a third Fréchet space G ends up with the fact that the triple (E,G,F) has an infinite dimensional closed common nuclear Köthe subspace, provided that F has the property (y).

Dedicated to the memory of Prof. Dr. Tosun Terzioğlu

# 1. Introduction

Let X and Y be locally convex spaces. A continuous linear map  $T: X \to Y$  is called bounded if there is a  $\theta$ -neighborhood U of X such that T(U) is bounded in Y. We say that a triple (X, Z, Y) has the bounded factorization property and write  $(X, Z, Y) \in \mathcal{BF}$  if each linear continuous operator  $T: X \to Y$  that factors over Y (that is,  $T = R_1 R_2$ , where  $R_2 : X \to Z$  and  $R_1 : Z \to Y$  are linear continuous operators) is bounded. Nurlu and Terzioglu [2] proved that under some conditions, existence of continuous linear unbounded operators between nuclear Köthe spaces causes existence of common basic subspaces. Djakov and Ramanujan [1] sharpened this work by removing nuclearity assumption and using a weaker splitting condition. In [6], it is shown that the existence of an unbounded factorized operator for a triple of Köthe spaces, under some assumptions, implies the existence of a common basic subspace for at least two of the spaces. Concerning the class of general Fréchet spaces, the existence of an unbounded operator inbetween is studied in [5]. It is proved that there is an infinite

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dimensional closed common nuclear subspace when the range space has a basis, and admits a continuous norm. When the range space has the property (y), that implies the existence of a common nuclear Köthe quotient as proved in [4]. Combining these two results, when the range space has the property (y), common nuclear Köthe subspace is obtained in [3, Proposition 1]. The aim of this note is to prove the Fréchet space analogue of [6, Proposition 6], that is, under the condition that F has property (y), and  $(E, G, F) \notin \mathcal{BF}$  then there is a common nuclear subspace for all three spaces. We rule out the condition where G can be written as  $G = \omega \times X$ , where X is a Banach space to avoid the case T becomes almost bounded [8].

The locally convex space E with neighborhood base  $\mathcal{U}(E)$  is said to have property (y) if there is a neighborhood  $U_1 \in \mathcal{U}(E)$  such that

$$E' = \bigcup_{U \in \mathscr{U}(E)} \overline{E'[U_1^\circ] \cap U^\circ}.$$

### 2. Main Result

**Theorem 2.1.** Let E, F, G be Fréchet spaces where F has property (y). Assume there is a continuous, linear, unbounded operator  $T: E \to F$  which factors through G such as T = RS. Then, there exists an infinite dimensional nuclear Köthe subspace M of E such that the restriction  $T|_M$  and the restriction  $R|_{S(M)}$  are isomorphisms.

*Proof.* Let  $T: E \to F$  be an unbounded operator factoring through  $G \neq \omega \times X$ , for any Banach space X.

$$E \xrightarrow{T} F$$

$$S \downarrow \nearrow_R$$

By [3, Proposition 1], there exists an infinite dimensional closed nuclear Köthe subspace M of E such that the restriction  $T|_M$  is an isomorphism onto T(M). Since T is injective on M, R is injective on S(M) and maps S(M) onto T(M) = R(S(M)). Using the fact that T is an isomorphism, it is easy to verify that  $R|_{S(M)}$  is one-to-one. Now let  $y \in \overline{S(M)}$ . So find a sequence  $(S(m_n))_{n \in \mathbb{N}}$  in S(M) such that  $\lim_{M \to \infty} S(m_n) = y$ . R is continuous at y, then  $RS(m_n) = T(m_n) = Ry \in \overline{T(M)} = T(M)$ , since T(M) is closed. Thus  $\lim_{M \to \infty} T(m_n) = T(m) = Ry$  for some  $m \in M$ . Since T is an isomorphism on M,  $\lim_{M \to \infty} T^{-1}T(m_n) = T^{-1}T(m)$ , that is,  $\lim_{M \to \infty} m_n = m$ . S is continuous at m, and that implies  $\lim_{M \to \infty} S(m_n) = S(m) = y \in S(M)$ . Therefore S(M) is closed. Hence

 $R: S(M) \to R(S(M))$  is an isomorphism by the Open Mapping Theorem.

As proved in [7, Lemma 2.1] and [7, Theorem 2.3], property (y), which is assumed to be enjoyed by F can be replaced by being locally closed, or being isomorphic to a closed subspace of a Köthe space. It is shown that these conditions are equivalent to have the property (y).

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Ersin Kizgut, Department of Mathematics, Middle East Technical University, 06800 Ankara Turkey

E-mail address: kizgut@metu.edu.tr

Murat Yurdakul, Department of Mathematics, Middle East Technical University, 06800 Ankara Turkey

E-mail address: myur@metu.edu.tr