Møller's Energy in the Dyadosphere of a Charged Black Hole

OKTAY AYDOGDU[‡] **and MUSTAFA SALTI**[†]
Department of Physics, Faculty of Art and Science,
Middle East Technical University, 06531, Ankara-Turkey
[‡]oktay231@yahoo.com, [†]musts6@yahoo.com

Abstract. We use the Møller energy-momentum complex both in general relativity and teleparallel gravity to evaluate energy distribution (due to matter plus fields including gravity) in the dyadosphere region for Reissner-Nordström black hole. We found the same and acceptable energy distribution in these different approaches of the Møller energy-momentum complex. Our teleparallel gravitational result is also independent of the teleparallel dimensionless coupling constant, which means that it is valid in any teleparallel model. This paper sustains (a) the importance of the energy-momentum definitions in the evaluation of the energy distribution of a given space-time and (b) the viewpoint of Lessner that the Møller energy-momentum complex is a powerful concept for energy and momentum.

Keywords: Energy, dyadosphere, black hole, teleparallel gravity.
PAC Numbers: 04.40.-q; 04.20.Jb; 04.50.+h; 04.70.-s

1. Introduction

Einstein used the principle of equivalence and the conservation laws of energy-momentum to formulate the covariant field equations [1]. He defined the energy and momentum conservation law in the form given below.

$$[\sqrt{-g}(T^{\mu}_{\nu} + t^{\mu}_{\nu})]_{,\mu} = 0 \tag{1}$$

where $\mu, \nu = 0, 1, 2, 3$ and T^{μ}_{ν} is the stress energy density of matter. Einstein identified t^{μ}_{ν} as representing the stress energy density of gravitational fields. He also noted that t^{μ}_{ν} was not a tensor, but concluded that the equations given above hold good in all coordinate systems since they were directly obtained from the principle of general relativity. Bondi [2] expressed that a non-localizable form of energy is inadmissible in relativity and its location can in principle be found. Cooperstock [3] hypothesized that in a curved space-time energy-momentum is/are confined to the region of non-vanishing energy-momentum tensor $T_{\mu\nu}$ and consequently the gravitational waves are not carriers of energy and/or momentum in vacuum space-times. This hypothesis has neither been proved nor disproved. There are many results that provide support this hypothesis

[4]. It would be interesting to investigate the cylindrical gravitational waves in vacuum space-times.

In literature, after Einstein's expression was used for the energy and momentum distributions of the gravitational field, many attempts have been proposed to resolve the gravitational energy problem [5, 6, 7, 8, 9, 10, 11, 12, 13]. Except for the definition of Møller, these definitions only give meaningful results if the calculations are performed in "Cartesian" coordinates. Møller constructed an expression which enables one to evaluate energy and momentum in any coordinate system.

Several examples of particular space-times have been investigated and different energy-momentum pseudo-tensor are known to give the same energy distribution for a given space-time [14, 15, 16, 17, 18, 19, 20, 21]. Virbhadra [16], using the energy and momentum complexes of Einstein, Landau-Lifshitz, Papapetrou and Weinberg for a general non-static spherically symmetric metric of the Kerr-Schild class, showed that all of these energy-momentum formulations give the same energy distribution as in the Penrose energy-momentum formulation. Recently, the problem of energy-momentum localization has also been considered in alternative gravitation theory, namely teleparallel gravity [13, 22, 23, 24]. The authors found that energy-momentum also localize in this alternative theory, and their results agree with some of the previous papers which were studied in the general theory of relativity. Vargas [13], using the definitions of Einstein and Landau-Lifshitz in teleparallel gravity, found that the total energy is zero in Friedmann-Robertson-Walker space-times and the result is the same as calculated in general relativity [25].

In this paper, we evaluate the energy in the dyadosphere of a charged black hole by using the Møller definition in both general relativity and teleparallel gravity, and then we compare the result.

Notations and conventions: $c = h = 1$, metric signature $(+, -, -, -)$, Greek indices and Latin ones run from 0 to 3. Throughout this paper, Latin indices $(i,\ j,\ ...)$ represent the vector number, and Greek indices $(\mu,\ \nu, ...)$ represent the vector components.

2. The Dyadosphere of a Charged Black Hole

The event horizon of a charged black hole is, according to Ruffini [26] and Preparata *et. al.* [27], surrounded by a special region called the dyadosphere where the electromagnetic field exceeds the Euler-Heisenberg critical value for electron-positron pair production. The new concept of Dyadosphere of an electromagnetic black hole was introduced by Ruffini to explain gamma ray bursts. Ruffini defined the dyadosphere as the region just outside the horizon of a charged black hole whose electromagnetic field strength is larger than the well-known Heisenberg-Euler critical value

$$\Xi_{critical} = \frac{m_e^2 c^3}{r\hbar} \tag{2}$$

where m_e and e denote mass and charge of an electron, respectively. For a Reissner-Nordström black hole, the dyadosphere is defined by the radial interval $r_+ \leq r \leq r_{ds}$

where the horizon

$$r_{+} = \frac{GM}{c^2}(1 + \sqrt{1 - \frac{q^2}{GM^2}}) \tag{3}$$

forms the inner radius of the dyadosphere, while its outer radius is given by

$$r_{ds} = \sqrt{\frac{\hbar}{cm_e}\frac{GM}{c^2}\frac{m_p}{m_e}\frac{e}{q_p}\frac{Q}{M\sqrt{G}}} \tag{4}$$

where M, Q, $m_p = \sqrt{\frac{\hbar c}{G}}$ and $q_c = \sqrt{\hbar c}$ are mass, charge parameters, the Planck mass and the Planck charge, respectively. The total energy of electron-positron pairs converted from static electric energy and deposited within the dydosphere is obtained [28] as

$$E_{dya} = \frac{Q^2}{2r_{+}}(1 - \frac{r_{-}}{r_{ds}})(1 - \frac{r_{+}^2}{r_{ds}^2}) \tag{5}$$

De Lorenci, Figueiredo, Fliche and Novello [29] found the correction for the Reissner-Nordström metric from the first contribution of the Euler-Heisenberg Lagrangian and obtained the following line-element

$$\begin{aligned} ds^2 = {} & (1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\sigma Q^4}{5r^6})dt^2 - (1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\sigma Q^4}{5r^6})^{-1}dr^2 \\ & - r^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned} \tag{6}$$

By writing $\sigma = 0$, we obtain the Reissner-Nordström space-time. De Lorenci *et al.* showed that the correction term $\frac{\sigma Q^4}{5R^6}$ is of the same order of magnitude as the Reissner-Nordström charge term $\frac{Q^2}{2R^2}$.

3. Møller's Energy in General Relativity

In general relativity, the energy-momentum complex of Møller [9] is given by

$$M_\mu^\nu = \frac{1}{8\pi}\chi_{\mu,\alpha}^{\nu\alpha} \tag{7}$$

satisfying the local conservation laws:

$$\frac{\partial M_\mu^\nu}{\partial x^\nu} = 0 \tag{8}$$

where the antisymmetric super-potential $\chi_\mu^{\nu\alpha}$ is

$$\chi_\mu^{\nu\alpha} = \sqrt{-g}[g_{\mu\beta,\gamma} - g_{\mu\gamma,\beta}]g^{\nu\gamma}g^{\alpha\beta}. \tag{9}$$

The locally conserved energy-momentum complex M_μ^ν contains contributions from the matter, non-gravitational and gravitational fields. M_0^0 is the energy density and M_a^0 are the momentum density components. The momentum four-vector of Møller is given by

$$P_\mu = \int\int\int M_\mu^0 dxdydz. \tag{10}$$

Using Gauss's theorem, this definition transforms into

$$P_\mu = \frac{1}{8\pi}\int\int \chi_\mu^{0\alpha}\mu_\alpha dS. \tag{11}$$

where μ_α (where $\alpha = 1, 2, 3$) is the outward unit normal vector over the infinitesimal surface element dS. P_i give momentum components P_1, P_2, P_3 and P_0 gives the energy.

For the line-element given by equation (6), the matrix of the $g_{\mu\nu}$ is given as

$$\begin{pmatrix} (1-\frac{2M}{r}+\frac{Q^2}{r^2}-\frac{\sigma Q^4}{5r^6}) & 0 & 0 & 0 \\ 0 & -(1-\frac{2M}{r}+\frac{Q^2}{r^2}-\frac{\sigma Q^4}{5r^6})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{12}$$

and its inverse matrix $g^{\mu\nu}$ is defined by

$$\begin{pmatrix} (1-\frac{2M}{r}+\frac{Q^2}{r^2}-\frac{\sigma Q^4}{5r^6})^{-1} & 0 & 0 & 0 \\ 0 & -(1-\frac{2M}{r}+\frac{Q^2}{r^2}-\frac{\sigma Q^4}{5r^6}) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{13}$$

Thus, the required non-vanishing component of $\chi_\mu^{\nu\alpha}$ is

$$\chi_0^{01} = 2\sin\theta(M - \frac{Q^2}{r} + \frac{\sigma Q^4}{5r^5}) \tag{14}$$

using equations (11) and (14), we obtain the energy distribution as

$$E(r) = M - \frac{Q^2}{r} + \frac{\sigma Q^4}{5r^5}. \tag{15}$$

4. Møller's Energy in the Teleparallel Gravity

The teleparallel theory of gravity (the tetrad theory of gravitation) is an alternative approach to gravitation and corresponds to a gauge theory for the translation group based on Weitzenböck geometry [30]. In the theory of teleparallel gravity, gravitation is attributed to torsion [31], which plays the role of a force [32], and the curvature tensor vanishes identically. The essential field is acted by a nontrivial tetrad field, which gives rise to the metric as a by-product. The translational gauge potentials appear as the nontrivial item of the tetrad field, so induces on space-time a teleparallel structure which is directly related to the presence of the gravitational field. The interesting place of teleparallel theory is that, due to its gauge structure, it can reveal a more appropriate approach to consider some specific problem. This is the situation, for example, in the energy and momentum problem, which becomes more transparent.

Møller modified general relativity by constructing a new field theory in teleparallel space [33]. The aim of this theory was to overcome the problem of the energy-momentum complex that appears in Riemannian Space [34]. The field equations in this new theory were derived from a Lagrangian which is not invariant under local tetrad rotation. Saez [35] generalized Møller theory into a scalar tetrad theory of gravitation. Meyer [36] showed that Møller theory is a special case of Poincare gauge theory [37, 38].

In teleparallel gravity, the super-potential of Møller is given by Mikhail *et al.* [12] as

$$U_\mu^{\nu\beta} = \frac{(-g)^{1/2}}{2\kappa} P_{\chi\rho\sigma}^{\tau\nu\beta}[\Phi^\rho g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu}\xi^{\chi\rho\sigma} - (1-2\lambda) g_{\tau\mu}\xi^{\sigma\rho\chi}] \tag{16}$$

where $\xi_{\alpha\beta\mu} = h_{i\alpha} h^{i}{}_{\beta;\mu}$ is the con-torsion tensor and $h_i{}^{\mu}$ is the tetrad field and defined uniquely by $g^{\alpha\beta} = h_i^{\alpha} h_j^{\beta} \eta^{ij}$ (here η^{ij} is the Minkowski space-time). κ is the Einstein constant and λ is free-dimensionless coupling parameter of teleparallel gravity. For the teleparallel equivalent of general relativity, there is a specific choice of this constant.

Φ_ρ is the basic vector field given by

$$\Phi_\mu = \xi^{\rho}{}_{\mu\rho} \tag{17}$$

and $P^{\tau\nu\beta}_{\chi\rho\sigma}$ can be found by

$$P^{\tau\nu\beta}_{\chi\rho\sigma} = \delta^{\tau}_{\chi} g^{\nu\beta}_{\rho\sigma} + \delta^{\tau}_{\rho} g^{\nu\beta}_{\sigma\chi} - \delta^{\tau}_{\sigma} g^{\nu\beta}_{\chi\rho} \tag{18}$$

with $g^{\nu\beta}_{\rho\sigma}$ being a tensor defined by

$$g^{\nu\beta}_{\rho\sigma} = \delta^{\nu}_{\rho}\delta^{\beta}_{\sigma} - \delta^{\nu}_{\sigma}\delta^{\beta}_{\rho}. \tag{19}$$

The energy-momentum density is defined by

$$\Xi^{\beta}_{\alpha} = U^{\beta\lambda}_{\alpha,\lambda} \tag{20}$$

where comma denotes ordinary differentiation. The energy is expressed by the surface integral;

$$E = \lim_{r\to\infty} \int_{r=constant} U_0^{0\zeta} \eta_\zeta dS \tag{21}$$

where η_ζ is the unit three-vector normal to surface element dS.

The general form of the tetrad, h_i^{μ}, having spherical symmetry was given by Robertson [39]. In the Cartesian form it can be written as

$$h_0^0 = iA, \qquad h_a^0 = Cx^a, \qquad h_0^\alpha = iDx^\alpha, \qquad h_a^\alpha = B\delta_a^\alpha + Ex^\alpha x^\alpha + \epsilon_{a\alpha\beta} F x^\beta \tag{22}$$

where A, B, C, D, E, and F are functions of t and $r = \sqrt{x^\alpha x^\alpha}$, and the zeroth vector h_0^μ has the factor $i^2 = -1$ to preserve Lorentz signature. We consider an asymptotically flat space-time in this paper, and impose the boundary condition that for $r \to \infty$ the tetrad given by (22) approaches the tetrad of Minkowski space-time, $h_a^\mu = \mathtt{diag}(i, \delta_a^\alpha)$.

Using the general coordinate transformation

$$h_{a\mu} = \frac{\partial \mathbf{X}^{\nu'}}{\partial \mathbf{X}^{\mu}} h_{a\nu} \tag{23}$$

where $\{\mathbf{X}^\mu\}$ and $\left\{\mathbf{X}^{\nu'}\right\}$ are, respectively, the isotropic and Schwarzschild coordinates (t, r, θ, ϕ). In the spherical, static and isotropic coordinate system $\mathbf{X}^1 = r\sin\theta\cos\phi$, $\mathbf{X}^2 = r\sin\theta\sin\phi$, $\mathbf{X}^3 = r\cos\theta$. We obtain the tetrad components of $h_a{}^\mu$ as

$$h_a{}^\mu = \begin{pmatrix} i(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\sigma Q^4}{5r^6})^{-1/2} & 0 & 0 & 0 \\ 0 & (1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\sigma Q^4}{5r^6})^{1/2} s\theta c\phi & \frac{1}{r} c\theta c\phi & -\frac{s\phi}{rs\theta} \\ 0 & (1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\sigma Q^4}{5r^6})^{1/2} s\theta s\phi & \frac{1}{r} c\theta s\phi & \frac{c\phi}{rs\theta} \\ 0 & (1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\sigma Q^4}{5r^6})^{1/2} c\theta & -\frac{1}{r} s\theta & 0 \end{pmatrix} \tag{24}$$

where we have introduced the following notation: $s\theta = \sin\theta$, $c\theta = \cos\theta$, $s\phi = \sin\phi$ and $c\phi = \cos\phi$. We can now construct the con-torsion tensor, whose non-vanishing components are

$$
\begin{aligned}
\xi^{0}{}_{01} &= \left(\ln\sqrt{1-\frac{2M}{r}+\frac{Q^2}{r^2}-\frac{\sigma Q^4}{5r^6}}\right)_{,r} \\
\xi^{1}{}_{11} &= \left(\ln(\frac{1}{\sqrt{1-\frac{2M}{r}+\frac{Q^2}{r^2}-\frac{\sigma Q^4}{5r^6}}})\right)_{,r} \\
\frac{1}{\sin^2\theta}\xi^{1}{}_{33} = \xi^{1}{}_{22} &= \frac{-r}{\sqrt{1-\frac{2M}{r}+\frac{Q^2}{r^2}-\frac{\sigma Q^4}{5r^6}}} \\
\xi^{2}{}_{21} = \xi^{3}{}_{31} &= \frac{1}{r} \\
\xi^{3}{}_{32} = \xi^{3}{}_{23} &= \cot\theta \\
\xi^{2}{}_{33} &= -\sin\theta\cos\theta \\
\xi^{2}{}_{12} = \xi^{3}{}_{13} &= \frac{1}{r\sqrt{1-\frac{2M}{r}+\frac{Q^2}{r^2}-\frac{\sigma Q^4}{5r^6}}}
\end{aligned}
\tag{25}
$$

where a comma followed by a coordinate denotes a derivative in relation to that coordinate. Next, the required non-vanishing component of $U_{\mu}^{\nu\beta}$ is [40, 41]

$$U_{0}^{01} = \frac{2\sin\theta}{\kappa}(M-\frac{Q^2}{r}+\frac{\sigma Q^4}{5r^5}) \tag{26}$$

From this point of view, using equation (21) with this result the energy of the dyadosphere of a charged black hole is found as

$$E(r) = M-\frac{Q^2}{r}+\frac{\sigma Q^4}{5r^5} \tag{27}$$

5. Discussion

There are several calculations on the energy(due to matter plus fields) distribution of charged black holes. The energy distribution was found that it depends on the mass M and the charge Q. For example; Chamorro-Virbhadra [15] and Xulu [18] showed, considering the general relativity analogs of Einstein and Møller's definitions, that the energy of a charged dilation black hole depends on the value ω which controls the coupling between the dilation and the Maxwell fields.

$$E_{Einstein} = M-\frac{Q^2}{2r}(1-\omega^2) \tag{28}$$

$$E_{Moller} = M-\frac{Q^2}{r}(1-\omega^2) \tag{29}$$

Also, Virbhadra [14] and Xulu [18] obtained that the energy distribution in the sense of Einstein and Møller disagree in general.

In this study, to calculate the energy distribution associated with the dyadosphere of a charged black hole, we investigated the Møller energy-momentum definition in both general relativity and teleparallel gravity. The energy is found the same in both approximations. It is also independent of the teleparallel dimensionless coupling constant, which means that it is valid not only in teleparallel equivalent of general relativity but also in any teleparallel model. In the previous paper Xulu [42], using the Einstein, Landau-Lifshitz, Papapetrou and Weinberg prescriptions found that the energy associated with the dyadosphere of a charged black hole is

$$E_{Eins} = E_{LL} = E_{Pap} = E_{Wein} = M - \frac{Q^2}{2r} + \frac{\sigma Q^4}{10r^5} \tag{30}$$

It is obvious that in the dyadosphere region (where r is small) the last term of energy definition plays a very important role. As expected, $\sigma = 0$ gives the energy distribution

$$E_{Eins} = E_{LL} = E_{Pap} = E_{Wein} = M - \frac{Q^2}{2r}, \qquad E_{Mol} = M - \frac{Q^2}{r} \tag{31}$$

which is for the Reissner-Nordström metric. At the large distances, ours and Xulu's results give the same energy distribution given below.

$$\lim_{r\to\infty} E(r) = M \tag{32}$$

According to the Cooperstock hypothesis [43], the energy is confined to the region of non-vanishing energy-momentum tensor of matter and all non-gravitational fields.

Finally, this paper sustains (a) the importance of the energy-momentum definitions in the evaluation of the energy distribution of a given space-time and (b) the viewpoint of Lessner [44] that the Møller energy-momentum complex is a powerful concept for energy and momentum.

Acknowledgments

We would The Turkish Scientific and Technical Research Council (Tübitak)-Feza Gürsey Institute, Istanbul, for the hospitality we received in summer terms 2002-2005. The work of MS was supported by Tübitak.

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