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Integral Equations and Operator Theory

Toeplitz Operators on Arveson and Dirichlet Spaces

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Abstract. We define Toeplitz operators on all Dirichlet spaces on the unit ball of $\mathbb{C}^N$ and develop their basic properties. We characterize bounded, compact, and Schatten-class Toeplitz operators with positive symbols in terms of Carleson measures and Berezin transforms. Our results naturally extend those known for weighted Bergman spaces, a special case applies to the Arveson space, and we recover the classical Hardy-space Toeplitz operators in a limiting case; thus we unify the theory of Toeplitz operators on all these spaces. We apply our operators to a characterization of bounded, compact, and Schatten-class weighted composition operators on weighted Bergman spaces of the ball. We lastly investigate some connections between Toeplitz and shift operators.

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1. Introduction

The theory of Toeplitz operators on Bergman spaces on the unit ball in one and several variables is a well-established subject. Weighted Bergman spaces A^2_q with $q > -1$ are naturally imbedded in Lebesgue classes L^2_q by the *inclusion* i, and there are sufficiently many Bergman projections from Lebesgue classes onto Bergman spaces. Then one defines the Toeplitz operator $T_\phi : A^2_q \to A^2_q$ with symbol ϕ by $T_\phi = P_q M_\phi i$, where M_ϕ is the operator of multiplication by ϕ and P_q is the orthogonal projection from L^2_q onto A^2_q, a Bergman projection. Investigating the boundedness and compactness of these Toeplitz operators with symbols in various

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classes of functions has been an active area of research. A good source, especially for positive ϕ, is [37, Chapter 6].

By contrast, there is not one single definition of a Toeplitz operator that is agreed upon even on the classical Dirichlet space of the disc. The papers [11], [12], [14], [20], [26], [32], [35], [36] discuss several different kinds of Toeplitz operators on the Dirichlet space. The connections among them, and between them and the Toeplitz operators on Bergman spaces are not clear. Only [26] deals with the Dirichlet space on the ball, and only [32] and [35] can handle the more general Dirichlet spaces $\mathcal{D}_q$ but for limited values of q, those between the Dirichlet space and the Hardy space. To the best of our knowledge, there is no work on Toeplitz operators on the Arveson space, not to mention one that can encompass all Dirichlet spaces $\mathcal{D}_q$ on the unit ball.

There are some difficulties with Toeplitz operators on Dirichlet spaces that are not Bergman spaces, and these are the causes for discrepancies in various definitions used. The first is that inclusion does not imbed these spaces in the most appropriate Lebesgue classes. The second is to decide which projections to use from which Lebesgue classes. Thus one sees in literature Toeplitz operators $T_\phi f$ defined via an integral that involve f or its derivatives, or ϕ or its derivatives, or the Bergman, Hardy, or Dirichlet kernels or their derivatives. A third difficulty is that reproducing kernels of $\mathcal{D}_q$ for a large range of q are bounded and their normalized forms are not weakly convergent. This makes them impossible to use for obtaining a Berezin transform and perhaps explains why this range of q is never touched upon.

The difficulties are resolved by recognizing Dirichlet spaces $\mathcal{D}_q$ on the ball as the Besov spaces B_q^2, where $q \in \mathbb{R}$ is adjusted so that $\mathcal{D}_q = A_q^2$ when $q > -1$. These spaces are defined by imbedding them into Lebesgue classes via the linear maps $I_s^t f(z) = (1-|z|^2)^t D_s^t f(z)$, where D_s^t is a radial differential operator of sufficiently high order t with $q + 2t > -1$. Extended Bergman projections P_s that map Lebesgue classes boundedly onto Dirichlet spaces can be precisely identified as in the case of weighted Bergman spaces by $q + 1 < 2(s+1)$. Then I_s^t is a right inverse to P_s. This is all done in [22].

Now for *all* $q \in \mathbb{R}$, we define the Toeplitz operator ${}_sT_\phi : \mathcal{D}_q \to \mathcal{D}_q$ with symbol ϕ by ${}_sT_\phi = P_s M_\phi I_s^{-q+s}$. When $q > -1$, the case of weighted Bergman spaces, $s = q$ is classical, but when $q \leq -1$, s must satisfy $-q + 2s > -1$, so $s \neq q$. It is possible to take $s \neq q$ also when $q > -1$. So we have more general Toeplitz operators defined via I_s^{-q+s} strictly on Bergman spaces too. It turns out that the properties of ${}_sT_\phi$ studied in this paper are independent of s and q. The results we obtain on the boundedness, compactness, and membership in Schatten classes of ${}_sT_\phi$ for $\phi \geq 0$ specialize to what is known for weighted Bergman spaces when $s = q$. Our main tools are Carleson measures and Berezin transforms. The first is defined via I_s^t rather than i; the second is defined via weakly convergent families in all $\mathcal{D}_q$ that are actually Bergman reproducing kernels with different normalizations. These Carleson measures and weakly convergent families for all $\mathcal{D}_q$ are studied first in [23].

More is true. The space $\mathcal{D}_{-1}$ is the Hardy space H^2. Now $s > -1$ must hold, so $s \neq -1$, and hence ${}_sT_\phi$ is not the classical Toeplitz operator on H^2. However, as $s \to -1^+$, we indeed recover the classical Toeplitz operators on H^2. We thereby present a unified theory of Toeplitz operators on all Dirichlet and Bergman spaces, the Arveson space, and the Hardy space.

The paper is organized as follows. The notation and some preliminary material are summarized in Section 2. Section 3 is for groundwork on Dirichlet spaces, Bergman projections on them, their imbeddings, and the differential operators between them, on which so much of this work rests. In Section 4, we define Toeplitz operators on all $\mathcal{D}_q$ and develop several of their elementary properties. An intertwining relation between Toeplitz operators on $\mathcal{D}_q$ and the classical ones on weighted Bergman spaces turns out to be versatile. We introduce the Berezin transforms in Section 5 and obtain some of their immediate consequences. We then explore the connection with the classical Hardy-space Toeplitz operators. Our main results are in Section 6. We characterize bounded, compact, and Schatten-class Toeplitz operators with positive symbols. We work more generally with Toeplitz operators whose symbols are positive measures. The results in Sections 4, 5, and 6 attest to the fact that the Toeplitz operators on general $\mathcal{D}_q$ are natural extensions of classical Bergman-space Toeplitz operators. Section 7 describes an important application of Toeplitz operators on $\mathcal{D}_q$. We readily obtain characterizations of bounded, compact, and Schatten-class weighted composition operators on weighted Bergman spaces on the ball in terms of Carleson measures and Berezin transforms. The paper concludes with some remarks on the relationship between Toeplitz and shift operators in Section 8.

2. Notation and Preliminaries

The unit ball of $\mathbb{C}^N$ is denoted $\mathbb{B}$, and the volume measure ν on it is normalized with $\nu(\mathbb{B}) = 1$. When $N = 1$, it is the unit disc $\mathbb{D}$. For $c \in \mathbb{R}$, we define on $\mathbb{B}$ also the measures

$$d\nu_c(z) = (1 - |z|^2)^c \, d\nu(z),$$

which are finite only for $c > -1$, where $|z|^2 = \langle z, z\rangle$ and $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_N\overline{w}_N$. In particular, we set $\tau = \nu_{-(N+1)}$. The associated Lebesgue classes are L^p_c, and L^∞ simply is the class of bounded measurable functions on $\mathbb{B}$.

If X is a set, then $\overline{X}$ denotes its closure and ∂X its boundary. We let $\mathcal{C}$ be the space of continuous functions on $\overline{\mathbb{B}}$ and $\mathcal{C}_0$ its subspace whose members vanish on $\partial\mathbb{B}$. If T is a Hilbert-space operator, then $\sigma(T)$ denotes its spectrum and $\sigma_p(T)$ its point spectrum.

In multi-index notation, $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ is an N-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$, and $0^0 = 1$. The symbol δ_{nm} denotes the Kronecker delta.

Constants in formulas are all denoted by unadorned C although each might have a different value. They might depend on certain parameters, but are always independent of the functions that appear in the formulas.

We use the convenient Pochhammer symbol defined by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$$

when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function Γ. For fixed a, b, Stirling formula gives

$$\frac{\Gamma(c+a)}{\Gamma(c+b)} \sim c^{a-b} \qquad \text{and} \qquad \frac{(a)_c}{(b)_c} \sim c^{a-b} \qquad (c \to \infty), \tag{2.1}$$

where $x \sim y$ means that both $|x| \leq C\,|y|$ and $|y| \leq C\,|x|$, and above such C are independent of c. The hypergeometric function is

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k} \frac{x^k}{k!} \qquad (|x| < 1).$$

The Bergman metric on $\mathbb{B}$ is

$$d(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|} = \tanh^{-1} |\varphi_z(w)| \qquad (z, w \in \mathbb{B}),$$

where $\varphi_z(w)$ is the Möbius transformation on $\mathbb{B}$ that exchanges z and w; see [33, §2.2]. The ball centered at w with radius $0 < r < \infty$ in the Bergman metric is denoted $b(w, r)$. The Bergman ball $b(0, r)$ is also the Euclidean ball with the same center and radius $0 < \tanh r < 1$. The Bergman metric is invariant under compositions with the automorphisms of $\mathbb{B}$, hence $\psi(b(w, r)) = b(\psi(w), r)$ for any $\psi \in \mathrm{Aut}(\mathbb{B})$. Bergman balls have the following properties, whose proofs can be found in [24, §2].

Lemma 2.1. *Given $c \in \mathbb{R}$ and r, we have*

$$\nu_c(b(w, r)) \sim (1 - |w|^2)^{N+1+c} \qquad (w \in \mathbb{B}).$$

Given also $w \in \mathbb{B}$, we have

$$1 - |z|^2 \sim 1 - |w|^2 \qquad \text{and} \qquad |1 - \langle z, w \rangle| \sim 1 - |w|^2 \qquad (z \in b(w, r)).$$

Lemma 2.2. *Given $c \in \mathbb{R}$ and r, there is a constant C such that for all $0 < p < \infty$, $g \in H(\mathbb{B})$, and $w \in \mathbb{B}$, we have*

$$|g(w)|^p \leq \frac{C}{\nu_c(b(w, r))} \int_{b(w,r)} |g|^p \, d\nu_c.$$

Let's note that the measure τ is also invariant under compositions with the members of $\mathrm{Aut}(\mathbb{B})$; see [33, Theorem 2.2.6].

Given $0 < r < \infty$, we call a sequence $\{a_n\}$ of points in $\mathbb{B}$ an *r-lattice* in $\mathbb{B}$ if the union of the balls $\{b(a_n, r)\}$ cover $\mathbb{B}$ and $d(a_n, a_m) \geq r/2$ for $n \neq m$. The second condition controls the amount of cover so that any point in $\mathbb{B}$ belongs to at most M of the balls $\{b(a_n, 2r)\}$ for some M that does not depend on anything. That r-lattices exist is proved for the unit disc in [7, Lemma 3.5].

A twice differentiable function f on $\mathbb{B}$ satisfying $\Delta(f \circ \varphi_z)(0) = 0$ for all $z \in \mathbb{B}$ is called $\mathcal{M}$-*harmonic*, where Δ is the usual Laplacian on $\mathbb{R}^{2N}$, and φ_z is the Möbius transformation of $\mathbb{B}$ mentioned above. If f is $\mathcal{M}$-harmonic, so is $f \circ \psi$ for any $\psi \in \mathrm{Aut}(\mathbb{B})$. If f is $\mathcal{M}$-harmonic, then the mean value of f on a sphere of radius less than 1 is equal to $f(0)$; see [33, p. 52]. If additionally $f \in L^1_c$ for $c > -1$, it follows that

$$f(\psi(0)) = \frac{(1+c)_N}{N!} \int_{\mathbb{B}} (f \circ \psi)\, d\nu_c \qquad (\psi \in \mathrm{Aut}(\mathbb{B}))$$

by polar coordinates. Now we pick $\psi = \varphi_w$, make a change of variables in the integral using formula [33, Theorem 2.2.6 (6)] for the Jacobian of ϕ_w, and use identity [33, Theorem 2.2.2 (iv)] to simplify. The result is

$$f(w) = \frac{(1+c)_N}{N!} (1-|w|^2)^{N+1+c} \int_{\mathbb{B}} \frac{(1-|z|^2)^c}{|1-\langle z,w\rangle|^{(N+1+c)2}} f(z)\, d\nu(z). \tag{2.2}$$

The right hand side is seen to be a Berezin transform of f in Section 5.

3. Dirichlet Spaces

Dirichlet spaces are Hilbert spaces of holomorphic functions on $\mathbb{B}$. We give three equivalent definitions each of which has its use. The index $q \in \mathbb{R}$ is everywhere unrestricted.

Definition 3.1a. The *Dirichlet space* $\mathcal{D}_q$ is the reproducing kernel Hilbert space on $\mathbb{B}$ with reproducing kernel

$$K_q(z,w) = \begin{cases} \dfrac{1}{(1-\langle z,w\rangle)^{N+1+q}} = \displaystyle\sum_{k=0}^{\infty} \frac{(N+1+q)_k}{k!} \langle z,w\rangle^k, & \text{if } q > -(N+1); \\ \dfrac{{}_2F_1(1,1;1-N-q;\langle z,w\rangle)}{-N-q} = \displaystyle\sum_{k=0}^{\infty} \frac{k!\,\langle z,w\rangle^k}{(-N-q)_{k+1}}, & \text{if } q \le -(N+1). \end{cases}$$

Thus $\mathcal{D}_q$ for $q > -1$ are the weighted Bergman spaces A^2_q, $\mathcal{D}_{-1}$ is the Hardy space H^2, $\mathcal{D}_{-N}$ is the Arveson space $\mathcal{A}$ (see [1] and [4]), and $\mathcal{D}_{-(N+1)}$ is the classical Dirichlet space $\mathcal{D}$ since

$$K_{-(N+1)}(z,w) = \frac{1}{\langle z,w\rangle} \log \frac{1}{1-\langle z,w\rangle}.$$

The hypergeometric kernels appear in [10, p. 13]. The kernels K_q are complete Nevanlinna-Pick kernels if and only if $q \le -N$ as explained in [5]. Further, they are bounded if and only if $q < -(N+1)$.

The reproducing kernel K_q is sesqui-holomorphic, $\mathcal{D}_q$ consists of functions in $H(\mathbb{B})$, and monomials are dense in $\mathcal{D}_q$. By (2.1), we have

$$K_q(z,w) \sim \sum_{k=0}^{\infty} k^{N+q} \langle z,w\rangle^k = \sum_{k=0}^{\infty} k^{N+q} \sum_{|\alpha|=k} \frac{k!}{\alpha!}\, z^\alpha \overline{w}^\alpha = \sum_{\alpha} \frac{|\alpha|^{N+q}\,|\alpha|!}{\alpha!}\, z^\alpha \overline{w}^\alpha$$

for any q. Thus

$$\|z^\alpha\|^2_{\mathcal{D}_q} \sim \frac{\alpha!}{|\alpha|^{N+q}\,|\alpha|!} \qquad (\alpha \in \mathbb{N}^N) \tag{3.1}$$

by [6, Theorem 3.3.1]. The norms (3.1) lead to the second equivalent definition of Dirichlet spaces.

Definition 3.1b. The *Dirichlet space* $\mathcal{D}_q$ is the space of $f(z) = \sum_\alpha c_\alpha\, z^\alpha$ in $H(\mathbb{B})$ for which

$$\sum_{\alpha\neq 0} |c_\alpha|^2 \frac{\alpha!}{|\alpha|^{N+q}\,|\alpha|!} < \infty.$$

If $N=1$, the growth rate of the norms in (3.1) is $\|z^n\|_{\mathcal{D}_q} \sim n^{-(1+q)/2}$. For this reason, the $\mathcal{D}_q$ defined here is often named $\mathcal{D}_{-(1+q)}$ or $\mathcal{D}_{-(1+q)/2}$ elsewhere.

The third equivalent definition recognizes that the Dirichlet space $\mathcal{D}_q$ as the Besov space B^2_q as described in [21] and [22]. For comparison, it is also the holomorphic Sobolev space $A^2_{1+q+2t,t}$ of [10], but this must not be confused with the Bergman-space notation A^2_q of ours. But we need to introduce some radial derivatives first.

Let $f \in H(\mathbb{B})$ be given by its homogeneous expansion $f = \sum_{k=0}^{\infty} f_k$, where f_k is a homogeneous polynomial of degree k. Then its radial derivative at z is $Rf(z) = \sum_{k=1}^{\infty} k\, f_k(z)$. In [22, Definition 3.1], for any s,t, the radial differential operator D^t_s is defined on $H(\mathbb{B})$ by $D^t_s f = \sum_{k=0}^{\infty} ({}^t_s d_k) f_k$, where

$${}^t_s d_k = \begin{cases} \dfrac{(N+1+s+t)_k}{(N+1+s)_k}, & \text{if } s > -(N+1),\ s+t > -(N+1); \\ \dfrac{(N+1+s+t)_k\,(-(N+s))_{k+1}}{(k!)^2}, & \text{if } s \le -(N+1),\ s+t > -(N+1); \\ \dfrac{(k!)^2}{(N+1+s)_k\,(-(N+s+t))_{k+1}}, & \text{if } s > -(N+1),\ s+t \le -(N+1); \\ \dfrac{(-(N+s))_{k+1}}{(-(N+s+t))_{k+1}}, & \text{if } s \le -(N+1),\ s+t \le -(N+1). \end{cases}$$

What is important is that

$${}^t_s d_k \neq 0 \quad (k=0,1,2,\ldots) \qquad \text{and} \qquad {}^t_s d_k \sim k^t \quad (k\to\infty)$$

for any s,t. Clearly D^0_s is the identity for any s,

$$D^u_{s+t} D^t_s = D^{u+t}_s, \qquad \text{and} \qquad D^t_s(1) = {}^t_s d_0 \tag{3.2}$$

for any s,t,u. It turns out that each D^t_s is a continuous invertible operator of order t on $H(\mathbb{B})$ with two-sided inverse

$$(D^t_s)^{-1} = D^{-t}_{s+t}. \tag{3.3}$$

Other useful properties are that $D^1_{-N} = I + R$ and $D^t_s(z^\beta) = {}_s^t d_{|\beta|} z^\beta$. The parameters s and t can be complex numbers too; then we just need to replace them with their real parts in inequalities as done in [22].

A script $\mathcal{D}_q$ with only a lower index represents a Dirichlet space while an upper case D^t_s with a lower and an upper index represents a radial differential operator. They should not be confused.

Another property of D^t_s we use without further mention is that it always acts on the holomorphic variable. Hence the series expansion of K_q shows that always

$$D^t_q K_q(z, w) = K_{q+t}(z, w). \tag{3.4}$$

Now we define the linear transformations I^t_s that are essential to this work by

$$I^t_s f(z) = (1 - |z|^2)^t \, D^t_s f(z) \qquad (f \in H(\mathbb{B})).$$

Definition 3.1c. The *Dirichlet space* $\mathcal{D}_q$ is the space of $f \in H(\mathbb{B})$ for which the function $I^t_s f$ belongs to L^2_q for some s and t satisfying

$$q + 2t > -1. \tag{3.5}$$

The L^2_q norm of any such $I^t_s f$ is an equivalent $\mathcal{D}_q$ norm of f.

It is shown in [10, Theorem 5.12 (i)] and [22, Theorem 4.1] that Definition 3.1c is independent of s, t, and that the L^2_q norms of $I^t_s f$ and $I^{t_1}_{s_1} f$ are equivalent, both as long as (3.5) is satisfied by t and t_1. To obtain the equivalence of this definition to the first two definitions of $\mathcal{D}_q$, it suffices to compute the norm of z^α in $\mathcal{D}_q$ in Definition 3.1c and to observe that it has the same growth rate as that of (3.1) as $|\alpha| \to \infty$; see also [10, pp. 13–14]. We use [22, Proposition 2.1] in such norm computations.

Thus $I^t_s : \mathcal{D}_q \to L^2_q$ with t satisfying (3.5) is an isometric imbedding modulo the equivalences of norms in $\mathcal{D}_q$.

Definition 3.1c yields explicit equivalent forms for the inner product of $\mathcal{D}_q$ as

$${}_q[f, g]^t_s = \int_{\mathbb{B}} I^t_s f \, \overline{I^t_s g} \, d\nu_q = [I^t_s f, I^t_s g]_{L^2_q} \qquad (f, g \in \mathcal{D}_q)$$

with t satisfying (3.5). The reproducing property ${}_q[f, K_q(\cdot, w)]^t_s = C\, f(w)$ written explicitly takes the form

$$\int_{\mathbb{B}} D^t_s f(z) \, \overline{D^t_s K_q(z, w)} \, d\nu_{q+2t}(z) = C\, f(w)$$

for the same t, which can be further simplified for $s = q$ using (3.4). We need a constant C in order to accomodate the variation due to s, t. Let's show the norm on $\mathcal{D}_q$ associated to ${}_q[\cdot, \cdot]^t_s$ by ${}_q\| \cdot \|^t_s$.

The following is easy to show, but a proof can be found in [25, §3].

Proposition 3.2. *For any q, s, t, $D^t_s(\mathcal{D}_q) = \mathcal{D}_{q+2t}$ is an isometric isomorphism with appropriate norms on the two spaces; for example, when $\mathcal{D}_q$ has ${}_q\| \cdot \|^u_s$ and $\mathcal{D}_{q+2t}$ has ${}_{q+2t}\| \cdot \|^{u-t}_{s+t}$ while (3.5) is satisfied with u in place of t.*

We would like to know the *adjoint* of $D_s^t : \mathcal{D}_q \to \mathcal{D}_{q+2t}$. Because each Dirichlet space has several equivalent inner products, let's state it explicitly by showing the particular inner products used. It is the operator $(D_s^t)^* : \mathcal{D}_{q+2t} \to \mathcal{D}_q$ satisfying ${}_{q+2t}[D_s^t f, g]_{s+t}^{u-t} = {}_q[f, (D_s^t)^* g]_s^u$ with $q + 2u > -1$ for $f \in \mathcal{D}_q$ and $g \in \mathcal{D}_{q+2t}$. Writing this out in integrals, by the uniqueness of the adjoint and using (3.3) and (3.2), we obtain the somewhat surprising result that

$$(D_s^t)^* = D_{s+t}^{-t} = (D_s^t)^{-1}. \tag{3.6}$$

Bergman projections, as extended in [22], are the linear transformations

$$P_s f(z) = \int_{\mathbb{B}} K_s(z, w)\, f(w)\, d\nu_s(w) \qquad (z \in \mathbb{B})$$

defined for all s with suitable f. The next result is contained in [22, Theorem 1.2].

Theorem 3.3. *The operator $P_s : L_q^2 \to \mathcal{D}_q$ is bounded if and only if*

$$-q + 2s > -1. \tag{3.7}$$

Given an s satisfying (3.7)*, if t satisfies* (3.5)*, then*

$$P_s I_s^t f = \frac{N!}{(1+s+t)_N}\, f =: \frac{1}{C_{s+t}}\, f \qquad (f \in \mathcal{D}_q).$$

The second statement clearly shows that P_s is onto whenever it is bounded. Note that (3.7) and (3.5) together imply $s + t > -1$ so that $1 + s + t$ does not hit a pole of Γ and $C_{s+t} > 0$. If $q > -1$, we can take $t = 0$, then $I_s^0 = i$, and Theorem 3.3 reduces to the classical result on Bergman spaces. The next result is proved in [25, §5].

Proposition 3.4. *If $P_s : L_q^2 \to \mathcal{D}_q$ is bounded and the norm on $\mathcal{D}_q$ is ${}_q\|\cdot\|_s^t$, then*

$$\|P_s\| = \frac{N!\,\sqrt{\Gamma(1-q+2s)\,\Gamma(1+q+2t)}}{\Gamma(N+1+s+t)}.$$

We often write the inequalities (3.7) and (3.5) in the form $q + 1 < p(s+1)$ and $q + pt > -1$ when we consider the general family of B_q^p or A_q^p spaces and Lebesgue classes L_q^p.

Theorem 3.3 states that the composition $P_s I_s^t : \mathcal{D}_q \to \mathcal{D}_q$ is a constant times the identity with s, t satisfying (3.7) and (3.5). The composition $I_s^t P_s : L_q^2 \to L_q^2$ in reverse order is also important in our analysis of Toeplitz operators. Starting with differentiation under the integral sign and (3.4), the following result is compiled from [22, §5] and [19, Theorem 1.9].

Theorem 3.5. *The operator $I_s^t P_s : L_q^2 \to L_q^2$ is bounded if and only if s, t satisfy* (3.7) *and* (3.5)*, and in that case, it is the operator*

$$V_s^t f(z) = (1 - |z|^2)^t \int_{\mathbb{B}} \frac{(1 - |w|^2)^s}{(1 - \langle z, w\rangle)^{N+1+s+t}}\, f(w)\, d\nu(w) \qquad (f \in L_q^2).$$

Note again that (3.7) and (3.5) together imply $s + t > -1$ so that K_{s+t} is binomial. Now we have the operator equalities

$$C_{s+t} P_s I_s^t = I, \quad I_s^t P_s = V_s^t, \quad C_{s+t} V_s^t I_s^t = I_s^t, \quad \text{and} \quad C_{s+t} P_s V_s^t = P_s. \tag{3.8}$$

Analogous equalities appear, for example, in [38, Lemma 20] for $q > -1$.

The adjoint $(V_s^t)^* : L_q^2 \to L_q^2$ of V_s^t is computed using Fubini theorem and is

$$(V_s^t)^* = V_{q+t}^{-q+s}. \tag{3.9}$$

Hence V_s^t is self-adjoint on L_q^2 if and only if

$$s - t = q. \tag{3.10}$$

Let q be given. If s satisfies (3.7), then the value of t obtained from (3.10) satisfies (3.5). Conversely, if t satisfies (3.5), then the value of s obtained from (3.10) satisfies (3.7).

Notation 3.6. Henceforth given a q, we select s so as to satisfy (3.7), and put

$$Q = -q + 2s \qquad \text{and} \qquad u = -q + s. \tag{3.11}$$

in the remaining part of the paper. Note that

$$Q = s + u = q + 2u > -1$$

so that $\mathcal{D}_Q = A_Q^2$. We use only the self-adjoint V_s^u in order to have Toeplitz operators that are direct extensions of classical Bergman-space Toeplitz operators and to have exact equalities as much as possible. Also we use only the inner product $[\cdot,\cdot]_{D_q} = {}_q[\cdot,\cdot]_s^u$ and the corresponding norm

$$\|f\|_{\mathcal{D}_q}^2 = [f,f]_{D_q} = [I_s^u f, I_s^u f]_{L_q^2} = \|I_s^u f\|_{L_q^2}^2 = \|D_s^u f\|_{L_Q^2}^2 = \int_{\mathbb{B}} |D_s^u f|^2 \, d\nu_Q \tag{3.12}$$

in $\mathcal{D}_q$. This is a genuine norm, that is, the only function whose norm is 0 is the one that is identically 0. If $q > -1$, it is standard to use $u = 0$. Finally, we redefine the Bergman projections $P_s : L_q^2 \to \mathcal{D}_q$ by multiplying them by C_Q as done in [16, (7)]. Then (3.8) takes the form

$$P_s I_s^u = I, \quad I_s^u P_s = C_Q V_s^u, \quad C_Q V_s^u I_s^u = I_s^u, \quad \text{and} \quad C_Q P_s V_s^u = P_s. \tag{3.13}$$

Lastly $\|P_s\| = 1$ now by Proposition 3.4.

The adjoint $P_s^* : \mathcal{D}_q \to L_q^2$ of P_s can now be computed. If $g \in L_q^2$ and $f \in \mathcal{D}_q$, then

$$[P_s g, f]_{\mathcal{D}_q} = [I_s^u P_s g, I_s^u f]_{L_q^2} = C_Q\,[V_s^u g, I_s^u f]_{L_q^2} = C_Q\,[g, V_s^u I_s^u f]_{L_q^2} = [g, I_s^u f]_{L_q^2}$$

by (3.12), (3.13), (3.9), and (3.10). Thus $P_s^* = I_s^u$. The same computation read backwards shows that the adjoint $(I_s^u)^* : L_q^2 \to \mathcal{D}_q$ of I_s^u is $(I_s^u)^* = P_s$. More generally, the Banach space adjoints of $P_s : L_q^p \to B_q^p$ are computed with respect to more general asymmetric pairings in Besov spaces in [22, Theorem 5.3]. Summarizing,

$$(V_s^u)^* = V_s^u, \qquad P_s^* = I_s^u, \qquad \text{and} \qquad (I_s^u)^* = P_s. \tag{3.14}$$

In particular, with the inclusion $i = I_s^0 : A_Q^2 \to L_Q^2$, we have

$$P_Q^* = i \qquad \text{and} \qquad i^* = P_Q. \tag{3.15}$$

This might seem unusual, but we remind that the target space of P_Q here is A_Q^2, and not L_Q^2 as it is commonly taken.

Let $M_\phi : L_q^2 \to L_q^2$ be the operator of multiplication by a suitable measurable, say L^∞, function ϕ on $\mathbb{B}$. Its adjoint $M_\phi^* : L_q^2 \to L_q^2$ is clearly $M_\phi^* = M_{\overline{\phi}}$. What is more interesting is that the adjoint $M^*_{(1-|z|^2)^u} : L_q^2 \to L_Q^2$ of the particular multiplication operator $M_{(1-|z|^2)^u} : L_Q^2 \to L_q^2$ turns out to be

$$M^*_{(1-|z|^2)^u} = M_{(1-|z|^2)^{-u}}$$

simply by writing out the definition of the adjoint. Now we have one more way to compute the adjoint of $I_s^u = M_{(1-|z|^2)^u} i D_s^u : \mathcal{D}_q \to L_q^2$, where $D_s^u : \mathcal{D}_q \to A_Q^2$, i is the inclusion $i : A_Q^2 \to L_Q^2$, and the multiplication is as just discussed. Then by (3.6), (3.15), the above remarks, differentiating under the integral sign, and (3.4), we reobtain that

$$\begin{aligned}(I_s^u)^* f(z) &= (D_s^u)^* i^* M^*_{(1-|z|^2)^u} f(z) = D_Q^{-u} P_Q M_{(1-|z|^2)^{-u}} f(z) \\ &= C_Q\, D_Q^{-u} \int_{\mathbb{B}} \frac{(1-|w|^2)^{Q-u}}{(1-\langle z,w\rangle)^{N+1+Q}}\, f(w)\, d\nu(w) \\ &= C_Q \int_{\mathbb{B}} K_s(z,w)\, f(w)\, d\nu_s(w) = P_s f(z).\end{aligned}$$

Example 3.7. We repeat [24, Remark 4.8] in our notation. We need it when we define Berezin transforms in Section 5. Given a q, pick an s satisfying (3.7), recall that $Q > -1$, let $w \in \mathbb{B}$, and put

$${}_q g_w(z) = \frac{K_s(z,w)}{\|K_s(\cdot,w)\|_{\mathcal{D}_q}} = \sqrt{C_Q}\,(1-|w|^2)^{(N+1+Q)/2} K_s(z,w) \qquad (z \in \mathbb{B}).$$

Then obviously $\|{}_q g_w\|_{\mathcal{D}_q} = 1$ for all $w \in \mathbb{B}$. Thus ${}_q g_w$ is essentially a normalized reproducing kernel; but although the kernel K_s is that of $\mathcal{D}_s$, the normalization is done with respect to the norm of $\mathcal{D}_q$.

The kernels $K_q(\cdot,w)$ and $K_s(\cdot,w)$ have the reproducing properties

$$[f, K_q(\cdot,w)]_{\mathcal{D}_q} = C\, f(w) \qquad \text{and} \qquad [f, K_s(\cdot,w)]_{\mathcal{D}_q} = \frac{1}{C_Q}\, D_s^u f(w)$$

in $\mathcal{D}_q$. The second property parallels the fact that ${}_q g_w \to 0$ weakly in $\mathcal{D}_q$ by [24, Theorems 4.3 and 4.4], which relate weak convergence in $\mathcal{D}_q$ to convergence of certain derivatives. This relationship is further mirrored in

$$D_s^u({}_q g_w)(z) = \sqrt{C_Q}\, \frac{(1-|w|^2)^{(N+1+Q)/2}}{(1-\langle z,w\rangle)^{N+1+Q}} = \frac{K_Q(z,w)}{\|K_Q(\cdot,w)\|_{\mathcal{D}_Q}} =: {}_Q k_w(z),$$

which defines ${}_Q k_w \in A_Q^2$.

When $q > -1$, then $s = q$ satisfies (3.7), and ${}_q g_w(z)$ is nothing but the normalized reproducing kernel of the Bergman space A^2_q. When $q \leq -1$, we can use $s = 0$ or $Q = 0$ for simplicity in ${}_q g_w(z)$.

4. Toeplitz Operators

In this section, we define the Toeplitz operators on all $\mathcal{D}_q$ and obtain their several elementary properties. The main theme is that they extend and preserve the character of classical Toeplitz operators on weighted Bergman spaces. Theorem 3.3 forces us to define them as follows.

Definition 4.1. Let q, an s satisfying (3.7), and a measurable function ϕ on $\mathbb{B}$ be given. We define the *Toeplitz operator* ${}_sT_\phi : \mathcal{D}_q \to \mathcal{D}_q$ with *symbol* ϕ as the composition ${}_sT_\phi = P_s M_\phi I^u_s$ of linear operators, where u is as in (3.11).

When $q > -1$, a value of s satisfying (3.7) is $s = q$, whence $u = 0$. Then I^0_q is inclusion, and ${}_sT_\phi$ reduces to the classical Toeplitz operator ${}_qT_\phi = P_q M_\phi i$ on the Bergman space $A^2_q = \mathcal{D}_q$. We use the term *classical* to mean a Toeplitz operator with $i = I^0_q$. The value $s = q$ does not work when $q \leq -1$, but we can use $s = 0$ or $Q = 0$ for simplicity for any such q, and for the latter $C_0 = 1$. So by introducing ${}_sT_\phi$ in Definition 4.1, we not only are able to handle all Dirichlet spaces, but also study several generalized Toeplitz operators indexed by s even on a single Bergman space. One of our aims below is to show that the essential features of ${}_sT_\phi$ are unaffected by any s satisfying (3.7).

Hankel-Toeplitz operators with analytic symbols on weighted Bergman spaces of the unit disc that employ Cauchy-Riemann operators resembling I^u_s are investigated in [36].

Explicitly,

$$\begin{aligned}{}_sT_\phi f(z) &= C_Q \int_{\mathbb{B}} K_s(z,w)\,\phi(w)\,(1-|w|^2)^{2u}\,D^u_s f(w)\,d\nu_q(w)\\ &= C_Q \int_{\mathbb{B}} K_s(z,w)\,\phi(w)\,D^u_s f(w)\,d\nu_Q(w) \qquad (f \in \mathcal{D}_q).\end{aligned}$$

We see that ${}_sT_\phi f$ makes sense if $\phi \in L^1_Q$ and f is a polynomial. Hence ${}_sT_\phi$ is a densely defined possibly unbounded operator on $\mathcal{D}_q$ for such ϕ, because polynomials are dense in each $\mathcal{D}_q$. It is also clear that the map $\phi \mapsto {}_sT_\phi$ is linear.

Proposition 4.2. *If $\phi \in L^\infty$, then ${}_sT_\phi$ is bounded with $\|{}_sT_\phi\| \leq \|\phi\|_{L^\infty}$.*

Proof. Taking $f \in \mathcal{D}_q$ and using $\|P_s\| = 1$,

$$\|{}_sT_\phi f\|_{\mathcal{D}_q} = \|P_s M_\phi I^u_s f\|_{\mathcal{D}_q} \leq \|\phi\, I^u_s f\|_{L^2_q} \leq \|\phi\|_{L^\infty}\,\|I^u_s f\|_{L^2_q} = \|\phi\|_{L^\infty}\,\|f\|_{\mathcal{D}_q},$$

as desired. ∎

Remark 4.3. If $f \in \mathcal{D}_q$, then $D_s^u f \in \mathcal{D}_Q = A_Q^2 \subset L_Q^2$ by Proposition 3.2. If $\phi \in L^\infty$, from its integral form, we surmise that ${}_sT_\phi f$ makes sense even when $D_s^u f$ belongs to the larger space L_Q^1 since also $\phi\, D_s^u f \in L_Q^1$. This is typical of objects defined through Bergman projections, because $K_s(z,\cdot)$ is bounded for each z for any s.

Having obtained the integral form for ${}_sT_\phi$, we can now define Toeplitz operators on $\mathcal{D}_q$ with symbols that are measures on $\mathbb{B}$. If μ is Borel measure on $\mathbb{B}$ and u is as in (3.11), we let

$$d\kappa(w) = (1-|w|^2)^{2u}\, d\mu(w),$$

and define

$$\begin{aligned} {}_sT_\mu f(z) &= C_Q \int_{\mathbb{B}} K_s(z,w)\,(1-|w|^2)^{2u}\, D_s^u f(w)\, d\mu(w) \\ &= C_Q \int_{\mathbb{B}} K_s(z,w)\, D_s^u f(w)\, d\kappa(w) \qquad (f \in \mathcal{D}_q). \end{aligned}$$

The operator ${}_sT_\mu$ is more general and reduces to ${}_sT_\phi$ when $d\mu = \phi\, d\nu_q$. It makes sense when κ is finite and f is a polynomial. Like ${}_sT_\phi$, it is a densely defined possibly unbounded operator on $\mathcal{D}_q$ for finite κ. Note that μ need not be finite in conformity with that q is unrestricted.

We develop basic properties of ${}_sT_\phi$ and ${}_sT_\mu$ in this section. We can assume ϕ and μ are such that the corresponding Toeplitz operators are bounded. First, if $\phi \equiv \lambda$, then ${}_sT_\lambda = \lambda\, I$ for any s by (3.13). Next,

$${}_sT_\phi^* = (I_s^u)^* M_\phi^* P_s^* = P_s M_{\overline{\phi}} I_s^u = {}_sT_{\overline{\phi}}$$

by (3.14). So ${}_sT_\phi$ is self-adjoint if ϕ is real-valued a.e. in $\mathbb{B}$.

By (3.14) again,

$$\begin{aligned} [{}_sT_\phi f, f]_{\mathcal{D}_q} &= [P_s M_\phi I_s^u f, f]_{\mathcal{D}_q} = [M_\phi I_s^u f, I_s^u f]_{L_q^2} \\ &= \int_{\mathbb{B}} \phi\, |D_s^u f|^2\, d\nu_Q \qquad (f \in \mathcal{D}_q). \end{aligned} \tag{4.1}$$

Also $\big|[{}_sT_\phi f, f]_{\mathcal{D}_q}\big| \le \|\phi\|_{L^\infty}\, \|f\|_{\mathcal{D}_q}^2$ if $\phi \in L^\infty$. Similarly,

$$[{}_sT_\mu f, f]_{\mathcal{D}_q} = \int_{\mathbb{B}} |D_s^u f|^2\, d\kappa \qquad (f \in \mathcal{D}_q). \tag{4.2}$$

Proposition 4.4. *If $\phi \ge 0$ a.e. in $\mathbb{B}$, then ${}_sT_\phi$ is a positive operator. If μ is a positive measure, then ${}_sT_\mu$ is a positive operator.*

We now present a very useful intertwining relation for transforming certain problems for Toeplitz operators on Besov spaces to similar problems for classical Toeplitz operators on Bergman spaces.

Theorem 4.5. *We have* $D_s^u({}_sT_\phi) = ({}_QT_\phi)D_s^u$ *and* $D_s^u({}_sT_\mu) = ({}_QT_\kappa)D_s^u$*, where* ${}_QT_\phi = P_QM_\phi i$ *and* ${}_QT_\kappa = C_Q \int_{\mathbb{B}} K_Q(z,w)\, f(w)\, d\kappa(w)$ *are classical Toeplitz operators on* A_Q^2*. Consequently*

$$ {}_sT_\phi = D_Q^{-u}({}_QT_\phi)D_s^u, \qquad {}_sT_\mu = D_Q^{-u}({}_QT_\kappa)D_s^u, $$

and

$$ {}_QT_\phi = D_s^u({}_sT_\phi)D_Q^{-u}, \qquad {}_QT_\kappa = D_s^u({}_sT_\mu)D_Q^{-u}, $$

where

$$ D_Q^{-u} = (D_s^u)^{-1} = (D_s^u)^* $$

by (3.6)*. In other words,* ${}_sT_\phi : \mathcal{D}_q \to \mathcal{D}_q$ *and* ${}_QT_\phi : A_Q^2 \to A_Q^2$ *are unitarily equivalent, and so are* ${}_sT_\mu$ *and* ${}_QT_\kappa$*. Said differently, the following diagrams commute:*

$$ \begin{array}{ccc} A_Q^2 & \xrightarrow{{}_QT_\phi} & A_Q^2 \\ {\scriptstyle D_s^u}\uparrow & & \uparrow{\scriptstyle D_s^u} \\ \mathcal{D}_q & \xrightarrow{{}_sT_\phi} & \mathcal{D}_q \end{array} \qquad\qquad \begin{array}{ccc} A_Q^2 & \xrightarrow{{}_QT_\kappa} & A_Q^2 \\ {\scriptstyle D_s^u}\uparrow & & \uparrow{\scriptstyle D_s^u} \\ \mathcal{D}_q & \xrightarrow{{}_sT_\mu} & \mathcal{D}_q \end{array} $$

Proof. By differentiation under the integral sign and (3.4), if $\phi \in L^\infty$, then

$$ \begin{aligned} D_s^u({}_sT_\phi f)(z) &= C_Q \int_{\mathbb{B}} \frac{\phi(w)}{(1-\langle z,w\rangle)^{N+1+Q}}\, D_s^u f(w)\, d\nu_Q(w) \\ &= P_QM_\phi(D_s^u f)(z) \qquad (f \in \mathcal{D}_q), \end{aligned} $$

because $Q > -1$ so that K_Q is binomial. But $D_s^u f \in A_Q^2$ by Proposition 3.2, where $t = u$, which means that A_Q^2 has norm $\|\cdot\|_{L_Q^2}$. This is the first intertwining relation; the second is identical.

For the second assertion, we note that $(D_s^u)^{-1} = D_Q^{-u}$ by (3.3). The third assertion follows from Proposition 3.2. □

Similar relations can be found in [36, §1] and [12, Lemma 3.1]. They are more limited than ours since $N = 1$ for both, the first is only for Bergman spaces, and the second is only with first-order derivatives.

One property of classical Toeplitz operators on Bergman spaces is that if ϕ is holomorphic, then ${}_QT_\phi = M_\phi$. Theorem 4.5 shows that the corresponding relationship for Toeplitz operators on Besov spaces is not so simple; we have instead ${}_sT_\phi = D_Q^{-u}M_\phi D_s^u$ when ϕ is holomorphic. These are related to Cesàro operators and considered in [24, §11].

Here is an interesting consequence of Theorem 4.5. Recall ${}_sT_\phi = (I_s^u)^*M_\phi I_s^u$ by definition, where $I_s^u : \mathcal{D}_q \to L_q^2$. A similar relationship holds for ${}_sT_\mu$ too when the target space of I_s^u is chosen appropriately.

Theorem 4.6. *Let* $\check{I}_s^u$ *be the operator* $\check{I}_s^u : \mathcal{D}_q \to L^2(\mu)$ *defined by the same formula as* I_s^u*. Then* ${}_sT_\mu = (\check{I}_s^u)^*\check{I}_s^u$*.*

Proof. Let $f, g \in \mathcal{D}_q$. Then $[(\breve{I}_s^u)^* \breve{I}_s^u f, g]_{\mathcal{D}_q} = [\breve{I}_s^u f, \breve{I}_s^u g]_{L^2(\mu)}$, and $D_s^u g \in A_Q^2$ by Proposition 3.2. On the other hand, Theorem 4.5, (3.7), Fubini theorem, and Theorem 3.3 with $t = 0$ yield

$$\begin{aligned}
[{}_sT_\mu f, g]_{\mathcal{D}_q} &= [D_Q^{-u}({}_QT_\kappa) D_s^u f, g]_{\mathcal{D}_q} = [({}_QT_\kappa) D_s^u f, D_s^u g]_{L_Q^2} \\
&= \int_{\mathbb{B}} C_Q \int_{\mathbb{B}} \frac{D_s^u f(w)}{(1 - \langle z, w\rangle)^{N+1+Q}} \, d\kappa(w) \, \overline{D_s^u g(z)} \, d\nu_Q(z) \\
&= \int_{\mathbb{B}} D_s^u f(w) \, \overline{C_Q \int_{\mathbb{B}} \frac{D_s^u g(z)}{(1 - \langle w, z\rangle)^{N+1+Q}} \, d\nu_Q(z)} \, d\kappa(w) \\
&= \int_{\mathbb{B}} D_s^u f(w) \, \overline{D_s^u g(w)} \, d\kappa(w) = [\breve{I}_s^u f, \breve{I}_s^u g]_{L^2(\mu)}.
\end{aligned}$$

By the uniqueness of the adjoint, we are done. □

As a matter of fact, Carleson measures on $\mathcal{D}_q$ are defined in [23] using this $\breve{I}_s^u : \mathcal{D}_q \to L^2(\mu)$, and we use those Carleson measures to characterize ${}_sT_\mu$ with positive μ in Section 6. The classical Bergman-space version of Theorem 4.6 is in [27, §1], where the inclusion $R : A_0^2 \to L^2(\mu)$ is used in place of $\breve{I}_s^u$.

The effects of the choice for u are evident in the results obtained so far. Other t would not yield these expected properties. We see more effects below.

Every property of Toeplitz operators obtained above can also be derived from Theorem 4.5 and the corresponding property of classical Bergman-space Toeplitz operators. We prove several other properties employing the same instrument.

Proposition 4.7. *If $\psi \in H(\mathbb{B})$, then $({}_sT_\phi)({}_sT_\psi) = {}_sT_{\phi\psi}$ and $({}_sT_{\overline{\psi}})({}_sT_\phi) = {}_sT_{\overline{\psi}\phi}$.*

Proof. By Theorem 4.5, a similar result on Bergman-space Toeplitz operators, and Theorem 4.5 again,

$${}_sT_\phi({}_sT_\psi) = D_Q^{-u}({}_QT_\phi) D_s^u D_Q^{-u}({}_QT_\psi) D_s^u = D_Q^{-u}({}_QT_{\phi\psi}) D_s^u = {}_sT_{\phi\psi}.$$

The second identity follows by taking adjoints. □

It also follows that $({}_sT_\psi)({}_sT_\psi) = {}_sT_{\psi^2}$ for $\psi \in H(\mathbb{B})$ or $\overline{\psi} \in H(\mathbb{B})$. We are now in a position to prove a result about the commutants of Toeplitz operators with holomorphic symbols on the disc.

Theorem 4.8. *Suppose $N = 1$. If $\phi \in L^\infty$, $\psi \in H^\infty$ is nonconstant, and ${}_sT_\phi$ and ${}_sT_\psi$ commute on $\mathcal{D}_q$, then $\phi \in H^\infty$.*

Proof. Let $P_Q(\phi) = f$; then $f \in A_Q^2 \cap H^\infty$ and $\phi = f + g$ with g in the orthogonal complement of A_Q^2 in L_Q^2. We let $k = 0, 1, 2, \ldots$ and compute the successive actions of the given Toeplitz operators on $1 \in \mathcal{D}_q$ ordered in two ways. By Theorem 4.5, (3.2), and the proof of [8, Theorem] which is equally valid for weighted Bergman spaces, we obtain

$${}_sT_{\psi^k}({}_sT_\phi)1 = D_Q^{-u}({}_QT_{\psi^k})({}_QT_\phi) D_s^u 1 = D_Q^{-u}(f\,\psi^k)$$

and

$$ {}_sT_\phi({}_sT_{\psi^k})1 = D_Q^{-u}({}_QT_{\phi\psi^k})1 = D_Q^{-u}(f\,\psi^k) + D_Q^{-u}P_Q(g\,\psi^k). $$

Thus $P_Q(g\,\psi^k) = 0$ by (3.3). Let $h \in \mathcal{D}_q$. Then again by the proof of [8, Theorem], we have $g = 0$ and $\phi = f \in H^\infty$. □

Obviously, if $f \equiv 0$, then ${}_sT_\phi f = 0$. And it is clear from the integral form of ${}_sT_\phi$ that if $\phi = 0$ a.e. in $\mathbb{B}$, then ${}_sT_\phi$ is the zero operator. The converses are also true.

Proposition 4.9. *If $\phi \in H(\mathbb{B})$ and $\phi \not\equiv 0$, then ${}_sT_\phi$ is one-to-one on $\mathcal{D}_q$. The map $\phi \mapsto {}_sT_\phi$ is one-to-one.*

Proof. These follow from their classical Bergman-space counterparts, which are in [3], and Theorem 4.5. □

We have already shown that a bounded ϕ gives rise to a bounded ${}_sT_\phi$. It is reasonable to expect that a more restricted ϕ gives rise to a compact ${}_sT_\phi$.

Proposition 4.10. *If $\phi \in L^\infty$ has compact support in $\mathbb{B}$, then ${}_sT_\phi$ is compact. Similarly, if μ is finite and has compact support in $\mathbb{B}$, then ${}_sT_\mu$ is compact. If $\phi \in \mathcal{C}$, then ${}_sT_\phi$ is compact if and only if $\phi \in \mathcal{C}_0$.*

Proof. These all follow from the same classical Bergman-space results (see [37, §6.1], for example), Theorem 4.5, and the fact that a composition of a compact operator with a bounded one is compact. □

5. Berezin Transforms

To develop the theory of Toeplitz operators further, we need to introduce the Berezin transforms.

Definition 5.1. Let $\{{}_qg_w\}$ be the family of functions in $\mathcal{D}_q$ described in Example 3.7, and let T be a linear operator on $\mathcal{D}_q$. We define the *Berezin transform* of T as the function $\widetilde{T}(w) = [T({}_qg_w), {}_qg_w]_{\mathcal{D}_q}$ on $\mathbb{B}$.

It is clear that $\widetilde{T^*}(w) = \overline{\widetilde{T}(w)}$, that $|\widetilde{T}(w)| \le \|T\|$ for all $w \in \mathbb{B}$ if T is bounded, and that $\widetilde{T}(w)$ is a continuous function of w since ${}_qg_w$ depends on w continuously.

When T is a Toeplitz operator, we also use the common notation ${}_s\widetilde{\phi}_q$ for ${}_s\widetilde{T}_\phi$ and ${}_s\widetilde{\mu}_q$ for ${}_s\widetilde{T}_\mu$, and call them the *Berezin transforms* of ϕ and μ. Equation (4.1), Example 3.7, and Theorem 4.5 yield the explicit forms

$$ \begin{aligned} {}_s\widetilde{\phi}_q(w) &= \int_{\mathbb{B}} \phi(z)\,|{}_Qk_w(z)|^2\,d\nu_Q(z) \\ &= C_Q\,(1-|w|^2)^{N+1+Q} \int_{\mathbb{B}} \frac{(1-|z|^2)^{2u}}{|1-\langle z,w\rangle|^{(N+1+Q)2}}\,\phi(z)\,d\nu_q(z) \\ &= [{}_QT_\phi({}_Qk_w), {}_Qk_w]_{L^2_Q} = \widetilde{\phi}_Q(w) \qquad (w \in \mathbb{B}), \end{aligned} $$

which is valid for any $\phi \in L^1_Q$, where $\widetilde{\phi}_Q$ is the classical Bergman-space Berezin transform of ϕ. Hence, when $N = 1$, ${}_s\widetilde{\phi}_q = C_Q\, \mathbf{B}_Q\phi$ of [19, §2.1] since $Q > -1$. Analogously, by (4.2),

$$\begin{aligned} {}_s\widetilde{\mu}_q(w) &= \int_{\mathbb{B}} |{}_Qk_w(z)|^2 \, d\kappa(z) \\ &= C_Q\, (1-|w|^2)^{N+1+Q} \int_{\mathbb{B}} \frac{(1-|z|^2)^{2u}}{|1-\langle z,w\rangle|^{(N+1+Q)2}}\, d\mu(z) \\ &= [{}_QT_\kappa({}_Qk_w), {}_Qk_w]_{L^2_Q} = \widetilde{\kappa}_Q(w) \qquad (w \in \mathbb{B}) \end{aligned} \tag{5.1}$$

for those μ for which the integral converges. Hence ${}_s\widetilde{\mu}_q = C_Q\, \mathbf{B}^Q_q\mu$ of [24, §5]. It is now clear that if $\phi \geq 0$ a.e. in $\mathbb{B}$, then ${}_s\widetilde{\phi}_q \geq 0$ on $\mathbb{B}$, and if μ is a positive measure, then ${}_s\widetilde{\mu}_q \geq 0$ on $\mathbb{B}$.

Clearly, if ${}_sT_\phi = 0$ or $\phi = 0$ a.e. in $\mathbb{B}$, then ${}_s\widetilde{T}_\phi = {}_s\widetilde{\phi}_q = 0$ on $\mathbb{B}$. The converse of this property justifies Definition 5.1.

Proposition 5.2. *The maps* ${}_sT_\phi \mapsto {}_s\widetilde{T}_\phi$ *and* $\phi \mapsto {}_s\widetilde{\phi}_q$ *are one-to-one.*

Proof. The first claim is an obvious consequence of the second, which can be proved, because $Q > -1$, as in [19, Proposition 2.6] by taking more partial derivatives since now N is arbitrary. □

Definition 4.1, Example 3.7, and Definition 5.1 depend on the action on $\mathcal{D}_q$ of the reproducing kernel K_s with s satisfying (3.7), which can be chosen as K_q if and only if $q > -1$. In other words, in many instances on Toeplitz operators on general $\mathcal{D}_q$, the parameter s replaces the parameter q. Here's one more result in this direction.

Proposition 5.3. *If* $\phi \in H(\mathbb{B})$, *then* ${}_sT^*_\phi({}_qg_w) = \overline{\phi(w)}\, {}_qg_w$.

Proof. We have

$$\begin{aligned} {}_sT_{\overline{\phi}}({}_qg_w)(z) &= D_Q^{-u}({}_QT_{\overline{\phi}})D_s^u({}_qg_w)(z) \\ &= \sqrt{C_Q}\,(1-|w|^2)^{(N+1+Q)/2}\, D_Q^{-u}({}_QT_{\overline{\phi}})K_Q(z,w) \\ &= \overline{\phi(w)}\,\sqrt{C_Q}\,(1-|w|^2)^{(N+1+Q)/2}\, D_Q^{-u}K_Q(z,w) \\ &= \overline{\phi(w)}\,\sqrt{C_Q}\,(1-|w|^2)^{(N+1+Q)/2}\, K_s(z,w) = \overline{\phi(w)}\, {}_qg_w(z) \end{aligned}$$

by Theorem 4.5, Example 3.7, the classical Bergman-space result, and (3.4). □

Therefore if $\phi \equiv \lambda$, then λ is an eigenvalue for ${}_sT_\lambda$ with eigenvector ${}_qg_w$. As expected, this is the only possibility for the point spectrum of ${}_sT_\phi$ as we show next, where we also determine the spectrum of ${}_sT_\phi$.

Theorem 5.4. *If* $\phi \in H^\infty$, *then* $\sigma({}_sT_\phi) = \overline{\phi(\mathbb{B})}$, *and* $\sigma_p({}_sT_\phi) = \emptyset$ *unless* ϕ *is identically constant.*

Proof. Again this is a straightforward consequence of the unitary equivalence stated in Theorem 4.5 and the well-known Bergman-space result which can be found in [37, Chapter 6]. □

We do not pursue spectral theory any further in this work. Let's finally give some general equivalent conditions for the boundedness and compactness of ${}_sT_\phi$.

Proposition 5.5. *Suppose $\phi \in L^1_Q$ is $\mathcal{M}$-harmonic. Then ${}_sT_\phi$ is bounded if and only ϕ is bounded. And ${}_sT_\phi$ is compact if and only if $\phi = 0$ on $\mathbb{B}$.*

Proof. The if part of the first statement is Proposition 4.2, and the if part of the second statement is obvious. If ${}_sT_\phi$ is bounded, then by (2.2) and Example 3.7,

$$|\phi(w)| = |{}_s\widetilde{\phi}_q(w)| = \left|[{}_sT_\phi({}_qg_w), {}_qg_w]_{\mathcal{D}_q}\right| \le \|{}_sT_\phi({}_qg_w)\|_{\mathcal{D}_q} \|{}_qg_w\|_{\mathcal{D}_q} \le \|{}_sT_\phi\|$$

for all $w \in \mathbb{B}$. Hence ϕ is bounded. If ${}_sT_\phi$ is compact, then

$$|\phi(w)| \le \|{}_sT_\phi({}_qg_w)\|_{\mathcal{D}_q} \to 0 \qquad \text{as} \qquad |w| \to 1.$$

That is, the restriction of ϕ to $\partial\mathbb{B}$ vanishes. By the maximum principle, ϕ vanishes on all of $\mathbb{B}$. □

We summarize the basic formulas for the Arveson space $\mathcal{A} = \mathcal{D}_{-N}$. The parameter s is chosen so that $Q = N + 2s > -1$. Then $s > -(N+1)/2 > -(N+1)$ and the kernel K_s is always binomial. Also $u = N + s > 0$, and thus a strictly positive-order derivative is required in all definitions and formulas. If $f \in \mathcal{A}$, then

$$\|f\|^2_{\mathcal{D}_{-N}} = \int_{\mathbb{B}} (1 - |z|^2)^{N+2s} \, |D_s^{N+s} f(z)|^2 \, d\nu(z).$$

We write only those formulas in which the symbol of the Toeplitz operator is a function; for the formulas when the symbol is a measure, we just substitute $d\mu(w)$ for $(1 - |w|^2)^{-N} \, d\nu(w)$. The Toeplitz operator is

$${}_sT_\phi f(z) = \frac{(N+1+2s)_N}{N!} \int_{\mathbb{B}} \frac{\phi(w) \, (1 - |w|^2)^{N+2s}}{(1 - \langle z, w\rangle)^{N+1+s}} \, D_s^{N+s} f(w) \, d\nu(w).$$

The weakly convergent family in $\mathcal{A}$ we use in defining the Berezin transform is

$${}_qg_w(z) = \sqrt{\frac{(N+1+2s)_N}{N!}} \, \frac{(1 - |w|^2)^{(2N+1+2s)/2}}{(1 - \langle z, w\rangle)^{N+1+s}}.$$

The Berezin transform is

$${}_s\widetilde{\phi}_{-N}(w) = \frac{(N+1+2s)_N}{N!} \, (1 - |w|^2)^{2N+1+2s} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{N+2s}}{|1 - \langle z, w\rangle|^{(2N+1+2s)2}} \, \phi(z) \, d\nu(z).$$

A value of Q that gives simpler formulas is $Q = N + 2s = 0$, because the factors $(1 - |\cdot|^2)^{N+2s}$ disappear, and then $s = -N/2$ and $u = N/2$. Another case that might be of interest is $s = 0$ in which $Q = u = N$.

When $N = 1$, the Arveson space becomes one with the Hardy space H^2. Setting $N = 1$ above, it is clear that the Toeplitz operators studied in this paper are not the classical Toeplitz operators on H^2. The ones here depend on an imbedding of H^2 in L^2_{-1} by way of I^u_s rather than its usual imbedding in $L^2(\partial\mathbb{D})$ by way of inclusion, and require a radial derivative of positive order u.

Remark 5.6. However, let's take the limits as $u \to 0^+$, that is, as $s \to -1^+$, of the formulas for H^2 when $N = 1$. Let's assume ϕ has boundary values on $\partial\mathbb{D}$, also called ϕ, so that Hardy-space expressions make sense; $f \in H^2$ clearly has boundary values. It is known by weak-$*$ convergence of measures that

$$\lim_{s\to-1^+} \sqrt{2(1+s)}\, \|f\|_{\mathcal{D}_{-1}} = \|f\|_{H^2} \qquad (f \in H^2),$$

where $\|\cdot\|_{H^2}$ is the classical norm on H^2. For a detailed proof, [25, §3] can be consulted. With the same computation, we obtain

$$\lim_{s\to-1^+} \frac{1}{\sqrt{2(1+s)}}\, {}_q g_w(z) = \mathbf{k}_w(z),$$

where $\mathbf{k}_w$ is the classical normalized reproducing kernel of H^2. Next we obtain

$$\lim_{s\to-1^+} {}_sT_\phi f(z) = \mathbf{T}_\phi f(z) \qquad (f \in H^2),$$

where $\mathbf{T}_\phi f = \mathbf{P}(\phi f)$ is the classical Toeplitz operator on H^2 defined via the Szegő projection $\mathbf{P}$. We also obtain

$$\lim_{s\to-1^+} ({}_s\widetilde{\phi}_{-1})(w) = \widetilde{\Phi}(w),$$

where $\widetilde{\Phi}$ is the classical Berezin transform on H^2, which is the Poisson transform of the boundary values of ϕ. No extra factor is required for ${}_sT_\phi$ or ${}_s\widetilde{\phi}_{-1}$, because the factor $C_Q = 2(1+s)$ is built into them.

The same conclusions hold on $\mathcal{D}_{-1}$ also when $N > 1$; no change is necessary for ${}_sT_\phi$ or ${}_s\widetilde{\phi}_{-1}$; in $\|\cdot\|_{\mathcal{D}_{-1}}$ and ${}_qg_w$ we just replace $2(1+s)$ by $(2(1+s))_N/N!$. Therefore the classical Toeplitz operators on H^2 are limiting cases of the Toeplitz operators on $\mathcal{D}_{-1}$ studied in this paper as the order of the radial derivative in their definition tends to 0.

6. Toeplitz Operators with Positive Symbols

Throughout this section we assume $\phi \geq 0$ and $\mu \geq 0$ so that the resulting Toeplitz operators ${}_sT_\phi$ and ${}_sT_\mu$ on $\mathcal{D}_q$ are positive. We then give equivalent conditions for the boundedness, compactness, and membership in Schatten classes of these Toeplitz operators. Our main tools are the Berezin transform and Carleson measures. The only exception to positivity is Theorem 6.7, where ϕ is bounded instead.

Definition 6.1. A positive Borel measure μ on $\mathbb{B}$ is called a *q-Carleson measure* if the ratio

$${}_q\widehat{\mu}_r(w) = \frac{\mu(b(w,r))}{\nu_q(b(w,r))}$$

is bounded for $w \in \mathbb{B}$ for some $0 < r < \infty$. The measure μ is called a *vanishing q-Carleson measure* if the same ratio tends to 0 as $|w| \to 1$ for some $0 < r < \infty$.

The following characterization of q-Carleson and vanishing q-Carleson measures is given in [24, Theorem 5.9], actually in slightly more general form. Its corollary also appears in the same source.

Theorem 6.2. *Fix q. Let r, an r-lattice $\{a_n\}$, and s satisfying* (3.7) *be given. The following conditions are equivalent for a positive Borel measure μ on $\mathbb{B}$.*

(i) *The measure μ is a q-Carleson (resp. vanishing q-Carleson) measure.*
(ii) *The sequence $\{_q\widehat{\mu}_r(a_n)\}$ is bounded (resp. has limit* 0*).*
(iii) *The imbedding $\breve{I}^u_s : \mathcal{D}_q \to L^2(\mu)$ is bounded (resp. compact).*
(iv) *The Berezin transform $_s\widetilde{\mu}_q$ is bounded on $\mathbb{B}$ (resp. in $\mathcal{C}_0$).*

Thus the property of being a (vanishing) q-Carleson measure is independent of r, $\{a_n\}$, and s under (3.7), but depends on q. In accordance with that q is unrestricted, a (vanishing) q-Carleson measure need not be finite.

Corollary 6.3. *A positive Borel measure μ on $\mathbb{B}$ is a q-Carleson (resp. vanishing q-Carleson) measure if and only if κ is a Q-Carleson (resp. vanishing Q-Carleson) measure.*

Now we can state our main theorem.

Theorem 6.4. *Suppose μ is a positive Borel measure on $\mathbb{B}$. Then $_sT_\mu$ is bounded (resp. compact) on $\mathcal{D}_q$ if and only if μ is a q-Carleson (resp. vanishing q-Carleson) measure.*

Proof. With all the preparation done in earlier sections, we give two related very short proofs.

By Theorem 4.6, $_sT_\mu$ is bounded or compact on $\mathcal{D}_q$ if and only if $\breve{I}^u_s$ has the same property. By Theorem 6.2, either property is equivalent to a q-Carleson-measure property for μ.

Or, by Theorem 4.5, $_sT_\mu$ is bounded or compact if and only if $_QT_\kappa$ has the same property. By [37, Theorems 6.4.4 and 6.4.5], either property translates to a Q-Carleson-measure property for κ. By Corollary 6.3, we fall back to a q-Carleson-measure property for μ. □

It is among the consequences of Theorem 6.2 that if μ is a q-Carleson measure, then κ is finite; see [24, §1]. In the light of Theorem 6.4, the finiteness of κ, which is stated for $_sT_\mu$ to make sense when it is first defined in Section 4, is as natural a condition as possible.

Corollary 6.5. *Suppose $\phi \geq 0$ is a measurable function on $\mathbb{B}$. Then $_sT_\phi$ is bounded (resp. compact) on $\mathcal{D}_q$ if and only if $\phi\, d\nu_q$ is a q-Carleson (resp. vanishing q-Carleson) measure.*

It is clear from Theorem 6.2 that the results of Theorem 6.4 and Corollary 6.5 are independent of the particular value of s used in the definition of the Toeplitz operator or the particular weakly convergent family $\{_qg_w\}$ used in the definition of its Berezin transform or the particular value of the radius r used in the definition of ${}_q\widehat{\mu}_r$. We next show that the results are also independent of the Dirichlet space D_q that the Toeplitz operator acts on when the operator in question is ${}_sT_\phi$. So suppose $d\mu(z) = \phi(z)\,d\nu_q(z)$. Then by Lemma 2.1,

$$
\begin{aligned}
{}_q\widehat{\mu}_r(w) &\sim \frac{1}{(1-|w|^2)^{N+1+q}} \int_{b(w,r)} \phi(z)\,(1-|z|^2)^q\,d\nu(z) \\
&\sim \frac{1}{\nu(b(w,r))} \int_{b(w,r)} \phi(z)\,d\nu(z) =: \widehat{\phi}_r(w),
\end{aligned}
$$

which defines the *averaging function* $\widehat{\phi}_r$ on Bergman balls independently of q.

Corollary 6.6. *Suppose $\phi \geq 0$ is a measurable function on $\mathbb{B}$. Let r, an r-lattice $\{a_n\}$, and s satisfying* (3.7) *be given. The following are equivalent.*

(i) *The Toeplitz operator ${}_sT_\phi : \mathcal{D}_q \to \mathcal{D}_q$ is bounded (resp. compact).*
(ii) *The Berezin transform ${}_s\widetilde{\phi}_q$ is bounded on $\mathbb{B}$ (resp. in $\mathcal{C}_0$).*
(iii) *The averaging function $\widehat{\phi}_r$ is bounded on $\mathbb{B}$ (resp. in $\mathcal{C}_0$).*
(iv) *The sequence $\{\widehat{\phi}_r(a_n)\}$ is bounded (resp. has limit* 0*).*

We make an excursion from our main line of development to insert a result on the compactness of Toeplitz operators whose symbols are not necessarily positive.

Theorem 6.7. *Let $N = 1$ and $\phi \in L^\infty$. Then ${}_sT_\phi$ on $\mathcal{D}_q$ is compact if and only if ${}_s\widetilde{\phi}_q$ lies in $\mathcal{C}_0$.*

Proof. Pick u so that $Q = 0$. By Theorem 4.5, ${}_sT_\phi$ is compact if and only if the classical Toeplitz operator ${}_0T_\phi$ on A_0^2 is compact, which in turn holds if and only if ${}_0\widetilde{\phi}_0$ is in $\mathcal{C}_0$ by [9, Corollary 2.5]. But ${}_s\widetilde{\phi}_q = {}_0\widetilde{\phi}_0$ by our choice of Q. □

Unfortunately, the methods of [9] do not immediately generalize to dimensions $N > 1$ or to classical Toeplitz operators ${}_qT_\phi = P_q M_\phi i$ on weighted Bergman spaces A_q^2 with $q \neq 0$. There are some extensions to non-Hilbert Bergman spaces A_0^p with $p > 1$ in [29], but with extra assumptions.

Example 6.8. Let's illustrate Corollaries 6.5 and 6.6 and Theorem 6.7 by picking $Q = 0$ and $\phi(z) = (1-|z|^2)^c$ when $N = 1$. By Corollary 6.5, ${}_sT_\phi$ is compact if and only if $c > 0$. Its Berezin transform is

$$
{}_q\widetilde{\phi}_s(w) = (1-|w|^2)^2 \int_{\mathbb{D}} \frac{(1-|z|^2)^c}{|1-\langle z,w\rangle|^4}\,d\nu(z).
$$

By [33, Proposition 1.4.10], ${}_q\widetilde{\phi}_s(w) \sim (1-|w|^2)^b$, where the power b depends on c but is always positive so that ${}_q\widetilde{\phi}_s \in \mathcal{C}_0$ in all cases. This is as predicted by Corollary 6.6 or Theorem 6.7.

We return to positive symbols and now investigate the conditions under which the operators ${}_sT_\phi$ or ${}_sT_\mu$ belong to the *Schatten-von Neumann ideal* $\mathcal{S}^p$ of $\mathcal{D}_q$. For $0 < p < \infty$, a compact operator T on a Hilbert space H with inner product $[\cdot,\cdot]$ is said to belong to to $\mathcal{S}^p$ of H if its sequence of singular values lies in ℓ^p. We refer to

[18, Chapter III] for relevant definitions and basic properties of Schatten ideals. If T is a compact operator or an operator in $\mathcal{S}^1$, then the value of the sum $\sum_j [Te_j, e_j]$ is the same for any orthonormal basis $\{e_j\}_{j\in J}$ in H, and is called the *trace* $\mathrm{tr}(T)$ of T. The sum is finite in the latter case whence we call T a *trace-class* operator. If T is a positive compact operator on H, then T^p is uniquely defined, and $T \in \mathcal{S}^p$ if and only if $T^p \in \mathcal{S}^1$. An operator in $\mathcal{S}^2$ is called a *Hilbert-Schmidt operator.* A compact operator T belongs to $\mathcal{S}^p$ if and only if $|T|^p$ defined as $(T^*T)^{p/2}$ belongs to $\mathcal{S}^1$, which holds if and only if T^*T belongs to $\mathcal{S}^{p/2}$. We have $\mathcal{S}^1 \subset \mathcal{S}^p \subset \mathcal{S}^\infty$ for $1 < p < \infty$. Further, for operators on H, $T_1 \le T_2$ means that $[T_1 f, f] \le [T_2 f, f]$ for all $f \in H$.

We are interested in $H = \mathcal{D}_q$ for any $q \in \mathbb{R}$. We need a few lemmas before we characterize the Toeplitz operators with positive symbols that are in Schatten ideals $\mathcal{S}^p$ of $\mathcal{D}_q$ for $1 \le p < \infty$. Recall that ϕ, μ, ${}_sT_\phi$, and ${}_sT_\mu$ are all positive in this section.

Lemma 6.9. *If T is a positive or a trace-class operator on $\mathcal{D}_q$, then*

$$\mathrm{tr}(T) = \mathrm{tr}(D_s^u T D_Q^{-u}) = C_Q \int_{\mathbb{B}} (D_s^u T D_Q^{-u})^\sim \, d\tau,$$

where $(D_s^u T D_Q^{-u})^\sim$ is the classical Bergman-space Berezin transform of the operator $D_s^u T D_Q^{-u} : A_Q^2 \to A_Q^2$.

Proof. Let $\{\, e_\alpha : \alpha \in \mathbb{N}^N \,\}$ be an orthonormal basis for $\mathcal{D}_q$ with respect to the inner product $[\cdot,\cdot]_{\mathcal{D}_q}$. Put $f_\alpha = D_s^u e_\alpha$. Then $\{\, f_\alpha : \alpha \in \mathbb{N}^N \,\}$ is an orthonormal basis for $\mathcal{D}_Q = A_Q^2$ with respect to the inner product $[\cdot,\cdot]_{L_Q^2}$ by Proposition 3.2. Then

$$\mathrm{tr}(T) = \sum_\alpha [Te_\alpha, e_\alpha]_{\mathcal{D}_q} = \sum_\alpha [D_s^u T e_\alpha, D_s^u e_\alpha]_{L_Q^2} = \sum_\alpha \big[(D_s^u T D_Q^{-u}) f_\alpha, f_\alpha\big]_{L_Q^2},$$

which proves the first equality. The second equality follows by modifying the proof of [37, Proposition 6.3.2] for the ball and for weighted Bergman spaces. □

Lemma 6.10. *We have*

$$\mathrm{tr}({}_sT_\mu) = C_Q \int_{\mathbb{B}} {}_s\widetilde{\mu}_q \, d\tau = C_Q \int_{\mathbb{B}} K_Q(z,z)\, d\kappa(z) = C_Q \int_{\mathbb{B}} \frac{d\mu(z)}{(1-|z|^2)^{N+1+q}}$$

and

$$\mathrm{tr}({}_sT_\phi) = C_Q \int_{\mathbb{B}} {}_s\widetilde{\phi}_q \, d\tau = C_Q \int_{\mathbb{B}} \phi(z)\, K_Q(z,z)\, d\nu_Q(z) = C_Q \int_{\mathbb{B}} \phi \, d\tau.$$

Proof. By Lemma 6.9 and (5.1), we have

$$\mathrm{tr}({}_sT_\mu) = C_Q \int_{\mathbb{B}} {}_Q\widetilde{T}_\kappa \, d\tau = C_Q \int_{\mathbb{B}} {}_s\widetilde{\mu}_q \, d\tau.$$

The rest now follows by modifying the proof of the Corollary to [37, Proposition 6.3.2] to suit the weighted Bergman spaces and the ball. □

Lemma 6.11. *If $1 \le p < \infty$ and $\phi \in L^p(\tau)$, then ${}_sT_\phi \in \mathcal{S}^p$.*

Proof. Let $\{e_\alpha\}$ be any orthonormal basis for $\mathcal{D}_q$. By Lemma 6.9, we have

$$\operatorname{tr}({}_sT_\phi) = \sum_\alpha [{}_sT_\phi e_\alpha, e_\alpha]_{\mathcal{D}_q} = \sum_\alpha [{}_QT_\phi f_\alpha, f_\alpha]_{L^2_Q} = \operatorname{tr}({}_QT_\phi),$$

where ${}_QT_\phi$ is a classical Bergman-space Toeplitz operator. So ${}_sT_\phi \in \mathcal{S}^p$ if and only if ${}_QT_\phi \in \mathcal{S}^p$. We are done by [37, Lemma 6.3.4]. □

Lemma 6.12. *Given r, there is a C such that* ${}_sT_\mu \le C\,({}_sT_{{}_q\widehat{\mu}_r})$.

Proof. Let $f \in \mathcal{D}_q$. We compute using (4.1), Lemma 2.1, Fubini theorem, Lemma 2.2, (4.2), and obtain

$$\begin{aligned}
[{}_sT_{{}_q\widehat{\mu}_r} f, f]_{\mathcal{D}_q} &= \int_{\mathbb{B}} \frac{\mu(b(z,r))}{\nu_q(b(z,r))}\, |D^u_s f(z)|^2\, d\nu_Q(z) \\
&\sim \int_{\mathbb{B}} \frac{|D^u_s f(z)|^2}{(1-|z|^2)^{N+1-2u}} \int_{\mathbb{B}} \chi_{b(z,r)}(w)\, d\mu(w)\, d\nu(z) \\
&= \int_{\mathbb{B}} \int_{b(w,r)} \frac{|D^u_s f(z)|^2}{(1-|z|^2)^{N+1-2u}}\, d\nu(z)\, d\mu(w) \\
&\sim \int_{\mathbb{B}} \frac{1}{\nu_q(b(w,r))} \int_{b(w,r)} (1-|z|^2)^{2u}\, |D^u_s f(z)|^2\, d\nu_q(z)\, d\mu(w) \\
&\ge C \int_{\mathbb{B}} (1-|w|^2)^{2u}\, |D^u_s f(w)|^2\, d\mu(w) = [{}_sT_\mu f, f]_{\mathcal{D}_q},
\end{aligned}$$

which is what is wanted. □

The classical Bergman-space versions of Lemmas 6.9–6.12 can be found in [37, §6.3].

Now we are ready for a characterization of Toeplitz operators in $\mathcal{S}^p$.

Theorem 6.13. *Suppose μ is a positive Borel measure on $\mathbb{B}$. Let $1 \le p < \infty$, r, an r-lattice $\{a_n\}$, and s satisfying* (3.7) *be given. The following are equivalent.*

(i) *The Toeplitz operator* ${}_sT_\mu : \mathcal{D}_q \to \mathcal{D}_q$ *belongs to* $\mathcal{S}^p$.
(ii) *The Berezin transform* ${}_s\widetilde{\mu}_q$ *belongs to* $L^p(\tau)$.
(iii) *The averaging function* ${}_q\widehat{\mu}_r$ *belongs to* $L^p(\tau)$.
(iv) *The sequence* $\{{}_q\widehat{\mu}_r(a_n)\}$ *belongs to* ℓ^p.

Proof. (i) $\Longrightarrow$ (ii): By positivity, if ${}_sT_\mu$ is in $\mathcal{S}^p$, then ${}_sT^p_\mu$ is in $\mathcal{S}^1$ so that $\operatorname{tr}({}_sT^p_\mu)$ is finite. Now by definition and [37, Proposition 6.3.3],

$$\int_{\mathbb{B}} {}_s\widetilde{\mu}_q{}^p\, d\tau = \int_{\mathbb{B}} [{}_sT_\mu({}_qg_w), {}_qg_w]^p_{\mathcal{D}_q}\, d\tau(w) \le \int_{\mathbb{B}} [{}_sT^p_\mu({}_qg_w), {}_qg_w]_{\mathcal{D}_q}\, d\tau(w).$$

But the last term is just $\operatorname{tr}({}_sT^p_\mu)$.

(ii) $\Longrightarrow$ (iii): Lemma 2.1 shows that

$$\begin{aligned} {}_q\widehat{\mu}_r(w) &\sim \frac{1}{(1-|w|^2)^{N+1+q}} \int_{b(w,r)} d\mu \\ &\sim (1-|w|^2)^{N+1+Q} \int_{b(w,r)} \frac{(1-|z|^2)^{2u}}{|1-\langle z,w\rangle|^{(N+1+Q)2}}\, d\mu(z) \\ &\leq (1-|w|^2)^{N+1+Q} \int_{\mathbb{B}} \frac{d\kappa(z)}{|1-\langle z,w\rangle|^{(N+1+Q)2}} = {}_s\widetilde{\mu}_q(w). \end{aligned}$$

(iii) $\Longrightarrow$ (i): Suppose ${}_q\widehat{\mu}_r \in L^p(\tau)$. Then ${}_sT_{{}_q\widehat{\mu}_r} \in \mathcal{S}^p$ by Lemma 6.11. By positivity and [37, Theorem 1.4.7], $\sum_\alpha [{}_sT_{{}_q\widehat{\mu}_r} e_\alpha, e_\alpha]^p_{\mathcal{D}_q} < \infty$ for any orthonormal set $\{e_\alpha\}$ in $\mathcal{D}_q$. Then $\sum_\alpha [{}_sT_\mu e_\alpha, e_\alpha]^p_{\mathcal{D}_q} < \infty$ too by Lemma 6.12. We are done by applying [37, Theorem 1.4.7] again.

(iii) $\Longleftrightarrow$ (iv): This is in [24, §5] and has an independent proof. □

As observed above, the conclusions of Theorem 6.13 do not depend on s, r, $\{a_n\}$, or $\{{}_qg_w\}$, but do depend on q. When we specialize to ${}_sT_\phi$, that dependence disappears too in the same way as in Corollary 6.6.

Corollary 6.14. *Suppose $\phi \geq 0$ is a measurable function on $\mathbb{B}$. Let $1 \leq p < \infty$, r, an r-lattice $\{a_n\}$, and s satisfying* (3.7) *be given. The following are equivalent.*

(i) *The Toeplitz operator ${}_sT_\phi : \mathcal{D}_q \to \mathcal{D}_q$ belongs to $\mathcal{S}^p$.*
(ii) *The Berezin transform ${}_s\widetilde{\phi}_q$ belongs to $L^p(\tau)$.*
(iii) *The averaging function $\widehat{\phi}_r$ belongs to $L^p(\tau)$.*
(iv) *The sequence $\{\widehat{\phi}_r(a_n)\}$ belongs to ℓ^p.*

The classical Bergman-space versions ($q > -1$ with $i = I^0_q$) of Theorems 6.4 and 6.13 on $\mathbb{D}$ can be found in [37, Chapter 6]. What is new here are that the results now hold for all Dirichlet spaces ($q \in \mathbb{R}$), that they hold although Toeplitz operators here are defined via I^u_s for all q rather than i, and thus they give a unified picture of Toeplitz operators on weighted Bergman and other Dirichlet spaces.

Thus, when $\phi \geq 0$, the Toeplitz operator ${}_sT_\phi$ on the Arveson space is bounded, compact, or in $\mathcal{S}^p$ precisely when the classical Toeplitz operator ${}_0T_\phi$ on the Bergman space A^2_0 is bounded, compact, or in $\mathcal{S}^p$, which occurs precisely when the averaging function $\widehat{\phi}_r$ is bounded, in $\mathcal{C}_0$, or in $L^p(\tau)$, respectively.

Remark 6.15. We continue Remark 5.6 by letting $q = -1$ and taking limits as $s \to -1^+$ in Corollary 6.5. We take $N = 1$ for simplicity, and recall that $\phi \geq 0$. We know ${}_sT_\phi$ becomes the classical Toeplitz operator $\mathbf{T}_\phi$ on H^2 in the limit.

By Theorem 6.2 (iii), the condition that $\phi\, d\nu_{-1}$ is a (-1)-Carleson measure means

$$\int_{\mathbb{D}} (1-|z|^2)^{1+2s}\, |D^{1+s}_s f(z)|^2\, \phi(z)\, d\nu(z) \leq C\, \|f\|^2_{\mathcal{D}_{-1}} \qquad (f \in H^2).$$

After multiplying both sides by $2(1+s)$, as $s \to -1^+$, it takes the form

$$\int_0^{2\pi} |f(e^{i\theta})|^2 \, \phi(e^{i\theta}) \, \frac{d\theta}{2\pi} \le C \int_0^{2\pi} |f(e^{i\theta})|^2 \, \frac{d\theta}{2\pi} \qquad (f \in H^2)$$

in the same way as in Remark 5.6. Since this is true for all $f \in H^2$, it is equivalent to that ϕ is bounded by C a.e. on $\partial\mathbb{D}$. By Theorem 6.2 (iv), the condition that $\phi \, d\nu_{-1}$ is a vanishing (-1)-Carleson measure means that ${}_s\widetilde{\phi}_q$ is in $\mathcal{C}_0$. This is the same as having $2(1+s)\,({}_s\widetilde{\phi}_q)$ in $\mathcal{C}_0$. As $s \to -1^+$, by Remark 5.6, it is equivalent to having $\widetilde{\Phi}$, the Poisson transform of the boundary values of ϕ, in $\mathcal{C}_0$. This holds if and only if $\phi = 0$ a.e. in $\mathbb{D}$, or equivalently, $\phi = 0$ a.e. on $\partial\mathbb{D}$.

As in Remark 5.6, when $N > 1$, $2(1+s)$ is replaced by $(2(1+s))_N/N!$ in intermediate steps with no effect on conclusions.

Thus we recover the characterizations of the boundedness and compactness of the classical $\mathbf{T}_\phi$ on H^2 (see [37, Propositions 9.1.2 and 9.1.3]) in the limiting case $s \to -1^+$ of ${}_sT_\phi$ on $\mathcal{D}_{-1}$, supplying further evidence that ${}_sT_\phi$ unifies Toeplitz operators on Hardy, weighted Bergman, and Dirichlet spaces.

7. Weighted Composition Operators on Weighted Bergman Spaces

Definition 7.1. Let $f, \eta, \varphi \in H(\mathbb{B})$ and φ have range in $\mathbb{B}$. The operator $M_\eta C_\varphi$ defined by $M_\eta C_\varphi f = \eta(f \circ \varphi)$ is called a *weighted composition operator*.

We are interested in weighted composition operators $M_\eta C_\varphi : A_Q^2 \to A_Q^2$ for $Q > -1$. Suppose Q, q, and s are related as in (3.11). Consider $D_s^u : \mathcal{D}_q \to A_Q^2$ which is an isometry. We also know $(D_s^u)^{-1} = D_Q^{-u}$. If $f, g \in \mathcal{D}_q^2$, then $F = D_s^u f$ and $G = D_s^u g$ are in A_Q^2. We now define ${}_\eta E_\varphi : \mathcal{D}_q \to \mathcal{D}_q$ by ${}_\eta E_\varphi = D_Q^{-u} M_\eta C_\varphi D_s^u$. Operators resembling ${}_\eta E_\varphi$ are used in [28] and [39] in similar contexts. Then

$$\begin{aligned}
\big[({}_\eta E_\varphi)^*({}_\eta E_\varphi)f, g\big]_{\mathcal{D}_q} &= \big[({}_\eta E_\varphi)f, ({}_\eta E_\varphi)g\big]_{\mathcal{D}_q} \\
&= [D_s^u D_Q^{-u} M_\eta C_\varphi D_s^u f, D_s^u D_Q^{-u} M_\eta C_\varphi D_s^u g]_{L_Q^2} \\
&= [M_\eta C_\varphi F, M_\eta C_\varphi G]_{L_Q^2} \\
&= \int_{\mathbb{B}} F(\varphi(z)) \, \overline{G(\varphi(z))} \, |\eta(z)|^2 \, (1-|z|^2)^Q \, d\nu(z) \\
&= \int_{\mathbb{B}} F(\varphi(z)) \, \overline{G(\varphi(z))} \, d({}_\eta\nu_Q)(z) \\
&= \int_{\mathbb{B}} F(\zeta) \, \overline{G(\zeta)} \, d({}_\eta\nu_Q \circ \varphi^{-1})(\zeta) \\
&= \int_{\mathbb{B}} (D_s^u f)(\zeta) \, \overline{(D_s^u g)(\zeta)} \, d\kappa(\zeta) = [{}_sT_\mu f, g]_{\mathcal{D}_q},
\end{aligned}$$

as in the proof of Theorem 4.6. Thus ${}_sT_\mu = ({}_\eta E_\varphi)^*({}_\eta E_\varphi)$. Here $\kappa = {}_\eta\nu_Q \circ \varphi^{-1}$ is the *pull-back measure* that assigns the value

$$\kappa(\Omega) = \int_{\varphi^{-1}(\Omega)} |\eta(z)|^2 \, (1-|z|^2)^Q \, d\nu(z)$$

to each Borel subset Ω of $\mathbb{B}$. As in Section 4, $d\mu(\zeta) = (1-|\zeta|^2)^{q-Q} \, d\kappa(\zeta)$. Note that both κ and μ are positive Borel measures.

Theorem 7.2. *The weighted composition operator $M_\eta C_\varphi$ is bounded (resp. compact) on the weighted Bergman space A^2_Q of $\mathbb{B}$ if and only if the Berezin transform ${}_s\widetilde{\mu}_q$ is bounded on $\mathbb{B}$ (resp. in $\mathcal{C}_0$).*

Proof. For compactness, we use the fact that a composition of a bounded operator and a compact one is compact. The operator $M_\eta C_\varphi$ is bounded (resp. compact) on A^2_Q if and only if ${}_\eta E_\varphi$ is bounded (resp. compact) on $\mathcal{D}_q$ if and only if ${}_sT_\mu$ is bounded (resp. compact) on $\mathcal{D}_q$ if and only if μ is a q-Carleson (resp. vanishing q-Carleson) measure by Theorem 6.4. By Theorem 6.2, these conditions are equivalent to the stated conditions on the Berezin transform.

We can restate the equivalent conditions more explicitly in terms of the parameters η and φ of the operator $M_\eta C_\varphi$. By (5.1) and the definition of μ above,

$$\begin{aligned} {}_s\widetilde{\mu}_q(w) &= C_Q \, (1-|w|^2)^{N+1+Q} \int_{\mathbb{B}} \frac{1}{|1-\langle \zeta, w\rangle|^{(N+1+Q)2}} \, d({}_\eta\nu_Q \circ \varphi^{-1})(\zeta) \\ &= C_Q \, (1-|w|^2)^{N+1+Q} \int_{\mathbb{B}} \frac{|\eta(z)|^2}{|1-\langle \varphi(z), w\rangle|^{(N+1+Q)2}} \, d\nu_Q(z). \end{aligned} \tag{7.1}$$

Thus $M_\eta C_\varphi$ is bounded (resp. compact) if and only if the quantity in (7.1) as a function of w is bounded in $\mathbb{B}$ (resp. in $\mathcal{C}_0$). □

When $N = 1$, this theorem is proved in [13, Proposition 2] using a characterization of Carleson measures via a derivative of disc automorphisms, a tool not readily available for $N > 1$. (Incidentally, the so-called weighted φ-Berezin transform $B_{\varphi,\alpha}$ in [13] should have the measure dA_α instead of dA in its definition.) Yet we are able to prove Theorem 7.2 with great ease once the theory of Carleson measures on Besov and Toeplitz operators on Dirichlet spaces are developed.

Our next result on the Schatten-ideal membership of $M_\eta C_\varphi$ follows from Theorem 6.13 with a proof very similar to that of Theorem 7.2.

Theorem 7.3. *Let $2 \le p < \infty$. The weighted composition operator $M_\eta C_\varphi$ belongs to $\mathcal{S}^p$ of the weighted Bergman space A^2_Q of $\mathbb{B}$ if and only if the Berezin transform ${}_s\widetilde{\mu}_q$ lies in $L^{p/2}(\tau)$.*

For $N = 1$ and $Q = 0$, Theorem 7.3 is contained in [13, Theorem 3] with a similar proof. The following corollary follows similarly too.

Corollary 7.4. *The weighted composition operator $M_\eta C_\varphi : A^2_Q \to A^2_Q$ on the ball is Hilbert-Schmidt if and only if*

$$\int_{\mathbb{B}} \frac{|\eta(z)|^2}{(1-|\varphi(z)|^2)^{N+1+Q}}\, d\nu_Q(z) < \infty.$$

8. Shift Operators

In this section, we always take $N = 1$, so our operators act on function spaces on the disc $\mathbb{D}$, and the constant C_Q is equal to $1 + Q$.

We need explicit orthonormal bases for $\mathcal{D}_q$. Definition 3.1a and [6, Theorem 3.3.1] imply the following. On each $\mathcal{D}_q$, there is an inner product $[\![\cdot,\cdot]\!]_{\mathcal{D}_q}$ with respect to which $\{z^k\}_{k\in\mathbb{N}}$ is a complete orthogonal set, and the corresponding norm $|||z^k|||_{\mathcal{D}_q}$ of z^k is the square root of the reciprocal of the coefficient of $(z\overline{w})^k$ in the Taylor expansion of $K_q(z,w)$. This inner product and its norm are equivalent to the ones in (3.12). Explicitly,

$$|||z^k|||^2_{\mathcal{D}_q} = [\![z^k, z^k]\!]_{\mathcal{D}_q} = \begin{cases} \dfrac{k!}{(2+q)_k}, & \text{if } q > -2; \\ \dfrac{(-1-q)_{k+1}}{k!}, & \text{if } q \le -2. \end{cases} \tag{8.1}$$

On the other hand, by (3.12) and [22, Proposition 2.1],

$$\|z^k\|^2_{\mathcal{D}_q} = \begin{cases} \dfrac{(2+Q)_k\, k!}{(1+Q)\,(2+s)^2_k}, & \text{if } s > -2; \\ \dfrac{(2+Q)_k\,(-1-s)^2_{k+1}}{(1+Q)\,(k!)^3}, & \text{if } s \le -2. \end{cases} \tag{8.2}$$

In either case, none of the norms are 0 and the norm of z^k is $\sim k^{(-1-q)/2}$ as $k \to \infty$. Consequently, $\{\,{}_q e_k(z) = z^k/\|z^k\|_{\mathcal{D}_q} : k \in \mathbb{N}\,\}$ is an orthonormal basis for $\mathcal{D}_q$ with respect to the norm $\|\cdot\|_{\mathcal{D}_q}$, and $\{\,{}_q E_k(z) = z^k/|||z^k|||_{\mathcal{D}_q} : k \in \mathbb{N}\,\}$ is an orthonormal basis for $\mathcal{D}_q$ with respect to the norm $|||\cdot|||_{\mathcal{D}_q}$. Note also that if $q > -1$ and $u = 0$, then $Q = q = s$ and

$$|||z^k|||_{A^2_Q} = \sqrt{C_Q}\,\|z^k\|_{A^2_Q} \qquad (k = 0, 1, \ldots). \tag{8.3}$$

Definition 8.1. We call $M_z : \mathcal{D}_q \to \mathcal{D}_q$ the *q-shift.*

So the (-1)-shift is the unilateral shift on the Hardy space H^2, the 0-shift is the Bergman shift, and the (-2)-shift is the Dirichlet shift. There are intimate connections between Toeplitz operators, multiplication operators, and shift operators.

Recall that ${}_QT_\phi$ is the classical Toeplitz operator ($u = 0$) on the weighted Bergman space A^2_Q, and for that, ${}_QT_z = M_z$. For the general Toeplitz operators defined via I^u_s with $u \ne 0$ on general $\mathcal{D}_q$ considered in this paper, ${}_sT_z$ is not a fixed multiple of M_z any more. To see how ${}_sT_z$ behaves on $\mathcal{D}_q$, it suffices to check its action on z^k. By [22, Proposition 2.1],

$$ {}_sT_z(z^k) = \begin{cases} \dfrac{2+s+k}{2+Q+k}\, z^{k+1}, & \text{if } s > -2; \\ \dfrac{(1+k)^2}{(2+Q+k)(-s+k)}\, z^{k+1}, & \text{if } s \le -2; \end{cases} \qquad (k = 0, 1, 2, \ldots). $$

Thus

$$ {}_sT_z({}_qe_k) = \sqrt{\frac{1+k}{2+Q+k}}\,({}_qe_{k+1}) $$

and ${}_sT_z$ on $\mathcal{D}_q$ is a weighted shift operator with weight sequence $\left\{ W_k = \sqrt{\frac{1+k}{2+Q+k}} \right\}$ with respect to the orthonormal basis $\{{}_qe_k\}$. No W_k is 0, $\{W_k\}$ is bounded, but does not tend to 0; hence ${}_sT_z$ is one-to-one, bounded, but not compact. Noncompactness of ${}_sT_z$ can also be deduced via Theorem 6.7 by a laborious computation of ${}_s\widetilde{z}_q$ using the methods of [33, Proposition 1.4.10]. Hence by [34, Theorem 2 (b)], ${}_sT_z$ and M_z with respect to either orthonormal basis are similar operators. Moreover, by [34, Proposition 7], ${}_sT_z$ on $\mathcal{D}_q$ with respect to $\{{}_qe_k\}$ is unitarily equivalent to M_z acting on the space of holomorphic functions on $\mathbb{D}$ in which the norm of z^k is $W_0 \cdots W_{k-1} = \sqrt{\frac{k!}{(2+Q)_k}}$. Recalling (8.1) and that $Q = -q + 2s > -1$, this space is familiar. Moreover, we have (8.3). Let's sum up.

Theorem 8.2. *The operator ${}_sT_z$ on the Dirichlet space $\mathcal{D}_q$ with respect to the orthonormal basis $\{{}_qe_k\}$ is unitarily equivalent to the Q-shift M_z on the weighted Bergman space A^2_Q with the norm $|||\cdot|||_{\mathcal{D}_Q}$ or the norm $\|\cdot\|_{\mathcal{D}_Q}$.*

This theorem also follows from Theorem 4.5 and the discussion following it. Let's note that ${}_sT_z$ with respect to $\{{}_qE_k\}$ and M_z with respect to either orthonormal basis are also weighted shifts.

The unilateral shift M_z has a special place for the classical Toeplitz operators $\mathbf{T}_\phi = \mathbf{P}M_\phi$ on H^2, where $\mathbf{P} : L^2(\partial\mathbb{D}) \to H^2$ is the Szegő projection. An operator $T : H^2 \to H^2$ is the classical Toeplitz operator $\mathbf{T}_\phi$ for some $\phi \in L^\infty(\partial\mathbb{D})$ if and only if $M_z^* T M_z = T$. This property fails for classical Toeplitz operators on Bergman spaces.

With ${}_sT_\phi$, the more relevant equation is $({}_sT_z^*)T({}_sT_z) = T$. If $T = {}_sT_\phi$ satisfies this equation, then $({}_sT_z^*)({}_sT_\phi)({}_sT_z) = {}_sT_{\phi|z|^2} = {}_sT_\phi$ by Proposition 4.7. Then by linearity ${}_sT_{\phi(1-|z|^2)} = 0$, and by Proposition 4.9, $\phi(z)(1-|z|^2) = 0$ for almost every $z \in \mathbb{D}$. Thus $\phi = 0$ a.e. in $\mathbb{D}$ and ${}_sT_\phi = 0$. (We have promised to have $N = 1$ in this section, but this last result clearly holds for all N.) More is true.

Theorem 8.3. *The equation $({}_sT_z^*)T({}_sT_z) = T$ has no bounded nonzero solution $T : \mathcal{D}_q \to \mathcal{D}_q$.*

Proof. We adapt the proofs of [17, Theorems 3 and 5 (a)] to our situation and sketch the parts that are different only in the case $s > -2$. The B and w_k of [17] correspond to our T and $|||z^k|||^2_{\mathcal{D}_q}$.

We set

$$h_k(z) = \frac{1+s+k}{1+Q+k}\,\frac{z_k}{|||z^k|||^2_{\mathcal{D}_Q}} \in \mathcal{D}_q,$$

and define T on $\mathcal{D}_q$ by

$$T(z^k) = \frac{1+Q+k}{1+s+k}\,h_k(z),$$

with the understanding that $h_0 = 1$ and $T(1) = 1$. Also

$${}_sT^*_z(h_{k+1}) = \frac{1+Q+k}{1+s+k}\,h_k.$$

Combining these with the ${}_sT_z(z^k)$ computed above, we see that T satisfies the operator equation in the statement of the theorem. However, T is bounded if and only if

$$\sum_{k=0}^{\infty} \frac{1}{|||z^k|||^2_{\mathcal{D}_q}} \left| \left[\!\!\left[\frac{1+Q+k}{1+s+k}\,h_k, g \right]\!\!\right]_{\mathcal{D}_q} \right|^2 \leq C\,|||g|||^2_{\mathcal{D}_q}$$

for any $g \in \mathcal{D}_q$. Substituting in the details of the norm and the inner product yields that T is bounded if and only if $\{|||z^k|||^{-4}_{\mathcal{D}_Q} \sim k^{(1+Q)2}\}$ is bounded. But since $Q > -1$, this is impossible. □

Theorem 8.3 is a little surprising, because it is proved in [17, Theorem 5 (a)] that the similar operator equation $M^*_z T M_z = T$ has bounded solutions on $\mathcal{D}_q$ with $q \leq -1$, that is, if $\mathcal{D}_q$ is not a Bergman space, and some solutions are of the form $T(z^k) = k^{1+q}\,z^k$. However, ${}_sT_z$ is not a constant multiple of M_z on $\mathcal{D}_q$ with $q \leq -1$, and the fact that a derivative is used within I^u_s in defining ${}_sT_z$ effectively sends the case into the Bergman space $\mathcal{D}_Q$. Note the $|||z^k|||^2_{\mathcal{D}_Q}$ in the definition of $h_k(z)$, for example. We do not know whether the solutions given above to $M^*_z T M_z = T$ are Toeplitz operators, but such an equation cannot be satisfied by all Toeplitz operators ${}_sT_\phi$ on $\mathcal{D}_q$, as our next result implies.

Theorem 8.4. *Let L, N be nonzero bounded operators on $\mathcal{D}_q$. If $L({}_sT_\phi)N = {}_sT_\phi$ for all $\phi \in L^\infty$, then L and N are both scalar multiples of the identity.*

Proof. This time, we adapt the proof in [15] to our situation, and again give only a sketch in the case $s > -2$.

Initially proceeding as in [15], and using additionally (3.14) and that D^u_s is invertible, we conclude that N commutes with ${}_sT_z$.

Next let $h = N(1) = \sum_{k=0}^{\infty} h_k z^k \in \mathcal{D}_q$ and $H = D^u_s h \in A^2_Q$; then $h = D^{-u}_Q H$. We compute $N({}_sT^m_z 1)$ in two ways. First, by Theorem 4.5,

$$N({}_sT^m_z 1) = N({}_sT^m_z) D^{-u}_Q 1 = N D^{-u}_Q M_{z^m} 1 = \frac{(2+s)_m}{(2+Q)_m}\,N(z^m).$$

Second, by the commutativity just stated and Theorem 4.5,

$$\begin{aligned} N({}_sT_z^m 1) &= {}_sT_z^m N1 = {}_sT_z^m h = {}_sT_z^m D_Q^{-u} H = D_Q^{-u} M_{z^m} H = D_Q^{-u}(z^m H) \\ &= \sum_{k=0}^{\infty} h_k \frac{(2+Q)_k}{(2+s)_k} D_Q^{-u} M_{z^k}(z^m) = \sum_{k=0}^{\infty} h_k \frac{(2+Q)_k}{(2+s)_k} {}_sT_{z^k} D_Q^{-u}(z^m) \\ &= \frac{(2+s)_m}{(2+Q)_m} \sum_{k=0}^{\infty} h_k \frac{(2+Q)_k}{(2+s)_k} {}_sT_{z^k}(z^m). \end{aligned}$$

Thus

$$N(z^m) = \sum_{k=0}^{\infty} h_k \frac{(2+Q)_k}{(2+s)_k} {}_sT_{z^k}(z^m) = {}_sT_H(z^m)$$

at each $z \in \mathbb{D}$. By the density of polynomials in $\mathcal{D}_q$, we have $N = {}_sT_H$. That $L = {}_sT_{\overline{G}}$ for some $G \in A_Q^2$ follows by taking adjoints.

The rest of the proof is identical to that in [15] and omitted. □

The characterization by $M_z^* T M_z = T$ of the classical Toeplitz operators on H^2 and its failure on A_q^2 rely on the fact that M_z is an isometry on $H^2 = \mathcal{D}_{-1}$ while it is not on A_q^2, all with respect to the classical norms of the spaces. There is also the following weaker notion; see [2].

Definition 8.5. Let m be a positive integer. A bounded linear operator T on a Hilbert space H with norm $\|\cdot\|$ is called an *m-isometry* if it satisfies either of the equivalent conditions

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} (T^*)^j T^j = 0 \qquad \text{or} \qquad \sum_{j=0}^{m} (-1)^j \binom{m}{j} \|T^j f\|^2 = 0,$$

the second for all $f \in H$.

A 1-isometry is an isometry, and an $(m-1)$-isometry is also an m-isometry. It is shown in [31, Theorem 3.7] that the Dirichlet shift, which is not an isometry, is a 2-isometry with respect to some norm on $\mathcal{D}_{-2}$. Our last aim in this paper is to extend this result to other Dirichlet spaces $\mathcal{D}_q$ with q a negative integer and thus give concrete examples of natural spaces and norms for which the shift operator is an m-isometry and not an $(m-1)$-isometry.

Theorem 8.6. *For a positive integer m, the $(-m)$-shift M_z on $\mathcal{D}_{-m}$ is an m-isometry with respect to the norm $|||\cdot|||_{\mathcal{D}_{-m}}$, but not an $(m-1)$-isometry.*

Proof. Considering the orthogonality of monomials and the series expansion of f in $\mathcal{D}_{-m}$, it suffices to check the second equality defining an m-isometry only on $\{z^k\}$. Here $q = -m$ and $M_z^j z^k = M_{z^j} z^k = z^{k+j}$. If $m = 1$, (8.1) gives $|||z^{k+j}|||_{\mathcal{D}_{-1}} = 1$ for any k and j, and this means nothing but that M_z is an isometry on H^2. If $m = 2, 3, \ldots$, (8.1) gives

$$|||z^k|||^2_{\mathcal{D}_{-m}} = \frac{(m-1)_{k+1}}{k!} = (m-1)\binom{m-1+k}{m-1}.$$

Thus we need to know the value of

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{m-1+k+j}{m-1} \tag{8.4}$$

for all k when $n = m$ and $n = m-1$. The formula [30, 4.2.5 (47)] with $a = m-1+k$ and $b = 1$ says that (8.4) is equal to $(-1)^n \, \delta_{n,m-1}$ for $0 \le m-1 \le n$ and for any k, which is 0 if $n = m$ and nonzero if $n = m-1$. This proves both assertions of the theorem. □

We have another similar partial result with $\|\cdot\|_{\mathcal{D}_q}$. This time let $q = -2m$ with $m = 1, 2, \dots$ and $Q = 0$. Then $s = -m$, and (8.2) gives $\|z^k\|^2_{\mathcal{D}_{-2}} = k+1$ and

$$\|z^k\|^2_{\mathcal{D}_{-2m}} = (m-1)^2 \, (k+1) \binom{m-1+k}{m-1}^2 = (m-1)^2 \, (k+1) \, (k+1)^2_m$$

for $m = 2, 3, \dots$. If

$$\sum_{j=0}^{2m} (-1)^j \binom{2m}{j} (k+j+1) \binom{m-1+k+j}{m-1}^2 = 0 \tag{8.5}$$

for all k, then M_z is a $2m$-isometry on $\mathcal{D}_{-2m}$ with respect to the norm $\|\cdot\|_{\mathcal{D}_{-2m}}$. We have checked that it is true for $q = -2, -4, -6, -8$, and a computation of random cases on a computer algebra software gives results in the desired direction, but we do not know if (8.5) is true in general, nor do we know if M_z is not a $(2m-1)$-isometry. The corresponding result for q odd and negative seems to be wrong; for example, there is no Q for $q = -1$ or -3 that can make it true.

Comparing the cases $q = -2$ of Theorem 8.6 and of the above computation with the case $\mu = \nu$ of [31, Theorem 3.7], we see that if f is in the classical Dirichlet space $\mathcal{D}_{-2}$, then $|||f|||^2_{\mathcal{D}_{-2}} = \|f\|^2_{\mathcal{D}_{-2}} = \|f\|^2_{H^2} + \int_{\mathbb{D}} |f'|^2 \, d\nu$, where $\|\cdot\|_{\mathcal{D}_{-2}}$ is with $Q = 0$.

As a final remark, we have considered whether M_z on, say, the Bergman space A^2_1 or the Dirichlet space $\mathcal{D}_{-3/2}$, could be a (-1)-isometry or a $(3/2)$-isometry with respect to one of the norms considered, where a c-isometry for $c \in \mathbb{R}$ is defined appropriately through the infinite binomial expansion of $(1-x)^c$, but the few cases we have checked have not yielded a positive answer.

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