Energy-Momentum of a Stationary Beam of Light in Teleparallel Gravity‡

Oktay Aydoğdu[1] **and Mustafa Saltı**[2]
Department of Physics, Graduate School of Natural and Applied Science,
Middle East Technical University, 06531, Ankara-Turkey
E-mails: [1]oktay231@yahoo.com, [2]musts6@yahoo.com

Abstract. In this paper, we utilize the teleparallel gravity analogs of the energy and momentum definitions of Bergmann-Thomson and Landau-Lifshitz in order to explicitly evaluate the energy distribution(due to matter and fields including gravity) based on the Bonnor space-time. it is shown that for a stationary beam of light, these energy-momentum definitions give the same result. Furthermore, this result supports the viewpoint of Cooperstock and also agree with the previous works by Bringley and Gad.

Keywords: Energy-momentum; Bonnor; Teleparallel gravity.
PAC Numbers: 04.20.-q; 04.20.Jb; 04.50.+h

1. Introduction

Einstein's theory of general relativity is a superb theory of space-time and gravitation but some of its features are not without difficulties. For example; the localization of energy and momentum has been a problematic issue since the outset of this theory[1]. Recently; some researchers have moved this problem in the direction of an alternative theory, namely teleparallel gravity[2, 3].

In general theory of relativity, many physicist have devoted considerable attention to the problem of obtaining the conserved quantities such as energy-momentum that include the contribution from gravity. This has been applied to cosmological models as well. Einstein used the principle of equivalence and the conservation laws of energy-momentum to formulate the covariant field equations[4]. He defined the energy and momentum conservation law in the form given below.

$$\frac{\partial}{\partial x^\mu}[\sqrt{-g}(T^\mu_\nu + t^\mu_\nu)] = 0 \tag{1}$$

where $\mu, \nu = 0, 1, 2, 3$ and T^μ_ν is the stress energy density of matter. Einstein identified t^μ_ν as representing the stress energy density of gravitational fields. He also noted that t^μ_ν was not a tensor, but concluded that the equations given above hold good in all coordinate systems since they were directly obtained from the principle of general

‡ Accepted for publication in "Astrophysics and Space Science"

relativity. Bondi[5] expressed that a non-localizable form of energy is inadmissible in relativity and its location can in principle be found. Cooperstock[6] hypothesized that in a curved space-time energy-momentum is/are confined to the region of non-vanishing energy-momentum tensor $T_{\mu\nu}$ and consequently the gravitational waves are not carriers of energy and/or momentum in vacuum space-times. This hypothesis has neither been proved nor disproved. After pioneering work by Einstein, many energy-momentum definitions have been introduced in the literature. For instance; Møller[7], Papapetrou[8], Landau-Liftshitz[9], Tolman[10], Weinberg[11] and Qadir-Sharif's[12]. The energy and momentum prescriptions cited above give the meaningful results when the line-element is transformed to the cartesian coordinates. However for the Møller's prescriptions it is not necessary to use of Cartesian coordinates. In the literature, Virbhadra and his collaborators have considered many space-times and have shown that several energy-momentum complexes give the same and acceptable results[13, 14, 15].

Aguirregabiria *et. al.*[16] showed that several energy-momentum complexes give the same result for any Kerr-Shild class metric. This results derives from the Gürses-Gürsey discovery[17] that the energy-momentum pseudo-tensor vanishes globally for all Kerr-Schild metrics and hence the distribution of energy-momentum becomes tensorially significant. Gürses and Gürsey pointed out that for all Kerr-Schild class metrics the pseudo-tensors of Einstein and Landau-Lifshitz coincide. Recently, Chang *et. al.*[18] showed that every energy-momentum complex can be associated with a particular Hamilton boundary term, therefore the energy-momentum complexes may also be considered as quasi-local which provides a connection of the different complexes with a Hamiltonian aspect. It does not direct one to a preferred choice of complex. In Gen. Relat. Gravit. 36, 1255(2004); Vargas, using the definitions of Einstein and Landau-Lifshitz in teleparallel gravity, found that the total energy is zero in Friedmann-Robertson-Walker space-times. Therefore his result is the same as calculated in general relativity[19, 20, 21]. After these work, there some papers that show different energy-momentum complexes give the same energy-momentum for a given geometry in both general relativity and teleparallel gravity[22, 23, 24].

The paper is organized as follow: In section 2 and 3, we introduce the Kerr-Schild class and the Bonnor space-times, respectively. In section 4, firstly, we give the teleparallel gravity analog of the energy-momentum definitions of Bergmann-Thomson and Landau-Lifshitz and then obtain energy-momentum distributions associated with the Bonnor metric. Finally, in section 5 we discussed our results. Throughout this paper we use the convention that the indices take the values from 0 to 3 and $G = 1$, $c = 1$ units.

2. The Kerr-Schild Class Space-times

The Kerr-Schild class space-times are defined by the metric g_{ij} which are given below.

$$g_{ij} = \eta_{ij} - R l_i l_j \tag{2}$$

where $\eta_{ij} = \mathtt{diag}(1,-1,-1,-1)$ is the Minkowski metric, R is the scalar field and l_i is a null, geodesic and shear-free vector field in the Minkowski space-time, which are respectively expressed as

$$\eta_{ij} l_i l_i = 0 \tag{3}$$

$$\eta_{ij} l_{a,i} l_i = 0 \tag{4}$$

$$(l_{i,j} + l_{j,i}) l^i_{,a} \eta^{ja} - (l^i_{,i})^2 = 0 \tag{5}$$

An interesting characteristic of the Kerr-Schild class metric g_{ij} in equation (2) is that the vector field l_i remains null, geodesic and shear-free with the metric g_{ij}. Equations (3), (4) and (5) lead to

$$g_{ij} l_i l_i = 0 \tag{6}$$

$$g_{ij} l_{a,i} l_i = 0 \tag{7}$$

$$(l_{i,j} + l_{j,i}) l^i_{,a} g^{ja} - (l^i_{,i})^2 = 0 \tag{8}$$

There are various well-known space-time models of the Kerr-Schild Class, for instance, Schwarzschild, Reisner-Nordsröm, Kerr, Kerr-Newman, Vaidya, Dybney, Kinnersley, Bonnor-Vaidya, and Vaidya-Patel. The energy momentum complexes of Einstein Θ^c_a, Landau-Lifshitz L^{ac}, Papapetrou Ω^{ac}, and Weinberg W^{ac} "coincide" for any Kerr-Schild class metric[16].

These energy and/or momentum complexes for any Kerr-Schild class metric are defined by

$$\Theta^c_a = \eta_{ab} L^{bc} \tag{9}$$

$$L^{ac} = \Omega^{ac} = W^{ac} = \frac{1}{16\pi} \Lambda^{acbd}{}_{,bd} \tag{10}$$

where

$$\Lambda^{acbd} = 2R(\eta^{ac} l^b l^d + \eta^{bd} l^a l^c - \eta^{ab} l^c l^d - \eta^{cd} l^a l^b) \tag{11}$$

here Θ^0_0, L^{00}, Ω^{00}, and W^{00} represent energy distribution(due to matter plus gravitational fields) and Θ^0_β, $L^{0\beta}$, $\Omega^{0\beta}$, and $W^{0\beta}$ represent momentum density in the x^β direction. The energy momentum components are:

$$P^a = \frac{1}{16\pi} \int\int \Lambda^{a0bd}{}_{,d} \tau_b dS \tag{12}$$

3. The Bonnor Space-time

The Bonnor space-time which is describe a stationary beam of light flowing in the z direction is defined by the line element[25]

$$ds^2 = -dx^2 - dy^2 - (1-\xi)dz^2 - 2\xi dt dz + (1+\xi)dt^2 \tag{13}$$

where ξ is a function of x and y,

$$\nabla^2 \xi = 16\pi\rho \tag{14}$$

$$\rho = -T_3^3 = -T_3^0 = T_0^3 = T_0^0 \tag{15}$$

here T_b^a is describe the energy and momentum tensor. The line-element which is describe the Bonnor space-time can be written as

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \xi(dt - dz)^2 \tag{16}$$

which is the required form of a Kerr-Schild class space-times with

$$R = \frac{\xi}{2}, \qquad l_0 = 1, \qquad l_3 = -1. \tag{17}$$

Both components of l_0 and l_3 are constant so l is trivially geodesic and shear-free. It can be easily shown to be null which proves that the Bonnor space-time is of Kerr-Schild class[26].

The contravariant components of the metric tensor:

$$g^{\mu\nu} = (1-\xi)\delta_0^\mu\delta_0^\nu - \delta_1^\mu\delta_1^\nu - \delta_2^\mu\delta_2^\nu - (1-\xi)\delta_3^\mu\delta_3^\nu - \xi(\delta_3^\mu\delta_0^\nu + \delta_0^\mu\delta_3^\nu) \tag{18}$$

The non-trivial tetrad field induces a teleparallel structure on space-time which is directly related to the presence of the gravitational field, and the Riemannian metric arises as

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu \tag{19}$$

Using this relation, we obtain the tetrad components:

$$h^i{}_\mu = (1+\xi)^{\frac{1}{2}}\delta_0^i\delta_\mu^0 + \delta_1^i\delta_\mu^1 + \delta_2^i\delta_\mu^2 + (1+\xi)^{-\frac{1}{2}}\delta_3^i\delta_\mu^3 - \xi(1+\xi)^{-\frac{1}{2}}\delta_0^i\delta_\mu^3 \tag{20}$$

and its inverse is

$$h_i{}^\mu = (1+\xi)^{-\frac{1}{2}}\delta_i^0\delta_0^\mu + \delta_i^1\delta_1^\mu + \delta_i^2\delta_2^\mu + (1+\xi)^{\frac{1}{2}}\delta_i^3\delta_3^\mu + \xi(1+\xi)^{-\frac{1}{2}}\delta_i^3\delta_0^\mu \tag{21}$$

4. Energy and Momentum Associated with the Bonnor Metric in the Teleparallel Gravity

An alternative approach to gravitation is the so-called teleparallel gravity [27] which corresponds to a gauge theory for the translation group based on the Weitzenböck geometry[28]. In this theory gravitation is attributed to torsion[29], which plays the role of a force[30], whereas the curvature tensor vanishes identically. The fundamental field is represented by a nontrivial tetrad field, which gives rise to the metric as a by-product. The last translational gauge potentials appear as the nontrivial part of the tetrad field, thus induces on space-time a teleparallel structure which is directly related to the presence of the gravitational field. The interesting point of teleparallel gravity is that, due to gauge structure, it can reveal a more appropriate approach to consider same specific problem. This is the case, for example, of the energy-momentum problem, which becomes more transparent when considered from teleparallel point of view.

The energy-momentum definitons of Bergmann-Thomson and Landau-Lifshitz in teleparallel gravity[3] are given by the following equations respectively:

$$hB^{\mu\nu} = \frac{1}{4\pi}\partial_\lambda(g^{\mu\beta}U_\beta{}^{\nu\lambda}) \tag{22}$$

$$hL^{\mu\nu} = \frac{1}{4\pi}\partial_\lambda(hg^{\mu\beta}U_\beta{}^{\nu\lambda}) \tag{23}$$

where $h = \mathtt{det}(h^a{}_\mu)$ and $U_\beta{}^{\nu\lambda}$ is the Freud's super-potential, which is given by:

$$U_\beta{}^{\nu\lambda} = hS_\beta{}^{\nu\lambda}. \tag{24}$$

Here $S^{\mu\nu\lambda}$ is the tensor

$$S^{\mu\nu\lambda} = m_1 T^{\mu\nu\lambda} + \frac{m_2}{2}(T^{\nu\mu\lambda} - T^{\lambda\mu\nu}) + \frac{m_3}{2}(g^{\mu\lambda}T^{\beta\nu}{}_\beta - g^{\nu\mu}T^{\beta\lambda}{}_\beta) \tag{25}$$

with m_1, m_2 and m_3 the three dimensionless coupling constants of teleparallel gravity[29]. For the teleparallel equivalent of general relativity the specific choice of these three constants are:

$$m_1 = \frac{1}{4}, \qquad m_2 = \frac{1}{2}, \qquad m_3 = -1 \tag{26}$$

To calculate this tensor firstly we must calculate Weitzenböck connection:

$$\Gamma^\alpha{}_{\mu\nu} = h_a{}^\alpha \partial_\nu h^a{}_\mu \tag{27}$$

and after this calculation we get torsion of the Weitzenböck connection:

$$T^\mu{}_{\nu\lambda} = \Gamma^\mu{}_{\lambda\nu} - \Gamma^\mu{}_{\nu\lambda} \tag{28}$$

For the Bergmann-Thomson and Landau-Lifshitz complexes, we have the relations,

$$P^B_\mu = \int_\Sigma hB^0{}_\mu dxdydz \tag{29}$$

$$P^L_\mu = \int_\Sigma hL^0{}_\mu dxdydz \tag{30}$$

where P_i give momentum components P_1, P_2, P_3 while P_0 gives the energy and the integration hyper-surface Σ is described by $x^0 = t =$constant.

Considering equations (20) and (21), we find following non-vanishing Weitzenböck connection components:

$$\Gamma^0{}_{01} = \frac{\dot\xi}{2(1+\xi)} \qquad \Gamma^0{}_{02} = \frac{\xi'}{2(1+\xi)} \qquad \Gamma^0{}_{31} = -\frac{\dot\xi}{1+\xi} \tag{31}$$

$$\Gamma^0{}_{32} = -\frac{\xi'}{1+\xi} \qquad \Gamma^3{}_{31} = -\frac{\dot\xi}{2(1+\xi)} \qquad \Gamma^3{}_{32} = -\frac{\xi'}{2(1+\xi)} \tag{32}$$

where dot and prime indicates derivative with respect to x and y, respectively. The corresponding non-vanishing torsion components are found

$$T^{001} = -T^{010} = T^{331} = -T^{313} = \frac{\dot\xi}{2} \tag{33}$$

$$T^{002} = -T^{020} = T^{332} = -T^{323} = \frac{\xi'}{2} \tag{34}$$

$$T^{031} = -T^{013} = \frac{\dot\xi(2+\xi)}{2(1+\xi)} \tag{35}$$

$$T^{032} = -T^{023} = \frac{\xi'(2+\xi)}{2(1+\xi)} \tag{36}$$

Using these results into equation (25), the non-vanishing components of the tensor $S_\mu{}^{\nu\lambda}$ are found as:

$$S^{001} = -S^{313} = -\frac{\dot{\xi}}{4}, \qquad S^{002} = -S^{002} = -\frac{\xi'}{4} \tag{37}$$

$$S^{301} = -S^{103} = -S^{013} = \frac{\dot{\xi}(2+\xi)}{8(1+\xi)} \tag{38}$$

$$S^{302} = -S^{203} = -S^{023} = \frac{\xi'(2+\xi)}{8(1+\xi)} \tag{39}$$

From equation (23) with (24) we obtain following energy-momentum distributions:

$$hB^{00} = hB^{30} = hL^{00} = hL^{30} = \frac{1}{16\pi}\left[\ddot{\xi} + \xi''\right] \tag{40}$$

Then, from equations (14) and (15)

$$hB^{00} = hB^{30} = hL^{00} = hL^{30} = \rho = T_0^0 = T_0^3 \tag{41}$$

taking these results into equations (29) and (30) we get the energy associated with a stationary beam of light as

$$P_0^B = P_0^L = P_3^B = P_3^L = M \tag{42}$$

these results are exactly the same as obtained by Bringley[26] and Gad[31].

5. Discussion

The energy and momentum distribution associated with the Bonnor space-time found from the teleparallel analog of Bergmann-Thomson and Landau-Lifshitz definitions.

The common result props the Cooperstock hypothesis which states that energy localized to the region where the energy-momentum tensor is non-vanishing. This hypothesis states that there is no energy-momentum contribution from "vacuum" regions of space-time. If true, this hypothesis would have broad implications. For example, this hypothesis suggests that gravitational waves have no energy and that current attempts to detect these waves are doomed to failure.

In this paper, we have obtained the energy and momentum for a stationary beam of light found that the results are physically meaningful, exactly the same as the results which are obtained by Bringley and Gad and props the viewpoint of Cooperstock. Surely, the questions of energy-momentum localization in gravitation theories as well as the Cooperstock hypothesis are far from resolved, but the Bonnor space-time provides one more example where the teleparallel analog of the energy-momentum definitions of Bergmann-Thomson and Landau-Lifshitz.

Acknowledgment

We would like to thank Turkish Scientific Research Council (Tübitak)-Feza Gürsey Institute, Istanbul, for the hospitality our received in summer terms.

References

[1] C.W. Misner, K.S. Thorne and J.A.Wheeler, Gravitation (Freeman, New York) (1973).

[2] F.I. Mikhail, M.I. Wanas, A. Hindawi and E.I. Lashin, Int. J. Theor. Phys. **32**, 1627 (1993).

[3] T. Vargas, Gen. Rel. and Grav. **36**, 1255 (2004).

[4] A. Einstein, Sitzungsber. Preus. Akad. Wiss. Berlin (Math. Phys.) 778 (1915), Addendum-ibid. 799 (1915).

[5] H. Bondi, Proc. R. Soc. London **A427**, 249 (1990)

[6] F. I. Cooperstock, Gen. Rel. Grav. **26**, 323 (1994); Mod. Phys. Lett. **A 14**, 1531 (1999)

[7] C. Møller, Ann. Phys. (NY) **4**, 347 (1958); C. Møller, Ann. Phys. (NY) **12**, 118 (1961).

[8] A. Papapetrou, Proc. R.Irish. Acad. **A52**, 11 (1948).

[9] L.D. Landau and E.M. Lifshitz, *The Classical theory of fields* (Pergamon Press, 4th Edition, Oxford, 2002)

[10] R.C. Tolman, *Relativity, Thermodinamics and Cosmology* (Oxford Univ. Pres. London, P.227) (1934).

[11] S. Weinberg, *Gravitation and Cosmology: Principle and Applications of General Theory of Relativity* (John Wiley and Sons, Inc., New York, 1972)

[12] A. Qadir and M. Sharif, Phys. Lett. **A 167**, 331 (1992).

[13] K.S. Virbhadra, Phys. Rev. **D42**, 1066 (1990); 2919 (1990); Phys. Lett **A157**, 195 (1991); Mathematics Today **9**, 39 (1991); Pramana-J. Phys. **38**, 31 (1992)

[14] K.S. Virbhadra and N. Rosen, Gen. Rel. Grav. **25**, 429 (1993); Pramana-J. Phys. **45**, 215 (1995); Gen. Rel. Grav. **26**, 319 (1994); Phys. Rev. **D60**, 104041 (1999).

[15] S.S. Xulu, Int. J. Mod. Phys. **A 15**, 4849-4856 (2000); I. Radinschi, Mod. Phys. Lett., Vol.15 No.35, 2171-2175 (2000); E.C. Vagenas, Int. J. Mod. Phys. A**18**, 5781-5794 (2003)

[16] J. M. Aguirregabiria, A. Chamorro, and K.S Virbhadra, Gen. Rel. Grav. **28**, 1393 (1996).

[17] M. Gürses and F. Gürsey, J. Math. Phys. **10**, 2385 (1975)

[18] C.C. Chang, J. M. Nester and C. Chen, Phys. Rev. Lett. **83**, 1897 (1999).

[19] N. Rosen, Gen. Rel. Gravit. **26**, 319 (1994)

[20] V.B. Johri, D. Kalligas, G.P. Singh and C.W.F. Everitt, Gen. Rel. Grav. **27**, 323 (1995)

[21] N. Banerjee and S. Sen, Pramana J. Phys. **49**, 609 (1997)

[22] M. Saltı, Nuovo Cimento B, Vol.120 No.1, 53-60 (2005), Astrophs. Space Sci., Vol.299 No.1, 159-166 (2005); Mod. Phys. Lett.A, Vol.20 No.28, 2175-2782 (2005).

[23] M. Saltı and A. Havare, Int. J. Mod. Phys. A, Vol.20 No.10, 2169-2177 (2005)

[24] O. Aydogdu and M. Saltı, Astrophys. Space Sci., to appear [gr-qc/0505079]

[25] P.C. Vaidya, Pramana-J. Phys. **3**, 28 (1974)

[26] T. Bringley, Mod. Phys. Lett. **A 17**, 157-162 (2002)

[27] The basic references on teleparallel gravity can be found in F. Gronwald and F.W. Hehl (1996) in Proc. School of Cosmology and Gravitation on Quantum Gravity, Erice (1995), ed P.G.Bergmann, V. de Sabbata and H.J. Treder, (Singapore: World scientific); de V.C. Andrade, L.C.T. Guillen and J.G. Pereira (2000) in contribution to the IX Marcel Grossmans Meeting, Rome, Italy, July/2000

[28] R.Weitzenböck, "Invariantent theorie" (Groningen: Noordhoff)(1923)

[29] K. Hayashi and T. Shirafuji Phys.Rev D **19**, 3524 (1978)

[30] V.V. de Andrade and J.G. Pereira, Phys. Rev. D **56**, 4689 (1997)

[31] R.M. Gad, preprint: gr-qc/0307010