ENERGY IN THE SCHWARZSCHILD-de SITTER SPACETIME‡

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The energy (due to matter and fields including gravitation) of the Schwarzschild-de Sitter spacetime is investigated by using the Møller energy-momentum definition in both general relativity and teleparallel gravity. We found the same energy distribution for a given metric in both of these different gravitation theories. It is also independent of the teleparallel dimensionless coupling constant, which means that it is valid in any teleparallel model. Our results sustain that (a) the importance of the energy-momentum definitions in the evaluation of the energy distribution of a given spacetime and (b) the viewpoint of Lessner that the Møller energy-momentum complex is a powerful concept of energy and momentum.

Keywords: Energy; Schwarzschild-de Sitter spacetime; general relativity; teleparallel gravity; Møller's prescription.

1. INTRODUCTION

Choosing the Schwarzschild gauge, the line-element can be written in the following form.

$$ds^2 = \Delta(r)dt^2 - \Delta^{-1}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

where $\Delta(r)$ is an arbitrary (continuous, differentiable) function of r. For spacetime with a single horizon (like Schwarzschild, de Sitter), $\Delta(r)$ vanishes at one point, say, $r = r_0$. Near r_0, $\Delta(r)$ can be $\Delta(r) = \Lambda(r_0)(r - r_0)$, where $\Lambda(r_0)$ is twice the surface gravity of the horizon. For spacetime with multiple horizons (like Schwarzschild-de Sitter), $\Delta(r)$ vanishes at more than one point, say, $r = r_i$ (where $i = 1, 2, 3, .., n$). Around each of these points one can expand $\Delta(r) = \Lambda(r_i)(r - r_i)$, where $\frac{\Lambda(r_i)}{2}$ is the surface gravity of each of these horizons.

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The spherically coordinate of the Schwarzschild-de Sitter spacetime is defined by line-element given above, where

$$\Delta(r) = 1 - \frac{2M}{r} - \frac{r^2}{l^2} \tag{2}$$

here M is the mass of the black hole, and l^2 is related to the positive cosmological constant. The spacetime model has more than one horizon if $0 < \chi < \frac{1}{27}$ where $\chi = \frac{M^2}{l^2}$.

The black hole horizon r_h and the cosmological horizon r_c are located[1], respectively, at

$$r_h = \frac{2M}{\sqrt{3\chi}} \cos\frac{\pi + \Xi}{3}, \tag{3}$$

$$r_c = \frac{2M}{\sqrt{3\chi}} \cos\frac{\pi - \Xi}{3} \tag{4}$$

where

$$\Xi = \arccos(3\sqrt{3\chi}). \tag{5}$$

It is of interest to investigate the energy distribution associated with this black hole model. We hope to calculate the same energy distribution in both general relativity and teleparallel gravity. This is the motivation of the present paper.

After the pioneering expression by Einstein[2] for the energy and momentum distributions of the gravitational field, many attempts have been proposed to resolve the gravitational energy problem[3, 4]. Except the definition of Møller, these definitions give meaningful results only if the calculations are performed in "Cartesian" coordinates. Møller constructed an expression which enables one to evaluate energy and momentum in any coordinate system. In general relativity, Virbhadra[5] using the energy and momentum complexes of Einstein, Landau-Lifshitz, Papapetrou and Weinberg for a general non-static spherically symmetric metric of the Kerr-Schild class, showed that all of these energy-momentum formulations give the same energy distribution as in the Penrose energy-momentum formulation. Vargas in [4] using the definitions of Einstein and Landau-Lifshitz in teleparallel gravity, found that the total energy is zero in Friedmann-Robertson-Walker spacetimes and his result is the same as calculated in general relativity[6]. Further, several examples of particular spacetimes have been investigated and different energy-momentum pseudo-tensors are known to give the same energy distribution for a given spacetime[7, 8, 9].

Notations and conventions: $c = h = 1$, metric signature $(+, -, -, -)$, Greek indices and Latin ones run from 0 to 3. Throughout this paper, Latin indices $(i, j, ...)$ represent the vector number, and Greek indices $(\mu, \nu, ...)$ represent the vector components.

2. THE ENERGY-MOMENTUM DEFINITION OF MØLLER

2.1. The Møller Energy in General Relativity

In general relativity, the energy-momentum complex of Mϕller[3] is given by

$$M_\mu^\nu = \frac{1}{8\pi} \Sigma_{\mu,\alpha}^{\nu\alpha} \tag{6}$$

satisfying the local conservation laws:

$$\frac{\partial M_\mu^\nu}{\partial x^\nu} = 0 \tag{7}$$

where the antisymmetric super-potential $\Sigma_\mu^{\nu\alpha}$ is

$$\Sigma_\mu^{\nu\alpha} = \sqrt{-g}[g_{\mu\beta,\gamma} - g_{\mu\gamma,\beta}]g^{\nu\gamma}g^{\alpha\beta}. \tag{8}$$

The locally conserved energy-momentum complex M_μ^ν contains contributions from the matter, non-gravitational and gravitational fields. M_0^0 is the energy density and M_a^0 are the momentum density components. The momentum four-vector of Møller is given by

$$P_\mu = \int\int\int M_\mu^0 dxdydz. \tag{9}$$

Using Gauss's theorem, this definition transforms into

$$P_\mu = \frac{1}{8\pi}\int\int \Sigma_\mu^{0\alpha}\mu_\alpha dS. \tag{10}$$

where μ_α (where $\alpha = 1, 2, 3$) is the outward unit normal vector over the infinitesimal surface element dS. P_i give momentum components P_1, P_2, P_3 and P_0 gives the energy.

2.2. The Møller Energy in Teleparallel Gravity

The teleparallel theory of gravity(the tetrad theory of gravitation) is an alternative approach to gravitation and corresponds to a gauge theory for the translation group based on Weitzenböck geometry[10]. In the theory of teleparallel gravity, gravitation is attributed to torsion[11], which plays the role of a force[12], and the curvature tensor vanishes identically. The essential field is acted by a nontrivial tetrad field, which gives rise to the metric as a by-product. The translational gauge potentials appear as the nontrivial item of the tetrad field, so induces on spacetime a teleparallel structure which is directly related to the presence of the gravitational field. The interesting place of teleparallel theory is that, due to its gauge structure, it can reveal a more appropriate approach to consider some specific problem. This is the situation, for example, in the energy and momentum problem, which becomes more transparent.

Mϕller modified general relativity by constructing a new field theory in teleparallel space. The aim of this theory was to overcome the problem of the energy-momentum complex that appears in Riemannian Space[13]. The field equations in this new theory were derived from a Lagrangian which is not invariant under local tetrad rotation. Saez[14] generalized Mϕller theory into a scalar tetrad theory of gravitation. Meyer[15] showed that Mϕller theory is a special case of Poincare gauge theory[16, 17].

In teleparallel gravity, the super-potential of Mϕller is given by Mikhail *et al.*[4] as

$$U_\mu^{\nu\beta} = \frac{(-g)^{1/2}}{2\kappa}P_{\chi\rho\sigma}^{\tau\nu\beta}[\Phi^\rho g^{\sigma\chi}g_{\mu\tau} - \lambda g_{\tau\mu}\xi^{\chi\rho\sigma} - (1-2\lambda)g_{\tau\mu}\xi^{\sigma\rho\chi}] \tag{11}$$

where $\xi_{\alpha\beta\mu} = h_{i\alpha}h^i{}_{\beta;\mu}$ is the con-torsion tensor and $h_i{}^\mu$ is the tetrad field and defined uniquely by $g^{\alpha\beta} = h_i^\alpha h_j^\beta \eta^{ij}$ (here η^{ij} is the Minkowski spacetime). κ is the Einstein constant and λ is free-dimensionless coupling parameter of teleparallel gravity. For the teleparallel equivalent of general relativity, there is a specific choice of this constant.

Φ_ρ is the basic vector field given by

$$\Phi_\mu \doteq \xi^\rho{}_{\mu\rho} \tag{12}$$

and $P_{\chi\rho\sigma}^{\tau\nu\beta}$ can be found by

$$P_{\chi\rho\sigma}^{\tau\nu\beta} = \delta_\chi^\tau g_{\rho\sigma}^{\nu\beta} + \delta_\rho^\tau g_{\sigma\chi}^{\nu\beta} - \delta_\sigma^\tau g_{\chi\sigma}^{\nu\beta} \tag{13}$$

with $g_{\rho\sigma}^{\nu\beta}$ being a tensor defined by

$$g_{\rho\sigma}^{\nu\beta} = \delta_\rho^\nu \delta_\sigma^\beta - \delta_\sigma^\nu \delta_\rho^\beta. \tag{14}$$

The energy-momentum density is defined by

$$\Xi_\alpha^\beta = U_{\alpha,\lambda}^{\beta\lambda} \tag{15}$$

where comma denotes ordinary differentiation. The energy is expressed by the surface integral;

$$E = \lim_{r\to\infty} \int_{r=constant} U_0^{0\zeta} \eta_\zeta dS \tag{16}$$

where η_ζ is the unit three-vector normal to surface element dS.

3. CALCULATIONS

3.1. Solution in General Relativity

The matrix of the $g_{\mu\nu}$ is defined by

$$\begin{pmatrix} (1-\frac{2M}{r}+\frac{r^2}{l^2}) & 0 & 0 & 0 \\ 0 & -(1-\frac{2M}{r}+\frac{r^2}{l^2})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{17}$$

and its inverse matrix $g^{\mu\nu}$ is given by

$$\begin{pmatrix} (1-\frac{2M}{r}+\frac{r^2}{l^2})^{-1} & 0 & 0 & 0 \\ 0 & -(1-\frac{2M}{r}+\frac{r^2}{l^2}) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{18}$$

Using the matrices given above, the required non-vanishing component of $\Sigma_\mu^{\nu\alpha}$ is

$$\Sigma_0^{01} = 2\sin\theta \left[M + \frac{r^3}{l^2}\right]. \tag{19}$$

From this point of view, the energy of the Schwarzschild-de Sitter black holes in general relativity is found as given below.

$$E(r) = M + \frac{r^3}{l^2} \tag{20}$$

3.2. Solution in Teleparallel Gravity

The general form of the tetrad, h_i^μ, having spherical symmetry was given by Robertson[18]. In the Cartesian form it can be written as

$$h_0^0 = iA, h_a^0 = Cx^a, h_0^\alpha = iDx^\alpha, h_a^\alpha = B\delta_a^\alpha + Ex^\alpha x^\alpha + \epsilon_{a\alpha\beta}Fx^\beta \tag{21}$$

where A, B, C, D, E, and F are functions of t and $r = \sqrt{x^\alpha x^\alpha}$, and the zeroth vector h_0^μ has the factor $i = \sqrt{-1}$ to preserve Lorentz signature. We impose the boundary condition that in the case of $r \to \infty$ the tetrad given above approaches the tetrad of Minkowski spacetime, $h_a^\mu = diag(i, \delta_a^\alpha)$.

Using the general coordinate transformation

$$h_{a\mu} = \frac{\partial \mathbf{X}^\nu}{\partial \mathbf{X}^\mu} h_{a\nu} \tag{22}$$

where $\mathbf{X}^\mu$ and $\mathbf{X}^\nu$ are, respectively, the isotropic and Schwarzschild coordinates (t, r, θ, ϕ), we obtain the tetrad ${h_a}^\mu$ as

$$\begin{pmatrix} \frac{i}{\sqrt{1-\frac{2M}{r}+\frac{r^2}{l^2}}} & 0 & 0 & 0 \\ 0 & \sqrt{1-\frac{2M}{r}+\frac{r^2}{l^2}}s\theta c\phi & \frac{1}{r}c\theta c\phi & -\frac{s\phi}{rs\theta} \\ 0 & \sqrt{1-\frac{2M}{r}+\frac{r^2}{l^2}}s\theta s\phi & \frac{1}{r}c\theta s\phi & \frac{c\phi}{rs\theta} \\ 0 & \sqrt{1-\frac{2M}{r}+\frac{r^2}{l^2}}c\theta & -\frac{1}{r}s\theta & 0 \end{pmatrix} \tag{23}$$

where we have introduced the following notation: $s\theta = \sin\theta$, $c\theta = \cos\theta$, $s\phi = \sin\phi$ and $c\phi = \cos\phi$. Now, we are interested in to find the total energy distribution. Since the intermediary mathematical exposition are length, we give only the final result. After making the required calculations[19, 20], the required non-vanishing component of $U_\mu^{\nu\beta}$ is

$$U_0^{01} = \frac{2\sin\theta}{\kappa}\left[M + \frac{r^3}{l^2}\right]. \tag{24}$$

Substituting this result in energy integral, we have the following energy distribution

$$E(r) = M + \frac{r^3}{l^2} \tag{25}$$

4. DISCUSSION

Since the advent of Einstein's theory of general relativity, several papers have been devoted to investigate the energy distribution for a given spacetime. Several black hole model considered to obtain the energy distribution. For example; Chamorro-Virbhadra[21] and Xulu[22] showed, considering the general relativity analogs of Einstein and Møller's definitions, that the energy of a charged dilation black hole depends on the value λ which controls the coupling between the dilation and the Maxwell fields.

$$E_{Einstein} = M - \frac{Q^2}{2r}(1-\lambda^2) \tag{26}$$

$$E_{Moller} = M - \frac{Q^2}{r}(1-\lambda^2). \tag{27}$$

Also, Virbhadra[5] and Xulu[22] obtained that the energy distribution in the sense of Einstein and Møller disagree in general. Next, Lassner[23] showed that the Møller energy-momentum complex is a powerful concept of energy and momentum.

Møller showed that a tetrad description of a gravitational field equation allows a more satisfactory treatment of the energy-momentum complex than does general relativity. Therefore, we have applied the super-potential method by Mikhail *et. al.*[4] to calculate the energy of the central gravitating body.

Recently, one of us[24] has considered the Møller energy-momentum definition in both general relativity and teleparallel gravity for the viscous Kasner-type metric and calculated the same energy. Also the result of that paper agree with some of the previous papers by Cooperstock and Israelit, Rosen, Johri *et al.*, Banerjee-Sen in general relativity and by Vargas in teleparallel gravity.

In this paper, in order to investigate the energy associated with the Schwarzschild-de Sitter black hole, we considered the Møller energy-momentum formulation in both general relativity and teleparallel gravity. We obtained that the energy distribution is the same in both of these different gravitation theories and also found that the energy depends on the mass of the Schwarzschild-de Sitter black hole,

$$E(r) = M + \frac{r^3}{l^2}. \tag{28}$$

It is also independent of the teleparallel dimensionless coupling constant, which means that it is valid not only in teleparallel equivalent of general relativity but also in any teleparallel model. In the special case where there is no cosmological constant, we obtain the energy distribution as

$$\lim_{l\to\infty} E(r) = M. \tag{29}$$

Finally, this paper sustains (a) the importance of the energy-momentum definitions in the evaluation of the energy distribution of a given spacetime and (b) the viewpoint of Lessner[23].

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