Dualisation of the Salam-Sezgin D=8 Supergravity

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Abstract

The first-order formulation of the Salam-Sezgin $D = 8$ supergravity coupled to N vector multiplets is discussed. The non-linear realization of the bosonic sector of the $D = 8$ matter coupled Salam-Sezgin supergravity is introduced by the dualisation of the fields and by constructing the Lie superalgebra of the symmetry group of the doubled field strength.

1 Introduction

It is possible to formulate the scalar sectors of a wide class of supergravity theories as non-linear sigma models based on G/K coset parametrization maps. In particular the scalar sectors of all the pure and the matter coupled $N > 2$ extended supergravities in $D = 4, 5, 6, 7, 8, 9$ dimensions as well as the maximally extended supergravities in $D \leq 11$ can be formulated as symmetric space sigma models. The first-order bosonic field equations of the maximal supergravities for $D \leq 11$ are regained as a twisted self-duality equation by the non-linear realization of the bosonic sector of these theories in [1,2]. Therefore the non-linear nature of the scalar fields has been modified to cover the bosonic field content as well. In [3,4,5] the non-linear realization has been enlarged further to include the graviton for the $D = 11$, IIB and IIA supergravities. In these later works including the gravity the entire bosonic sector is formulated as a coset model by constructing the algebra which generates the coset and by taking the Lorentz group as the local symmetry group. Furthermore in the same works it is also discussed that a larger non-linear realization of the bosonic sector can be considered to include a Kac-Moody algebra identified as E_{11}.

In this work we formulate the bosonic sector of the $D = 8$, Salam-Sezgin supergravity which is coupled to N vector multiplets [6] as a non-linear realization. The 2N vector multiplet scalars of the theory parameterize the coset $SO(N,2)/SO(N) \times SO(2)$ which is a Riemannian globally symmetric space [7] and $SO(N,2)$ is a semi-simple, non-compact real form while $SO(N) \times SO(2)$ is its maximal compact subgroup. The construction presented here is parallel with the one of [1,2]. We will define the Lie superalgebra which will be used to formulate the bosonic fields as a non-linear coset model. The algebra structure will be constructed in such a way that it will yield the integrated first-order bosonic equations of the theory as a twisted self-duality equation of the Cartan form of the coset representative [1,2].

Section two is the introduction of the $D = 8$, matter coupled Salam-Sezgin supergravity. We will derive the equations of motion and then locally integrate them to find the first-order field equations. Whereas in section three by following the outline of [1,2] we will introduce dual fields for the bosonic field content of the theory excluding the graviton and we will also define new algebra generators for the original fields and their duals. A coset element will then be constructed whose Cartan form is intended to yield the correct second-order equations when inserted in the Cartan-Maurer equation. In order to calculate the Cartan form one needs to know the algebra structure of the generators which parameterize the coset representatives when coupled to the fields. The algebra which generates the coset representatives is a differential graded algebra and it is composed of the differential forms and the field generators. It covers the Lie superalgebra of the field generators which is the Lie algebra of the symmetry group of the Cartan form. The correct choice of the commutators and the anti-commutators of the Lie superalgebra is a result of the direct comparison of the field equations of section two and the Cartan-Maurer equation. Finally we will show that the twisted self-duality equation $*\mathcal{G} = \mathcal{S}\mathcal{G}$, [1,2] when applied on the Cartan form leads to the first-order formulation of the equations of motion achieved in section two.

2 The $D = 8$ Matter Coupled Salam-Sezgin Supergravity

The eight dimensional Salam-Sezgin supergravity with matter coupling is constructed in [6]. N vector multiplets $\{\lambda, A, \varphi^i\}$ (λ is a Fermion and for $i = 1, 2$ $\{\varphi^i\}$ are the scalars whereas A is a one-form field) are coupled to the original field content $\{e_\mu^m, \psi^\mu, \chi, B_{\mu\nu}, A_i^\mu, \sigma\}$ where $\{e_\mu^m\}$ is the graviton, $\{\psi^\mu, \chi\}$ are the fermionic fields, $\{\sigma\}$ is the dilaton, $B_{\mu\nu}$ is a two-form field and for $i = 1, 2$ $\{A_i^\mu\}$ are the one-form fields. We will combine the original one-forms $\{A_i^\mu; i = 1, 2\}$ and the N vector multiplet one-forms $\{A\}$ into a single set and we will denote them as $\{A^j; j = 1, ...,$N+2$\}$. Later we will also classify the 2N vector multiplet scalars $\{\varphi^i\}$ as dilatons and axions as a result of the solvable Lie algebra parametrization [8]. The single dilaton $\{\sigma\}$ of the original theory and the 2N scalars $\{\varphi^i\}$ of the vector multiplet which parameterize the $SO(N,2)/SO(N) \times SO(2)$ Kähler coset manifold constitute the scalar sector of the matter coupled theory. An explicit representation parametrization for the scalar coset $SO(N,2)/SO(N) \times SO(2)$ is given in [6] where the scalar fields are not classified as dilatons and axions. Since $SO(N,2)$ is a non-compact real form of some semi-simple Lie group (for odd N $so(N,2)$ is the non-compact real form of $B_{\frac{N+1}{2}}$ and for even N it is the non-compact real form of $D_{\frac{N+2}{2}}$ and in general depending on N, $SO(N,2)$ is not necessarily a split real form (maximally non-compact) [7,9,10]) we will use the solvable Lie algebra parametrization [8]. This parametrization introduces the dilatons which are coupled to the generators of $\mathbf{h}_k$, the Lie algebra of the maximal R-split torus of $SO(N,2)$ [9] and the axions which are coupled to the positive root generators of $SO(N,2)$ which do not commute with the elements of $\mathbf{h}_k$ [7,9]. Due to the nature of the semi-simple, non-compact real form $SO(N,2)$ and the fact that $SO(N) \times SO(2)$ is the maximal compact subgroup of $SO(N,2)$ by using the Iwasawa decomposition [7] the coset $SO(N,2)/SO(N)\times SO(2)$ can be parameterized as

$$
\begin{aligned}
L &= \mathbf{g}_H \mathbf{g}_N \\
&= e^{\frac{1}{2}\phi^i(x)H_i} e^{\chi^m(x)E_m}
\end{aligned}
\tag{2.1}
$$

where $\{H_i\}$ for $i = 1,...,$ $\dim\mathbf{h}_k$ are the generators of $\mathbf{h}_k$ and $\{E_m\}$ for $m \in \Delta^+_{nc}$ are the positive root generators which generate the orthogonal complement of $\mathbf{h}_k$ within the solvable Lie algebra $\mathbf{s}_0$ of $SO(N,2)$, [7,8,9]. Also the 2N scalars are now classified as $\{\phi^i\}$, the dilatons for $i = 1,...,$ $\dim\mathbf{h}_k$ and $\{\chi^m\}$, the axions for $m \in \Delta^+_{nc}$. In other words if we denote $\dim\mathbf{h}_k$ by r and label the roots which are elements of Δ^+_{nc} from 1 to n then we have $r + n =$ 2N. By using the generalized transpose #, [1,10] we can introduce the internal metric $\mathcal{M} = L^{\#}L$. For $SO(N,2)$ since the subgroup generated by the compact generators of the Lie algebra $so(N,2)$ is an orthogonal group # coincides with the matrix transpose in the fundamental representation [1]. Thus the Lagrangian of the matter scalar sector of the eight dimensional Salam-Sezgin supergravity [6] can be written as

$$
\mathcal{L}_{scalar} = \frac{1}{16} tr(d\mathcal{M}^{-1} \wedge *d\mathcal{M}) \tag{2.2}
$$

where $\mathcal{M} = L^T L$. The bosonic Lagrangian of the $D = 8$, matter coupled Salam-Sezgin supergravity can now be given as [6]

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{4} R * 1 + \frac{3}{8} d\sigma \wedge *d\sigma - \frac{1}{2} e^{2\sigma} G \wedge *G \\
&+ \frac{1}{16} tr(d\mathcal{M}^{-1} \wedge *d\mathcal{M}) - \frac{1}{2} e^{\sigma} F \wedge \mathcal{M} * F
\end{aligned}
\tag{2.3}
$$

where we have assumed the (N+2)-dimensional matrix representation of $SO(N,2)$. We should also imply that the last term in (2.3) above can be explicitly written as $-\frac{1}{2}e^{\sigma}\mathcal{M}_{ij}F^i \wedge *F^j$ since $\mathcal{M}$ is a symmetric matrix having the components $\mathcal{M}_{ij} = L^a_i L^a_j$ for $i,j,a = 1,...,$N+2. The (N+2) two-forms $\{F^i\}$ are the field strengths of $\{A^i\}$, $F^i = dA^i$ and the Chern-Simons term G is defined as

$$
G = dB + \eta_{ij} F^i \wedge A^j. \tag{2.4}
$$

Here B is the two-form field, the indices $\{i,j\}$ are running from 1 to N+2 and η is the metric corresponding to $SO(N,2)$ explicitly $\eta = (-,-,+,+,+,...)$. We also have the standard formulas

$$L^T \eta L = \eta \qquad , \qquad L^{-1} = \eta L \eta \tag{2.5}$$

of the orthogonal matrix groups for $SO(N,2)$. The second identity is due to the fact that the coset representatives L can be locally chosen as symmetric matrices $L^T = L$ which is evident from the explicit parametrization given in [6] and the transformation relating the parametrization of [6] and the solvable Lie algebra parametrization which can be shown to exist locally [11].

We will now give an identity for the Cartan generators $\{H_i\}$ which we will make use of in deriving the first-order equations from the second-order ones. Firstly we should observe that

$$\partial_i L \equiv \frac{\partial L}{\partial \phi^i} = \frac{1}{2} H_i L \qquad , \qquad \partial_i L^T \equiv \frac{\partial L^T}{\partial \phi^i} = \frac{1}{2} L^T H_i^T,$$

$$\partial_i L^{-1} \equiv \frac{\partial L^{-1}}{\partial \phi^i} = -\frac{1}{2} L^{-1} H_i \qquad , \qquad \partial_i \mathcal{M} \equiv \frac{\partial \mathcal{M}}{\partial \phi^i} = \frac{1}{2} L^T (H_i + H_i^T) L. \tag{2.6}$$

By differentiating the equations in (2.5) with respect to $\{\phi^i\}$ and by using the identities (2.5) and (2.6) effectively we can show that $H_i^T \eta = -\eta H_i$ and moreover

$$(H_i L)^T = H_i L \tag{2.7}$$

for $i = 1, ..., \dim \mathbf{h}_k$. In obtaining the symmetry of $H_i L$, namely the equation (2.7), we have implicitly used the fact that the coset representatives can be chosen as symmetric matrices. The identities (2.6) and (2.7) are matrix equations and apparently they are valid in the (N+2)-dimensional representation we have assumed. In order to find the field equations for the non-gravitational fields we first vary the Lagrangian (2.3) with respect to the fields σ, B and $\{A^i\}$ and we find the corresponding field equations as

$$\frac{3}{4}d(*d\sigma) = -e^{2\sigma}G \wedge *G - \frac{1}{2}e^{\sigma}\mathcal{M}_{ij}F^i \wedge *F^j,$$

$$d(e^{2\sigma} * G) = 0,$$

$$d(e^{\sigma}\mathcal{M}_{ij} * F^j) = -2e^{2\sigma}(\eta_{ij}F^j \wedge *G). \tag{2.8}$$

One can follow the standard steps in [9,10] to find the corresponding field equations for the scalars where the formulation is done for a generic coset. Here we will not present the complete derivation but we will only give the resulting equations. The field equations for $\{\phi^i\}$ and $\{\chi^m\}$ are

$$\begin{aligned} d(e^{\frac{1}{2}\gamma_i\phi^i} * U^\gamma) &= -\frac{1}{2}\gamma_j e^{\frac{1}{2}\gamma_i\phi^i} d\phi^j \wedge *U^\gamma \\ &+ \sum_{\alpha-\beta=-\gamma} e^{\frac{1}{2}\alpha_i\phi^i} e^{\frac{1}{2}\beta_i\phi^i} N_{\alpha,-\beta} U^\alpha \wedge *U^\beta, \\ d(*d\phi^i) &= \frac{1}{2}\sum_{\alpha\in\Delta_{nc}^+} \alpha_i e^{\frac{1}{2}\alpha_i\phi^i} U^\alpha \wedge e^{\frac{1}{2}\alpha_i\phi^i} * U^\alpha \\ &- 2e^{\sigma}((H_i)^a_n L^n_m L^a_j) * F^m \wedge F^j \end{aligned} \tag{2.9}$$

where the indices associated with the dilatons and the Cartan generators are from 1 to $\dim\mathbf{h}_k$ also $\alpha, \beta, \gamma \in \Delta_{nc}^+$ [9]. The matrices $\{(H_i)^a_n\}$ are the ones corresponding to the generators $\{H_i\}$ in the (N+2)-dimensional representation chosen. Our formulation for the dilaton equation namely the second equation of (2.9) is different from the one in [9] where the contribution from the coupling of $\{A^i\}$ in the second equation of (2.9) was expressed by using the weights of the representation. We keep the original fields instead of their weight expansions. We have also introduced $U^\alpha = \mathbf{\Omega}^\alpha_\beta\, d\chi^\beta$ which is not explicitly calculated in [9,10]. Here $\mathbf{\Omega}$ is the matrix

$$\begin{aligned}\mathbf{\Omega} &= \sum_{n=0}^{\infty} \frac{\omega^n}{(n+1)!} \\ &= (e^{\omega} - I)\,\omega^{-1}\end{aligned} \tag{2.10}$$

with $\omega_{\beta}^{\gamma} = \chi^{\alpha} K_{\alpha\beta}^{\gamma}$. The structure constants $K_{\alpha\beta}^{\gamma}$ are defined as $[E_{\alpha}, E_{\beta}] = K_{\alpha\beta}^{\gamma} E_{\gamma}$ or since $\{E_{\alpha}\}$ are the generators corresponding to a subset, Δ_{nc}^{+} of the roots of $so(N,2)$ we have $[E_{\alpha}, E_{\beta}] = N_{\alpha,\beta} E_{\alpha+\beta}$. In other words $K_{\beta\beta}^{\alpha} = 0$, $K_{\beta\gamma}^{\alpha} = N_{\beta,\gamma}$ if $\beta + \gamma = \alpha$ and $K_{\beta\gamma}^{\alpha} = 0$ if $\beta + \gamma \neq \alpha$ in the root sense.

One can locally integrate the bosonic field equations (2.8) and (2.9) by introducing dual fields and by using the fact that locally a closed form is an exact one. By integration we mean to extract an exterior derivative on both sides of the equations. It is straightforward to write the first-order equations for (2.8). On the other hand for the equations in (2.9) the first-order formulation of the scalars must be treated separately due to their non-linear nature. The integration of the terms in (2.9) which are related to (2.2) can be done by an application of the dualisation method of [1,2] only for the scalar sector separately that is the method we have followed. If we introduce the dual four-form $\widetilde{B}$, the set of five-forms $\{\widetilde{A}^j\}$ and the six-forms $\{d\widetilde{\sigma}, d\widetilde{\phi}^i, d\widetilde{\chi}^m\}$ we can locally derive the first-order equations as

$$e^{2\sigma} * G = d\widetilde{B},$$

$$e^{\sigma} \mathcal{M}_j^i * F^j = -d\widetilde{A}^i + 2d\widetilde{B} \wedge \eta_j^i A^j,$$

$$*d\sigma = d\widetilde{\sigma} - \frac{4}{3} B \wedge d\widetilde{B} + \frac{2}{3}\delta_{ij} A^i \wedge d\widetilde{A}^j,$$

$$e^{\frac{1}{2}\alpha_i \phi^i} (\mathbf{\Omega})_l^{\alpha} * d\chi^l = (e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}})_j^{\alpha+r} \widetilde{S}^j,$$

$$\frac{1}{2} * d\phi^m = (e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}})_j^m \widetilde{S}^j + (H_m)_{ji} A^j \wedge d\widetilde{A}^i + \eta_i^k (H_m)_{jk} A^j \wedge A^i \wedge d\widetilde{B} \tag{2.11}$$

where we have denoted $r \equiv \dim \mathbf{h}_k$. The vector $\overset{\rightharpoonup}{\widetilde{S}}$ is $\widetilde{S}^j = \frac{1}{2} d\widetilde{\phi}^j$ for $j = 1, ..., r$ and $\widetilde{S}^{\alpha+r} = d\widetilde{\chi}^{\alpha}$ for $\alpha \in \Delta_{nc}^{+}$. The matrices $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ are introduced as

$\mathbf{\Gamma}_n^k = \frac{1}{2}\phi^i\, \widetilde{g}_{in}^k$ and $\mathbf{\Lambda}_n^k = \chi^m \widetilde{f}_{mn}^k$.The coefficients $\{\widetilde{g}_{in}^k\}$ and $\{\widetilde{f}_{mn}^k\}$ are as follows

$$\widetilde{f}_{\alpha m}^n = 0, \qquad m \leq r \quad , \quad \widetilde{f}_{\alpha,\alpha+r}^i = \frac{1}{4}\alpha_i, \qquad i \leq r$$

$$\widetilde{f}_{\alpha,\alpha+r}^i = 0, \qquad i > r \quad , \quad \widetilde{f}_{\alpha,\beta+r}^i = 0, \qquad i \leq r,\ \alpha \neq \beta$$

$$\widetilde{f}_{\alpha,\beta+r}^{\gamma+r} = N_{\alpha,-\beta}, \qquad \alpha - \beta = -\gamma,\ \alpha \neq \beta$$

$$\widetilde{f}_{\alpha,\beta+r}^{\gamma+r} = 0, \qquad \alpha - \beta \neq -\gamma,\ \alpha \neq \beta. \tag{2.12}$$

We should remind the reader of the fact that $\alpha, \beta \in \Delta_{nc}^+$ and we assume that these roots are enumerated from 1 to 2N$-r$. The conditions on α, β, γ in the last two lines of (2.12) must be taken as root equations. In the next section we will see that these coefficients are in fact the structure constants of the commutators of the scalar and the dual scalar generators in other words $[E_\alpha, \widetilde{T}_m] = \widetilde{f}_{\alpha m}^n \widetilde{T}_n$, $[H_i, \widetilde{T}_m] = \widetilde{g}_{im}^n \widetilde{T}_n$ where $\widetilde{T}_i = \widetilde{H}_i$ for $i = 1, ..., r$ and $\widetilde{T}_{\alpha+r} = \widetilde{E}_\alpha$ for $\alpha \in \Delta_{nc}^+$. Here $\{\widetilde{H}_i\}$ are the generators which we will assign to the fields $\{\widetilde{\phi}^i\}$ and $\{\widetilde{E}_\alpha\}$ are the generators which will be assigned to the fields $\{\widetilde{\chi}^\alpha\}$. The last term in the last equation of (2.11) needs attention since if its exterior derivative is taken, in order to obtain the corresponding term in the second-order equations, one needs to make use of the fact that as, $H_i^T \eta = -\eta H_i$ and since η is a diagonal matrix, $H_i \eta$ must have anti-symmetric matrix representatives. One also needs to make use of the identities (2.7) to reach the corresponding second-order equation in (2.9) when the last equation of (2.11) is differentiated. We are using the Euclidean metric to raise and lower the indices whenever necessary for notational purposes.

These first-order equations will be re-formulated as a twisted self-duality equation after we introduce the dualisation of the theory resulting in a coset model in the next section.

3 The Coset Formulation

In this section by following the outline of [1,2] we will generalize the coset formulation of the scalar manifold $SO(N,2)/SO(N) \times SO(2)$ to the entire bosonic sector of the $D = 8$ matter coupled Salam-Sezgin supergravity. Our aim is to construct a coset representative which will be used to realize the original field equations. The group which is generated by the generators we introduce to construct the coset representative within the dualisation, in general becomes the symmetry group of the Cartan form $\mathcal{G}$ which is generated by the coset representative map. We need to construct a new algebra which will lead to the non-linear realization of the bosonic field equations. The first-order equations (2.11) derived in the last section will be formulated as a twisted self-duality equation $*\mathcal{G} = \mathcal{S}\mathcal{G}$ where $\mathcal{S}$ is a pseudo-involution of the dualized Lie superalgebra. The full symmetry group of the first-order equations is in general bigger than the symmetry group of the Cartan form.

We will assign a generator for each field. The untilded original generators are $\{K, V_i, Y, H_j, E_m\}$ for the fields $\{\sigma, A^i, B, \phi^j, \chi^m\}$ respectively. Their duals are introduced as $\{\widetilde{K}, \widetilde{V}_i, \widetilde{Y}, \widetilde{H}_j, \widetilde{E}_m\}$ for the dual fields $\{\widetilde{\sigma}, \widetilde{A}^i, \widetilde{B}, \widetilde{\phi}^j, \widetilde{\chi}^m\}$. We require that the Lie superalgebra to be constructed from the generators has the $\mathbb{Z}_2$ grading as usual. For this reason the generators will be chosen as odd if the corresponding potential is an odd degree differential form and otherwise even. In particular $\{V_i, \widetilde{V}_i\}$ are odd generators and the rest of the generators are even. We will also consider a differential graded algebra generated by the differential forms and the generators which are coupled to the fields. This algebra covers the Lie superalgebra of the field generators. Therefore the odd (even) generators behave like odd (even) degree differential forms under this graded differential algebra structure when they commute with the exterior product. The odd generators obey anti-commutation relations while the even ones and the mixed couples obey commutation relations.

As it will be clear later the structure constants of this new algebra will be chosen so that they will lead to the correct second-order equations (2.8) and (2.9). Firstly let us consider the map

$$\nu = e^{\frac{1}{2}\phi^j H_j} e^{\chi^m E_m} e^{\sigma K} e^{A^i V_i} e^{\frac{1}{2}BY}. \tag{3.1}$$

The corresponding Cartan form $\mathcal{G} = d\nu\nu^{-1}$ can be expanded in terms of the original generators and it satisfies the Cartan-Maurer equation $d\mathcal{G} - \mathcal{G}\wedge\mathcal{G} = 0$ owing to its definition. The coset representative by including the dual generators as well will be chosen as

$$\nu' = e^{\frac{1}{2}\phi^j H_j} e^{\chi^m E_m} e^{\sigma K} e^{A^i V_i} e^{\frac{1}{2}BY} e^{\frac{1}{2}\widetilde{B}\widetilde{Y}} e^{\widetilde{A}^i\widetilde{V}_i} e^{\widetilde{\sigma}\widetilde{K}} e^{\widetilde{\chi}^m\widetilde{E}_m} e^{\frac{1}{2}\widetilde{\phi}^j\widetilde{H}_j}. \tag{3.2}$$

The Cartan form $\mathcal{G}' = d\nu'\nu'^{-1}$ can also be calculated in the expansion of the original and the dual generators. For the calculation of both of the Cartan forms $\mathcal{G}$ and $\mathcal{G}'$ we need to know the algebra structure that the generators obey. This structure will be constructed in a way that the twisted self-duality equation $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ will lead to the correct first-order equations. The action of the pseudo-involution $\mathcal{S}$ on the generators is as follows

$$\mathcal{S}Y = \widetilde{Y} \quad , \quad \mathcal{S}K = \widetilde{K} \quad , \quad \mathcal{S}E_m = \widetilde{E}_m \quad , \quad \mathcal{S}H_i = \widetilde{H}_i,$$

$$\mathcal{S}\widetilde{Y} = Y \quad , \quad \mathcal{S}\widetilde{K} = K \quad , \quad \mathcal{S}\widetilde{E}_m = E_m \quad , \quad \mathcal{S}\widetilde{H}_i = H_i,$$

$$\mathcal{S}V_i = \widetilde{V}_i \quad , \quad \mathcal{S}\widetilde{V}_i = -V_i. \tag{3.3}$$

The Cartan form $\mathcal{G}' = d\nu'\nu'^{-1}$ also obeys the Cartan-Maurer equation

$$d\mathcal{G}' - \mathcal{G}' \wedge \mathcal{G}' = 0 \tag{3.4}$$

One can primarily use the twisted self-duality condition $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ to construct the generator expansion of $\mathcal{G}'$ and obtain an expression only in terms of the original fields then calculate (3.4). This equation must lead to the second-order field equations (2.8) and (2.9) [2]. In the calculation of the generator expansion of the Cartan form $\mathcal{G}'$ we firstly calculate $\mathcal{G}$ in terms of the unknown structure constants of the original field generators. $\mathcal{G}$ constitutes the part of $\mathcal{G}'$ which is composed of the original generators. We can use the twisted self-duality condition we propose on $\mathcal{G}'$ to generate the other part of $\mathcal{G}'$ which is composed of the dual generators. This is possible since when one inspects the Lie superalgebra structure in general, in order to obtain the correct field equations one finds that the commutation or the anti-commutation relations of the original generators must lead to the original generators and a pair of an original and a dual generator would lead to a dual generator under the algebra product while the product of two dual generators would vanish. When one calculates $\mathcal{G}'$ as explained above and then insert it in the Cartan-Maurer equation, the result can be compared with (2.8) and (2.9) to read the desired commutation and the anti-commutation relations of the Lie superalgebra of the field generators. We will not give the details of this long calculation but only present the results. The resulting commutation and the anti-commutation relations of the original and the dual generators apart from the purely scalar commutators are

$$[K,V_i]=\frac{1}{2}V_i \quad , \quad [K,Y]=Y \quad , \quad [K,\widetilde{Y}]=-\widetilde{Y},$$

$$[\widetilde{V}_k,K]=\frac{1}{2}\widetilde{V}_k \quad , \quad \{V_i,V_j\}=\eta_{ij}Y \quad , \quad [H_l,V_i]=(H_l)_i^k V_k$$

$$[E_m,V_i]=(E_m)_i^j V_j \quad , \quad \{V_l,\widetilde{V}_k\}=\frac{2}{3}\delta_{lk}\widetilde{K}+\sum_i (H_i)_{lk}\widetilde{H}_i,$$

$$[V_k,\widetilde{Y}]=-4\,\eta_k^l\,\widetilde{V}_l \quad , \quad [Y,\widetilde{Y}]=-\frac{16}{3}\widetilde{K} \quad , \quad [H_i,\widetilde{V}_k]=-(H_i^T)_k^m\widetilde{V}_m,$$

$$[E_\alpha,\widetilde{V}_k]=-(E_\alpha^T)_k^m\widetilde{V}_m. \tag{3.5}$$

The matrices $((H_m)_i^j,\,(E_\alpha)_i^j)$ are the images of the corresponding generators (H_m,E_α) under the representation chosen respectively. Also the matrices $((H_m^T)_i^j,\,(E_\alpha^T)_i^j)$ are the matrix transpose of $((H_m)_i^j,\,(E_\alpha)_i^j)$. The scalar generators and the generators coupled to the 6-form dual fields of the scalars namely $\{H_i,E_m,\widetilde{E}_m,\widetilde{H}_i\}$ constitute a subalgebra with the following commutators

$$[H_j,E_\alpha]=\alpha_j E_\alpha \quad , \quad [E_\alpha,E_\beta]=N_{\alpha,\beta}E_{\alpha+\beta},$$

$$[H_j,\widetilde{E}_\alpha]=-\alpha_j\widetilde{E}_\alpha \quad , \quad [E_\alpha,\widetilde{E}_\alpha]=\frac{1}{4}\sum_{j=1}^{r}\alpha_j\widetilde{H}_j,$$

$$[E_\alpha,\widetilde{E}_\beta]=N_{\alpha,-\beta}\widetilde{E}_\gamma, \qquad \alpha-\beta=-\gamma,\alpha\neq\beta, \tag{3.6}$$

where $i,j=1,...,r$ and $\alpha,\beta,\gamma\in\Delta_{nc}^+$. Here Δ_{nc}^+ is a subset of the positive roots of $SO(N,2)$ whose corresponding generators do not commute with the elements of $\mathbf{h}_k$ in $so(N,2)$ [9]. The remaining commutators and the anti-commutators which are not listed in (3.5) and (3.6) vanish indeed. We observe that the structure constants of the commutators of the scalar and the dual scalar generators in (3.6) are the coefficients introduced in (2.12).

We can now calculate the doubled field strength $\mathcal{G}' = d\nu'\nu'^{-1}$ explicitly by using the above commutation and the anti-commutation relations. From the definition of the coset element in (3.2) by using (3.5) and (3.6) the calculation of $\mathcal{G}' = d\nu'\nu'^{-1}$ yields

$$
\begin{aligned}
\mathcal{G}' = &\frac{1}{2}d\phi^i H_i + e^{\frac{1}{2}\alpha_i\phi^i}U^\alpha E_\alpha + d\sigma K + e^{\frac{1}{2}\sigma}L_i^k dA^i V_k + \frac{1}{2}e^\sigma GY \\
&+ \frac{1}{2}e^{-\sigma}d\widetilde{B}\widetilde{Y} + (-\frac{4}{3}B\wedge d\widetilde{B} + \frac{2}{3}A^j\wedge d\widetilde{A}^i\delta_{ij} + d\widetilde{\sigma})\widetilde{K} + \overset{\rightharpoonup}{\widetilde{\mathbf{T}}}(\overset{\rightharpoonup}{\mathbf{J}} + e^{\mathbf{\Gamma}}e^{\mathbf{\Lambda}}\overset{\rightharpoonup}{\widetilde{\mathbf{S}}}) \\
&+ (e^{-\frac{1}{2}\sigma}((L^T)^{-1})^l_k d\widetilde{A}^k + 2e^{-\frac{1}{2}\sigma}((L^T)^{-1})^l_k\,\eta^k_i\,A^i\wedge d\widetilde{B})\widetilde{V}_l.
\end{aligned}
\tag{3.7}
$$

In the derivation of (3.7) we have used the matrix identities $de^Xe^{-X} = dX + \frac{1}{2!}[X,dX] + \frac{1}{3!}[X,[X,dX]] + ...$ and $e^XYe^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + ...$. We have defined the set $\{U^\alpha\}$ in section two. Although L is a symmetric matrix we keep L^T in the above expression for an easier comparison of the first-order equations we will obtain from (3.7) with the ones previously calculated in (2.11). Here the row vector $\overset{\rightharpoonup}{\widetilde{\mathbf{T}}}$ is defined as $\widetilde{\mathbf{T}}_i = \widetilde{H}_i$ for $i = 1,...,r$ and $\widetilde{\mathbf{T}}_{\alpha+r} = \widetilde{E}_\alpha$ for $\alpha\in\Delta^+_{nc}$. As defined in section two the column vector $\overset{\rightharpoonup}{\widetilde{\mathbf{S}}}$ is $\widetilde{\mathbf{S}}^i = \frac{1}{2}d\widetilde{\phi}^i$ for $i = 1,...,r$ and $\widetilde{\mathbf{S}}^{\alpha+r} = d\widetilde{\chi}^\alpha$ for $\alpha\in\Delta^+_{nc}$ (we have already assumed that the roots in Δ^+_{nc} are labelled by integer indices from 1 to n therefore $r + n = 2$N). The $\overset{\rightharpoonup}{\mathbf{J}}$ term in (3.7) is a result of the coupling between the scalars and the one-form potentials in (2.3) and we define it as

$$
\begin{aligned}
\mathbf{J}^m = (H_m)_{ji}A^j\wedge d\widetilde{A}^i + \eta^k_i(H_m)_{jk}A^j\wedge A^i\wedge d\widetilde{B} \quad &, \quad m = 1,...,r \\
\mathbf{J}^{\alpha+r} = 0 \quad &, \quad \alpha\in\Delta^+_{nc}.
\end{aligned}
\tag{3.8}
$$

In (3.7) more explicitly we have

$$
\begin{aligned}
\overset{\rightharpoonup}{\widetilde{\mathbf{T}}}(\overset{\rightharpoonup}{\mathbf{J}} + e^{\mathbf{\Gamma}}e^{\mathbf{\Lambda}}\overset{\rightharpoonup}{\widetilde{\mathbf{S}}}) = &\sum_{m=1}^{r}((e^{\mathbf{\Gamma}}e^{\mathbf{\Lambda}})^m_j\widetilde{\mathbf{S}}^j + (H_m)_{ji}A^j\wedge d\widetilde{A}^i \\
&+ \eta^k_i(H_m)_{jk}A^j\wedge A^i\wedge d\widetilde{B})\widetilde{H}_m + \sum_{\alpha\in\Delta^+_{nc}}(e^{\mathbf{\Gamma}}e^{\mathbf{\Lambda}})^{\alpha+r}_j\widetilde{\mathbf{S}}^j\widetilde{E}_\alpha.
\end{aligned}
\tag{3.9}
$$

The next step is to show that if we apply the twisted self-duality equation $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ on (3.7) by using (3.3) we get the correct first-order equations (2.11). Thus the twisted self-duality equation $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ gives

$$\frac{1}{2}e^{\sigma} * G = \frac{1}{2}e^{-\sigma}d\widetilde{B},$$

$$e^{\frac{1}{2}\sigma}\text{Ł}^i_j * dA^j = -e^{-\frac{1}{2}\sigma}((L^T)^{-1})^i_j d\widetilde{A}^j + 2e^{-\frac{1}{2}\sigma}((L^T)^{-1})^i_j\, \eta^j_k\, d\widetilde{B} \wedge A^k,$$

$$*d\sigma = d\widetilde{\sigma} - \frac{4}{3}B \wedge d\widetilde{B} + \frac{2}{3}\delta_{ij}A^j \wedge d\widetilde{A}^i,$$

$$e^{\frac{1}{2}\alpha_i\phi^i}(\mathbf{\Omega})^{\alpha}_l * d\chi^l = (e^{\mathbf{\Gamma}}e^{\mathbf{\Lambda}})^{\alpha+r}_j\widetilde{\mathbf{S}}^j,$$

$$\frac{1}{2} * d\phi^m = (e^{\mathbf{\Gamma}}e^{\mathbf{\Lambda}})^m_j\widetilde{\mathbf{S}}^j + (H_m)_{ji}A^j \wedge d\widetilde{A}^i + \eta^k_i(H_m)_{jk}A^j \wedge A^i \wedge d\widetilde{B}. \quad (3.10)$$

These equations are the same with the first-order equations (2.11) which are obtained by directly integrating the second-order equations (2.8) and (2.9). We should also state that in (2.11) the first-order formulation of the scalar sector (2.2) excluding the matter coupling is done implicitly and separately by using the dualisation method only on the scalars. The second equation in (3.10) may seem to be different however it is the second equation of (2.11) multiplied by $(L^T)^{-1}$ from the left. This result also separately justifies the proper choice of the commutation and the anti-commutation relations in (3.5) and (3.6).

4 Conclusion

We have given the non-linear realization of the Salam-Sezgin $D = 8$ supergravity which is coupled to N vector multiplets [6]. After obtaining the second and the first-order equations of motion in section two we have doubled the non-gravitational bosonic field content by defining dual fields and constructed a Lie superalgebra which leads to a coset formulation of the bosonic sector of the $D = 8$ matter coupled Salam-Sezgin supergravity in section three.

This work is an example of the dualisation method [1,2] which was used to formulate the maximal supergravities in $D \leq 11$ as non-linear coset models. The formulation presented here shows that the bosonic sectors of the matter coupled supergravities also exhibit non-linear sigma model structures with a symmetry group which covers the rigid symmetry group of the scalar Lagrangian. The symmetry group of the doubled field strength is generated by the Lie superalgebra introduced in section three, however, the symmetry group of the twisted self-duality equation in other words the first-order equations is still to be determined.

The non-linear realization of the bosonic sector may be enlarged to include the gravity as well. A full dualisation of the gravity and the other bosonic fields would lead to a larger symmetry group. This group would cover the Lorentz group and the Lie supergroup constructed here. The complete non-linear realization of the bosonic sector may then be performed by taking the Lorentz group as the local symmetry group also by considering the simultaneous non-linear realization of the conformal group. Furthermore the method presented in [3,4,5] may be applied to detect the Kac-Moody symmetries of the matter coupled Salam-Sezgin $D = 8$ supergravity.

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