Wigner rotations in laser cavities

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Abstract

The Wigner rotation is a key word in many branches of physics, chemistry and engineering sciences. It is a group theoretical effect resulting from two Lorentz boosts. The net effect is one boost followed or preceded by a rotation. This rotation can therefore be formulated as a product of three boosts. In relativistic kinematics, it is a rotation in the Lorentz frame where the particle is at rest. This rotation does not change its momentum, but it rotates the direction of the spin. The Wigner rotation is not confined to relativistic kinematics. It manifests itself in physical systems where the underlying mathematics is the Lorentz group. It is by now widely known that this group is the basic scientific language for quantum and classical optics. It is shown that optical beams perform Wigner rotations in laser cavities.

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The Wigner rotation is a kinematical effect resulting from two successive Lorentz boosts along different directions. The result is not another Lorentz boost, but a boost followed by a rotation. This rotation is commonly called the Wigner rotation. In his 1939 paper on the Lorentz group [1], Wigner indeed emphasized the importance of the rotation subgroup of the Lorentz group and its physical significance. Since then, the word "Wigner rotation" mentioned frequently in many branches of physics.

The earliest manifestation of the Wigner rotation is the Thomas precession which we observe in atomic spectra. Thomas formulated this problem thirteen years before the appearance of Wigner's 1939 paper [2]. The Thomas effect in nuclear spectroscopy is mentioned in Jackson's book on electrodynamics [3]. Recently, as the relativistic effects play more prominent roles, the Wigner rotation has become of the key issues in field theory of extended objects [4], electron beams [5], relativistic quark model [6,7], nuclear scattering [8], neutrino physics [9], as well as many other areas of physics, chemistry and engineering sciences [10].

It is important to note that special relativity is not the only field of physics where the Lorentz group plays as the fundamental scientific language. For instance, in the physics of phase space, the symmetry group governing linear canonical transformations is the symplectic group which consists of rotations and squeeze operations. For the two-dimensional phase space consisting of one coordinate and one momentum variables, the group governing linear canonical transformation the symplectic group Sp(2). This group is locally isomorphic to the Lorentz group O(2, 1) applicable to two space and one time-dimensions. The squeeze transformation in phase space is like the Lorentz boost in special relativity. Here also we can consider two successive squeezes which result in one squeeze followed by a rotation. This is clearly another form of the Wigner rotation [11]. The physics of phase space covers not only classical mechanics but also squeezed states of light [12]

Another recent trend is that the Lorentz group is becoming prominent in classical optics, including polariazation optics [13], interferometers [14], multilayer optics [15,16]. As for lens optics, the formalism starts with two-by-two matrices representing a lens with its focal length and a translation. Repeated applications of these matrices lead to a two-by-two matrix representing the Sp(2). Thus, the fundamental scientific language in lens optics is clearly the group Sp(2) [17,18]. Thus, it would not be surprising to see another form of the Wigner rotation in lens optics.

Let us note that the geomentrical optics of laser cavities is a from of lens optics. In this paper, we would like to report that light waves in a laser cavity are performing Wigner rotations. We consider in this paper a cavity bounded by two identical mirrors. Then the problem can be translated into an optical system consisting of a chains of identical lenses separated by the same distance. One complete cycle consists of two lenses and two translations. We shall show that this complete cycle performs two repeated Wigner rotations.

For this purpose, let us define precisely the Wigner rotation. This rotation is necessary because a product of two boost matrices in different directions is a boost followed or preceded by a rotation matrix. Here, there are three boosts and one rotation. Thus, the simplest way to construct a Wigner rotation is to arrange three boost matrices leading to a rotation matrix [19]. For this purpose, let us perform three boosts as illustrated in Fig. 1.

Let us start with a particle at rest, with its four momentum

$$P_a = (m, 0, 0, 0), \tag{1}$$

where we use the metric convention (ct, z, x, y). Let us next boost this four-momentum along the z direction using the matrix

$$B_{1} = \begin{pmatrix} \cosh \eta & \sinh \eta & 0 & 0\\ \sinh \eta & \cosh \eta & 0 & 0\\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
(2)

resulting in the four-momentum

$$P_b = m(\cosh\eta, \sinh\eta, 0, 0). \tag{3}$$



FIG. 1. Closed Lorentz boosts. Initially, a massive particle is at rest with its four momentum P_a . The first boost B_1 brings P_a to P_b . The second boost B_2 transforms P_b to P_c . The third boost B_3 brings P_c back to P_a . The particle is again at rest. The net effect is a rotation around the axis perpendicular to the plane containing these three transformations. We may assume for convenience that P_b is along the z axis, and P_c in the zx plane. The rotation is then made around the y axis.

Let us rotate this vector around the y axis by an angle θ . Then the resulting fourmomentum is

$$P_c = m \left(\cosh\eta, (\sinh\eta)\cos\theta, (\sinh\eta)\sin\theta, 0\right). \tag{4}$$

The rotation matrix which performs this operation is

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
 (5)

Instead of this rotation, we propose to obtain this four-vector by boosting the fourmomentum of Eq.(3). The boost matrix in this case is

$$B_2 = \begin{pmatrix} \cosh \lambda & -\sin(\theta/2) \sinh \lambda & \cos(\theta/2) \sinh \lambda & 0\\ -\sin(\theta/2) \sinh \lambda & 1 + \sin^2(\theta/2) (\cosh \lambda - 1) & -\sin \theta \sinh^2(\lambda/2) & 0\\ \cos(\theta/2) \sinh \lambda & -\sin \theta \sinh^2(\lambda/2) & 1 + \cos^2(\theta/2) (\cosh \lambda - 1) & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
 (6)

with

$$\lambda = 2 \tanh^{-1} \left\{ \left[\sin(\theta/2) \right] \tanh \eta \right\}.$$
(7)

A detailed calculation of this matrix is given the paper by Han, et al. [13]

Next, we boost the four-momentum of Eq.(4) to that of Eq.(1). The particle is again at rest. The boost matrix is

$$B_3 = R(\theta) B_1^{-1} R(-\theta) \tag{8}$$

The net result of these transformations is

$$B_3 B_2 B_1.$$
 (9)

This leaves the initial four-momentum of Eq.(1) invariant. Is it going to be an identity matrix? The answer is No. The result of the matrix multiplications is

$$R(\Omega) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos\Omega & -\sin\Omega & 0\\ 0 & \sin\Omega & \cos\Omega & 0\\ 0 & 0 & 0 & 1 \end{pmatrix},$$
(10)

with

$$\Omega = 2 \sin^{-1} \left\{ \frac{(\sin \theta) \sinh^2(\eta/2)}{\sqrt{\cosh^2 \eta - \sinh^2 \eta \sin^2(\theta/2)}} \right\}.$$
(11)

This matrix performs a rotation around the y axis and leaves the four-momentum of Eq.(1) invariant. This rotation is an element of Wigner's little group whose transformations leave the four-momentum invariant. This is precisely the Wigner rotation.

Indeed, Wigner's little group is the maximum subgroup of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. The Wigner rotation is associated with the little group for a particle at rest. Then, how about the little group which leaves the four-momentum P_b of Eq.(3)?

As Wigner noted, this four-vector can be brought to P_a of Eq.(1) by the inverse of the matrix of B_1 . Then the rotation matrix

$$R(\Theta) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos\Theta & -\sin\Theta & 0\\ 0 & \sin\Theta & \cos\Theta & 0\\ 0 & 0 & 0 & 1 \end{pmatrix},$$
 (12)

leaves the four-momentum P_a invariant. Thus, the transformation



FIG. 2. Lorentz-boosted rotation. If the particle moves along the z direction, it can be brought to its rest frame by the inverse of the boost matrix B_1 . When it is at rest, we can rotate the system without changing while its momentum. Under this rotation, the spin of the particle will change its direction. The particle can then be brought to its initial state by the boost matrix B_1 . The initial four-momentum can also be rotated by first as indicated in this figure. It can then be boosted back to its initial momentum state. The net result is a matrix which does not change the momentum. This can also be achieved by a Lorentz-boosted rotation around the y axis.

$$B_1 R(\Theta) B_1^{-1} \tag{13}$$

will leave the four-momentum P_b invariant. Clearly the rotation of Eq.(12) is a Wigner rotation [20].

According to Wigner's definition based on the O(3)-like little group, both $R(\Omega)$ of Eq.(10) and $R(\Theta)$ are Wigner rotations. In the case of $R(\Omega)$, the rotation angle is determined by the kinematical parameters η and θ . On the other hand, the angle Θ is arbitrary. In order to see those two seemingly different rotations are equivalent, we shall convert the kinematics of Θ into that of Ω .

For this purpose, we note first that there are two ways of transforming P_b to P_c . One is the rotation of Eq.(5), and the other is the boost B_2 of Eq.(6). Thus the transformation

$$B_2^{-1} R(\theta) \tag{14}$$

will leave the four-vector P_b invariant. If we put the restriction that the transformations of Eq.(13) and Eq.(14) be equal:

$$B_1 R(\Theta) B_1^{-1} = B_2^{-1} R(\theta), \tag{15}$$

then the result is

$$R(\theta) \ R(\Theta) = B_3 \ B_2 \ B_1, \tag{16}$$

$$\Theta = \Omega - \theta. \tag{17}$$

In 1986 [20] and 1999 [13], Han *et al.* performed exactly the same calculation using two-by-two formalism applicable to the Jones matrix formalism in polarization optics. They of course used the correspondence between the O(2, 1) and Sp(2) groups. The rotation of Eq.(5) is translated into a two-by-two rotation matrix with θ replaced by $\theta/2$.

The two-by-two squeeze matrix corresponding to the boost matrix B_1 of Eq.(2) is

$$S_1 = \begin{pmatrix} e^{\eta/2} & 0\\ 0 & e^{-\eta/2} \end{pmatrix}.$$
 (18)

The two-by-two rotation matrix corresponding to the four-by-four rotation matrix of Eq.(5) is

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$
 (19)

After the matrix multiplication, the squeeze matrix S_2 corresponding to B_2 of Eq.(6) becomes [13]

$$S_2 = \begin{pmatrix} \cosh(\lambda/2) - \sin(\theta/2) \sinh(\lambda/2) & \cos(\theta/2) \sinh(\lambda/2) \\ \cos(\theta/2) \sinh(\lambda/2) & \cosh(\lambda/2) + \sin(\theta/2) \sinh(\lambda/2) \end{pmatrix}.$$
 (20)

This is a matrix which squeezes along the direction which makes the angle $(\pi + \theta)/2$ with the z axis. The two-by-two squeeze matrix corresponding to B_3 of Eq.(8) is

$$S_3 = \begin{pmatrix} \cosh(\eta/2) - \cos\theta \sinh(\eta/2) & -\sin\theta \sinh(\eta/2) \\ -\sin\theta \sinh(\eta/2) & \cosh(\eta/2) + \cos\theta \sinh(\eta/2) \end{pmatrix}.$$
 (21)

Now the matrix multiplication $S_3S_2S_1$ corresponds to the closure of the kinematical triangle given in Fig. 1. The result is

$$S_3 S_2 S_1 = \begin{pmatrix} \cos(\Omega/2) & -\sin(\Omega/2) \\ \sin(\Omega/2) & \cos(\Omega/2) \end{pmatrix},$$
(22)

where Ω is given in Eq.(11).

Even though the above two-by-two formalism is contained in a paper on polarization optics [13], it is applicable to other subjects of physics having the Sp(2) symmetry. Cavity optics is a case in point. It is an extension of lens optics governed by the two-by-two matrices of Sp(2).

Before discussing cavities, let us go back to the definition of the Wigner rotation. In his original paper, Wigner introduced the O(3) group as the subgroup of the rotation group which leaves the four-momentum of a rest particle invariant. In the kinematical configuration of Fig. 1 and in Eq.(9), the net transformation leaves the four-momentum P_a of Eq.(1) invariant. It is a rotation matrix.

With this point in mind, we can write Eq.(13) as

$$\begin{pmatrix} e^{\eta/2} & 0\\ 0 & e^{-\eta/2} \end{pmatrix} \begin{pmatrix} \cos(\Theta/2) & -\sin(\Theta/2)\\ \sin(\Theta/2) & \cos(\Theta/2) \end{pmatrix} \begin{pmatrix} e^{-\eta/2} & 0\\ 0 & e^{\eta/2} \end{pmatrix}.$$
 (23)

Now, these three matrices can be combined into one matrix:

$$\begin{pmatrix} \cos(\Theta/2) & -e^{\eta}\sin(\Theta/2) \\ e^{-\eta}\sin(\Theta/2) & \cos(\Theta/2) \end{pmatrix}.$$
 (24)

If we repeat the same operation N times, the angle Θ becomes $N\Theta$.

We are now ready to discuss what is happening in a laser cavity. Let us consider for simplicity a cavity consisting of two identical concave mirrors separated by by a distance d. Then the *ABCD* matrix for a round trip of one beam is

$$\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix},$$
(25)

where R is the radius of the mirror. This form is quite familiar to us from the laser literature [21–23]. However, the crucial question is what happens when this process is repeated many times. This question was also addressed in the literature. For this purpose, Haus replaces one of the concave mirrors with a flat mirror and repeats the process in order to complete the cycle [22]. We note that Haus's procedure is equivalent to starting the cycle from the midpoint between the mirrors. This procedure can be simplified if we introduce a group theoretical notion of equivalent class. This procedure is simple. We translate the system by d/2 using a translation matrix. We thus write the *ABCD* matrix of Eq.(25) as

$$\begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} \begin{pmatrix} 1-d/R & d-d^2/2R \\ -2/f & 1-d/R \end{bmatrix}^2 \begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix}.$$
 (26)

Furthermore,

$$\begin{pmatrix} 1-d/R & d-d^2/2R \\ -2/R & 1-d/R \end{pmatrix} = \begin{pmatrix} \sqrt{d} & 0 \\ 0 & 1/\sqrt{d} \end{pmatrix} \begin{pmatrix} 1-d/R & 1-d/2R \\ -2d/R & 1-d/R \end{pmatrix} \begin{pmatrix} 1/\sqrt{d} & 0 \\ 0 & \sqrt{d} \end{pmatrix}.$$
 (27)

The purpose of this decomposition was to write the matrix in the middle in terms of dimensionless quantities.

Now, the ABCD matrix of Eq.(25) can be written as

$$\begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{d} & 0 \\ 0 & 1/\sqrt{d} \end{pmatrix} \begin{bmatrix} \begin{pmatrix} 1-d/R & 1-d/2R \\ -2d/R & 1-d/R \end{bmatrix}^2 \begin{pmatrix} 1/\sqrt{d} & 0 \\ 0 & \sqrt{d} \end{pmatrix} \begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix}.$$
 (28)

If the beam makes N round trips, the ABCD matrix becomes

$$\begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{d} & 0 \\ 0 & 1/\sqrt{d} \end{pmatrix} \begin{bmatrix} \begin{pmatrix} 1-d/R & 1-d/2R \\ -2d/R & 1-d/R \end{bmatrix}^{2N} \begin{pmatrix} 1/\sqrt{d} & 0 \\ 0 & \sqrt{d} \end{pmatrix} \begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix}.$$
 (29)

Thus, we can thus decompose this expression into a core matrix C, and the escort matrix E and its inverse E^{-1} in the following manner.

$$E C^{2N} E^{-1},$$
 (30)

with

$$C = \begin{pmatrix} 1 - d/R & 1 - d/2R \\ -2d/R & 1 - d/R \end{pmatrix},$$

$$E = \begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{d} & 0 \\ 0 & 1/\sqrt{d} \end{pmatrix}.$$
(31)

With this expression, we can concentrate on the core matrix C, and write this in the form

$$C = \begin{pmatrix} \cos\phi & -e^{\xi}\sin\phi \\ e^{-\xi}\sin\phi & \cos\phi \end{pmatrix},\tag{32}$$

with

$$\cos\phi = 1 - \frac{d}{R}, \quad e^{2\xi} = \frac{R}{2d} - \frac{1}{4}.$$
 (33)

Here both d and R are positive, and the restriction on them is that d be greater than 2R. This is the stability condition frequently mentioned in the literature [22,23].

Let us next write the core matrix as

$$C = \begin{pmatrix} e^{\eta/2} & 0\\ 0 & e^{-\eta/2} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi\\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} e^{-\eta/2} & 0\\ 0 & e^{\eta/2} \end{pmatrix}.$$
 (34)

Here, a rotation matrix is sandwiched between a squeeze matrix and its inverse. This expression is exactly of the form of Eq.(23) for the Wigner rotation. In the above expression also, the rotation matrix in the middle is the Wigner rotation matrix.

If the light beam makes one cycle, the effect is C^2 , and the its expression is

$$C = \begin{pmatrix} e^{\eta/2} & 0\\ 0 & e^{-\eta/2} \end{pmatrix} \begin{pmatrix} \cos(2\phi) & -\sin(2\phi)\\ \sin(2\phi) & \cos(2\phi) \end{pmatrix} \begin{pmatrix} e^{-\eta/2} & 0\\ 0 & e^{\eta/2} \end{pmatrix}.$$
 (35)

Indeed, the beam makes a Wigner rotation of 2ϕ when it completes one cycle.

If the light beam makes N round trips, we have to compute C^{2N} , and the result is

$$C^{2N} = \begin{pmatrix} e^{\eta/2} & 0\\ 0 & e^{-\eta/2} \end{pmatrix} \begin{pmatrix} \cos(2N\phi) & -\sin(2N\phi)\\ \sin(2N\phi) & \cos(2N\phi) \end{pmatrix} \begin{pmatrix} e^{-\eta/2} & 0\\ 0 & e^{\eta/2} \end{pmatrix},$$
(36)

or

$$C^{2N} = \begin{pmatrix} \cos(2N\phi) & -e^{\eta}\sin(2N\phi) \\ e^{-\eta}\sin(2N\phi) & \cos(2N\phi) \end{pmatrix}.$$
(37)

In this paper, we noted first that the matrices in lens/mirror optics can be formulated in terms of the three-parameter Sp(2) group. We exploited the isomorphism between Sp(2)and SO(2, 1) which is the Lorentz group for the particles moving in a two-dimensional plane. The Wigner rotation, with its group theoretical origin, manifests itself in special relativity and optical sciences including cavity optics. It is gratifying to note that laser beams perform many Wigner rotations before they leave the cavity.

In this paper, we considered only the simplest cavity consisting of two identical mirrors. We note that there are more general approaches for cavities consisting of two different mirrors [21]. It would be an interesting project to exploit the Lorentz-group content of this and other general cases.

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