# $W_{\infty}$-Covariance of the Weyl-Wigner-Groenewold-Moyal Quantization 

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#### Abstract

The differential structure of operator bases used in various forms of the Weyl-Wigner-GroenewoldMoyal (WWGM) quantization is analyzed and a derivative-based approach, alternative to the conventional integral-based one is developed. Thus the fundamental quantum relations follow in a simpler and unified manner. An explicit formula for the ordered products of the Heisenberg-Weyl algebra is obtained. The $W_{\infty}$-covariance of the WWGM-quantization in its most general form is established. It is shown that the group action of $W_{\infty}$ that is realized in the classical phase space induces on bases operators in the corresponding Hilbert space a similarity transformation generated by the corresponding quantum $W_{\infty}$ which provides a projective representation of the former $W_{\infty}$. Explicit expressions for the algebra generators in the classical phase space and in the Hilbert space are given. It is made manifest that this $W_{\infty}$-covariance of the WWGM-quantization is a genuine property of the operator bases.


## I. INTRODUCTION

The Weyl-Wigner-Groenewold-Moyal (WWGM) quantization [1] that is usually called the phase space formulation of quantum mechanics has gained a wide popularity in many different areas of physics including statistical mechanics [2], quantum optics [3], collission theory [4], and classically chaotic nonlinear systems [55, [6]. This quantization scheme can be simply stated as an association between classical observables (c -number functions defined on a classical phase space) and quantum observables (operators acting in the corresponding Hilbert space $\mathcal{H}$ ). In the mathematical literature it is developed as the theory of pseudodifferential operators where the c-number functions determined by the WWGM-quantization are referred as the symbols of the corresponding Hilbert space operators [7]. In fact all the existing methods of quantization can be seen as association processes obeying certain rules [8]. Therefore, the search for possible covariances and hence the invariance properties that these associations may posses are of fundamental importance. That is to say, when a member of the associated pair is transformed, determination of the transformation rule of the other one in a well defined manner must be the first step of a systematic investigation. In the context of WWGM-quantization it is unfortunate that only a very restricted class of covariance properties are specified so far. The main goal of the present paper is to uncover the covariance properties of the WWGM-quantization in its as general form as possible.

The WWGM-quantization associates the usual product of two operators $\hat{F}_{1} \hat{F}_{2}$, not with the usual commutative product of functions $f_{1} f_{2}$, but rather with an associative star product $f_{1} \star f_{2}$ that is in general noncommutative, where $f_{1}$ and $f_{2}$ are the c-number functions corresponding under the WWGM-quantization to the operators $\hat{F}_{1}$ and $\hat{F}_{2}$, respectively. Henceforth operators and functions of operators will be denoted by hat ^over letters. Associativity of the $\star$ product is inherited from the associativity of usual product of the operators. This in turn means that it is not the Poisson brackets (PB), but its unique $\hbar$ (Planck's constant) deformation (9]

$$
\begin{align*}
(i \hbar)^{-1}\left\{f_{1}, f_{2}\right\}_{M B} & =(i \hbar)^{-1}\left(f_{1} \star f_{2}-f_{2} \star f_{1}\right) \\
& =\left\{f_{1}, f_{2}\right\}_{P B}+O(\hbar) \tag{1}
\end{align*}
$$

called the Moyal brackets (MB) that corresponds to the Lie bracket of operators under the WWGM-association. Due to the associativity of the $\star$ product MB obeys Jacobi identity. Hence, WWGM-quantization sets up a Lie algebra isomorphism between the Lie algebra of quantum observables and the resulting Lie algebra of classical observables with
respect to MB. More precisely, the association depends on a certain rule of ordering of functions of noncommuting operators 10,11. For this reason the star product and therefore the corresponding MB must be labeled with a parameter specifying the chosen rule of ordering. The $O(\hbar)$ terms in Eq.(1) that are called "quantum corrections" can be computed to any desired order of $\hbar$ within the classical regime. Hence, the quantization itself can be understood as a deformation of classical observables without any need for introducing a Hilbert space on which the operators act. This leads to the fact that one can define a "pseudomechanics" which has the star product and MB as its principle ingredients which reduces to classical mechanics in the limit $\hbar \rightarrow 0$ [9, 12]. In this case the dynamics of a classical observable $f$ associated with a classical Hamiltonian system described by the Hamilton function $H$ is governed by the "equation of motion"

$$
\begin{equation*}
\frac{d f}{d t}=(i \hbar)^{-1}\{H, f\}_{M B} \tag{2}
\end{equation*}
$$

The WWGM- quantization has a "simple covariance" with respect to affine canonical (i.e., inhomogeneous symplectic) transformations 13 15]. This affine canonical covariance follows from the structure of the automorphism group of the Heisenberg-Weyl (HW) group $W_{1}$ and the Stone-von Neumann theorem (7) which in essence states that, upto a central element generated by the unit element of the HW-algebra every irreducible representation of $W_{1}$ is unitarily equivalent to the Schrödinger representation $\hat{D}$ (given by Eq.(3) below). On the other hand, in addition to the inner automorphisms, automorphism group of $W_{1}$ contains the inhomogeneous symplectic group $I S p(2)$ which is the semidirect product of the translation group and the symplectic group $\operatorname{Sp}(2)$. Thus, we can combine $\hat{D}$ with an element $\phi \in I S p(2)$ to obtain another representation $\hat{D} \circ \phi$ which is unitarily equivalent to the Schrödinger representation. This unitary equivalence provides a double-valued metaplectic representation of ISp(2). The affine canonical covariance of the WWGM-quantization is a simple consequence of this fact.

To the best of our knowledge, the only known covariance of the WWGM-quantization is the above mentioned metaplectic covariance. Even this is investigated only for particular cases of WWGM-quantization. In the following we prove that this quantization scheme has an infinity of covariances described by the recently found $W_{\infty}$ algebra and $W_{\infty}$ group [16]. We wish to warn that there are two $W_{\infty}$ 's here. The first one describes the Lie algebra of deformed classical canonical diffeomorphisms and it is explicitly realized in the tangent space of the phase space. The other one that acts in the corresponding Hilbert space with the usual commutator is the Lie algebra of ordered monomials of the HW-algebra generators. Let us call these the classical $W_{\infty}$ and quantum $W_{\infty}$, respectively. Just as it is in the case of the metaplectic covariance, it should be emphasized that this is also a direct consequence of the $W_{\infty}$-covariance of the operator bases that are parametrized Schrödinger displacement operators $\hat{D}$ and their Fourier transforms (see Eqs.(4)-(6) below).

We consider only systems with one degree of freedom for the sake of simplicity. However, the structure of the underlying algebra (HW-algebra) allows a straightforward generalization to systems with finite or denumerably infinite number of degrees of freedom. Our approach here is distinct from the conventional one in that differential structure of the bases are given primary status and are investigated in detail. As an alternative to the integral-based conventional approach we suggest to call this approach the derivative-based approach. Except for those that occur in the definitions integrals rarely occur in our investigation. Wherever they do occur integrals should be understood as double integrations over the whole phase space that is topologically equivalent to $\mathbf{R}^{2}$.

The organization of the paper is as follows. In Section 2 a review of the WWGM-quantization and the definition of ordered products are given. This section fixes the notation and includes formulas and definitions needed for the subsequent analyses. In Section 3 the differential structure of the Weyl basis is obtained and an explicit formula for the ordered products is developed. Parametrized Bopp operators are introduced. Section 4 contains the differential structure of the generalized Wigner basis and the corresponding Bopp operators. The announced $W_{\infty}$-covariance of the Weyl and Wigner bases are explicitly established in Section 5. Quantum deformation of the canonical diffeomorphisms is also found and the generalized star product and Moyal brackets are obtained. The final Section 6 contains a summary of results.

## II. THE WWGM QUANTIZATION AND ORDERED PRODUCTS

Let us consider the Schrödinger representation of the HW-algebra: $[\hat{q}, \hat{p}]=i \hbar \hat{I}$, where $\hat{q}$ and $\hat{p}$ are the hermitian position and momentum operators, respectively. In terms of boson annihilation $(\hat{a})$ and creation $\left(\hat{a}^{\dagger}\right)$ operators defined by $\hat{a}=\left(a_{0} \hbar \sqrt{2}\right)^{-1}\left(\hbar \hat{q}+i a_{0}^{2} \hat{p}\right)$, the defining commutation relation is $\left[\hat{a}, \hat{a^{\dagger}}\right]=\hat{I} . \dagger$ stands for the Hermitian conjugation and in terms of a frequency $\omega$ and mass $m$ a length constant $a_{0}=(\hbar / m \omega)^{1 / 2}$ is used. The identity operator $\hat{I}$ of the algebra generates a $U(1)$ group that is the center of $W_{1}$. Then the so called displacement operators

$$
\begin{equation*}
\hat{D}(\xi, \eta)=\operatorname{expi}(\xi \hat{q}+\eta \hat{p}) \quad ; \quad \hat{D}(z, \bar{z})=\exp \left(z \hat{a}^{\dagger}-\bar{z} \hat{a}\right) \tag{3}
\end{equation*}
$$

which act irreducibly in $\mathcal{H}$ are the representatives of the coset space $W_{1} / U(1)$ in the real $(\xi, \eta)$ and complex $(z, \bar{z})$ parametrization of the group space $W_{1}$, respectively. $\bar{z}$ denotes the complex conjugation of $z=-\left(a_{0} \sqrt{2}\right)^{-1}\left(\hbar \eta-i a_{0}^{2} \xi\right)$. The displacement operators, or their suitable parametrizations form complete operator bases, in the sense that, any operator obeying certain conditions can be expanded in terms of them 10, 11. Each basis is closely connected with the ordering of noncommuting $\hat{q}$ and $\hat{p}$ (or $\hat{a}$ and $\hat{a}^{\dagger}$ ) in the expansion of operators. Therefore, not only the symbols thus obtained but also the resulting phase spaces are distinct. In essence, the WWGM-quantization comes into play by considering the parameters of the group space as the coordinate functions of a phase space. In this sense, the basis elements play a dual role. On the one hand they are operators parametrized by the coordinate functions of a phase space and acting in $\mathcal{H}$, and on the other hand they behave as operator-valued c-number functions defined on the same phase space.

A unified approach to different quantization rules is achieved by using s-parametrized ( $s \in \mathbf{C}$ ) displacement operators 10, 17

$$
\begin{equation*}
\hat{D}(s)=e^{-i \hbar s \xi \eta / 2} \hat{D}(\xi, \eta) \quad ; \quad \hat{D}(z, s)=e^{s|z|^{2} / 2} \hat{D}(z, \bar{z}) \tag{4}
\end{equation*}
$$

and their Fourier transforms

$$
\begin{align*}
& \hat{\Delta}_{q p}(s)=(\hbar / 2 \pi) \iint e^{-i(\xi q+\eta p)} \hat{D}(s) d \xi d \eta  \tag{5}\\
& \hat{\Delta}_{Z}(s)=(\pi)^{-1} \iint e^{-(z \bar{Z}-\bar{z} Z) / a_{0} \sqrt{2}} \hat{D}(z, s) d^{2} z \tag{6}
\end{align*}
$$

where, $z_{R}$ and $z_{I}$ being the real and the imaginary parts of $z, d^{2} z=d z_{R} d z_{I}$ and $Z=q+i\left(a_{0}^{2} / \hbar\right) p, \bar{Z}$ are the complex coordinates for the $(q, p)$ phase space. The $\hat{\Delta}(s)$ operators are called the s-parametrized displaced parity operators.

It is easy to verify the following trace and unitarity properties

$$
\begin{align*}
\hat{D}^{\dagger}(s) & =\hat{D}^{\dagger}(0) e^{i \hbar \xi \eta \bar{s} / 2}=\hat{D}^{-1}(\bar{s})  \tag{7}\\
\hat{D}^{\dagger}(z, s) & =\hat{D}^{\dagger}(z) e^{\bar{s}|z|^{2} / 2}=\hat{D}^{-1}(z,-\bar{s})  \tag{8}\\
\operatorname{Tr}[\hat{D}(s)] & =(2 \pi / \hbar) \delta(\xi) \delta(\eta)  \tag{9}\\
\operatorname{Tr}[\hat{D}(z, s)] & =\pi \delta^{2}(z) \tag{10}
\end{align*}
$$

Another important property, which is independent of $s$, is the so called displacement property

$$
\begin{align*}
\hat{D}(s) \hat{f}(\hat{q}, \hat{p}) \hat{D}^{-1}(s) & =\hat{f}(\hat{q}+\hbar \eta, \hat{p}-\hbar \xi)  \tag{11}\\
\hat{D}(z, s) \hat{f}\left(\hat{a}, \hat{a}^{\dagger}\right) \hat{D}^{-1}(z, s) & =\hat{f}\left(\hat{a}-z, \hat{a}^{\dagger}-\bar{z}\right) \tag{12}
\end{align*}
$$

Making use of the relations given above and the definitions (5) and (6) one can easily obtain

$$
\begin{align*}
\iint \hat{\Delta}_{q p}(s) d q d p & =h=h\left(2 \pi a_{0}^{2}\right)^{-1} \iint \hat{\Delta}_{Z}(s) d^{2} Z  \tag{13}\\
\operatorname{Tr}\left[\hat{\Delta}_{q p}(s)\right] & =1=\operatorname{Tr}\left[\hat{\Delta}_{Z}(s)\right]  \tag{14}\\
\hat{\Delta}_{q p}^{\dagger}(s) & =\hat{\Delta}_{q p}(-\bar{s}) \quad, \quad \hat{\Delta}_{Z}^{\dagger}(s)=\hat{\Delta}_{Z}(\bar{s}) \tag{15}
\end{align*}
$$

Now, two large class of associations can be defined as follows

$$
\begin{align*}
\hat{F}(\hat{q}, \hat{p}) & =h^{-1} \iint f^{(-s)}(q, p) \hat{\Delta}_{q p}(s) d q d p  \tag{16}\\
\hat{G}\left(\hat{a}^{\dagger}, \hat{a}\right) & =\left(2 \pi a_{0}^{2}\right)^{-1} \iint g^{(-s)}(Z, \bar{Z}) \hat{\Delta}_{Z}(s) d^{2} Z \tag{17}
\end{align*}
$$

whose inverse transformations are

$$
\begin{align*}
f^{(-s)}(q, p) & =\operatorname{Tr}\left[\hat{F} \hat{\Delta}_{q p}(-s)\right]  \tag{18}\\
g^{(-s)}(Z, \bar{Z}) & =\operatorname{Tr}\left[\hat{G} \hat{\Delta}_{Z}(-s)\right] \tag{19}
\end{align*}
$$

respectively, where $\operatorname{Tr}$ stands for the trace. For the special values $s=1,0,-1$, Eqs.(16) and (18) are known as the standart, Weyl, and antistandard rules of associations, respectively. On the other hand, Eqs.(17) and (19) are known as the normal, Weyl, and antinormal rules of associations for $s=1,0,-1$, respectively. If the density operator $\hat{\rho}$ of a quantum mechanical system is mapped by Eqs.(18) and (19), the resulting c-number functions

$$
\begin{equation*}
W_{\rho}(q, p,-s)=\operatorname{Tr}\left[\hat{\rho} \hat{\Delta}_{q p}(-s)\right] \quad, \quad W_{\rho}^{\prime}(Z, \bar{Z},-s)=\operatorname{Tr}\left[\hat{\rho} \hat{\Delta}_{Z}(-s)\right] \tag{20}
\end{equation*}
$$

are called, generically, the quasiprobability distribution functions (qdf). They enable us to carry out quantum mechanical calculations in classical manner in the corresponding phase space [6. 18]. In the real parametrization, the qdf's corresponding to $s=0$ and $s=\mp 1$ are called the Wigner and the Kirkwood qdf's, respectively. In the case of complex parametrization, the qdf corresponding to $s=1$ and $s=-1$ are known as the Glauber-Sudarshan P-functions and Q-functions. There is one more special association defined by

$$
\begin{align*}
\hat{F}(\hat{q}, \hat{p}) & =(\hbar / 2 \pi) \iint f(\xi, \eta) \hat{D}^{-1}(\xi, \eta) d \xi d \eta \\
f(\xi, \eta) & =\operatorname{Tr}[\hat{F} \hat{D}(\xi, \eta)] \tag{21}
\end{align*}
$$

This is known as the alternative Weyl association (or quantization), and the above mentioned Wigner quantization is simply the Fourier transform of it. The phase space resulting from Eq.(12) and having ( $\xi, \eta$ ) as canonically conjugate coordinates, is also known as the Weyl phase space. In the case of complex parametrization the alternative Weyl association takes the form

$$
\begin{equation*}
\hat{F}\left(\hat{a}^{\dagger}, \hat{a}\right)=\pi^{-1} \iint f(z, \bar{z}) \hat{D}^{-1}(z, \bar{z}) d^{2} z \quad ; \quad f(z, \bar{z})=\operatorname{Tr}[\hat{F} \hat{D}(z, \bar{z})] \tag{22}
\end{equation*}
$$

By using the properties of the $\hat{D}$ basis it can be easily verified that the Hilbert-Schmid norm of an operator defined by $\|\hat{F}\|=\left(\operatorname{Tr}\left[\hat{F}^{\dagger} \hat{F}\right]\right)^{1 / 2}$ is equal to the usual Hilbert space norm $\|f\|=(\langle f \mid f\rangle)^{1 / 2}$ of the corresponding c-number function. Thus, the alternative Weyl quantization is a norm preserving $1-1$ association between the space of bounded operators and the space of square integrable functions. Except for some particular values of $s$ that may give rise to singularities 10], the other associations are norm preserving $1-1$ associations as well.

The parametrized bases operators were for the first time, introduced by Cahill and Glauber [10] in order to interpolate among various types of orderings. Thus, the s-ordered products $\hat{y}_{n m}^{(s)} \equiv\left\{\left(\hat{a}^{\dagger}\right)^{n}(\hat{a})^{m}\right\}_{s}$ and $\hat{t}_{n m}^{(s)} \equiv\left\{(\hat{q})^{n}(\hat{p})^{m}\right\}_{s}$ are defined as follows:

$$
\begin{align*}
\hat{y}_{n m}^{(s)} & =\left.\partial_{z}^{n} \partial_{(-\bar{z})}^{m} \hat{D}(z, s)\right|_{z=0}  \tag{23}\\
\hat{t}_{n m}^{(s)} & =\left.(-i)^{n+m} \partial_{\xi}^{n} \partial_{\eta}^{m} \hat{D}(s)\right|_{\xi=0=\eta} \tag{24}
\end{align*}
$$

where, and henceforth, the notation $\partial_{x} \equiv \partial / \partial x$. will be used. By writing

$$
\begin{equation*}
\hat{D}(z, s)=e^{\left(s-s^{\prime}\right)|z|^{2} / 2} \hat{D}\left(z, s^{\prime}\right) \quad ; \quad \hat{D}(s)=e^{-i \hbar(s-s \prime) \xi \eta / 2} \hat{D}(s \prime) \tag{25}
\end{equation*}
$$

and differentiating, we obtain

$$
\begin{align*}
& \hat{y}_{n m}^{(s)}=\sum_{k=0}^{(n, m)} 2^{-k} b(k, n, m)\left[-\left(s-s^{\prime}\right)\right]^{k} \hat{y}_{n-k, m-k}^{\left(s^{\prime}\right)}  \tag{26}\\
& \hat{t}_{n m}^{(s)}=\sum_{k=0}^{(n, m)} 2^{-k} b(k, n, m)\left[i \hbar\left(s-s^{\prime}\right)\right]^{k} \hat{t}_{n-k, m-k}^{\left(s^{\prime}\right)} \tag{27}
\end{align*}
$$

where $(n, m)$ denotes the smaller of the integers n and m , and $\binom{n}{k}=n![(n-k)!k!]^{-1}$ being a binomial coefficient we set

$$
\begin{equation*}
b(k, n, m)=\binom{n}{k}\binom{m}{k} k! \tag{28}
\end{equation*}
$$

These relations express an arbitrary s-ordered product in terms of a polynomial in $s^{\prime}$-ordered products where $s^{\prime}$ is also arbitrary. Note that the minus sign in $\left[-\left(s-s^{\prime}\right)\right]$ in Eq.(20) and $i \hbar$ in Eq.(21). These are the remnants of the commutators of the corresponding operators there.

## III. DIFFERENTIAL STRUCTURE OF THE WEYL BASIS

The embedding of orderings in a continuum provides a natural context for viewing their differences and interrelationships in a continuous manner and enable us to carry out the related analyses in the most general form. However, the definitions (23) and (24) are quite implicit in contrast with the simple notion of ordering as a prescription about the arrangement of operators. Moreover, some complicated and long formulas that may be encountered in the formulation of the WWGM-quantization can be traced back to this implicit definition of ordering. The derivatives of the bases $\hat{D}(s)$ and $\hat{D}(z, s)$ with respect to the phase space coordinates are obtained in this section. An explicit formula for the ordered products which initiated the main observations of this report will be derived in the next section.

First, the following relations can be easily obtained from (11) and (12):

$$
\begin{align*}
\xi \hat{D}(s) & =\hbar^{-1}[\hat{p}, \hat{D}(s)] \quad ; \quad \eta \hat{D}(s)=-\hbar^{-1}[\hat{q}, \hat{D}(s)]  \tag{29}\\
z \hat{D}(z, s) & =[\hat{a}, \hat{D}(z, s)] \quad ; \quad \bar{z} \hat{D}(z, s)=\left[\hat{a}^{\dagger}, \hat{D}(z, s)\right] \tag{30}
\end{align*}
$$

They can be generalized as follows

$$
\begin{equation*}
\xi^{n} \eta^{m} \hat{D}(s)=\left(-\hbar^{-1} a d_{\hat{q}}\right)^{m}\left(\hbar^{-1} a d_{\hat{p}}\right)^{n} \hat{D}(s) \quad ; \quad z^{m} \bar{z}^{n} \hat{D}(z, s)=\left(a d_{\hat{a}^{\dagger}}\right)^{n}\left(a d_{\hat{a}}\right)^{m} \hat{D}(z, s) \tag{31}
\end{equation*}
$$

where $a d_{\hat{A}}$ denotes the adjoint action of $\hat{A}: a d_{\hat{A}} \hat{B} \equiv[\hat{A}, \hat{B}]$. Note that $\left[a d_{\hat{q}}, a d_{\hat{p}}\right]=a d_{[\hat{q}, \hat{p}]}=0$. The same relation remains valid if the pair $(\hat{q}, \hat{p})$ is replaced by $\left(\hat{a}^{\dagger}, \hat{a}\right)$. Thus, the ordering of ad operations in Eq.(31) is inessential.

By taking the derivatives of the various factorizations of $\hat{D}(s)$ and $\hat{D}(z, s)$ implied by the Baker-Campbell-Hausdorff $(\mathrm{BCH})$ formula we obtain the following identities:

$$
\begin{align*}
\partial_{\xi} \hat{D}(s) & =i\left(\hat{q}+\eta s^{-}\right) \hat{D}(s)=i \hat{D}(s)\left(\hat{q}-\eta s^{+}\right)=(i / 2)\left[\hat{q}-\frac{1}{2} \hbar \eta s, \hat{D}(s)\right]_{+} \\
\partial_{\eta} \hat{D}(s) & =i\left(\hat{p}-\xi s^{+}\right) \hat{D}(s)=i \hat{D}(s)\left(\hat{p}+\xi s^{-}\right)=(i / 2)\left[\hat{p}-\frac{1}{2} \hbar \xi s, \hat{D}(s)\right]_{+}  \tag{32}\\
\partial_{z} \hat{D}(z, s) & =\left[\hat{a}^{\dagger}-\left(\bar{z} s^{-} / \hbar\right)\right] \hat{D}(z, s)=\hat{D}(z, s)\left[\hat{a}^{\dagger}+\left(\bar{z} s^{+} / \hbar\right)\right]=(1 / 2)\left[\hat{a}^{\dagger}+\frac{1}{2} \bar{z} s, \hat{D}(z, s)\right]_{+} \\
\partial_{(-\bar{z})} \hat{D}(z, s) & =\left[\hat{a}-\left(z s^{+} / \hbar\right)\right] \hat{D}(z, s) \\
& =\hat{D}(z, s)\left[\hat{a}+\left(z s^{-} / \hbar\right)\right]=(1 / 2)\left[\hat{a}-\frac{1}{2} z s, \hat{D}(z, s)\right]_{+} \tag{33}
\end{align*}
$$

where $[,]_{+}$denotes the anticommutator and

$$
\begin{equation*}
s^{\mp}=\frac{1}{2} \hbar(1 \mp s) \tag{34}
\end{equation*}
$$

By making use of the Leibniz rule

$$
\begin{equation*}
\partial^{n}(u v)=\sum_{k=0}^{n}\binom{n}{k}\left(\partial^{k} u\right)\left(\partial^{n-k} v\right) \tag{35}
\end{equation*}
$$

it is possible to obtain generalizations of Eqs.(32)-(33) in several ways. That is, depending on the way the derivatives are taken, the definitions (23) and (24) may not yield the desired s-ordered product, but an equivalent one. In this sense the definitions (23) and (24) are implicit. The dual role played by the bases operators allows us to have the opposite sides of Eqs.(29) and (30) contain quantities living in different spaces. On the other hand Eqs.(32)-(33) are not written in the same way. The implicit nature of the above formulas is due this fact. Eqs.(32)-(33) can be rewritten in such a way that the quantities appearing on the opposite sides of the equalities live in different spaces. This we may achieve in two ways: (i) by taking the terms proportional to $\eta \hat{D}(s), \xi \hat{D}(s), z \hat{D}(z, s)$, and $\bar{z} \hat{D}(z, s)$ to the left, hence leaving all the Hilbert space quantities at the right, or (ii) by replacing $\hbar \eta \hat{D}(s), \hbar \xi \hat{D}(s), z \hat{D}(z, s)$, and $\bar{z} \hat{D}(z, s)$ in view of Eqs.(29)-(30) by $-a d_{\hat{q}}, a d_{\hat{p}}, a d_{\hat{a}}$, and $a d_{\hat{a}^{\dagger}}$, respectively. The first way leads to the introduction of s-parametrized Bopp operators that we are going to investigate at the end of this section. The second way allows us to rewrite Eqs.(32)-(33) in the following unique form:

$$
\begin{align*}
\partial_{\xi} \hat{D}(s) & =(i / 2) \hat{T}_{\left[\hat{]_{(s)}}\right.} \hat{D}(s) \quad ; \quad \partial_{\eta} \hat{D}(s)=(i / 2) \hat{T}_{[\hat{p}]_{(-s)}} \hat{D}(s)  \tag{36}\\
\partial_{z} \hat{D}(z, s) & =(1 / 2) \hat{T}_{\left[\hat{a}^{\dagger}\right]_{(s)}} \hat{D}(z, s) ; \partial_{(-\bar{z})} \hat{D}(z, s)=(1 / 2) \hat{T}_{[\hat{a}]_{(-s)}} \hat{D}(z, s) \tag{37}
\end{align*}
$$

where we define the Hilbert space operation

$$
\begin{equation*}
\hat{T}_{[\hat{A}]_{(s)}}=(1+s) \hat{L}_{\hat{A}}+(1-s) \hat{R}_{\hat{A}} . \tag{38}
\end{equation*}
$$

Here $\hat{L}_{\hat{A}}$ and $\hat{R}_{\hat{A}}$ are, respectively, the multiplication from left and from right by $\hat{A}$. In fact $\hat{T}_{[\hat{A}]_{(-s)}}$ is an s-deformation of $\hat{T}_{[\hat{A}]_{+}} \equiv \hat{L}_{\hat{A}}+\hat{R}_{\hat{A}}$. It is equal to $2 \hat{L}_{\hat{A}}, \hat{T}_{[\hat{A}]_{+}}$and $2 \hat{R}_{\hat{A}}$ for $s=1, s=0$, and $s=-1$, respectively. We observe that for an arbitrary operator $\hat{B}$

$$
\begin{equation*}
\left[\hat{T}_{\left.[\hat{q}]_{(s)}\right)}, \hat{T}_{\left.[\hat{p}]_{(-s)}\right]}\right] \hat{B}=0=\left[\hat{T}_{\left[\hat{a}^{\dagger}\right]_{(s)}}, \hat{T}_{\left.[\hat{a}]_{(-s)}\right]} \hat{B}\right. \tag{39}
\end{equation*}
$$

which simply follows from $\partial_{\xi} \partial_{\eta}=\partial_{\eta} \partial_{\xi}$ and the relations (36), (37). It can also be directly verified. Then we generalize Eqs.(36),(37) as follows:

$$
\begin{align*}
\partial_{\xi}^{n} \partial_{\eta}^{m} \hat{D}(s) & =(i / 2)^{n+m} \hat{T}_{\left[\hat{]_{(s)}}\right.}^{n} \hat{T}_{[\hat{p}]_{(-s)}^{m}}^{m} \hat{D}(s)  \tag{40}\\
\partial_{z}^{n} \partial_{(-\bar{z})}^{m} \hat{D}(z, s) & =2^{-(n+m)} \hat{T}_{\left[\hat{a}^{\dagger}\right]_{(s)}}^{n} \hat{T}_{[\hat{a}]_{(-s)}}^{m} \hat{D}(z, s) \tag{41}
\end{align*}
$$

In view of Eqs.(39)-(41), these can be rewritten in a finitely many different looking but equivalent forms.
We substitute Eqs.(40) and (41) in the definitions (23) and (24) to obtain

$$
\begin{align*}
& \hat{t}_{n m}^{(s)}=2^{-(n+m)} \hat{T}_{\left.\hat{q}^{( }\right]_{(s)}}^{n} \hat{T}_{[\hat{p}]_{(-s)}}^{m} \hat{I}=2^{-(n+m)} \hat{T}_{[\hat{p}]_{(-s)}}^{m} \hat{T}_{[\hat{q}]_{(s)}}^{n} \hat{I},  \tag{42}\\
& \hat{y}_{n m}^{(s)}=2^{-(n+m)} \hat{T}_{\left[\hat{a}^{\dagger}\right]_{(s)}}^{n} \hat{T}_{[\hat{a}]_{(-s)}}^{m} \hat{I}=2^{-(n+m)} \hat{T}_{[\hat{a}]_{(-s)}}^{m} \hat{T}_{\left[\hat{a} \dagger_{(s)}\right.}^{n} \hat{I} . \tag{43}
\end{align*}
$$

By making use of the binomial formula

$$
\begin{equation*}
\hat{T}_{[\hat{A}]_{(s)}}^{n} \equiv\left[(1+s) \hat{L}_{\hat{A}}+(1-s) \hat{R}_{\hat{A}}\right]^{n}=\sum_{j=0}^{n}\binom{n}{j}(1+s)^{j}(1-s)^{n-j} \hat{L}_{\hat{A}}^{j} \hat{R}_{\hat{A}}^{n-k} \tag{44}
\end{equation*}
$$

we can rewrite expressions in (44) more explicitly as

$$
\begin{align*}
\hat{t}_{n m}^{(s)} & =2^{-n} \sum_{j=0}^{n}\binom{n}{j}(1+s)^{j}(1-s)^{n-j} \hat{q}^{j} \hat{p}^{m} \hat{q}^{n-j}  \tag{45}\\
& =2^{-m} \sum_{k=0}^{m}\binom{m}{k}(1-s)^{k}(1+s)^{m-k} \hat{p}^{k} \hat{q}^{n} \hat{p}^{m-k} \tag{46}
\end{align*}
$$

A similar relation holds for $\hat{y}_{n m}^{(s)}$ if the pair $(\hat{q}, \hat{p})$ is replaced by ( $\left.\hat{a}^{\dagger}, \hat{a}\right)$ in these relations. In view of Eq.(39), it is possible to write many equivalent forms of the above relations. But for later use we have written only two of them. From these we get for $s= \pm 1$

$$
\begin{equation*}
\hat{t}_{n m}^{(1)}=\hat{L}_{\hat{q}}^{n} \hat{R}_{\hat{p}}^{m} \hat{I}=\hat{q}^{n} \hat{p}^{m} \quad ; \quad \hat{t}_{n m}^{(-1)}=\hat{L}_{\hat{p}}^{m} \hat{R}_{\hat{q}}^{n} \hat{I}=\hat{p}^{m} \hat{q}^{n} \tag{47}
\end{equation*}
$$

and for $s=0$

$$
\begin{align*}
\hat{t}_{n m}^{(0)} & =2^{-n} \sum_{j=0}^{n}\binom{n}{j} \hat{q}^{j} \hat{p}^{m} \hat{q}^{n-j}  \tag{48}\\
& =2^{-m} \sum_{k=0}^{m}\binom{m}{k} \hat{p}^{k} \hat{q}^{n} \hat{p}^{m-k} \tag{49}
\end{align*}
$$

The expressions (47) exhibit the standart and antistandart rules of ordering while (48) and (49) yield the two well known expressions for symmetrically (or Weyl) ordered products. In fact the usual expression known for the Weyl ordered form of $\hat{t}_{n m}^{(0)}$ is a totally symmetrized product containing n factors of $\hat{q}$ and m factors of $\hat{p}$, normalized by dividing by the number of terms in the symmetrized expression. In the literature [19] the equivalence of these three Weyl ordered forms is said to be verified by using the usual commutation relations. This requires long and tedious computations. In our formulation on the other hand, not only the above mentioned equivalences, but also explicit expressions for many forms of the Weyl ordered products arise naturally as a corollary to Eq.(39).

As an application, let us consider the traces of Eqs.(40) and (41) for $s=0$. By noting that $\operatorname{Tr} \hat{D}=(2 \pi / \hbar) \delta(\xi) \delta(\eta)$, $\operatorname{Tr} \hat{D}(z)=\pi \delta^{2}(z)$ where $\hat{D} \equiv \hat{D}(0)$; the well known Weyl associations follow:

$$
\begin{align*}
\frac{2 \pi}{\hbar} \partial_{\xi}^{n} \delta(\xi) \partial_{\eta}^{m} \delta(\eta) & \leftrightarrow(i)^{n+m} \hat{t}_{n m}^{(0)}  \tag{50}\\
\pi \partial_{z}^{n} \partial_{(-\bar{z})}^{m} \delta^{2}(z) & \leftrightarrow \hat{y}_{n m}^{(0)} \tag{51}
\end{align*}
$$

These expressions cannot be so easily obtained in other approaches.
We propose that the relation (45) (or alternatively (46)) can be given as the definition of s-ordered product. There are several reasons that support this suggestion: (i) These expressions are simpler and more explicit than the definitions given by (23) and (24). Neither the phase space coordinates nor the basis operators appear in these expressions. (ii) The ordered products of $\hat{a}^{\dagger}$ and $\hat{a}$ can be treated on an equal footing. In fact the disapperance of $\hbar$, or any multiple of it, in these expressions implies that all the relations and the remarks given above are valid for any algebra having $[\hat{A, \hat{B}]}=i \lambda \hat{I} ; \lambda \in \mathbf{C}$ [20]. A physical application may be the algebra of velocity operators of a charged particle moving in an external electromagnetic field. (iii) These definitions may be extended to the case when one or both of the integers $n$ and $m$ are negative. Furthermore, by using these relations the Hermiticity of a general s-ordered product can be easily decided. From (45) and (46) it follows that $\left[\hat{t}_{n m}^{(s)}\right]^{\dagger}=\hat{t}_{n m}^{(-\bar{s})}$. Thus, for every pair of integers n,m, $\hat{t}_{n m}^{(s)}$ are Hermitian provided $\bar{s}=-s$. In particular, the Weyl ordered products $\hat{t}_{n m}^{(0)}$ are Hermitian. In the case of $\hat{a}, \hat{a}^{\dagger}$ we have, from (45) and (46), $\left[\hat{y}_{n m}^{(s)}\right]^{\dagger}=\hat{y}_{m n}^{(\bar{s})}$. Hence, the s-ordered products $\hat{y}_{n m}^{(s)}$ are Hermitian if and only if $m=n$, and $\bar{s}=s$. For general $s \in \mathbf{C}$ one can find combinations such as

$$
\begin{equation*}
\hat{\kappa}_{n m}(s)=\alpha \hat{t}_{n m}^{(s)}+\bar{\alpha} \hat{t}_{n m}^{(-\bar{s})} \quad, \quad \hat{\kappa}_{n m}^{\prime}(s)=\alpha \hat{y}_{n m}^{(s)}+\bar{\alpha} \hat{y}_{m n}^{(\bar{s})} \tag{52}
\end{equation*}
$$

( $\alpha \in \mathbf{C}$ ) that are Hermitian.
We now consider the alternative way of writing the derivatives in $\hat{D}(s)$ so that the quantities appearing at opposite sides belong to different spaces as

$$
\begin{array}{ll}
\left(-i \partial_{\xi}-s^{-} \eta\right) \hat{D}(s)=\hat{q} \hat{D}(s) \\
\left(-i \partial_{\eta}+s^{+} \xi\right) \hat{D}(s)=\hat{p} \hat{D}(s) \quad, \quad & \left(-i \partial_{\xi}+s^{+} \eta\right) \hat{D}(s)=\hat{D}(s) \hat{q}  \tag{53}\\
\left(-i \partial_{\eta}-s^{-} \xi\right) \hat{D}(s)=\hat{D}(s) \hat{p}
\end{array}
$$

By defining the s-parametrized Bopp operators

$$
\begin{align*}
& Q_{L}(s)=-i \partial_{\xi}-s^{-} \eta \quad, \quad Q_{R}(s)=-i \partial_{\xi}+s^{+} \eta \\
& P_{L}(s)=-i \partial_{\eta}+s^{+} \xi \quad, \quad P_{R}(s)=-i \partial_{\eta}-s^{-} \xi \tag{54}
\end{align*}
$$

we can generalize Eqs.(46):

$$
\begin{array}{lll}
Q_{L}^{n}(s) \hat{D}(s)=\hat{q}^{n} \hat{D}(s) & , & Q_{R}^{n}(s) \hat{D}(s)=\hat{D}(s) \hat{q}^{n} \\
P_{L}^{n}(s) \hat{D}(s)=\hat{p}^{n} \hat{D}(s) & , & P_{R}^{n}(s) \hat{D}(s)=\hat{D}(s) \hat{p}^{n} \tag{55}
\end{array}
$$

Being defined in the tangent space of the $(\xi, \eta)$-phase space, the s-parametrized Bopp operators obey the commutation relations

$$
\begin{equation*}
\left[Q_{L}(s), P_{L}(s)\right]=-i \hbar=-\left[Q_{R}(s), P_{R}(s)\right] \tag{56}
\end{equation*}
$$

with all the other commutators equal to zero. These relations indicate that the s-parametrized Bopp operators provide a concrete coordinate realization of the direct sum of two copies of the HW-algebra, and for real $s$ they are hermitian on the Lebesque space defined on the $(\xi, \eta)$-phase space.

In the case of complex coordinates Eqs.(37) yield

$$
\begin{align*}
& Q_{L}^{\prime n}(s) \hat{D}(z, s)=\left(\hat{a}^{\dagger}\right)^{n} \hat{D}(z, s) \quad, \quad Q_{R}^{\prime n}(s) \hat{D}(z, s)=\hat{D}(s)\left(\hat{a}^{\dagger}\right)^{n} \\
& P_{L}^{\prime n}(s) \hat{D}(z, s)=(-\hat{a})^{n} \hat{D}(z, s) \quad, P_{R}^{\prime n}(s) \hat{D}(z, s)=\hat{D}(z, s)(-\hat{a})^{n} \tag{57}
\end{align*}
$$

where we have defined the s-parametrized complex Bopp operators

$$
\begin{array}{rlrl}
Q_{L}^{\prime}(s) & =\partial_{z}+\bar{z}\left(s^{-} / \hbar\right) \\
P_{L}^{\prime}(s) & =\partial_{\bar{z}}-z\left(s^{+} / \hbar\right) & , &  \tag{58}\\
Q_{R}^{\prime}(s)=\partial_{z}-\bar{z}\left(s^{+} / \hbar\right) \\
P_{R}^{\prime}(s)=\partial_{\bar{z}}+z\left(s^{-} / \hbar\right)
\end{array}
$$

The nonvanishing commutation relations they satisfy are

$$
\begin{equation*}
\left[Q_{L}^{\prime}(s), P_{L}^{\prime}(s)\right]=-I=-\left[Q_{R}^{\prime}(s), P_{R}^{\prime}(s)\right] \tag{59}
\end{equation*}
$$

We note that the complex Bopp operators are related with a coordinate realization of the bosonic annihilation and creation operators. We wish also to remark in passing that Eqs.(55) and (57) resemble eigenvalue equations.

The Bopp operators were originally defined only for the Wigner $(s=0)$ quantization [18, 21]. They play an important role in the derivative-based approach that we are using. So we generalize them for any quantization rule.

## IV. DIFFERENTIAL STRUCTURE OF THE WIGNER BASIS

Differential structure of the $\hat{\Delta}(s)$ bases are formally the Fourier transform of that obtained in the preceding sections for the $\hat{D}(s)$ bases. To put it more simply, they can be derived from the definition (5), (6) by elementary calculations. Indeed, making use of Eqs.(29) and (30), it is easy to verify that

$$
\begin{array}{ll}
\partial_{q} \hat{\Delta}_{q p}(s)=-\frac{i}{\hbar}\left[\hat{p}, \hat{\Delta}_{q p}(s)\right] \quad, \quad \partial_{p} \hat{\Delta}_{q p}(s)=\frac{i}{\hbar}\left[\hat{q}, \hat{\Delta}_{q p}(s)\right] \\
\partial_{Z} \hat{\Delta}_{Z}(s)=\frac{1}{a_{0} \sqrt{2}}\left[\hat{a}^{\dagger}, \hat{\Delta}_{Z}(s)\right] \quad, \quad \partial_{\bar{Z}} \hat{\Delta}_{Z}(s)=-\frac{1}{a_{0} \sqrt{2}}\left[\hat{a}, \hat{\Delta}_{Z}(s)\right] \tag{60}
\end{array}
$$

and from (36) and (38) it follows that

$$
\begin{array}{lr}
q \hat{\Delta}_{q p}(s)=\frac{1}{2} \hat{T}_{[\hat{q}]_{(s)}} \hat{\Delta}_{q p}(s) \quad, \quad p \hat{\Delta}_{q p}(s)=\frac{1}{2} \hat{T}_{\left[\hat{p}_{(-s)}\right.} \hat{\Delta}_{q p}(s) \\
Z \hat{\Delta}_{Z}(s)=\frac{a_{0}}{\sqrt{2}} \hat{T}_{[\hat{a}]_{(-s)}} \hat{\Delta}_{Z}(s) \quad, \quad \bar{Z} \hat{\Delta}_{Z}(s)=\frac{a_{0}}{\sqrt{2}} \hat{T}_{\left[\hat{a}^{\dagger}\right]_{(s)}} \hat{\Delta}_{Z}(s) \tag{61}
\end{array}
$$

This is the only place where we need partial integration. Recalling the commutation relations (39), the above expressions may be generalized as follows:

$$
\begin{align*}
q^{n} p^{m} \hat{\Delta}_{q p}(s) & =2^{-(n+m)} \hat{T}_{[\hat{q}]_{(s)}}^{n} \hat{T}_{[\hat{p}]_{(-s)}}^{m} \hat{\Delta}_{q p}(s) \\
\bar{Z}^{n} Z^{m} \hat{\Delta}_{Z}(s) & =\left(\frac{a_{0}}{\sqrt{2}}\right)^{n+m} \hat{T}_{\left[\hat{a}^{\dagger}\right]_{(s)}}^{n} \hat{T}_{[\hat{a}]_{(-s)}}^{m} \hat{\Delta}_{Z}(s) \tag{62}
\end{align*}
$$

As an example, by taking the traces of both sides of Eqs.(62) we obtain

$$
\begin{align*}
q^{n} p^{m} & =\operatorname{Tr}\left[\hat{t}_{n m}^{(s)} \hat{\Delta}_{q p}(-s)\right] \\
\bar{Z}^{n} Z^{m} & =\left(a_{0} \sqrt{2}\right)^{n+m} \operatorname{Tr}\left[\hat{y}_{n m}^{(s)} \hat{\Delta}_{Z}(-s)\right] \tag{63}
\end{align*}
$$

Alternatively, taking the integrals of both sides of the same equations we are led to

$$
\begin{align*}
& \hat{t}_{n m}^{(s)}=h^{-1} \iint q^{n} p^{m} \hat{\Delta}_{q p}(s) d q d p \\
& \hat{y}_{n m}^{(s)}=\left(2 \pi a_{0}^{2}\right)^{-1}\left(a_{0} \sqrt{2}\right)^{-(n+m)} \iint \bar{Z}^{n} Z^{m} \hat{\Delta}_{Z}(s) d^{2} Z \tag{64}
\end{align*}
$$

Eqs.(63) and (64) explicitly show that the s-quantization of the monomials $q^{n} p^{m}$ and $\bar{Z}^{n} Z^{m}\left(a_{0} \sqrt{2}\right)^{-(n+m)}$ are nothing but the s-ordered products $\hat{t}_{n m}^{(s)}$ and $\hat{y}_{n m}^{(s)}$, respectively. In our approach these well known results concerning the WWGM-quantization are obtained with ease in a unified manner.

From Eqs.(60) and (61) we have

$$
\left.\left.\begin{array}{rlll}
Q_{\Delta L}(s) \hat{\Delta}_{q p}(s) & =\hat{q} \hat{\Delta}_{q p}(s) & , & \\
P_{\Delta L}(s) \hat{\Delta}_{q p}(s) & =\hat{p} \hat{\Delta}_{q p}(s) & & \hat{\Delta}_{q p}(s) \tag{65}
\end{array}\right) \hat{\Delta}_{q p}(s) \hat{q}\right)
$$

where

$$
\begin{align*}
Q_{\Delta L}(s)=q-i s^{-} \partial_{p} & , Q_{\Delta R}(s)=q+i s^{+} \partial_{p} \\
P_{\Delta L}(s)=p+i s^{+} \partial_{q} & , \quad P_{\Delta R}(s)=p-i s^{-} \partial_{q} \tag{66}
\end{align*}
$$

are the s-parametrized Bopp operators for the $\hat{\Delta}_{q p}(s)$ bases. The only nonvanishing commutators for them are

$$
\begin{equation*}
\left[Q_{\Delta L}(s), P_{\Delta L}(s)\right]=-i \hbar=-\left[Q_{\Delta R}(s), P_{\Delta R}(s)\right] \tag{67}
\end{equation*}
$$

We also give the s-parametrized complex Bopp operators

$$
\begin{align*}
Q_{\Delta L}^{\prime}(s) & =2^{-1 / 2}\left[\frac{\bar{Z}}{a_{0}}+a_{0}(1-s) \partial_{Z}\right] & , & Q_{\Delta R}^{\prime}(s)=2^{-1 / 2}\left[\frac{\bar{Z}}{a_{0}}-a_{0}(1+s) \partial_{Z}\right] \\
P_{\Delta L}^{\prime}(s) & =2^{-1 / 2}\left[\frac{Z}{a_{0}}-a_{0}(1+s) \partial_{\bar{Z}}\right] & , & P_{\Delta R}^{\prime}(s)=2^{-1 / 2}\left[\frac{Z}{a_{0}}+a_{0}(1-s) \partial_{\bar{Z}}\right] \tag{68}
\end{align*}
$$

Their action on bases can be obtained from (60) and (61) as

$$
\begin{align*}
Q_{\Delta L}^{\prime n}(s) \hat{\Delta}_{Z}(s) & =\left(\hat{a}^{\dagger}\right)^{n} \hat{\Delta}_{Z}(s) \quad, \quad Q_{\Delta R}^{\prime n}(s) \hat{\Delta}_{Z}(s)=\hat{\Delta}_{Z}(s)\left(\hat{a}^{\dagger}\right)^{n} \\
P_{\Delta L}^{\prime n}(s) \hat{\Delta}_{Z}(s) & =(\hat{a})^{n} \hat{\Delta}_{Z}(s) \quad, \quad P_{\Delta R}^{\prime n}(s) \hat{\Delta}_{Z}(s)=\hat{\Delta}_{Z}(s)(\hat{a})^{n} \tag{69}
\end{align*}
$$

In the complex case the nonvanishing commutators are

$$
\begin{equation*}
\left[Q_{\Delta L}^{\prime}(s), P_{\Delta L}^{\prime}(s)\right]=1=-\left[Q_{\Delta R}^{\prime}(s), P_{\Delta R}^{\prime}(s)\right] \tag{70}
\end{equation*}
$$

We will finish this section by another important observation that generalizes a well known relation between the Wigner $(s=0)$ association and arbitrary s-association. It follows immediately from

$$
\begin{align*}
\hat{t}_{n m}^{(r)} \hat{\Delta}_{q p}(s) & =\left\{Q_{\Delta L}^{n}(s) P_{\Delta L}^{m}(s)\right\}_{-r} \hat{\Delta}_{q p}(s) \\
\hat{\Delta}_{q p}(s) \hat{t}_{n m}^{(r)} & =\left\{Q_{\Delta R}^{n}(s) P_{\Delta R}^{m}(s)\right\}_{r} \hat{\Delta}_{q p}(s) \tag{71}
\end{align*}
$$

by taking trace of both sides and making use of Eqns. (14), (45), (46) and (65),

$$
\begin{equation*}
\operatorname{Tr}\left[\hat{t}_{n m}^{(r)} \hat{\Delta}_{q p}(s)\right]=\left\{Q_{\Delta L}^{n}(s) P_{\Delta L}^{m}(s)\right\}_{-r} I=\left\{Q_{\Delta R}^{n}(s) P_{\Delta R}^{m}(s)\right\}_{r} I \tag{72}
\end{equation*}
$$

Thus for any arbitrary $r$ and $s$, the c-number function corresponding to an r-ordered product via the s-rule of association can be obtained by the action of the r-ordered s-parametrized Bopp operators on the phase space identity operator $I$. The extension of these observations to the case of complex coordinates and by linearity to arbitrary functions of operators that can be expanded to a series of ordered products is straightforward.

## V. $W_{\infty}$-COVARIANCE OF THE BASES OPERATORS

By following the same lines leading to Eqs.(71) from (45) (or (46)) and (55) we obtain

$$
\begin{equation*}
T_{n m}^{(r)}(s) \hat{D}(s)=\left[\hat{t}_{n m}^{(r)}, \hat{D}(s)\right] \tag{73}
\end{equation*}
$$

where

$$
\begin{equation*}
T_{n m}^{(r)}(s) \equiv\left\{Q_{L}^{n}(s) P_{L}^{m}(s)\right\}_{-r}-\left\{Q_{R}^{n}(s) P_{R}^{m}(s)\right\}_{r} \tag{74}
\end{equation*}
$$

Equation (73) reveals the important fact that each r-ordered product of $\hat{q}$ 's and $\hat{p}$ 's generates an infinitesimal transformation in the Weyl basis $\hat{D}(s)$ in the Hilbert space $\mathcal{H}$. This transformation corresponds to an infinitesimal transformation of the basis in the Weyl phase space that is built up in terms of $r$ and $-r$ ordered Bopp operators. The exponentiation of these transformations leads to

$$
\begin{equation*}
V_{n m}(r, s) \hat{D}(s)=\hat{U}_{n m}(r) \hat{D}(s) \hat{U}_{n m}^{-1}(r) \tag{75}
\end{equation*}
$$

where $\gamma_{n m} \in \mathbf{C}$ are the transformation parameters and

$$
\begin{equation*}
V_{n m}(r, s) \equiv \exp \left(i \gamma_{n m} T_{n m}^{(r)}(s)\right) \quad, \quad \hat{U}_{n m}(r) \equiv \exp \left(i \gamma_{n m} \hat{t}_{n m}^{(r)}\right) \tag{76}
\end{equation*}
$$

$\hat{U}_{n m}^{-1}(r)$ denotes the operator inverse of $\hat{U}_{n m}(r)$. Eq.(73) at the algebra level and Eq. $(75)$ at the group level explain what we mean in its full generality by $W_{\infty}$-covariance of the Weyl basis. Here we have two $W_{\infty}$ algebras (and
groups): First one is generated by the ordered products $\hat{t}_{n m}^{(r)} ; n, m \geq 0$ and is acting in $\mathcal{H}$. The second is generated by $T_{n m}^{(r)}(s) ; n, m \geq 0$ and is realized in the Weyl phase space.

Let $s=0$ and multiply both sides of Eqs.(73) and (75) by an arbitrary bounded operator $\hat{F}(\hat{q}, \hat{p})$. Then take the trace of the resulting equations. We thus obtain

$$
\begin{align*}
T_{n m}^{(r)} f(\xi, \eta) & =\operatorname{Tr}\left(\left[\hat{F}\left(\hat{q}, \hat{p}, \hat{t}_{n m}^{(r)}\right] \hat{D}\right)\right.  \tag{77}\\
V_{n m}(r, 0) f(\xi, \eta) & =\operatorname{Tr}\left\{\left[\hat{U}_{n m}^{-1}(r) \hat{F}(\hat{q}, \hat{p}) \hat{U}_{n m}(r)\right] \hat{D}\right\} \tag{78}
\end{align*}
$$

where $f=\operatorname{Tr}[\hat{F} \hat{D}], T_{n m}^{(r)} \equiv T_{n m}^{(r)}(0)$. These two equations describe the $W_{\infty}$-covariance of the Weyl quantization both at the algebra level (Eq.(77)) and at the group level (Eq.(78)). In other words, if $\hat{F}$ is the Weyl quantization of a c-number function $f$, then the $W_{\infty}$ transform of $\hat{F}$,

$$
\begin{equation*}
\hat{F}^{\prime}=\hat{U}_{n m}^{-1}(r) \hat{F} \hat{U}_{n m}(r) \tag{79}
\end{equation*}
$$

is the Weyl quantization of the c-number function

$$
\begin{equation*}
f^{\prime}(\xi, \eta)=V_{n m}(r, 0) f(\xi, \eta) . \tag{80}
\end{equation*}
$$

For the sake of brevity, in the case of complex coordinates we write out only the main equations:

$$
\begin{align*}
T_{n m}^{\prime(r)}(s) \hat{D}(z, s) & =\left[\hat{y}_{n m}^{(r)}, \hat{D}(z, s)\right] \\
V_{n m}^{\prime}(r, s) \hat{D}(z, s) & =\hat{U}_{n m}^{\prime}(r) \hat{D}(z, s) \hat{U}_{n m}^{\prime-1}(r) \tag{81}
\end{align*}
$$

and

$$
\begin{align*}
T_{n m}^{\prime(r)} f(z, \bar{z}) & =\operatorname{Tr}\left\{\left[\hat{F}\left(\hat{a}^{\dagger}, \hat{a}\right), \hat{y}_{n m}^{(r)}\right] \hat{D}(z)\right\} \\
V_{n m}^{\prime}(r, 0) f(z, \bar{z}) & =\operatorname{Tr}\left\{\left[\hat{U}_{n m}^{\prime-1}(r) \hat{F}\left(\hat{a}^{\dagger}, \hat{a}\right) \hat{U}_{n m}^{\prime}(r)\right] \hat{D}(z)\right\} . \tag{82}
\end{align*}
$$

These expressions describe the $W_{\infty}$-covariance of the $\hat{D}(z, s)$ basis and of the Weyl quantization, respectively. Here we used the abrreviations ( $\alpha_{n m} \in \mathbf{C}$ )

$$
\begin{align*}
T_{n m}^{\prime(r)}(s) & \equiv\left\{Q_{L}^{\prime n}(s) P_{L}^{\prime m}(s)\right\}_{-r}-\left\{Q_{R}^{\prime n}(s) P_{R}^{\prime m}(s)\right\}_{r} \\
V_{n m}^{\prime}(r, s) & \equiv \exp \left(i \alpha_{n m} T_{n m}^{\prime(r)}(s)\right) \\
\hat{U}_{n m}^{\prime}(r) & \equiv \exp \left(i \alpha_{n m} \hat{y}_{n m}^{r r}\right) \tag{83}
\end{align*}
$$

and $T_{n m}^{\prime(r)} \equiv T_{n m}^{\prime(r)}(0)$.
From Eq.(71) we have

$$
\begin{equation*}
\Gamma_{n m}^{(r)}(s) \hat{\Delta}_{q p}(s)=\left[\hat{t}_{n m}^{(r)}, \hat{\Delta}_{q p}(s)\right] \tag{84}
\end{equation*}
$$

where

$$
\begin{equation*}
\Gamma_{n m}^{(r)}(s)=\left\{Q_{\Delta L}^{n}(s) P_{\Delta L}^{m}(s)\right\}_{-r}-\left\{Q_{\Delta R}^{n}(s) P_{\Delta R}^{m}(s)\right\}_{r} . \tag{85}
\end{equation*}
$$

It is straightforward to verify that in the case of complex coordinates the corresponding relations are the following:

$$
\begin{align*}
\Gamma_{n m}^{\prime(r)}(s) \hat{\Delta}_{Z}(s) & =\left[\hat{y}_{n m}^{(r)}, \hat{\Delta}_{Z}(s)\right]  \tag{86}\\
\Gamma_{n m}^{\prime(r)}(s) & =\left\{Q_{\Delta L}^{\prime \prime}(s) P_{\Delta L}^{\prime m}(s)\right\}_{-r}-\left\{Q_{\Delta R}^{\prime n}(s) P_{\Delta R}^{\prime m}(s)\right\}_{r}
\end{align*}
$$

Exponentiating the actions (84) and (86) we are led to

$$
\begin{align*}
V_{n m}^{\Delta}(r, s) \hat{\Delta}_{q p}(s) & =\hat{U}_{n m}(r) \hat{\Delta}_{q p}(s) \hat{U}_{n m}^{-1}(r) \\
V_{n m}^{\prime \Delta}(r, s) \hat{\Delta}_{Z}(s) & =\hat{U}_{n m}^{\prime}(r) \hat{\Delta}_{Z} \hat{U}_{n m}^{\prime-1}(r) \tag{87}
\end{align*}
$$

where

$$
V_{n m}^{\Delta}(r, s) \equiv \exp \left(i \gamma_{n m} \Gamma_{n m}^{(r)}(s)\right) \quad, \quad V_{n m}^{\prime \Delta}(r, s) \equiv \exp \left(i \alpha_{n m} \Gamma_{n m}^{\prime(r)}\right)
$$

These expressions exhibit both at the algebra and at the group level, the $W_{\infty}$-covariance of the Wigner $(s=0)$ and the Kirkwood $(s= \pm 1)$ bases. Suppose $\hat{F}(\hat{q}, \hat{p})$ and $\hat{G}\left(\hat{a}^{\dagger}, \hat{a}\right)$ are two arbitrary bounded operators, and let $\hat{F}^{\prime}=\hat{U}_{n m}^{-1}(r) \hat{F} \hat{U}_{n m}(r)$ and $\hat{G}^{\prime}=\hat{U}_{n m}^{\prime-1}(r) \hat{G} \hat{U}_{n m}^{\prime}(r)$ be their $W_{\infty}$ transforms. Then from Eqs.(87) we get

$$
\begin{align*}
V_{n m}^{\Delta}(r, s) f^{(s)}(q, p) & =\operatorname{Tr}\left[\hat{F}^{\prime} \hat{\Delta}_{q p}(s)\right] \\
V_{n m}^{\prime \Delta}(r, s) g^{(s)}(Z, \bar{Z}) & =\operatorname{Tr}\left[\hat{G}^{\prime} \hat{\Delta}_{Z}(s)\right] \tag{88}
\end{align*}
$$

where $f^{(s)}(q, p)=\operatorname{Tr}\left[\hat{F} \hat{\Delta}_{q p}(s)\right]$ and $g^{(s)}(Z, \bar{Z})=\operatorname{Tr}\left[\hat{G} \hat{\Delta}_{Z}(s)\right]$ are the corresponding c-number functions. The infinitesimal version of (84) are

$$
\begin{align*}
\Gamma_{n m}^{(r)}(s) f^{(s)}(q, p) & =\operatorname{Tr}\left(\left[\hat{F}, \hat{t}_{n m}^{(r)}\right] \hat{\Delta}_{q p}(s)\right) \\
\Gamma_{n m}^{\prime(r)}(s) g^{(s)}(Z, \bar{Z}) & =\operatorname{Tr}\left(\left[\hat{G}, \hat{y}_{n m}^{(r)}\right] \hat{\Delta}_{Z}(s)\right) \tag{89}
\end{align*}
$$

Thus, the complete $W_{\infty}$-covariance of the WWGM-quantization is achieved. Explicit expressions giving the algebra generators for $n, m \leq 2$ are presented below.

$$
\begin{array}{ll}
\hat{t}_{00}^{(0)}=\hat{I} & , \quad \Gamma_{00}^{(0)}(s)=0 \\
\hat{t}_{10}^{(0)}=\hat{q} \quad, \quad & \Gamma_{10}^{(0)}(s)=-i \hbar \partial_{p} \\
\hat{t}_{01}^{(0)}=\hat{p} \quad, \quad \Gamma_{01}^{(0)}(s)=i \hbar \partial_{q} \\
\hat{t}_{11}^{(0)}=\frac{1}{2}(\hat{q} \hat{p}+\hat{p} \hat{q}) \quad, \quad \Gamma_{11}^{(0)}(s)=i \hbar\left(q \partial_{q}-p \partial_{p}\right) \\
\hat{t}_{20}^{(0)}=\hat{q}^{2} & , \quad \Gamma_{20}^{(0)}(s)=-2 i \hbar q \partial_{p}+s \hbar^{2} \partial_{p}^{2} \\
\hat{t}_{02}^{(0)}=\hat{p}^{2} \quad, \quad & \Gamma_{02}^{(0)}(s)=2 i \hbar p \partial_{q}-s \hbar^{2} \partial_{q}^{2} \tag{90}
\end{array}
$$

In order to see the connection between the algebra of canonical diffeomorphisms and the $W_{\infty}$-algebra found above in a different bases, we consider the relation 22]

$$
\begin{align*}
\Gamma_{n m}^{(s)}(-s) f(q, p) & =\left(q^{n} p^{m}\right) \star_{(-s)} f(q, p)-f(q, p) \star_{(-s)}\left(q^{n} p^{m}\right) \\
& =\left\{q^{n} p^{m}, f(q, p)\right\}_{M B}^{(-s)} \tag{91}
\end{align*}
$$

where $f$ is an arbitrary c-number function and the s-parametrized star product $\star_{(-s)}$ is defined to be

$$
\begin{equation*}
\star_{(-s)}=\exp \frac{1}{2} i \hbar\left[(1-s) \partial_{p}^{L} \partial_{q}^{R}-(1+s) \partial_{q}^{L} \partial_{p}^{R}\right] \tag{92}
\end{equation*}
$$

$\partial^{L}$ and $\partial^{R}$ denote partial derivatives acting to the left ( L ) and to the right ( R ), respectively. We have in particular $\exp \left(-i \hbar \partial_{q}^{L} \partial_{p}^{R}\right)$ for $s=1$, $\exp \left(\frac{1}{2} i \hbar\left(\partial_{p}^{L} \partial_{q}^{R}-\partial_{q}^{L} \partial_{p}^{R}\right)\right)$ for $s=0$ and $\exp \left(i \hbar \partial_{p}^{L} \partial_{q}^{R}\right)$ for $s=-1$. Thus the above definition unifies the different expressions given in the literature for the star product and Moyal brackets and generalizes them for an arbitrary s-ordering. The s-Moyal brackets of two arbitrary functions can also be written as

$$
\begin{equation*}
\left\{f_{1}, f_{2}\right\}_{M B}^{(-s)}=2 i f_{1}\left[\exp -\frac{1}{2} i \hbar s\left(\partial_{p}^{L} \partial_{q}^{R}+\partial_{q}^{L} \partial_{p}^{R}\right)\right] \sin \left[\frac{1}{2} \hbar\left(\partial_{p}^{L} \partial_{q}^{R}-\partial_{q}^{L} \partial_{p}^{R}\right)\right] f_{2} \tag{93}
\end{equation*}
$$

which reduces to the well known Moyal form when $s=0$.
Writing out the first three terms explicitly, the expansion of the s-Moyal brackets is

$$
\begin{align*}
\left\{f_{1}, f_{2}\right\}_{M B}^{(-s)}=i \hbar\left\{f_{1},\right. & \left.f_{2}\right\}_{P B}+\frac{1}{2!}(i \hbar / 2)^{2} 4 s\left[\left(\partial_{q}^{2} f_{1}\right)\left(\partial_{p}^{2} f_{2}\right)-\left(f_{1} \leftrightarrow f_{2}\right)\right] \\
+ & \frac{1}{3!}(i \hbar / 2)^{3}\left\{\left[(1-s)^{3}+(1+s)^{3}\right]\left(\partial_{p}^{3} f_{1}\right)\left(\partial_{q}^{3} f_{2}\right)-\right. \\
& \left.6\left(1-s^{2}\right)\left(\partial_{q} \partial_{p}^{2} f_{1}\right)\left(\partial_{p} \partial_{q}^{2} f_{2}\right)-\left(f_{1} \leftrightarrow f_{2}\right)\right\}+\ldots \tag{94}
\end{align*}
$$

where PB denotes the Poisson brackets. This formula generalizes to arbitrary values of $s$ the expansions for some discrete values of $s$ that previously appeared in the literature. In particular we would like to note that in the case of Wigner quantization $(s=0)$, the leading order correction to the PB is proportional to $\hbar^{2}$, while in all the other cases $(s \neq 0)$ the leading term is proportional to $\hbar$.

Taking $\hat{F}$ to be $\hat{t}_{k l}^{(s)}$ in Eq.(89) and using (91) we obtain

$$
\begin{align*}
\Gamma_{n m}^{(s)}(-s)\left(q^{k} p^{l}\right) & =\operatorname{Tr}\left\{\left[\hat{t}_{k l}^{(s)}, \hat{t}_{n m}^{(s)}\right] \hat{\Delta}_{q p}(-s)\right\} \\
& =\left\{q^{n} p^{m}, q^{k} p^{l}\right\}_{M B}^{(-s)} \tag{95}
\end{align*}
$$

The last equality sets up a Lie algebra isomorphism between the quantum $W_{\infty}$, that is the algebra generated by s-ordered products under the usual Lie bracket action, and the algebra generated by the monomials $q^{n} p^{m}$ for $n, m \geq 0$ under the s-MB action. Since $(i \hbar)^{-1}\{,\}_{M B}^{(-s)} \rightarrow\{,\}_{P B}$ as $\hbar \rightarrow 0$, the essence of the full quantum $W_{\infty}$ can be captured on the classical phase space by simply deforming the Poisson brackets to s-Moyal brackets. On the other hand we have another infinite algebra generated by the operators $\Gamma_{n m}^{(r)}(s) ; n, m \geq 0$ indexed by two ordering parameters $r, s$ and built up by the product of the Bopp operators, that are concretely realized in the tangent space of $\mathbf{R}^{2}$. This is the algebra that we referred to as the classical $W_{\infty}$ in the introduction. As is seen from (95), or more readily from

$$
\begin{equation*}
\left[\Gamma_{n m}^{(r)}(s), \Gamma_{k l}^{(r)}(s)\right] \hat{\Delta}_{q p}(s)=-\left[\left[\hat{t}_{n m}^{(r)}, \hat{t}_{k l}^{(r)}\right], \hat{\Delta}_{q p}(s)\right] \tag{96}
\end{equation*}
$$

the above mentioned isomorphic quantum $W_{\infty}$ algebras are central extensions of this classical $W_{\infty}$. The vanishing of the right hand side of (95) (or (96)) requires by the completeness of the basis, that $\hat{t}_{n m}^{(r)}$ (or $\left.\left[\hat{t}_{n m}^{(r)}, \hat{t}_{k l}^{(r)}\right]\right)$ has to be proportional to $\hat{I}$, while the vanishing of the left hand side requires $\Gamma_{n m}^{(r)}\left(\right.$ or $\left.\left[\Gamma_{n m}^{(r)}(s), \Gamma_{k l}^{r}(s)\right]\right)$ to be zero. Note that, as is apparent from Eq.(96), there is an overall sign difference between the structure constants of the classical and quantum $W_{\infty}$ algebras. This can be remedied by a simple redefinition of the generators. Thus, the group generated by the quantum $W_{\infty}$ provides a projective representation of the classical $W_{\infty}$.

On the other hand, it is known that [16] the space of monomials $t_{n m} \equiv q^{n} p^{m} ; n, m \geq 0$ form the Lie algebra of canonical diffeomorphisms of a phase space, that is topologically equivalent to $\mathbf{R}^{\mathbf{2}}$, under the usual Poisson brackets. This algebra is known as $w_{\infty}$, or since the area element and the symplectic form coincides in two dimensions, as the algebra of area preserving diffeomorphisms $\operatorname{Dif} f_{A} \mathbf{R}^{2}$. The $W_{\infty}$ algebras discussed above are the quantum (or, $\hbar$ ) deformation of this classical $w_{\infty}$. The one called the quantum $W_{\infty}$ provides an implementation of the general canonical transformations at the quantum level.

## VI. CONCLUSION

We have developed a derivative based approach to the WWGM-quantization as an alternative to the integral based conventional one. This enabled us in particular to obtain some fundamental associations in a unified way easily and to derive an explicit formula for the s-ordered products that led to further observations. It is argued that this formula can also be used for any pair of operators. In the case of operators belonging to a nilpotent algebra such as the Heisenberg-Weyl algebra, seemingly different but equivalent expressions of a given ordered product can be obtained.

In a given association the primary issue is to determine how a member of the association transforms when the other one is transformed in a well defined way. We have explicitly shown that the WWGM-quantization in its most general form has a $W_{\infty}$-covariance which includes the known metaplectic covariance as a subset. Eq.(90) contains for $n, m \leq 2$ the generators of the metaplectic algebra $\operatorname{Isp}(2)$ in the classical phase space and its central extension in $\mathcal{H}$. Moreover, we emphasize that like the metaplectic covariance, the $W_{\infty}$-covariance we had shown is a genuine property of the complete operator bases used. An important group theoretical outcome of this construction is that we have obtained a projective representation of the classical $W_{\infty}$ realized in the tangent space of the related phase space.
$W_{\infty}$ algebras are currently the subject of active investigations in two dimensional gravity 16], conformal field theories 23] and in connection with quantum Hall effect in condensed matter physics (see 24] and the references therein). For example, the notion of incompressibility which plays a fundamental role in the theoretical understanding of quantum Hall effect has been related to the existence of the $W_{\infty}$ symmetry. These exciting developments suggest that the structure of the Landau levels, or more generally the quantum Hall effect could also be investigated in the framework of the WWGM-quantization. We plan to take up a systematic study of these problems in our forthcoming papers.

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