# String-theory Realization of Modular Forms for Elliptic Curves with Complex Multiplication 

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#### Abstract

It is known that the $L$-function of an elliptic curve defined over $\mathbb{Q}$ is given by the Mellin transform of a modular form of weight 2. Does that modular form have anything to do with string theory? In this article, we address a question along this line for elliptic curves that have complex multiplication defined over number fields. So long as we use diagonal rational $\mathcal{N}=(2,2)$ superconformal field theories for the stringtheory realizations of the elliptic curves, the weight- 2 modular form turns out to be the Boltzmann-weighted ( $q^{L_{0}-c / 24}$-weighted) sum of $\mathrm{U}(1)$ charges with $F e^{\pi i F}$ insertion computed in the Ramond sector.


## 1 Introduction

For any elliptic curve defined over $\mathbb{Q}$, its Hasse-Weil $L$-function is given by the Mellin transform of a modular form of weight 2 (by Shimura-Taniyama conjecture, now a theorem). For many elliptic curves with complex multiplication defined over number fields, their HasseWeil $L$-functions are still given by multiplying the Mellin transforms of modular forms. In the meantime, when we define a string theory by using a geometry as a target space, its worldsheet formulation must have modular invariance; various observables computed in such a string theory are often modular forms. Are there any relations between those two kinds of modular forms? If there are, what are the relations precisely? We address this question ${ }^{1}$ in this article; the spirit is therefore similar to that of [21].

A relation between them, if there is any, cannot be as simple as "they are the same". The Hasse-Weil $L$-function is defined for an algebraic variety (where complex structure is specified, but metric is not), while there is no choice in formulating a string theory without specifying a metric on a target space $\sqrt[2]{ }$ At least we need to extract some information from string theory in a way the results do not depend on the choice of a metric, or to find a way to extract for any choice of metric, to say the least. One will also notice that the $L$-function is defined for individual models defined over number fields; in arithmetic geometry, two elliptic curves given by $y^{2}=x^{3}-x$ and $y^{2}=x^{3}-4 x$ are regarded different varieties defined over $\mathbb{Q}$. Those two elliptic curves are regarded the same, however, when we use them as a target space of string theory; coordinate reparametrization in $\mathbb{C}$ (rather than in $\mathbb{Q}$ ) washes out the difference between them. 3 How can a string theory with a given elliptic curve defined over $\mathbb{C}$ provide information of the $L$-functions of various models defined over number fields?

In this article, we deal with a class of elliptic curves - those with complex multiplication with one condition (stated in Lemma 4.2.4) - and provide an answer to the questions raised above. It is best to begin with a relation between character functions (given essentially by theta functions) of string theory with an elliptic curve as a target space, and the Dedekind zeta function $\zeta_{k}(s)$ of a number field $k$; the observation that the Mellin transform of the

[^0]former is the latter [Moore [17], §4] corresponds to the first line of the following diagram,
\[

$$
\begin{aligned}
\text { characters } & \Longleftrightarrow \zeta_{k}(s)=L\left(H_{e t}^{0}(E), s\right), \\
? ? & \Longleftrightarrow L\left(H_{e t}^{1}(E), s\right) .
\end{aligned}
$$
\]

In section 3.4, we elaborate on the relation in the first line, because this process serves as a warming up exercise in finding out the appropriate string-theory objects in the second line. We will see, in section (4.3), that the objects "??" are (62, 63), because of the observation (1021). Thms. 4.3.1 and 4.3.2 explain how the Hasse-Weil $L$-functions (i.e., $L\left(H_{e t}^{1}(E), s\right)$ ) of arithmetic models of an elliptic curve with complex multiplication can be obtained from those objects defined in string-theory realizations. The objects (62, 63) are modular forms of weight 2 for a subgroup of $\operatorname{SL}(2 ; \mathbb{Z})_{w s}$ acting on the complex structure parameter $\tau_{w s}$ of the worldsheet torus of string theory. The "how can a string theory... ?" question raised earlier in Introduction is also answered right after Thm. 4.3.2,

Background materials from string theory are explained in sections 2, 3.2 and 4.1, while sections 3.1, 3.3 and 4.2 provide a quick review on materials from algebraic number theory we need in this article. Novelty may be found only in section 4.3. If a reader already has introductory level knowledge on string theory and class field theory, he/she can proceed directly to section 4.3.

Despite the apparent style of presentation built up with "Theorem," "Proposition" and so forth, and despite frequent use of algebraic number theory, the main result reported in this article -what is the object "??" in the diagram above - is a statement on string theory. We have adopted a math-like style of presentation in this article for the purpose of making statements more precise, and implicit contexts explicit.

## 2 Rational CFT and Complex Multiplication

Refs. [16, 6] pointed out, roughly speaking, that an elliptic curve has complex multiplication if and only if its string-theory realization is given by a rational conformal field theory (CFT). See also [30]. This section provides a brief review on part of the results of [16, 6] for readers with math background, by adopting a math-style presentation. We have also inserted Remark 2.0.4, which tries to give an idea of what elliptic curves with complex multiplication are like, for those who have never heard of the jargon.

Think of a bosonic string theory with an elliptic curve as the target space. This is
equivalent to thinking of a $(c, \tilde{c})=(2,2)$ CFT on a worldsheet that has two holomorphic (leftmover) $\mathrm{U}(1)$ current operators and also two anti-holomorphic (right-mover) ones. We call such a CFT as a $T^{2}$-target CFT. $T^{2}$-target CFT's are parametrized by a pair $(z, \rho) \in \mathcal{H} \times \mathcal{H}$, where $\mathcal{H}$ is the upper complex half plane without the real axis. Here, $z$ stands for the complex structure parameter and $\rho$ for the complexified Kähler parameter of an elliptic curve.

Notation 2.0.1. For $z \in \mathcal{H}, E_{z},\left[E_{z}\right]$ or $\left[E_{z}\right]_{\mathbb{C}}$ stands for a $\mathbb{C}$-isomorphism class of elliptic curves that can be constructed by $\mathbb{C} /(\mathbb{Z} \oplus z \mathbb{Z})$.

The same elliptic curve characterized by a pair $(z, \rho) \in \mathcal{H} \times \mathcal{H}$ can also be used as a target space of Type II superstring. There, we think of a $(c, \tilde{c})=(3,3)$ CFT with $\mathcal{N}=(2,2)$ supersymmetry on a worldsheet that has two holomorphic (left-mover) $\mathrm{U}(1)$ current operators in addition to the $\mathrm{U}(1)$ current operator $J_{L}$ in the left-mover superconformal algebra, and also two anti-holomorphic (right-mover) ones in addition to $J_{R}$ in the right-mover superconformal algebra. We call such a CFT as a $T^{2}$-target $\mathcal{N}=(2,2)$ superconformal field theory (SCFT).

### 2.0.1 Statements from Refs. [16, 6]

References [16, 6] found the following, among other things.
Proposition 2.0.2. A $T^{2}$-target CFT is a rational CFT if and only if both of the following two conditions are satisfied: i) both $E_{z}$ and $E_{\rho}$ admit complex multiplications, and ii) $E_{z}$ and $E_{\rho}$ are isogenous.

Proposition 2.0.3. A rational $T^{2}$-target CFT is diagonal, when either $\rho \in \operatorname{Aut}\left(E_{z}\right)$, or $z \in \operatorname{Aut}\left(E_{\rho}\right)$.

In the rest of this note, we only think of the cases with $\rho \in \operatorname{Aut}\left(E_{z}\right)$. For non-string readers, additional information (incl. definition) on diagonal rational CFT's are provided in section [2.0.2. For readers with physics background, it is useful to know the following classic results in mathematics:

Remark 2.0.4. Elliptic curves with complex multiplication by an order of an imaginary quadratic field $K=\mathbb{Q}\left(\sqrt{-d_{0}}\right)$ (for some positive square-free integer $d_{0}$ ) are classified modulo isomorphisms over $\mathbb{C}$ by the set

$$
\begin{equation*}
\amalg_{f_{z} \in \mathbb{N}_{>0}} \operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right), \tag{1}
\end{equation*}
$$

whert 4

$$
\operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)=\left\{\left(\begin{array}{cc}
2 a & b \\
b & 2 c
\end{array}\right)\left|a, b, c \in \mathbb{Z},(a, b, c)=1, D_{z}:=4 a c-b^{2}=\left|D_{K}\right| f_{z}^{2}\right\} / \mathrm{SL}(2 ; \mathbb{Z})\right.
$$

Here, a matrix $h \in \mathrm{SL}(2 ; \mathbb{Z})$ acts on a matrix $g(a, b, c)$ parametrized by integers $a, b, c$ as above, by $g(a, b, c) \mapsto g\left(a^{\prime}, b^{\prime}, c^{\prime}\right)=h \cdot g(a, b, c) \cdot h^{T}$. The corresponding $\mathbb{C}$-isomorphism class in $\operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$ is given by $E_{z}$ with the solution $z \in \mathcal{H}$ of

$$
\begin{equation*}
a z^{2}+b z+c=0 \tag{2}
\end{equation*}
$$

A set of integers $(a, b, c)$ that gives rise to an elliptic curve $E_{z}$ in this way is denoted by $\left(a_{z}, b_{z}, c_{z}\right)$. Because the $\operatorname{SL}(2 ; \mathbb{Z})$ action on a set of integers $\left(a_{z}, b_{z}, c_{z}\right)$ corresponds to the $\mathrm{SL}(2 ; \mathbb{Z})$ transformation on $z \in \mathcal{H}$, an $\operatorname{SL}(2 ; \mathbb{Z})$ orbit of sets of integers $\left(a_{z}, b_{z}, c_{z}\right)$ specifies just one $\mathbb{C}$-isomorphism class of elliptic curves.

For any $\mathbb{C}$-isomorphism class $\left[E_{z}\right]$ of elliptic curves in $\operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$, the ring of endomorphism of $E_{z}$ is $\operatorname{Aut}\left(E_{z}\right) \cong \mathbb{Z} \oplus\left(a_{z} z\right) \mathbb{Z}=: \mathcal{O}_{f_{z}} \subset \mathcal{O}_{K} \subset K \subset \mathbb{C}$; here, $\mathcal{O}_{K}$ is the ring of algebraic integers in $K$. Such a subring $\mathcal{O}_{f_{z}}$ of $\mathcal{O}_{K}$ is called an order, and $\mathcal{O}_{K}=\mathcal{O}_{f_{z}=1}$ is called the maximal order of the imaginary quadratic field $K=\mathbb{Q}\left(\sqrt{-d_{0}}\right)$. We say that an elliptic curve $E$ whose $\mathbb{C}$-isomorphism class $[E]$ is in $\operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$ has complex multiplication by $\mathcal{O}_{f_{z}}$.

It is also known that the ideal class group of an order ${ }^{5} \mathcal{O}_{f_{z}}$, denoted by $\mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}}\right)$, is isomorphic to the set $\operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$. Table 1 shows examples of elliptic curves with complex multiplication. This article will not provide a review on class field theory beyond what is written here; instead, we will provide reference to math textbooks occasionally 6 -

Suppose that an elliptic curve $E_{z}$ has complex multiplication by an order $\operatorname{Aut}\left(E_{z}\right)=\mathcal{O}_{f_{z}}$ of an imaginary quadratic field, as in Propositions 2.0.2 and 2.0.3. Two string theory realizations with $(z, \rho)$ and $\left(z, \rho^{\prime}\right)$, with $\rho, \rho^{\prime} \in \mathcal{O}_{f_{z}}=\mathbb{Z} \oplus\left(a_{z} z\right) \mathbb{Z}$ (as indicated in Proposition 2.0.3) are regarded the same, when $\rho-\rho^{\prime} \in \mathbb{Z}$ (converting $\rho$ to $\rho^{\prime} \in \rho+\mathbb{Z}$ is a part of $\operatorname{SL}(2 ; \mathbb{Z})$ action on the parameter $\rho$ ). This means that we can always choose $\rho$ of a rational diagonal

[^1]| $\left\|D_{K}\right\|$ | $f_{z}=1$ | $f_{z}=2$ | $f_{z}=3$ | $f_{z}=4$ | $f_{z}=5$ |
| :---: | :--- | :--- | :--- | :--- | :--- |
| 3 | $[1,1,1]$ | $[1,0,3]$ | $[1,1,7]$ | $[1,0,12],[3,0,4]$ | $[1,1,19],[3,3,7]$ |
| 4 | $[1,0,1]$ | $[1,0,4]$ | $[1,0,9],[2,2,5]$ | $[1,0,16],[4,4,5]$ | $[1,0,25],[2,2,13]$ |
| 7 | $[1,1,2]$ | $[1,0,7]$ | $h\left(\mathcal{O}_{\left.f_{z}=3\right)=4}\right.$ | $h\left(\mathcal{O}_{f_{z}=4}\right)=2$ | $h\left(\mathcal{O}_{f_{z}=5}\right)=6$ |
| 8 | $[1,0,2]$ | $[1,0,8],[3,2,3]$ | $h\left(\mathcal{O}_{f_{z}=3}\right)=2$ | $h\left(\mathcal{O}_{f_{z}=4}\right)=4$ | $h\left(\mathcal{O}_{f_{z}=5}\right)=6$ |
| 11 | $[1,1,3]$ | $[1,0,11],[3, \pm 2,4]$ | $h\left(\mathcal{O}_{f_{z}=3}\right)=2$ | $h\left(\mathcal{O}_{f_{z}=4}\right)=6$ | $h\left(\mathcal{O}_{f_{z}=5}\right)=4$ |
| 20 | $[1,0,5],[2,2,3]$ | $h\left(\mathcal{O}_{f_{z}=2}\right)=4$ | $h\left(\mathcal{O}_{f_{z}=3}\right)=4$ | $h\left(\mathcal{O}_{f_{z}=4}\right)=8$ | $h\left(\mathcal{O}_{f_{z}=5}\right)=10$ |

Table 1: Representatives $[a, b, c]$ of individual elements of $\mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}}\right)$ for imaginary quadratic fields $K$ with small $\left|D_{K}\right|$. Representatives are chosen so that $0 \leq a \leq c$ and $-a<b \leq a$; when $c=a$, however, $0 \leq b \leq a$. In order to save space, just the cardinality $h\left(\mathcal{O}_{f_{z}}\right)$ of $\mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}}\right)$ is shown in part of this Table, instead of representatives of all the elements of $\mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}}\right)$. There are two more imaginary quadratic fields, $K=\mathbb{Q}(\sqrt{-15})$ and $K=\mathbb{Q}(\sqrt{-19})$, that would come in between $D_{K}=-11$ and $D_{K}=-20$, but we omitted them, just to save space.
$T^{2}$-target CFT to be an integer $\left(f_{\rho}\right)$ multiple of $a_{z} z: \rho=f_{\rho} a_{z} z$. From here, therefore, follows a consequence of Propositions 2.0.2 and 2.0.3 phrased in a way favorable for algebraic geometers:

Proposition 2.0.5. Think of a $\mathbb{C}$-isomorphism class of elliptic curves with complex multiplication by an order $\mathcal{O}_{f_{z}}$ of an imaginary quadratic field $K$, i.e., any element of (1). Then there is a family of bosonic string theory realizations in the form of a rational and diagonal $T^{2}$-target CFT, parametrized by $f_{\rho} \in \mathbb{N}_{>0}$. Also, there is a family of Type II superstring theory realizations parametrized by $f_{\rho} \in \mathbb{N}_{>0}$, each one of which is in the form of a rational and diagonal $T^{2}$-target $\mathcal{N}=(2,2)$ SCFT.

The parameter $f_{\rho}$ controls the choice of complexified Kähler form on the $\mathbb{C}$-isomorphism class of elliptic curves. Although algebraic geometry only deals with complex structure of a geometry and does not refer to the choice of a metric on it, yet there is no option in any stringtheory realization of an object in algebraic geometry not to specify a metric (complexified Kähler form) on it. The Proposition above says that the choice of complexified Kähler form is parametrized by one positive integer $f_{0} \in \mathbb{N}_{>0}$, when we impose a condition that the CFT on worldsheets be rational and diagonal $\frac{7}{7}$

[^2]
### 2.0.2 Rudiments on Rational CFT's

To describe various structures of diagonal rational $T^{2}$-target CFT's in terms of arithmetic properties of the elliptic curves with complex multiplication, we prepare a few important general properties of diagonal rational CFT's (that are not necessarily $T^{2}$-target).

Remark 2.0.6. When a rational diagonal CFT is given, there is an isomorphism $\varphi_{0}$ between the algebra $\mathcal{A}_{-}$of purely holomorphic vertex operators and that $\left(\mathcal{A}_{+}\right)$of purely antiholomorphic vertex operators,

$$
\begin{equation*}
\varphi_{0}: \mathcal{A}_{-} \cong \mathcal{A}_{+} \tag{3}
\end{equation*}
$$

The Hilbert space of closed string states $\mathcal{H}^{\text {closed }}$ in such a theory has a structure

$$
\begin{equation*}
\mathcal{H}^{\text {closed }}=\oplus_{\alpha \in i \text { Reps. }}\left(V_{\alpha}^{-} \otimes V_{\alpha}^{+}\right) \tag{4}
\end{equation*}
$$

here, $V_{\alpha}^{+}$is an irreducible representation of $\mathcal{A}_{+}$, and there are just finite number of distinct irreducible representations (by definition of rational CFT's). The index $\alpha$ runs over all the irreducible representations. $V_{\alpha}^{-}$is the irreducible representation of $\mathcal{A}_{-}$that is regarded as $V_{\alpha}^{+}$when the action of $\mathcal{A}_{-}$is identified with that of $\mathcal{A}_{+}$through the isomorphism $\varphi_{0}$ (by definition of the diagonality).

Remark 2.0.7. In a rational $T^{2}$-target CFT, there exists a sublattice $\left(\Gamma_{-} \oplus \Gamma_{+}\right) \subset \mathrm{I}_{2,2} ; \mathrm{I}_{2,2}$ is the even unimodular lattice of signature $(2,2), \Gamma_{-}\left[\right.$resp. $\left.\Gamma_{+}\right]$is an even integral negative [resp. positive] definite primitive sublattice of $\mathrm{II}_{2,2}$. When a rational $T^{2}$-target CFT is also diagonal, then the isomorphism $\varphi_{0}: \mathcal{A}_{-} \cong \mathcal{A}_{+}$also induces a lattice isometry $\varphi_{0}: \Gamma_{-}[-1] \cong \Gamma_{+}$. The set of irreducible representations are labeled by

$$
\begin{equation*}
i \text { Reps } \cong \Gamma_{+}^{\vee} / \Gamma_{+} \cong \mathrm{II}_{2,2} /\left(\Gamma_{-} \oplus \Gamma_{+}\right) \cong \Gamma_{-}^{\vee} / \Gamma_{-}, \tag{5}
\end{equation*}
$$

where $L^{\vee}$ stands for the dual lattice of an integral lattice $L$. •
In the language of string theory, $\mathrm{I}_{2,2}$ is the lattice of charges under the four $\mathrm{U}(1)$ currents in a $T^{2}$-target CFT. The sublattice $\Gamma_{-}\left[\right.$resp. $\left.\Gamma_{+}\right]$in a rational $T^{2}$-target CFT corresponds to the set of $\mathrm{U}(1)$ charges for which purely holomorphic [resp. purely anti-holomorphic] operators exist. 8

[^3]Remark 2.0.8. (we keep the notation as in the previous remark) In a rational diagonal CFT, $\varphi_{0^{-}}$Cardy states refer to a special class of choices of boundary conditions that can be imposed on worldsheets; they are in one-to-one correspondence with the set $i$ Reps. Furthermore, when a worldsheet is in the form of a long strip and both of the two edges of the strip are subject to Cardy states, say, $\alpha$ and $\beta \in i$ Reps., then the open string states on such a long strip form a representation of the algebra ${ }^{9}\left(\mathcal{A}_{-} \times \mathcal{A}_{+}\right) /\left(\varphi_{0}\right.$ at.bdry $) \cong \mathcal{A}_{+}$. Therefore, the Hilbert space of open string states with both of the two boundary conditions being $\varphi_{0}$-Cardy states consists of a direct sum of $V_{\alpha}^{o}$, the irreducible representation of the algebra that is isomorphic to $V_{\alpha}^{+}$when seen as a representation of $\mathcal{A}_{+}$.

### 2.0.3 More Statements from Ref. [6]

With the preparation in remark 2.0.4 and section 2.0.2, we are now ready to write down more statements in 6]:

Proposition 2.0.9. Think of a diagonal rational $T^{2}$-target CFT, with $\rho=f_{\rho} a_{z} z$. Then the characters of the irreducible representations of the algebra $\mathcal{A}_{+}$and $\mathcal{A}_{-}$are given by

$$
\begin{align*}
\operatorname{ch}_{V_{\alpha}^{+}}\left(\bar{q}=e^{-2 \pi i \bar{\tau}}\right) & :=\operatorname{Tr}_{V_{\alpha}^{+}}\left[e^{-2 \pi i \bar{\tau}\left(\tilde{L}_{0}-\tilde{c} / 24\right)}\right]=\chi_{\alpha}(\bar{q}),  \tag{6}\\
\operatorname{ch}_{V_{\alpha}^{-}}\left(q=e^{2 \pi i \tau}\right) & :=\operatorname{Tr}_{V_{\alpha}^{-}}\left[e^{2 \pi i \tau\left(L_{0}-c / 24\right)}\right]=\chi_{\alpha}(q), \tag{7}
\end{align*}
$$

where

$$
\begin{equation*}
\chi_{\alpha}\left(e^{2 \pi X}\right)=\frac{f_{0}(X ; \alpha)}{\left(\eta\left(e^{2 \pi X}\right)\right)^{2}}, \quad f_{0}(X ; \alpha):=\sum_{p \in \alpha} e^{2 \pi i X \frac{(p, p)}{2}} \tag{8}
\end{equation*}
$$

$\eta(q)=q^{\frac{1}{24}} \prod_{n=1}^{\infty}\left(1-q^{n}\right)$ is the Dedekind $\eta$-function, and $L_{0}$ [resp. $\left.\tilde{L}_{0}\right]$ is one of the generators of the Virasoro algebra in the left-moving [resp. right-moving] sector. $\alpha \in i R e p s . \cong \Gamma_{+}^{\vee} / \Gamma_{+}$ labels irreducible representations on the left-hand sides, but is regarded as the subset of $\Gamma_{+}^{\vee}$ on the right-hand side; the positive definite intersection form $(-,-)$ of the lattice $\Gamma_{+}^{\vee}$ is used in the exponent.

The intersection form of $\Gamma_{+}$is given by a matrix

$$
f_{\rho}\left[\begin{array}{cc}
2 a_{z} & b_{z}  \tag{9}\\
b_{z} & 2 c_{z}
\end{array}\right]=2 a_{z} f_{\rho}\left[\begin{array}{cc}
1 & -z_{1} \\
-z_{1} & |z|^{2}
\end{array}\right]
$$

[^4]when a set of generators of $\Gamma_{+}$is chosen appropriately; here, $\operatorname{Re}(z)=: z_{1}$ and $\operatorname{Im}(z)=: z_{2}>0$. It follows that
\[

$$
\begin{equation*}
\operatorname{discr}\left(\Gamma_{+}\right)=f_{\rho}^{2} D_{z}=f_{\rho}^{2} f_{z}^{2}\left|D_{K}\right| \tag{10}
\end{equation*}
$$

\]

Proposition 2.0.10. In a diagonal rational $T^{2}$-target CFT with $\rho=f_{\rho} a_{z} z$, the open string states subject to $\varphi_{0}$-Cardy state 10 are in the representation of the algebra $\varphi_{0}:\left(\mathcal{A}_{-} \times\right.$ $\mathcal{A}_{+} /\left(\varphi_{0}\right.$ at.bdry) $\cong \mathcal{A}_{+}$, and the irreducible representations are labeled by (see below for notation)

$$
\begin{equation*}
\alpha \in \Lambda_{\text {Cardy }} / \Lambda_{\text {winding }} \cong i \text { Reps. }=\Gamma_{+}^{\vee} / \Gamma_{+} \tag{11}
\end{equation*}
$$

The character of the irreducible representation $V_{\alpha}^{0}$ of such open string states is given by

$$
\begin{equation*}
\operatorname{ch}_{V_{\alpha}^{o}}\left(e^{-2 \pi t}\right):=\operatorname{Tr}_{V_{\alpha}^{o}}\left[e^{-2 \pi t\left(L_{0}-c / 24\right)}\right]=\chi_{\alpha}\left(e^{-2 \pi t}\right) \tag{12}
\end{equation*}
$$

$L_{0}$ is one of the Virasoro generators for the open string sector. -
Just for the purpose of keeping track of the set of all the irreducible representations of an algebra $\mathcal{A}_{+}$and anything isomorphic to it, it is not necessary to introduce yet another set $\Lambda_{\text {Cardy }} / \Lambda_{\text {winding }}$, which is isomorphic to the set $\Gamma_{+}^{\vee} / \Gamma_{+}$. In string theory, however, the set $\Lambda_{\text {Cardy }} / \Lambda_{\text {winding }}$ labeling the irreducible representations of open string states should be regarded as the character group of $\Gamma_{+}^{\vee} / \Gamma_{+}$labeling the irreducible representations of close string states; they are the same finite set (set theoretically), though.

In the language of string theory, the lattice $\Gamma_{+}\left[\right.$resp. $\left.\Gamma_{+}^{\vee}\right]$ can be regarded also as all the possible values of the right-mover momentum $\sqrt{\left(\alpha^{\prime} / 2\right)} k_{+}$of all the purely anti-holomorphic operators [resp. all the operators]. $\left[\sqrt{\left(\alpha^{\prime} / 2\right)} k_{+}^{\mathbb{C}}\right]: \Gamma_{+} \rightarrow \mathbb{C}$ is an embedding where the absolute-value-square $\left|\left[\sqrt{\left(\alpha^{\prime} / 2\right)} k_{+}^{\mathbb{C}}\right](p)\right|_{\mathbb{C}}^{2}$ in $\mathbb{C}$ for $p \in \Gamma_{+}$reproduces the self intersection $(p, p)$ in $\Gamma_{+}$.
$\Lambda_{\text {Cardy }}$ and $\Lambda_{\text {winding }}$ are also rank-2 lattices, and $\Lambda_{\text {Cardy }} \cong \Gamma_{+}^{\vee}, \Lambda_{\text {winding }} \cong \Gamma_{+}$as lattices. In string theory, the geometry of elliptic curve (target space) is given by

$$
\begin{equation*}
\left[\mathbb{C} / \Lambda_{\text {winding }}\right]_{\mathbb{C}} \cong\left[E_{z}\right]_{\mathbb{C}} ; \tag{13}
\end{equation*}
$$

[^5]$\Lambda_{\text {Cardy }}=\Lambda_{\text {winding }}^{\vee}$ is a lattice of torsion points of the elliptic curve $\mathbb{C} / \Lambda_{\text {winding. }}$. When we refer to open string states on a strip-shape worldsheet with two boundaries subject to $\varphi_{0^{-}}$ Cardy states, we think of states of an open string with one boundary fixed at the origin of $\Lambda_{\text {Cardy }} / \Lambda_{\text {winding }}$ and the other boundary at another point in $\Lambda_{\text {Cardy }} / \Lambda_{\text {winding }}$. The number of torsion points of $\Lambda_{\text {Cardy }} / \Lambda_{\text {winding }}$ is
\[

$$
\begin{equation*}
\left[\Lambda_{\text {Cardy }}: \Lambda_{\text {winding }}\right]=\left[\Gamma_{+}^{\vee}: \Gamma_{+}\right]=f_{\rho}^{2} D_{z}=\#[i \text { Reps. }] . \tag{14}
\end{equation*}
$$

\]

The combination of the isometry $\Lambda_{\text {Cardy }} \cong \Gamma_{+}^{\vee}$ and an embedding $\left[\sqrt{\left(\alpha^{\prime} / 2\right)} k_{+}^{\mathbb{C}}\right]: \Gamma_{+}^{\vee} \rightarrow \mathbb{C}$ gives an embedding of $\Lambda_{\text {Cardy }} \rightarrow \mathbb{C}$. It is natural for string theorists to use $\left[\sqrt{2 / \alpha^{\prime}} \Delta X^{\mathbb{C}} /(2 \pi)\right]:$ $\Lambda_{\text {Cardy }} \rightarrow \mathbb{C}$ as the notation of this embedding, but we use a simpler notation $\Omega^{\prime}: \Lambda_{\text {Cardy }} \rightarrow \mathbb{C}$ instead in the following. The following information is written down here, as we use it later on:

$$
\begin{align*}
\Omega^{\prime}\left(\Lambda_{\text {winding }}\right) & =\sqrt{2 a_{z} f_{\rho}}(\mathbb{Z} \oplus z \mathbb{Z})=\left[\sqrt{\left(\alpha^{\prime} / 2\right)} k_{R}^{\mathbb{C}}\right]\left(\Gamma_{+}\right) \subset \mathbb{C}  \tag{15}\\
\Omega^{\prime}\left(\Lambda_{\text {Cardy }}\right) & =\frac{\sqrt{2 a_{z} f_{\rho}}}{f_{\rho}}\left(\frac{2 a_{z} z+b_{z}}{D_{z}} \mathbb{Z} \oplus \frac{b_{z}+2 c_{z} z}{D_{z}} \mathbb{Z}\right) \subset \mathbb{C} . \tag{16}
\end{align*}
$$

## $3 \zeta_{k}(s)$ : the $L$-function for $H_{e t}^{0}(E)$

### 3.1 Field of Definition of Arithmetic Models

Definition 3.1.1. Let $X$ be an algebraic variety defined over ${ }^{11}$ a number field $k$. When we wish to emphasize the choice of the field of definition, we also write $X / k$. When $k^{\prime} / k$ is a field extension, and $\left[X^{\prime}\right]_{k^{\prime}}$ is an $k^{\prime}$-isomorphism class of algebraic varieties defined over $k^{\prime}$, $X / k$ is said to be a model of $\left[X^{\prime}\right]_{k^{\prime}}$, if the base change ${ }^{12}$ of $X, X \times_{\operatorname{Spec}(k)} \operatorname{Spec}\left(k^{\prime}\right)$, belongs to $\left[X^{\prime}\right]_{k^{\prime}}$.

Although we wish to deal with a $\mathbb{C}$-isomorphism class of a variety $[X]_{\mathbb{C}}$ as a target space in string theory, arithmetic geometry deals with its models $X / k$ defined over some number field $k$. From the perspective of arithmetic geometry, one might be interested in $X / k$ in the category of algebraic varieties defined over a number field $k$, but its string-theory realizations depend only on the $\mathbb{C}$-isomorphism class $[X]_{\mathbb{C}}$ of which $X / k$ is a model.

[^6]Let $X$ be an $n$-dimensional algebraic variety defined over a number field $k$. One can then definn ${ }^{13} 2 n$-functions for $X / k, L\left(H_{e t}^{i}(X), s\right)$ for $i=0,1, \cdots, 2 n$, each one of which is associated with the $i$-the cohomology (denoted symbolically) $H^{i}(X)$ of $X$. In the case $X / k$ is a curve, $n=1$, the $L$-function for $H^{0}(X)$ is the Dedekind zeta function $\zeta_{k}(s)$ of the field of definition, $k$. The $L$-function for $H^{2}(X)$ is $\zeta_{k}(s-1)$. When we refer to Hasse-Weil $L$-function, or simply the $L$-function, that is meant to be the $L$-function associated with $H^{1}(X)$. The notation $L(X / k, s)$ is also used for $L\left(H_{e t}^{1}(X), s\right)$.

The Dedekind zeta function $\zeta_{k}(s)$ of a number field $k$ is so well-understood an object for a given number field $k$ that few people will find it interesting to think of an elliptic curve $E$ defined over $k$, think of string-theory realizations of $[E / k]_{\mathbb{C}}$, and then write down $\zeta_{k}(s)$ in terms of characters of states in the string-theory realizations of $[E / k]_{\mathbb{C}}$. We still do so in this section as a warming-up exercise for section 4 where we study how the $L$-functions associated with $H^{1}(E)$ are related to string-theory realizations of $E$.

The following facts are well-known.
Theorem 3.1.2 (e.g., Shimura [25], Thm. 5.7). Let $[E]_{\mathbb{C}} \in \operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$ be a $\mathbb{C}$-isomorphism class of elliptic curves with complex multiplication by an order $\mathcal{O}_{f_{z}}$ of an imaginary quadratic field $K$. Then it has a model defined over the ring class field

$$
\begin{equation*}
L_{f_{z}}=K(j([E])) \tag{17}
\end{equation*}
$$

of $K$, where $j([E])$ is the $j$-invariant of $[E]$; this number field $L_{f_{z}}$ is determined uniquely from $K$ and $f_{z} \in \mathbb{N}_{>0}$, and is independent of choice of $[E]$ in $\operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$. An example of models over $L_{f_{z}}$ is a Weierstrass model given by (e.g., Silverman [28], Prop. III.1.4(c))

$$
\begin{equation*}
y^{2}+x y=x^{3}-\frac{36}{j([E])-1728} x-\frac{1}{j([E])-1728} \tag{18}
\end{equation*}
$$

although this is not the only possible model of $E$ defined over $L_{f_{z}}$.
Obviously, there exists a model of $E$ defined over $L$ for any extension field $L$ of $L_{f_{z}}$; we can use the base change of $E / L_{f_{z}}$ to obtain a model over $L$. •
Theorem 3.1.3. Let $[E]_{\mathbb{C}} \in \operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$ be a $\mathbb{C}$-isomorphism class of elliptic curves with complex multiplication by $\mathcal{O}_{f_{z}}$. Then it also has a model defined over a number field

$$
\begin{equation*}
F_{f_{z}}^{[E]}:=\mathbb{Q}(j([E])) ; \tag{19}
\end{equation*}
$$

[^7]the Weierstrass equation given by (18) is an example of such models. The field of definition $F_{f_{z}}^{[E]}$ is a degree- $h\left(\mathcal{O}_{f_{z}}\right)$ extension over $\mathbb{Q}$, and $L_{f_{z}}$ is a degree-2 extension of $F_{f_{z}}^{[E]}$. The field $F_{f_{z}}^{[E]}$ is not necessarily stable under conjugation in $\operatorname{Aut}(\overline{\mathbb{Q}} / \mathbb{Q})$; the subfield $F_{f_{z}}^{[E]} \subset \overline{\mathbb{Q}}$ depends on which one of $[E] \in \operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$ is used (although $L_{f_{z}} \subset \overline{\mathbb{Q}}$ does not). For any $[E] \in \operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$, however, there exists a unique element $\rho_{[E]} \in \operatorname{Gal}\left(L_{f_{z}} / \mathbb{Q}\right)$ that generates $\operatorname{Gal}\left(L_{f_{z}} / F_{f_{z}}^{[E]}\right) \cong \mathbb{Z} / 2 \mathbb{Z}$.

Notation 3.1.4. The extension field $L_{f_{z}}$ for $f_{z}=1$ of an imaginary quadratic field $K$ may also be denoted by $H_{K}$ or simply by $H$, and called the Hilbert class field of $K$. The subfield $F_{f_{z}}^{[E]} \subset H_{K}$ may also be denoted by $F_{K}^{[E]}, F_{K}, F^{[E]}$ or $F$, when $K$ and/or $[E]$ is fixed.

In this section, we will focus on the cases where $k / K$ is an abelian extension containing $L_{f_{z}} / K$ (including $k=L_{f_{z}}$ cases), and write down theh relation between characters of stringtheory realizations of $[E / k]_{\mathbb{C}}$ and $\zeta_{k}(s)$.

### 3.2 Preparation in String Theory

The embedding $\Omega^{\prime}: \Lambda_{\text {winding }} \otimes \mathbb{Q} \rightarrow \mathbb{C}$ was determined so that the absolute-value-square norm $|\quad|{ }_{\mathbb{C}}^{2}$ reproduces the intersection form of $\Gamma_{+} \cong \Lambda_{\text {winding }}$. It is more convenient to modify the embedding by rescaling it, so that the image of the lattices $\Gamma_{+} \cong \Lambda_{\text {winding }}$ and $\Gamma_{+}^{\vee} \cong \Lambda_{\text {Cardy }}$ fit within $K=\mathbb{Q}\left(\sqrt{-d_{0}}\right)=\mathbb{Q}(z) \subset \overline{\mathbb{Q}} \subset \mathbb{C}$. Let us take $\Omega:=\left[\left(C^{-1} \sqrt{2 a_{z} f_{\rho}} z_{2} i\right)\right] \cdot \Omega^{\prime}$, where $C \in \mathbb{Q}$ is some constant. Then

$$
\begin{align*}
\Omega\left(\Lambda_{\text {Cardy }}\right) & =C^{-1}(\mathbb{Z} \oplus z \mathbb{Z}) \subset \overline{\mathbb{Q}} \subset \mathbb{C}  \tag{20}\\
\mathfrak{b}_{z}:=\Omega\left(\Lambda_{\text {winding }}\right) & =C^{-1} f_{\rho}\left(\left(2 a_{z} z+b_{z}\right) \mathbb{Z} \oplus\left(b_{z} z+2 c_{z}\right) \mathbb{Z}\right) \subset \overline{\mathbb{Q}} \subset \mathbb{C} \tag{21}
\end{align*}
$$

As long as $C \in \mathbb{Q}, \mathfrak{b}_{z}$ and $\Omega\left(\Lambda_{\text {Cardy }}\right)$ are now regarded as rank-2 lattices within $K \subset \mathbb{C}$. We fix $C \in \mathbb{Q}$ so that $\mathcal{O}_{K}$ is contained in $\Omega\left(\Lambda_{\text {Cardy }}\right)$; we need to take

$$
\begin{equation*}
C \in \frac{f_{z}}{\operatorname{GCD}\left(a_{z}, \frac{f_{z}-b_{z}}{2}\right)} \mathbb{Z} \quad \text { if } D_{K} \text { is odd, } \quad C \in \frac{f_{z}}{\operatorname{GCD}\left(a_{z}, b_{z} / 2\right)} \mathbb{Z} \quad \text { if } D_{K} \text { is even; } \tag{22}
\end{equation*}
$$

in the rest of this article, we use the minimum positive value of $C$. One can also see that $C^{-1} \mathcal{O}_{f_{z}} \subset \Omega\left(\Lambda_{\text {Cardy }}\right)$.

Here are a couple of bilinear forms introduced on an imaginary quadratic field $K$.

$$
\begin{align*}
(-,-)_{K / \mathbb{Q}}: K \times K \ni(x, y) \mapsto & (x, y)_{K / \mathbb{Q}}:=\operatorname{Tr}_{K / \mathbb{Q}}(x y) \in \mathbb{Q},  \tag{23}\\
\langle-,-\rangle_{K / \mathbb{Q}}: K \times K \ni(x, y) \mapsto & \langle x, y\rangle_{K / \mathbb{Q}}:=(x \bar{y}+\bar{x} y) \in \mathbb{Q} \subset \mathbb{C},  \tag{24}\\
& \langle x, x\rangle_{K / \mathbb{Q}}=2|x|_{\mathbb{C}}^{2}  \tag{25}\\
(-,-)_{\Gamma_{+} \otimes \mathbb{Q}}: K \times K \ni(x, y) \mapsto & (x, y)_{\Gamma_{+} \otimes \mathbb{Q}}:=C^{2} \frac{a_{z}}{f_{\rho} D_{z}}\langle x, y\rangle_{K / \mathbb{Q}}  \tag{26}\\
& (x, x)_{\Gamma_{+} \otimes \mathbb{Q}}=C^{2} \frac{a_{z}}{f_{\rho} D_{z}}\langle x, x\rangle_{K / \mathbb{Q}} . \tag{27}
\end{align*}
$$

The ratio between the last two norms, $\langle-,-\rangle_{K / \mathbb{Q}}$ and $(-,-)_{\Gamma_{+} \otimes \mathbb{Q}}$, is due to the rescaling factor $C^{-1} \sqrt{2 a_{z} f_{\rho}} z_{2} i$ between $\Omega$ and $\Omega^{\prime} ;\left(2 a_{z} f_{\rho}\right) z_{2}^{2}=f_{\rho} D_{z} /\left(2 a_{z}\right)$. Using these relations, $\Omega\left(\Lambda_{\text {Cardy }}\right)$ can be characterized in terms of $\mathfrak{b}_{z}$ and vice versa as

$$
\begin{equation*}
\Omega\left(\Lambda_{\text {Cardy }}\right)=\frac{f_{\rho} D_{z}}{C^{2} a_{z}} \mathfrak{b}_{z}^{*}, \quad \mathfrak{b}_{z}=\frac{f_{\rho} D_{z}}{C^{2} a_{z}} \Omega\left(\Lambda_{\text {Cardy }}\right)^{*} . \tag{28}
\end{equation*}
$$

Here, $\mathfrak{a}^{*}$ for a full-rank $\mathbb{Z}$-lattice $\mathfrak{a} \subset K \subset \mathbb{C}$ is the dual lattice with respect to the intersection form $\langle-,-\rangle_{K / \mathbb{Q}}$. The notation $\mathfrak{a}^{\vee}$ is reserved for the dual lattice with respect to the intersection form $(-,-)_{K / \mathbb{Q}}$.

Modular invariance of CFT on worldsheet is an important principle in string theory. In a diagonal rational CFT, characters $\left\{\chi_{\alpha}\right\}_{\alpha \in i \text { Reps. }}$ can be regarded as a vector-valued modular form; by introducing a vector space $\mathbb{C}[i R e p s]=.\operatorname{Span}_{\mathbb{C}}\left\{e_{\alpha} \mid \alpha \in i R e p s.\right\}$ with a basis that consists of formal elements $e_{\alpha}$ 's in one-to-one correspondence with the set of irreducible representations iReps.,

$$
\begin{equation*}
\sum_{\alpha} e_{\alpha} \chi_{\alpha}(q) \in M_{w t=0}(\operatorname{SL}(2 ; \mathbb{Z}), \mathbb{C}[i \operatorname{Reps} .]) \tag{29}
\end{equation*}
$$

$\chi_{\alpha}$ 's can be regarded as characters of the left-mover chiral algebra (where the argument is $q=e^{2 \pi i \tau}$ ), those of the right movers, and those of open string states subject to $\varphi_{0}$-Cardy states (where the argument is $q=e^{-2 \pi t}$ ). The modular invariance of the closed string partition function is ensured by the cancellation between the left-mover $\mathbb{C}[i R e p s$.$] representation matrix$ of the worldsheet-SL $(2 ; \mathbb{Z})$ group and that of the right-mover.

The power series expansion of $\chi_{\alpha}(q)$ 's with respect to $q$, however, begins with a fractional power term, $q^{-\frac{c}{24}}$; the central charge is $c=2$ in $T^{2}$-target CFT's. An object mathematically nicer is $\sum_{\alpha} e_{\alpha}\left[\chi_{\alpha} \eta^{2}\right]$, when the Fourier expansion begins with the $1=q^{0}$ term. Now

$$
\begin{equation*}
\sum_{\alpha \in i \text { Reps. }} e_{\alpha}\left[\chi_{\alpha} \eta^{2}\right]=\sum_{\alpha \in i \text { Reps. }} f_{0}(\tau ; \alpha) \in M_{w t=1}(\mathrm{SL}(2 ; \mathbb{Z}), \mathbb{C}[i \text { Reps. }]) . \tag{30}
\end{equation*}
$$

It may look artificial to multiply $\eta^{2}$ for the purpose of getting the integer-power leading term in the Fourier expansion, but this nice object shows up naturally in superstring version of the diagonal rational $T^{2}$-target CFT's. In the closed string Ramond sector,

$$
\begin{equation*}
f_{0}\left(\tau_{w s} ; \alpha\right)=(-i) \operatorname{Tr}_{V_{\alpha}^{-} ; \mathrm{R}}\left[F_{L} e^{\pi i F_{L}} q^{L_{0}-c / 24}\right], \quad q=e^{2 \pi i \tau_{w s}} \tag{31}
\end{equation*}
$$

and in the open string Ramond sector,

$$
\begin{equation*}
f_{0}\left(i t_{w s} ; \alpha\right)=(-i) \operatorname{Tr}_{V_{\alpha}^{o} ; \mathrm{R}}\left[F e^{\pi i F} q^{L_{0}-c / 24}\right], \quad q=e^{-2 \pi t_{w s}} ; \tag{32}
\end{equation*}
$$

here, $\tau_{w s} \in \mathcal{H}$ is the complex structure parameter of worldsheet torus, and $t_{w s} \in \mathbb{R}_{>0}$ the parameter of the shape of a cylinder; $F_{L}$ and $F$ are the fermion number operators on the closed string left-moving sector and open string sector, respectively ${ }^{14}$ It is evident in the closed string language that $\sum_{\alpha} e_{\alpha} f_{0}\left(\tau_{w s} ; \alpha\right)$ is a vector-valued modular form of weight 1 ; on a worldsheet torus $\Sigma$ with the complex structure $\tau_{w s}$ with the odd spin structure, think of the partition function with the action modified from $S$ to $\widetilde{S}$ by a parameter $u \in \mathbb{C}$ :

$$
\begin{align*}
Z\left(\tau_{w s}, u\right) & :=\int_{\operatorname{Map}(\Sigma, E(z, \rho))} e^{i \widetilde{S}(u, \bar{u})}  \tag{33}\\
i \widetilde{S}(u, \bar{u}) & =i S+2 \pi i \int_{\Sigma} \frac{d^{2} \sigma}{\operatorname{Im}\left(\tau_{w s}\right)}\left(u J_{L}(\sigma)-\bar{u} J_{R}(\sigma)\right) . \tag{34}
\end{align*}
$$

This "partition function" transforms as [10, 2]

$$
\begin{equation*}
Z\left(\frac{a \tau_{w s}+b}{c \tau_{w s}+d}, \frac{u}{c \tau_{w s}+d}\right)=\mathbb{E}\left[\frac{(1 / 2) c u^{2}}{c \tau_{w s}+d}-\frac{(1 / 2) c \bar{u}^{2}}{c \bar{\tau}_{w s}+d}\right] Z\left(\tau_{w s}, u\right) \tag{35}
\end{equation*}
$$

where $\mathbb{E}[X]:=e^{2 \pi i X}$, under the worldsheet $\operatorname{SL}(2 ; \mathbb{Z})_{w s}$ transformation acting on $\tau_{w s}$ through $\tau_{w s}^{\prime}=\frac{a \tau_{w s}+b}{c \tau_{w s}+d}$. Because

$$
\begin{equation*}
\frac{-1}{(2 \pi)^{2}}\left[\frac{\partial}{\partial u} \frac{\partial}{\partial \bar{u}} Z\left(\tau_{w s}, u, \bar{u}\right)\right]_{u=\bar{u}=0}=\sum_{\alpha} f_{0}\left(\tau_{w s} ; \alpha\right) f_{0}\left(-\bar{\tau}_{w s} ; \alpha\right), \tag{36}
\end{equation*}
$$

we can use (35) to see,

$$
\begin{equation*}
\sum_{\alpha} f_{0}\left(\tau_{w s}^{\prime} ; \alpha\right) f_{0}\left(-\bar{\tau}_{w s}^{\prime} ; \alpha\right)=\left(c \tau_{w s}+d\right)\left(c \bar{\tau}_{w s}+d\right)\left[\sum_{\alpha} f_{0}\left(\tau_{w s} ; \alpha\right) f_{0}\left(-\bar{\tau}_{w s} ; \alpha\right)\right] \tag{37}
\end{equation*}
$$

[^8]a property satisfied by a vector-valued modular form of weight 1 .
As we are working on the $T^{2}$-target diagonal rational CFT's, we know a lot more about the functions $f_{0}(\alpha), \alpha \in i$ Reps.. The vector-valued modular form $\sum_{\alpha} e_{\alpha} f_{0}\left(\tau_{w s} ; \alpha\right)$ is given by congruent theta functions associated with the rank-2 even positive definite lattice $\Gamma_{+} \cong$ $\Lambda_{\text {winding }}$
\[

$$
\begin{equation*}
f_{0}\left(\tau_{w s} ; \alpha\right)=\sum_{\xi \in \alpha} e^{2 \pi i \tau_{w s} \frac{C^{2} a_{z}}{f_{p} D_{z}} \frac{\langle\xi, \xi\rangle_{K / Q}}{2}}, \quad \alpha \in \Omega\left(\Lambda_{\text {Cardy }}\right) / \mathfrak{b}_{z} \cong i R e p s . \cong \Gamma_{+}^{\vee} / \Gamma_{+}, \tag{38}
\end{equation*}
$$

\]

using the rescaled embedding $\Omega: \Lambda_{\text {winding }} \otimes \mathbb{Q} \rightarrow \mathfrak{b}_{z} \otimes \mathbb{Q} \subset K \subset \mathbb{C}$. They are none other than a vector-valued modular form in the Weil representation $\rho_{\Gamma_{+}}$of $\operatorname{SL}(2 ; \mathbb{Z})_{w s}$ associated with the lattice $\Gamma_{+} \cong \Lambda_{\text {winding }}$.

### 3.3 Theta Functions and Zeta Functions

The Mellin transform of the Riemann theta function yields the Riemann zeta function (e.g., [Koblitz [11], II §4], [Lang [12], XIII §1]). This classical result has been generalized for any number field $k$, and is found in textbooks (e.g., Lang [12], Chap. XIII). We wish to find a relation, however, between $\zeta_{k}(s)$ and such functions as $f_{0}\left(\tau_{w s} ; \alpha\right)$ obtained from stringtheory realizations of an arithmetic variety $E / k$. Here, we provide a brief review of known mathematical facts that serve for this purpose.

Let $k$ be an abelian extension of $K$ that contains the ring class field $L_{f_{z}} / K$ (where $f_{z} \in \mathbb{N}_{>0}$ ). The Dedekind zeta function $\zeta_{k}(s)$ can be reconstructed from functions that can be defined in terms of the imaginary quadratic field $K$, by (e.g., [Lang [12], Thm. XII.1], [Serre [24], Prop. VI.13])

$$
\begin{align*}
\zeta_{L_{f_{z}}}(s) & =\prod_{\chi \in \operatorname{Char}\left[\operatorname{Gal}\left(L_{f_{z}} / K\right)\right]}\left(\sum_{\mathfrak{K} \in \mathrm{Cl}_{K}\left(\left(f_{z}\right)\right)}\langle\chi, \mathfrak{K}\rangle \zeta_{K}(s, \mathfrak{K})\right),  \tag{39}\\
\zeta_{k}(s) & =\prod_{\chi \in \operatorname{Char}[\operatorname{Gal}(k / K)]}\left(\sum_{\mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right)}\langle\chi, \mathfrak{K}\rangle \zeta_{K}(s, \mathfrak{K})\right), \tag{40}
\end{align*}
$$

apart from the Euler factors for prime ideals $\mathfrak{P} \in \operatorname{Spec}\left(\mathcal{O}_{L_{f_{z}}}\right)\left[\right.$ resp. $\left.\in \operatorname{Spec}\left(\mathcal{O}_{k}\right)\right]$ in the fiber of the support of the modulus $\left(f_{z}\right)_{\mathcal{O}_{K}}\left[\right.$ resp. $\left.\mathfrak{m}_{f}\right]$; here, $\mathfrak{m}_{f}$ is an integral $\mathcal{O}_{K}$ idea 15 such that

[^9]the ray class field corresponding to $\mathfrak{m}_{f}$ contains $k$. Here, $\chi$ runs over the characters of the abelian group $\operatorname{Gal}\left(L_{f_{z}} / K\right)$ or $\operatorname{Gal}(k / K)$, which are quotient of the abelian group $\mathrm{Cl}_{K}\left(\left(f_{z}\right)_{\mathcal{O}_{K}}\right)$ or $\mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right)$. For a congruent ideal class $\mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right)$ for an integral $\mathcal{O}_{K}$ ideal $\mathfrak{m}_{f}$ (which includes the case of $\left.\mathfrak{m}_{f}=\left(f_{z}\right)_{\mathcal{O}_{K}}\right)$,
\[

$$
\begin{equation*}
\zeta_{K}(s, \mathfrak{K}):=\sum_{I \in \mathfrak{K}} \frac{1}{(\mathbb{N} I)^{s}} \tag{41}
\end{equation*}
$$

\]

By writing $I \in \mathfrak{K}$, we mean that the sum is over integral $\mathcal{O}_{K}$ ideals $I$ that belong to the class $\mathfrak{K} . \mathbb{N} I$ is the norm of the ideal $I$.

There is a particularly simple formula for $\zeta_{K}(s, \mathfrak{K})$ 's, when $f_{z}=1$ and $k=H_{K}$. Let us take one integral $\mathcal{O}_{K}$ ideal $\mathfrak{a}(\mathfrak{K})$ from each ideal class $-\mathfrak{K} \in \mathrm{Cl}_{K}$, and fix a set of these representatives $\left\{\mathfrak{a}(\mathfrak{K}) \mid \mathfrak{K} \in \mathrm{Cl}_{K}\right\}$ once and for all. Then 16 (e.g., [Lang [12], Chap. XIII])

$$
\begin{equation*}
\zeta_{K}(s, \mathfrak{K})=\frac{(\mathbb{N a}(\mathfrak{K}))^{s}}{\#\left[\mathcal{O}_{K}^{\times}\right]} \sum_{\xi \in \mathfrak{a}(\mathfrak{K})} \frac{1}{|\xi|_{\mathbb{C}}^{2 s}}, \tag{42}
\end{equation*}
$$

where one embedding $\sigma: K \hookrightarrow \mathbb{C}$ is fixed implicitly (as we have done already in section (3.2); an idea here is that integral ideals in the class $\mathfrak{K}$ can be listed up in the form of $(\xi)_{\mathcal{O}_{K}} \mathfrak{a}(\mathfrak{K})^{-1}$.

The functions $\zeta_{K}(s, \mathfrak{K})$ for $\mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right)$ are obtained as Mellin transforms of congruent theta functions associated with the number field $K$. To see this, let us start off by introducing congruent theta functions associated with a number field $L$. The ring of algebraic integers $\mathcal{O}_{L}$ in $L$ can be regarded as a lattice by using

$$
\begin{equation*}
\langle-,-\rangle_{L / \mathbb{Q}}: \mathcal{O}_{L} \times \mathcal{O}_{L} \ni(x, y) \mapsto \sum_{a=1}^{r_{1}} \rho_{a}(x y)+\sum_{b=1}^{r_{2}}\left(\sigma_{b}(x) \bar{\sigma}_{b}(y)+\bar{\sigma}_{b}(x) \sigma_{b}(y)\right) \tag{43}
\end{equation*}
$$

as the intersection form; here $\rho_{a}: L \hookrightarrow \mathbb{R}$ with $a=1, \cdots, r_{1}$ are real embeddings of $L$, and $\sigma_{b}$ and $\bar{\sigma}_{b}$ with $b=1, \cdots, r_{2}$ are imaginary embeddings $L \hookrightarrow \mathbb{C}$ forming $r_{2}$ pairs under complex conjugation in $\mathbb{C} ; r_{1}+2 r_{2}=[L: \mathbb{Q}]$. This lattice $\left(\mathcal{O}_{L},\langle-,-\rangle_{L / \mathbb{Q}}\right)$ is integral, and in particular, it is even when $L$ is a totally imaginary field. For a sublattice $\Lambda$ of this lattice $\left(\mathcal{O}_{L},\langle-,-\rangle_{L / \mathbb{Q}}\right)$ and $x \in \mathbb{R}^{[L: \mathbb{Q}]} / \Lambda$, we set

$$
\begin{equation*}
\vartheta_{L}(\Lambda, x)=\vartheta_{L}(\tau ; \Lambda, x):=\sum_{w \in x} e^{2 \pi i \tau \frac{\langle w, w\rangle_{L / Q}}{2}}=\sum_{w \in x} q^{\frac{\langle w, w\rangle_{L / Q}}{2}} . \tag{44}
\end{equation*}
$$

[^10]The definition of the functions $\vartheta_{L}(\Lambda, x)$ is possible, in fact, without referring to a number field $L$ or the lattice $\left(\mathcal{O}_{L},\langle-,-\rangle_{L / \mathbb{Q}}\right)$; we just need an integral lattice $\Lambda$ as a part of data for the definition. So, one may drop the subscript ${ }_{L}$. We will be interested in those theta functions for an imaginary quadratic field $L=K$ along with $x$ placed at torsion points of $L \otimes_{\mathbb{Q}} \mathbb{R} / \Lambda$, often within $\Lambda^{*} / \Lambda \subset L \otimes_{\mathbb{Q}} \mathbb{R} / \Lambda$.

Now, let us get started for the case of $k=H_{K} ; f_{z}=1$ and $\mathfrak{m}_{f}=\mathcal{O}_{K}$. The Mellin transformation of the congruent theta functions associated with $K$ (Lang [12], Chap. XIII),

$$
\begin{equation*}
\frac{1}{\#\left[\mathcal{O}_{K}^{\times}\right]} \int_{0}^{+\infty}\left[\vartheta_{K}(i t ; \mathfrak{a}(\mathfrak{K}), 0)-1\right] t^{s} \frac{d t}{t}=\frac{\Gamma(s)}{(2 \pi)^{s}} \frac{1}{\#\left[\mathcal{O}_{K}^{\times}\right]} \sum_{\xi \in \mathfrak{a}(\mathfrak{K})} \frac{1}{|\xi|_{\mathbb{C}}^{2 s}}=\frac{\Gamma(s)}{(2 \pi)^{s}} \frac{\zeta_{K}(s, \mathfrak{K})}{\mathbb{N a}(\mathfrak{K})^{s}}, \tag{45}
\end{equation*}
$$

provide all the components $\zeta_{K}(s, \mathfrak{K})$ (for $\mathfrak{K} \in \mathrm{Cl}_{K}$ ) necessary in reconstructing $\zeta_{H_{K}}(s)$ through (42, 39).

Here, we record a property of the functions $\vartheta_{K}(\Lambda, 0)$ as we use it in section 3.4.1:
Proposition 3.3.1 (Iwaniec [8, Thm. 10.8; Miyake [15], Cor. 4.9.5(2)). Let $L$ be a totally imaginary field. Let $\Lambda$ be a rank- $2 r_{0} \mathbb{Z}$-lattice in $\left(\mathcal{O}_{L},\langle-,-\rangle_{L / \mathbb{Q}}\right)$, and $N_{\Lambda} \in \mathbb{N}_{>0}$ be the leve ${ }^{17}$ of $\Lambda$. Then $\vartheta_{L}(\Lambda, 0)$ is a modular form of weight $r_{0}$ for $\Gamma_{0}\left(2 N_{\Lambda}\right)$ with a multiplier system (homomorphism) $\underline{\chi}: \Gamma_{0}\left(2 N_{\Lambda}\right) \rightarrow\{ \pm 1\}$. The subgroup $\Gamma_{0}\left(2 N_{\Lambda}\right) \subset \operatorname{SL}(2 ; \mathbb{Z})$ acts on the argument $\tau$ as used in the definition (444) of theta functions through $\operatorname{SL}(2 ; \mathbb{Z})$.

This means that $\zeta_{K}(s, \mathfrak{K})$ can be regarded as the Mellin transform of a modular form of weight 1 for $\Gamma_{0}(M)$ with some appropriately chosen integer $M$ (apart from subtraction of 1).

Let us now move on to more general cases, where $k=L_{f_{z}}$ with $f_{z}>1$, or $\left[k: L_{f_{z}}\right]>1$. In order to reconstruct $\zeta_{k}(s)$ from the information of how many ideals $\mathcal{O}_{K}$ has, the idea behind (42) needs to be generalized. First, we still choose a set of representatives $\{\mathfrak{a}(\mathfrak{K}) \mid \mathfrak{K} \in$ $\left.\mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right)\right\}$ satisfying $\mathfrak{a}(\mathfrak{K}) \in-\mathfrak{K}$; then $\mathfrak{a}(\mathfrak{K})$ is prime to $\mathfrak{m}_{f}$ by definition. Let us assume that all the representatives are integral $\mathcal{O}_{K}$ ideals (not just fractional ideals) although this assumption is only for the sake of simplifying the presentation here. Second, for integral $\mathcal{O}_{K}$ ideals $\mathfrak{a}$ and $\mathfrak{m}_{f}$ that are relatively prime, let us also introduce the following notations:

$$
\begin{align*}
\mathfrak{a}_{1}\left(\mathfrak{m}_{f}\right) & :=\left\{\left.\xi \in \mathfrak{a}\right|^{\exists} \epsilon \in \mathcal{O}_{K}^{\times} \text {s.t. } \epsilon \xi \equiv 1 \bmod \mathfrak{m}_{f}\right\},  \tag{46}\\
\mathfrak{a}_{G_{0}}\left(\mathfrak{m}_{f}\right) & :=\left\{\left.\xi \in \mathfrak{a}\right|^{\exists} \epsilon \in \mathcal{O}_{K}^{\times},{ }^{\exists} a \in G_{0} \text { s.t. } \epsilon \xi \equiv a \bmod \mathfrak{m}_{f}\right\} \tag{47}
\end{align*}
$$

[^11]where $G_{0}$ is a subgroup of $\left[\mathcal{O}_{K} / \mathfrak{m}_{f}\right]^{\times}$in the multiplication law. We will also use a notation $\mathfrak{a}_{\mathbb{Z}}\left(\mathfrak{m}_{f}\right)$ for $\mathfrak{m}_{f}=\left(f_{z}\right)_{\mathcal{O}_{K}}$ for some $f_{z} \in \mathbb{N}_{>0}$ in this article, which is meant to be $\mathfrak{a}_{G_{0}}\left(\left(f_{z}\right)_{\mathcal{O}_{K}}\right)$ introduced above, with $G_{0} \subset\left[\mathcal{O}_{K} /\left(f_{z}\right)_{\mathcal{O}_{K}}\right]^{\times}$chosen to be the image of all integers in $\mathbb{Z}$ prime to $f_{z}$. Now, with these preparations, a general version of the idea behind (42) is stated as follows: the list of integral ideals $I$ that are prime to $\mathfrak{m}_{f}$ and belong to $\mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right)$ are in one-to-one correspondenc ${ }^{18}$ with $[\xi] \in[\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right) / \mathcal{O}_{K}^{\times}$; the correspondence is given by $I \cdot \mathfrak{a}(\mathfrak{K})=(\xi)_{\mathcal{O}_{K}}$ in the ideal group of $K$ (e.g., [Lang [12], Chap. XIII]). So, the expression (42) for $\zeta_{K}(s, \mathfrak{K})$ with $\mathfrak{K} \in \mathrm{Cl}_{K}$ is replaced by
\[

$$
\begin{equation*}
\zeta_{K}(s, \mathfrak{K})=\frac{(\mathbb{N a}(\mathfrak{K}))^{s}}{\#\left[\mathcal{O}_{K}^{\times}\right]} \sum_{\xi \in[\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right)} \frac{1}{|\xi|_{\mathbb{C}}^{2 s}}, \quad \mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right) \tag{48}
\end{equation*}
$$

\]

These zeta functions $\zeta_{K}(s, \mathfrak{K})$ for $\mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right)$ are obtained as the Mellin transform as in (45), but now in the form of

$$
\begin{equation*}
\frac{1}{\#\left[\mathcal{O}_{K}^{\times}\right]} \int_{0}^{\infty} \frac{d t}{t} t^{s} \sum_{y \in\left[\{\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right)\right]_{\mathfrak{a}(\mathfrak{K})} \mathfrak{m}_{f}} \vartheta_{K}\left(i t ; \mathfrak{a}(\mathfrak{K}) \mathfrak{m}_{f}, y\right)=\frac{\Gamma(s)}{(2 \pi)^{s}} \frac{\zeta_{K}(s, \mathfrak{K})}{(\mathbb{N a}(\mathfrak{K}))^{s}} \tag{49}
\end{equation*}
$$

Here, $\left[[\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right)\right]_{\mathfrak{a}(\mathfrak{K}) \mathfrak{m}_{f}}$ stands for the image $\underbrace{19}$ of $[\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right)$ in the quotient map $\mathfrak{a}(\mathfrak{K}) \rightarrow$ $\mathfrak{a}(\mathfrak{K}) / \mathfrak{a}(\mathfrak{K}) \mathfrak{m}_{f}$.

Here, we record a property of the functions $\vartheta_{K}(\Lambda, y)$ for a torsion point $y \in \mathbb{R}^{[L: \mathbb{Q}]} / \Lambda$, as we use it in section 3.4.2.

Proposition 3.3.2 (Iwaniec [8], Cor. 10.7; Miyake [15], Cor. 4.9.4). Let $\Lambda$ be a rank- $2 r_{0}$ $\mathbb{Z}$-lattice in $\left(\mathcal{O}_{L},\langle-,-\rangle_{L / \mathbb{Q}}\right)$ of a totally imaginary field $L / \mathbb{Q}$ with $[L: \mathbb{Q}]=2 r_{0}$, and $N_{\Lambda}$ its level. Let $\underline{N} \in \mathbb{N}_{>0}$ be divisible by $N_{\Lambda}$. Then, for any $x \in \underline{N}^{-1} \Lambda, \vartheta_{L}(\Lambda, x)$ is a modular form of weight $r_{0}$ for $\Gamma(4 \underline{N})$ without a non-trivial multiplier system. The group $\Gamma(4 \underline{N}) \subset \mathrm{SL}(2 ; \mathbb{Z})$ acts on the argument $\tau$ as we used in (44) through $\operatorname{SL}(2 ; \mathbb{Z})$.

When $x \in \Lambda^{*} / \Lambda$, in particular, we can choose $\underline{N}=N_{\Lambda}$.
The same statement holds true, when $\Gamma(4 \underline{N})$ is replaced by $\Gamma_{1}(4 \underline{N})$ and a non-trivial multiplier system (homomorphism) $\underline{\chi}: \Gamma_{1}(\underline{N}) \rightarrow S^{1}$ is allowed. When $x \in \mathcal{O}_{L} / \Lambda$, we can take $\underline{N}=N_{\Lambda}$ (as above), and moreover, $(x, x) / 2 \in \mathbb{Z}$, from which one can conclude that the multiplier system $\underline{\chi}$ is trivial (cf [8, 15]).

[^12]This is not to say, however, that $\zeta_{K}(s, \mathfrak{K})$ for $\mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right)$ with a non-trivial modulus $\mathfrak{m}_{f}$ cannot be the Mellin transform of a modular form for $\Gamma_{0}\left(M_{*}\right)$ for any choice of $M_{*} \in \mathbb{N}_{>0}$. We do not pursue this question in detail in this article, but at least in Example 3.4.3, we see that the set $[\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right)$ can be decomposed into a sum of $\mathbb{Z}$-sublattices of $\mathcal{O}_{K}$ (not necessarily $\mathcal{O}_{K}$-ideals), so the Proposition 3.3.1 can be used, instead of Prop. 3.3.2.

### 3.4 Combining Them Together

Having done preparations in sections 3.2 and 3.3, let us now combine them together to find out how the Dedekind zeta function $\zeta_{k}(s)$ of the field of definition of an elliptic curve $E / k$ is reconstructed from functions obtained in the string-theory realizations of $E / k$. In section 3.4.1, we begin with the cases of elliptic curves with complex multiplication by the maximal order $\mathcal{O}_{K}$ of a quadratic imaginary field $K$, and the field of definition is $k=L_{f_{z}=1}=H_{K}$. More general cases are treated in section 3.4.2.

### 3.4.1 Cases with $k=H_{K}$

Note first that the lattices $\Lambda_{\text {winding }}$ and $\Lambda_{\text {Cardy }}$ have been embedded into $K$, which is also identified (by a fixed imaginary embedding $\sigma: K \hookrightarrow \mathbb{C}$ ) with a subset of $\mathbb{C}$. The character functions $f_{0}\left(\tau_{w s} ; \alpha\right)$ in a string-theory realization of $E / k$ is closely related to the congruent theta functions in (44). To be concrete, choose one elliptic curve $E / H_{K}$ which has complex multiplication by $\mathcal{O}_{K}$. We focus on its string realizations whose $f_{\rho}$ satisfies

$$
\begin{equation*}
\operatorname{LCM}\left(\mathfrak{a}(\mathfrak{K})_{\mathfrak{K} \in \mathrm{Cl}_{K}}\right) \mid \mathfrak{b}_{z} \tag{50}
\end{equation*}
$$

Then [Moore [17, §4]

$$
\begin{equation*}
\sum_{\pi(\alpha)=0} f_{0}\left(\tau_{w s} ; \alpha\right)=\vartheta_{K}(\tau ; \mathfrak{a}(\mathfrak{K}), 0), \quad \tau=\frac{C^{2} a_{z}}{f_{\rho} D_{z}} \tau_{w s} \tag{51}
\end{equation*}
$$

where $\pi: \Lambda_{\text {Cardy }} / \Lambda_{\text {winding }} \rightarrow \Omega\left(\Lambda_{\text {Cardy }}\right) / \mathfrak{a}(\mathfrak{K})$ is the quotient map, which is well-defined when the condition (50) is satisfied. Consequently, we see that $\zeta_{H_{K}}(s)$ is obtained by summing up the character functions $f_{0}\left(\tau_{w s} ; \alpha\right)$ of irreducible representations $\alpha$ of the chiarl algebra $\mathcal{A}_{-} \cong \mathcal{A}_{L}$ in those string realizations in a way specified by (51) first; the sum of $f_{0}(\alpha)$ 's is now a single component modular form of $\Gamma_{0}\left(2 N_{\mathfrak{a}(\mathfrak{K})}\right)$ (see Prop. 3.3.1), although $f_{0}(\alpha)$ 's as a whole forms a vector-valued modular form of the entire $\mathrm{SL}(2 ; \mathbb{Z})_{w s}$. Remaining steps are to take a Mellin transformation of the sum with respect to the imaginary part of the complex

| $-D_{K}$ | $z=w_{K}$ | $[a, b, c]$ | $C$ | $\Omega\left(\Lambda_{\text {Cardy }}\right)$ | $\mathfrak{b}_{z} / f_{\rho}$ | iReps.(if $\left.f_{\rho}=1\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | $\frac{-1+\sqrt{3} i}{2}$ | $[1,1,1]$ | 1 | $\mathbb{Z}+w_{K} \mathbb{Z}$ | $\left(2 w_{K}+1\right) \mathbb{Z}+\left(2+w_{K}\right) \mathbb{Z}$ | $\mathbb{Z} / 3 \mathbb{Z}$ |
| 4 | $i$ | $[1,0,1]$ | 1 | $\mathbb{Z}+w_{K} \mathbb{Z}$ | $2 \mathbb{Z}+2 w_{K} \mathbb{Z}$ | $\mathbb{Z} /(2 \mathbb{Z}) \times \mathbb{Z} /(2 \mathbb{Z})$ |
| 8 | $\sqrt{2} i$ | $[1,0,2]$ | 1 | $\mathbb{Z}+w_{K} \mathbb{Z}$ | $4 \mathbb{Z}+2 w_{K} \mathbb{Z}$ | $\mathbb{Z} /(4 \mathbb{Z}) \times \mathbb{Z} /(2 \mathbb{Z})$ |

Table 2: Embedding of the lattices $\Lambda_{\text {Cardy }}$ and $\Lambda_{\text {winding }}$ into $K \subset \mathbb{C}$, for the unique (trivial) element of $\operatorname{Ell}\left(\mathcal{O}_{K}\right)$ with $K=\mathbb{Q}\left(\sqrt{-d_{0}}\right), d_{0}=1,2,3$. For elliptic curves $\left[\mathbb{C} / \mathfrak{b}_{z}\right]_{\mathbb{C}}$ corresponding to the trivial element of $\operatorname{Ell}\left(\mathcal{O}_{K}\right)$, we can use the value of $z$ for one $\left(w_{K}\right)$ of the two generators of $\mathcal{O}_{K}$ over $\mathbb{Z}, \mathcal{O}_{K}=\mathbb{Z}+w_{K} \mathbb{Z}$. In all the three examples here, the relation $\Omega\left(\Lambda_{\text {Cardy }}\right)=\mathcal{O}_{K}$ holds, although this is not always the case (see Table 3).
structure parameter $\tau_{w s}$ of a worldsheet torus, and then to sum them and multiply them as in (39).

Example 3.4.1. For any one of the imaginary quadratic fields $K=\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-3})$, there is just one $\mathbb{C}$-isomorphism class of elliptic curves with complex multiplication by $\mathcal{O}_{K}\left(f_{z}=1\right)$. Put differently, $\mathrm{Cl}_{K}=\{[0]\}$. We choose an ideal $\mathfrak{a}([0])$ to be $\mathcal{O}_{K}$. In all these examples, one can confirm explicitly ${ }^{20}$ that $\mathcal{O}_{K} \subset \mathfrak{b}_{z} / f_{\rho}$. So, any string realization with $f_{\rho} \in \mathbb{N}_{>0}$ satisfies the condition (50). More detailed data are found in Table 2.

When we choose the minimal $f_{\rho}=1$ string realization,

$$
\begin{equation*}
\vartheta_{K}\left(\tau ; \mathcal{O}_{K}, 0\right)=\sum_{\alpha \in \text { iReps. }} f_{0}\left(\tau_{w s} ; \alpha\right), \quad \tau=\tau_{w s} /\left|D_{K}\right| \tag{52}
\end{equation*}
$$

for all these three examples, because $C=1, a_{z}=1$ and $\Omega\left(\Lambda_{\text {Cardy }}\right)=\mathcal{O}_{K} . \bullet$
Example 3.4.2. For the case $K=\mathbb{Q}(\sqrt{-5})$, there are two inequivalent $\mathbb{C}$-isomorphism classes of elliptic curves with complex multiplication by $\mathcal{O}_{K}$. The lattices $\Omega\left(\Lambda_{\text {Cardy }}\right)$ and $\mathfrak{b}_{z}$ in our embedding $\Omega$ are shown in Table 3.

For the two ideal classes $\mathrm{Cl}_{K} \cong \mathbb{Z} /(2 \mathbb{Z})=:\{[0],[1]\}$ in this case, we can choose $\mathfrak{a}([0])=$ $\mathcal{O}_{K}$, and $\mathfrak{a}([1])=(2 \mathbb{Z}+(1+\sqrt{5} i) \mathbb{Z})$, which is not a principal ideal. For this choice of representatives, $\{\mathfrak{a}([0]), \mathfrak{a}([1])\}$, the condition (50) is satisfied for any string realizations with $f_{\rho} \in \mathbb{N}_{>0}$ (see Figure $\mathbb{1}(\mathrm{a})$ ). So, let us use the string-theory realization with the minimum choice $f_{\rho}=1$. Note that we still think of two string realizations, one for each $[E] \in \operatorname{Ell}\left(\mathcal{O}_{K}\right)$.

[^13]| $z$ | $[a, b, c]$ | $C$ | $\Omega\left(\Lambda_{\text {Cardy }}\right)$ | $\mathfrak{b}_{z}$ | $\# 1$ | $\# 2$ | $\# 3$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\sqrt{5} i=w_{K}$ | $[1,0,5]$ | 1 | $\mathbb{Z}+w_{K} \mathbb{Z}$ | $10 \mathbb{Z}+2 w_{K} \mathbb{Z}$ | 1 | 20 | 10 |
| $\frac{-1+\sqrt{5} i}{2}=\frac{-1+w_{K}}{2}$ | $[2,2,3]$ | 1 | $\mathbb{Z}+\frac{1+w_{K}}{2} \mathbb{Z}$ | $2 w_{K} \mathbb{Z}+\left(5+w_{K}\right) \mathbb{Z}$ | 2 | 10 | 5 |

Table 3: Two $\mathbb{C}$-isomorphism classes of elliptic curves with complex multiplication by the maximal order $\mathcal{O}_{K}, K=\mathbb{Q}(\sqrt{-5})$. Embedded lattices $\Lambda_{\text {Cardy }}$ and $\Lambda_{\text {winding }}$ in $K \subset \mathbb{C}$ are shown here, relatively to $\mathcal{O}_{K}=\mathbb{Z}+w_{K} \mathbb{Z}$, where $w_{K}=\sqrt{5} i ; f_{\rho}=1$ is used for $\mathfrak{b}_{z}$. The last three columns are $\# 1=\left[\Omega\left(\Lambda_{\text {Cardy }}\right): \mathcal{O}_{K}\right], \# 2=\left[\mathcal{O}_{K}: \mathfrak{b}_{z}\right]=\left[\mathfrak{a}([0]): \mathfrak{b}_{z}\right]$ and $\# 3=\left[\mathfrak{a}([1]): \mathfrak{b}_{z}\right]$.

In order to obtain $\vartheta(\mathfrak{a}([0]), 0)$ [resp. $\vartheta(\mathfrak{a}([1]), 0)], f_{0}\left(\tau_{w s} ; \alpha\right)$ 's need to be summed up over $\# 2$ [resp. \#3] irreducible representations $\alpha \in i$ Reps. in those string-theory realizations; see Table 3. Although $\vartheta(\mathfrak{a}(\mathfrak{K}), 0)$ 's are expressed differently in terms of $f_{0}\left(\tau_{w s} ; \alpha\right)$ 's for the two string-theory realizations, the functions $f_{0}\left(\tau_{w s} ; \alpha\right)$ 's for the two realizations are also different. In the end, $\vartheta(\mathfrak{a}(\mathfrak{K}), 0)$ 's should not depend on which string-theory realization (which one of $\left.[E] \in \operatorname{Ell}\left(\mathcal{O}_{K}\right)\right)$ is used to reconstruct them. After all, one and the same zeta function $\zeta_{k}(s)$ is being re-constructed for the common Hilbert class field $k=K(j([E]))=\mathbb{Q}(\sqrt{-5}, \sqrt{-1})$.

We have seen that the Dedekind zeta function $\zeta_{k=H_{K}}(s)$ can be written down by using the combinations (51) of a string-theory realization of an arithmetic model $E / H_{K}$ (when $\left.[E] \in \operatorname{Ell}\left(\mathcal{O}_{K}\right)\right)$. Prop. 3.3.1 says that those combinations of $f_{0}$ 's are weight-1 modular forms of $\Gamma_{0}\left(2 N_{\mathfrak{a}(\mathfrak{K})}\right)$, where $N_{\mathfrak{a}(\mathfrak{K})}$ is the leve ${ }^{21}$ of the even lattice $\mathfrak{a}(\mathfrak{K})$, which is a sublattice of the even lattice $\left(\mathcal{O}_{K},\langle-,-\rangle_{K / \mathbb{Q}}\right)$. When we see those combinations as functions of $\tau_{w s}$, as natural in string-theory perspective, however, only the common subset of $\Gamma_{0}\left(2 N_{\mathfrak{a}(\mathfrak{\kappa})}\right)$ and the worldsheet $\mathrm{SL}(2 ; \mathbb{Z})_{w s}$ transformation,

$$
\operatorname{SL}(2 ; \mathbb{Z})_{\mathrm{ws}} \cap\left(\begin{array}{ll}
\left(f_{\rho} D_{z} / C^{2} a_{z}\right) &  \tag{53}\\
& 1
\end{array}\right) \Gamma_{0}\left(2 N_{\mathfrak{a}(\mathfrak{K})}\right)\left(\begin{array}{ll}
\left(C^{2} a_{z} / f_{\rho} D_{z}\right) & \\
& 1
\end{array}\right)
$$

is evident. So, in particular, there exists a congruence subgroup $\Gamma(M) \subset \mathrm{SL}(2 ; \mathbb{Z})_{\text {ws }}$ that is contained in all those groups, under which all of the combinations (51) for ${ }^{\forall} \mathfrak{K} \in \mathrm{Cl}_{K}$ of a string-theory realization are modular forms of weight 1 , and $\zeta_{H_{K}}(s)=L\left(H_{e t}^{0}(E), s\right)$ can be reproduced from them through the Mellin transformation, summation and multiplication.

The common subgroup is not of the form of $\Gamma_{0}(M)$ or $\Gamma_{1}(M)$ in $\operatorname{SL}(2 ; \mathbb{Z})_{w s}$, unless $C^{2} a_{z} /\left(f_{\rho} D_{z}\right) \in \mathbb{Z}$. It is isomorphic to the group $\Gamma_{0}\left(N_{*} \operatorname{LCM}\left(D_{*}, 2 N_{\mathfrak{a}(\mathfrak{R})}\right)\right)$ acting on $\tau / N_{*}$, where $N_{*}$ and $D_{*}$ are relatively prime integers satisfying $C^{2} a_{z} /\left(f_{\rho} D_{z}\right)=N_{*} / D_{*}$. It is in-

[^14]

Figure 1: (a) is an illustration for Example 3.4.2 and (b) for 3.4.3, In (a), $\mathcal{O}_{K}=\mathfrak{a}([0])$ and $\mathfrak{a}([1])$ in $K=\mathbb{Q}(\sqrt{-5}) \subset \mathbb{C}$ are shown by large black dots and large open circles, respectively. $\Omega\left(\Lambda_{\text {Cardy }}\right)=\mathcal{O}_{K}$ for $z=\sqrt{5} i$, whereas $\Omega\left(\Lambda_{\text {Cardy }}\right)$ for $z=(-1+\sqrt{5} i) / 2$ consists both of $\mathcal{O}_{K}$ and the tiny open circles in (a). The unit cell $\Omega\left(\Lambda_{\text {winding }}\right) / f_{\rho}$ is also indicated for the two inequivalent $\left[E_{z}\right]_{\mathbb{C}}$ 's in $\operatorname{Ell}\left(\mathcal{O}_{K}\right)$ in (a). In (b), where $K=\mathbb{Q}(\sqrt{-1})$ and $f_{z}=3$, $\mathcal{O}_{K}$ ideals $\left(f_{z}\right)_{\mathcal{O}_{K}}$ and $\mathfrak{a}([1])$ for $[1] \in \mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}=3}\right)$ are indicated by x and $\circ$, respectively. The set $\left[\mathcal{O}_{K}\right]_{\mathbb{Z}}\left(\left(f_{z}\right)\right)$ consists of dots. Also shown are the unit cell $\Omega\left(\Lambda_{\text {winding }}\right) / f_{\rho}$ for the two inequivalent $\left[E_{z}\right]_{\mathbb{C}}$ 's in $\operatorname{Ell}\left(\mathcal{O}_{f_{z}=3}\right)$. By enlarging those unit cells by $f_{\rho}=3$, all the points of $\Omega\left(\Lambda_{\text {winding }}\right)$ are to be contained in $f_{z} \mathfrak{a}([1])=(3+3 i)_{\mathcal{O}_{K}}$, the o-and-x points in (b).
teresting to note, from string-theory perspective, that there are modular transformations of $\Gamma_{0}\left(2 N_{\mathfrak{a}(\mathfrak{K})}\right)$ that are not captured within the worldsheet $\mathrm{SL}(2 ; \mathbb{Z})_{w s}$ transformation.

### 3.4.2 More General Cases: $k / K$ Is a Ramified Abelian Extension Containing $L_{f_{z}} / K$

Now think of an elliptic curve $[E]_{\mathbb{C}} \in \operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$ and its model $E / k$ over a number field $k$ that is an abelian extension of $K$ containing $L_{f_{z}} / K$. The field of definition $k$ is not necessarily the Hilbert class field $H_{K}$. We claim that $\zeta_{k}(s)=L\left(H_{e t}^{0}(E), s\right)$ is still obtained through a process as in section 3.4 .1 from the character functions $f_{0}$ 's of string-theory realizations of $[E]_{\mathrm{C}}$.

To see this, let us focus on string-theory realizations whose parameter $f_{\rho}$ on the complexified Kähler parameter satisfies

$$
\begin{equation*}
\operatorname{LCM}\left(\left(\mathfrak{m}_{f} \mathfrak{a}(\mathfrak{K})\right)_{\mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right)}\right) \supset \mathfrak{b}_{z} . \tag{54}
\end{equation*}
$$

Then the projection $\pi: \Omega\left(\Lambda_{\text {Cardy }}\right) / \mathfrak{b}_{z} \rightarrow \Omega\left(\Lambda_{\text {Cardy }}\right) / \mathfrak{a}(\mathfrak{K}) \mathfrak{m}_{f}$ is well-defined, and one finds that
the relation

$$
\begin{equation*}
\sum_{\pi(\alpha)=y} f_{0}\left(\tau_{w s} ; \alpha\right)=\vartheta_{K}\left(\tau ; \mathfrak{a}(\mathfrak{K}) \mathfrak{m}_{f}, y\right), \quad \tau_{w s} \frac{C^{2} a_{z}}{f_{\rho} D_{z}}=\tau \tag{55}
\end{equation*}
$$

holds. Now (49) and (39, 40) can be used to obtain $\zeta_{k}(s)$ from the data $f_{0}$ 's of the spectrum of a string-theory realization.

The combinations of character functions of a string realization appearing in the left-handside are weight-1 modular forms of $\Gamma\left(4 N_{\mathfrak{a}(\mathfrak{K})}\right)$ acting on $\tau$ (see Prop. 3.3.2); note that we use $y \in[\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right) \subset \mathfrak{a}(\mathfrak{K}) \subset \mathcal{O}_{K}$ in (49, 55), which means that $y \in \mathcal{O}_{K} \subset f_{z}^{-1} \mathcal{O}_{K}^{\vee} \subset\left(\mathfrak{m}_{f}\right)^{*} \subset$ $\left(\mathfrak{m}_{f} \mathfrak{a}(\mathfrak{K})\right)^{*}$. The common subset of this modular transformation and $\operatorname{SL}(2 ; \mathbb{Z})_{w s}$ acting on $\tau_{w s}$ can be worked out as in (53). It may also be possible to find an appropriate combination of $f_{0}(\alpha)$ 's so that it is a modular form for a group of the form $\Gamma_{0}(M)$ for some $M \in \mathbb{N}_{>0}$ instead of a group $\Gamma\left(4 N_{\mathfrak{a}(\mathfrak{R})}\right)$; we do this only in the example below, however.

Example 3.4.3. There are two $\mathbb{C}$-isomorphism classes of elliptic curves with complex multiplication by the order $\mathcal{O}_{f_{z}=3}$ in $K=\mathbb{Q}(\sqrt{-1}) ; \operatorname{Cl}_{K}\left(\mathcal{O}_{f_{z}=3}\right) \cong \mathbb{Z} /(2 \mathbb{Z})=:\{[0],[1]\}$. One is for $z=3 i$ and the other for $z=(-1+3 i) / 2$. We choose $\mathfrak{a}([0])=\mathcal{O}_{K}$ and $\mathfrak{a}([1])=(1+i)_{\mathcal{O}_{K}}$ for the two ideal classe 22 in $\mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}=3}\right)$. Using the data shown in Table 4, one can see that string-theory realizations with $3 \mid f_{\rho}$ satisfy the condition (54) for $\left(f_{z}\right)_{\mathcal{O}_{K}} \mathfrak{a}([1])=(3+3 i)_{\mathcal{O}_{K}}$.

One can work out, in such a string realization, the number of irreducible representations $\alpha$ of the chiral algebra contributing to $\zeta_{K}(s, \mathfrak{K})$, as follows. First, $\left[[\mathfrak{a}([0])]_{1}\left(\left(f_{z}\right)_{\mathcal{O}_{K}}\right)\right]_{f_{z} \mathfrak{a}([0])}$ consists of 4 elements represented by $y=1,2, i, 2 i \bmod f_{z} \mathfrak{a}([0])=(3)_{\mathcal{O}_{K}}$; for each one of $y$ 's, there are $\#\left[f_{z} \mathfrak{a}([0]) / \mathfrak{b}_{z}\right]$ irreducible representations $\alpha \in i R e p s$. of a string-theory realization (see Table 4) whose $f_{0}(\alpha)$ contributes to $\vartheta_{K}\left(f_{z} \mathfrak{a}([0]), y\right)$ in (55). Overall, $\zeta_{K}(s,[0])$ is obtained as the Mellin transform of a sum of $f_{0}(\alpha)$ 's of $\#\left[\mathcal{O}_{f_{z}}^{\times}\right] \times\left(4 / \#\left[\mathcal{O}_{f_{z}}^{\times}\right]\right) \times \#\left[f_{z} \mathfrak{a}([0]) / \mathfrak{b}_{z}\right]$ irreducible representations of the chiral algebra. Similarly, $\zeta_{K}(s,[1])$ is obtained as the Mellin transform of a sum of $f_{0}$ 's of $\#\left[\mathcal{O}_{f_{z}}^{\times}\right] \times\left(4 / \#\left[\mathcal{O}_{f_{z}}^{\times}\right]\right) \times \#\left[f_{z} \mathfrak{a}([1]) / \mathfrak{b}_{z}\right]$ irreducible representations.

The subset ${ }^{23}[\mathfrak{a}(\mathfrak{K})]_{\mathbb{Z}}\left(\left(f_{z}\right)_{\mathcal{O}_{K}}\right) \subset \mathcal{O}_{K}$, over which the sum (48) runs, is not a $\mathbb{Z}$-lattice, which is why the combination (55) is a modular form for $\Gamma\left(4 N_{f_{z} \mathfrak{a}(\Omega)}\right)$, but is not guaranteed to be one for a group of the form $\Gamma_{0}(M)$ for some choice of $M \in \mathbb{N}_{>0}$. In this example, however, the set $[\mathfrak{a}([0])]_{\mathbb{Z}}\left(\left(f_{z}\right)_{\mathcal{O}_{K}}\right)=\left[\mathcal{O}_{K}\right]_{\mathbb{Z}}\left((3)_{\mathcal{O}_{K}}\right)=[\mathfrak{a}([0])]_{1}\left(\left(f_{z}\right)_{\mathcal{O}_{K}}\right)$ can be decomposed into three $\mathbb{Z}$-lattices, $\mathcal{O}_{f_{z}}, i \mathcal{O}_{f_{z}}$ and $\left(f_{z}\right)_{\mathcal{O}_{K}}$, with multiplicities $+1,+1$ and -2 , respectively; one can see this decomposition most easily by a glance at Figure 1 (b). Similar decomposition

[^15]| $z$ | $[a, b, c]$ | $C$ | $\Omega\left(\Lambda_{\text {Cardy }}\right)$ | $\mathfrak{b}_{z} / f_{\rho}$ | $\left.\#\left[f_{z} \mathfrak{a}(0]\right) / \mathfrak{b}_{z}\right]$ | $\#\left[f_{z} \mathfrak{a}([1]) / \mathfrak{b}_{z}\right]$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $3 i$ | $[1,0,9]$ | 3 | $\mathbb{Z}+3 w_{K} \mathbb{Z}$ | $6 \mathbb{Z}+2 w_{K} \mathbb{Z}$ | $12\left(f_{\rho} / 3\right)^{2}$ | $6\left(f_{\rho} / 3\right)^{2}$ |
| $\frac{(-1+3 i)}{2}$ | $[2,2,5]$ | 3 | $\mathbb{Z}+\frac{-1+3 w_{K}}{2} \mathbb{Z}$ | $\left(3+w_{K}\right) \mathbb{Z}+2 w_{K} \mathbb{Z}$ | $6\left(f_{\rho} / 3\right)^{2}$ | $3\left(f_{\rho} / 3\right)^{2}$ |

Table 4: Two $\mathbb{C}$-isomorphism classes of elliptic curves in $\operatorname{Ell}\left(\mathcal{O}_{f_{z}=3}\right)$ where $K=\mathbb{Q}(\sqrt{-1})$. Their lattices $\Lambda_{\text {Cardy }}$ and $\Lambda_{\text {winding }}$ embedded in $K \subset \mathbb{C}$ are shown relatively to $\mathcal{O}_{K}=\mathbb{Z}+w_{K} \mathbb{Z}$, where $w_{K}=i$ for $K=\mathbb{Q}(\sqrt{-1})$ here.
of the set $[\mathfrak{a}([1])]_{\mathbb{Z}}\left(\left(f_{z}\right)_{\mathcal{O}_{K}}\right)$ into $\mathbb{Z}$-lattices will be more complicated; we did not work that out.

For cases with $k=L_{f_{z}}$ with $f_{z}>1$ (and possibly for more general cases), there is an alternative approach, which is to use a set of proper $\mathcal{O}_{f_{z}}$-ideals instead of $\mathcal{O}_{K}$-ideals as a set of representatives of the ideal class group $\mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}}\right)$. Instead of $\left\{\mathfrak{a}(\mathfrak{K})_{\mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}}\right)}\right\}$ introduced ${ }^{24}$ in section 3.3, we can use (e.g., [Shimura [25], Thm 4.11], [Lang [13], Chap. 8 §1, Thm. 4], [Moreland [19], §5]):

$$
\begin{equation*}
\{\mathfrak{b}(\mathfrak{K})\}_{\mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}}\right)}=\left\{C^{-1}\left(\mathfrak{a}(\mathfrak{K}) \cap \mathcal{O}_{f_{z}}\right)\right\}_{\mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}}\right)} \tag{56}
\end{equation*}
$$

Here, the factor $C^{-1}$ is not necessary for the purpose of finding a representative proper $\mathcal{O}_{f_{z}}$ ideal, but we include the factor $C^{-1}$, because $\mathfrak{b}(\mathfrak{K}) \subset C^{-1} \mathcal{O}_{f_{z}}$ still fits within $\Omega\left(\Lambda_{\text {Cardy }}\right)$. Now, instead of (48), we can use

$$
\begin{equation*}
\zeta_{K}(s, \mathfrak{K})=\frac{1}{\#\left[\mathcal{O}_{f_{z}}^{\times}\right]} \sum_{\left.C \eta \in[C \mathfrak{G}(\mathfrak{K})]_{z}\left(\left(f_{z}\right)\right)_{f_{z}}\right)} \frac{\left(C^{-2}\left[\mathcal{O}_{f_{z}}: C \mathfrak{b}(\mathfrak{K})\right]\right)^{s}}{|\eta|_{\mathbb{C}}^{2 s}}, \tag{57}
\end{equation*}
$$

where we recycle the notation $\mathfrak{a}_{\mathbb{Z}}\left(\mathfrak{m}_{f}\right)$ introduced in the discussion below (47) for $\mathcal{O}_{f_{z}}$-proper ideals $\mathfrak{a}$ and $\mathfrak{m}_{f}$.

The parameter $f_{\rho}$ of string-theory realizations should then be chosen so that

$$
\begin{equation*}
f_{z} \mathfrak{b}(\mathfrak{K}) \mid \mathfrak{b}_{z} \text { for }{ }^{\forall} \mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}}\right) \tag{58}
\end{equation*}
$$

as proper $\mathcal{O}_{f_{z}}$ ideals. When this condition is satisfied, the projection $\pi: \Omega\left(\Lambda_{\text {Cardy }}\right) / \mathfrak{b}_{z} \rightarrow$ $\Omega\left(\Lambda_{\text {Cardy }}\right) / f_{z} \mathfrak{b}(\mathfrak{K})$ is well-defined, so we can sum over the irreducible representations $\alpha \in$ iReps. $\cong \Omega\left(\Lambda_{\text {Cardy }}\right) / \mathfrak{b}_{z}$ in the fiber of this projection to obtain $\vartheta_{K}\left(f_{z} \mathfrak{b}(\mathfrak{K}), y\right)$. Summing

[^16]them over $y \in\left[C^{-1}\left([C \mathfrak{b}(\mathfrak{K})]_{\mathbb{Z}}\left(\left(f_{z}\right)_{\mathcal{O}_{z}}\right)\right)\right]_{f_{z} \mathfrak{b}(\mathfrak{K})}$ and carrying out the Mellin transformation, $\zeta_{K}(s, \mathfrak{K})$ can be reproduced through (57).

In the example we worked on above, the common periodicity of the lattices $f_{z} \mathfrak{b}(\mathfrak{K})$ 's for $\mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}}\right)$ is $(1+3 i, 1-3 i)_{\mathcal{O}_{f_{z}}}$, so the condition (58) reads $1 \mid f_{\rho}$ (i.e., any $\left.f_{\rho} \in \mathbb{N}_{>0}\right)$ for both of $\left[E_{z=3 i}\right]_{\mathbb{C}}$ and $\left[E_{z=(1+3 i) / 2}\right]_{\mathbb{C}}$ in $\operatorname{Ell}\left(\mathcal{O}_{f_{z}=3}\right)$ of $K=\mathbb{Q}(\sqrt{-1})$. That is more economical (than the requirement $3 \mid f_{\rho}$ in the approach above), in that string realizations with a fewer number of irreducible representations under the chiral algebra can be used in reproducing $\zeta_{k}(s)=L\left(H_{e t}^{0}(E), s\right)$.

## $4 \quad L$-functions for $H_{e t}^{1}(E)$

### 4.1 Preparation from String Theory

It is known that the $L$-functions for elliptic curves with complex multiplication are given by the Mellin transform of some modular forms of weight 2 , as we will review in section 4.2, Because the character functions $f_{0}\left(\tau_{w s} ; \alpha\right)$ of string-theory realizations are of weight 1 , they are not useful (at least immediately) in reconstructing the $L$-functions. Here, we prepare weight-2 observables in string-theory realizations, which are to be used in section 4.3 ,

Now think of the following set of functions of $t_{w s} \in \mathbb{R}_{>0}$ or of $\tau_{w s} \in \mathcal{H}$,

$$
\begin{align*}
f_{1}\left(i t_{w s} ; \alpha\right) & :=\operatorname{Tr}_{V_{\alpha}^{o}}\left[\Omega^{\prime} q^{\left(L_{0}-c / 24\right)}\right] \times[\eta(q)]^{2}, & q=e^{-2 \pi t_{w s}}  \tag{59}\\
f_{1}\left(\tau_{w s} ; \alpha\right) & =\operatorname{Tr}_{V_{\alpha}^{-}}\left[\Omega^{\prime} q^{\left(L_{0}-c / 24\right)}\right] \times[\eta(q)]^{2}, & q=e^{2 \pi i \tau_{w s}} \tag{60}
\end{align*}
$$

for $\alpha \in i$ Reprs. $\cong \Omega\left(\Lambda_{\text {Cardy }}\right) / \mathfrak{b}_{z} \cong \Gamma_{+}^{\vee} / \Gamma_{+}$, defined in the language of bosonic string theory. Newly inserted is

$$
\Omega^{\prime}= \begin{cases}\sqrt{\frac{\alpha^{\prime}}{2}} k_{-}^{\mathbb{C}} & \text { for } V_{\alpha}^{-}  \tag{61}\\ \sqrt{\frac{2}{\alpha^{\prime}} \frac{\Delta X^{\mathbb{C}}}{2 \pi}} & \text { for } V_{\alpha}^{o}\end{cases}
$$

which measures a combination of the four $U(1)$ charges of states in the bosonic string realizations.

In the language of superstring theory,

$$
\begin{align*}
f_{1}\left(\tau_{w s} ; \alpha\right) & =(-i) \operatorname{Tr}_{V_{\alpha}^{-} ; \mathrm{R}}\left[\Omega^{\prime} F_{L} e^{\pi i F_{L}} q^{L_{0}-c / 24}\right], & q=e^{2 \pi i \tau_{w s}}  \tag{62}\\
f_{1}\left(i t_{w s} ; \alpha\right) & =(-i) \operatorname{Tr}_{V_{\alpha} ; \mathrm{R}}\left[\Omega^{\prime} F e^{\pi i F} q^{\left(L_{0}-c / 24\right)}\right], & q=e^{-2 \pi t_{w s}} \tag{63}
\end{align*}
$$

In the path-integration formulation, we define the "partition function" $Z\left(\tau_{w s} ; u, \vec{\mu}\right)$ obtained by modifying $\widetilde{S}$ from the one in (34) to

$$
\begin{equation*}
i \widetilde{S}=i S+2 \pi i \int_{\Sigma} \frac{d^{2} \sigma}{\operatorname{Im}\left(\tau_{w s}\right)}\left(u J_{L}+\vec{\mu} \cdot i \sqrt{\frac{2}{\alpha^{\prime}}} \partial \vec{X}\right)+\text { right mover } \tag{64}
\end{equation*}
$$

$\vec{\mu}$ is a doublet of complex parameters associated with the doublet of currents $\partial \vec{X}$, and $\vec{\mu}$ its complex conjugates coupled to the currents $\bar{\partial} \vec{X}$ in the right-moving sector. Under the worldsheet $\operatorname{SL}(2 ; \mathbb{Z})_{w s}$ transformation [2],

$$
\begin{equation*}
Z\left(\frac{a \tau_{w s}+b}{c \tau_{w s}+d}, \frac{u}{c \tau_{w s}+d}, \frac{\vec{\mu}}{c \tau_{w s}+d}\right)=\mathbb{E}\left[\frac{(1 / 2) c\left(u^{2}+\vec{\mu}^{2}\right)}{c \tau_{w s}+d}-\frac{(1 / 2) c\left(\bar{u}^{2}+\vec{\mu}^{2}\right)}{c \bar{\tau}_{w s}+d}\right] Z\left(\tau_{w s}, u, \vec{\mu}\right) \tag{65}
\end{equation*}
$$

Taking derivatives with respect to $u$ once and $\vec{\mu}$ once, and setting $u=\vec{\mu}=0$, we find

$$
\begin{equation*}
\sum_{\alpha} f_{1}\left(\tau_{w s}^{\prime} ; \alpha\right) f_{1}\left(-\bar{\tau}_{w s}^{\prime} ; \alpha\right)=\left|\left(c \tau_{w s}+d\right)\right|^{4}\left[\sum_{\alpha} f_{1}\left(\tau_{w s} ; \alpha\right) f_{1}\left(-\bar{\tau}_{w s} ; \alpha\right)\right] \tag{66}
\end{equation*}
$$

a property satisfied by a vector-valued weight-2 modular form of $\mathrm{SL}(2 ; \mathbb{Z})_{w s}$.
As we think of $T^{2}$-target diagonal rational CFT's, we know the functions $f_{1}\left(\tau_{w s} ; \alpha\right)$ explicitly. They are in the form of

$$
\begin{equation*}
f_{1}\left(\tau_{w s} ; \alpha\right)=\sum_{w \in \alpha} \Omega^{\prime}(w) e^{2 \pi i \tau_{w s} \frac{(w, w)_{\Gamma_{+}}}{2}}=\sum_{w \in \alpha} \Omega^{\prime}(w) e^{2 \pi i \tau_{w s} \frac{C^{2} a_{z}}{f_{\rho} D_{z}} \frac{\langle w, w\rangle_{K / Q}}{2}} \tag{67}
\end{equation*}
$$

They are a variant of congruent theta functions, and can be organized into Hecke theta functions; more explanations (for string theorists) follow shortly.

### 4.2 L-functions of Elliptic Curves with Complex Multiplication

Here is a brief review on the Hasse-Weil $L$-function of elliptic curves with complex multiplication. We have already stated in section 3.1 that any $\mathbb{C}$-isomorphism class of elliptic curves with complex multiplication, $[E]_{\mathbb{C}} \in \operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$ for some imaginary quadratic field $K$ and $f_{z} \in \mathbb{N}_{>0}$, has a model $E / k$ over some number field $k$. The Hasse-Weil $L$-function is defined for such an object $E / k$, not for $[E]_{\mathbb{C}}$. Section 4.2.1 explains how the $L$-function can be computed for a given model $E / k$, and we will discuss in section 4.2.3 how to classify models $E / k$ for a given $\mathbb{C}$-isomorphism class $[E]_{\mathbb{C}}$. We will restrict our attention to models where the field of definition $k$ is either i) the ring class field $k=L_{f_{z}}$, ii) an abelian extension $k / K$ containing $L_{f_{z}} / K$, and iii) the field $k=F_{f_{z}}^{[E]}$.

### 4.2.1 Hecke $L$-functions and Hecke Theta Functions

The following three theorems translate computation of the $L$-function of an elliptic curve $E / k$ with complex multiplication into determination of the Hecke $L$-function associated with $E / k$.

Theorem 4.2 .1 (e.g., Shimura [25], Props. 7.40 and 7.41). 25 Let $E / k$ be an elliptic curve with complex multiplication by $\mathcal{O}_{f_{z}}$ where $k / K$ is an abelian extension containing $L_{f_{z}} / K$. A procedure described in the propositions of [25] referred to above associates a Hecke character of the idele class group of $k, \psi_{E / k}: \mathbb{A}_{k}^{\times} / k^{\times} \rightarrow \mathbb{C}^{\times}$, with the $k$-isomorphism class of $E / k . \psi_{E / k}$ is of type $[-1 / 2 ; \mathbf{1}, \mathbf{0}]$; we will explain what the type of a Hecke character means in Definition 4.2.12.

Theorem 4.2.2 (e.g., Shimura [25], Thm. 7.42). ${ }^{26}$ Let $E / k$ be an elliptic curve with complex multiplication by $\mathcal{O}_{f_{z}}$ where the field of definition $k / K$ is an abelian extension containing $L_{f_{z}} / K$. Then

$$
\begin{equation*}
L(E / k, s)=L\left(s, \psi_{E / k}\right) L\left(s, \bar{\psi}_{E / k}\right) \tag{68}
\end{equation*}
$$

we will have more words shortly on the Hecke L-functions on the right hand side. Here, the reduction of $E / k$ at a prime of $\mathcal{O}_{k}$ is either good, or cusp that is potentially good (because $E / k$ has complex multiplication (Silverman [28], Chap. VII §5)), and hence all the fibers (good or bad) are included in (68).

Theorem 4.2 .3 (e.g. Shimura [26], Thm. 7). Let $E / F_{f_{z}}^{[E]}$ be an elliptic curve with complex multiplication. Then its $L$-function is given by

$$
\begin{equation*}
L\left(E / F_{f_{z}}^{[E]}, s\right)=L\left(s, \psi_{E / L_{f_{z}}}\right) \tag{69}
\end{equation*}
$$

Here $\psi_{E / L_{f_{z}}}: \mathbb{A}_{L_{f_{z}}}^{\times} / L_{f_{z}}^{\times} \rightarrow \mathbb{C}^{\times}$is the Hecke character associated with the base change of $E / F_{f_{z}}^{[E]}$ with respect to $\operatorname{Spec}\left(L_{f_{z}}\right) \rightarrow \operatorname{Spec}\left(F_{f_{z}}^{[E]}\right)$.

In any of the models of elliptic curves discussed here, we need the $L$-function of a Hecke character $\psi_{E / k^{\prime}}: \mathbb{A}_{k^{\prime}}^{\times} / k^{\prime \times} \rightarrow \mathbb{C}^{\times}$for an abelian extension $k^{\prime} / K$ containing $L_{f_{z}} / K$. Those Hecke $L$-functions of a Hecke character of $\mathbb{A}_{k^{\prime}}^{\times}$can be written in terms of Hecke $L$-functions of Hecke characters of $\mathbb{A}_{K}^{\times}$, provided $E / k^{\prime}$ satisfies the condition $\left(^{*}\right)$ stated in the following.

[^17]Lemma 4.2.4 (Shimura [25], Thm. 7.44). Let $k^{\prime} / K$ be an abelian extension containing $L_{f_{z}} / K$ and $\psi_{E / k^{\prime}}: \mathbb{A}_{k^{\prime}}^{\times} / k^{\prime \times} \rightarrow \mathbb{C}^{\times}$a Hecke character associated with an elliptic curve $E / k^{\prime}$ with complex multiplication by an order of $K$. Suppose that all the points of finite order of $E$ are rational over $K_{a b}-\left(^{*}\right)$. Then there are $\left[k^{\prime}: K\right]$ Hecke characters of the idele class group of $K, \varphi: \mathbb{A}_{K}^{\times} / K^{\times} \rightarrow \mathbb{C}^{\times}$, that satisfy

$$
\begin{equation*}
\psi_{E / k^{\prime}}=\varphi \cdot \mathrm{Nm}_{k^{\prime} / K} \tag{70}
\end{equation*}
$$

All those $\varphi$ 's define one and the same character $[\varphi]$ on the image of $\mathrm{Nm}_{k^{\prime} / K}: \mathbb{A}_{k^{\prime}}^{\times} \rightarrow \mathbb{A}_{K}^{\times}$. All those $\varphi$ 's are of type $[-1 / 2 ; 1,0]$. Conversely, if such $\varphi^{\prime}$ 's exist for the Hecke character $\psi_{E / k^{\prime}}$ of a model $E / k^{\prime}$, then the model satisfies the condition $\left({ }^{*}\right)$.

It is known that any elliptic curve $\left[E_{z}\right]_{\mathbb{C}}$ with complex multiplication by an order of $K$ has models over some number fields that are abelian extensions of $K$ (incl. $K(j([E]))$ ) so that the condition $\left(^{*}\right)$ is satisfied; more information is found in pp.216-217 of [25], and also in section 4.2.3.

Proposition 4.2.5. (cf. [Serre [24], Prop. VI.13] and [Shimura [25], §7.9.A]) Here we use the same notation as in the previous Lemma, and suppose that a model $E / k^{\prime}$ satisfies the condition $(*)$. Then

$$
\begin{equation*}
L\left(s, \psi_{E / k^{\prime}}\right)=\prod_{\varphi \in[\varphi]} L(s, \varphi), \tag{71}
\end{equation*}
$$

where the product runs over all the $\left[k^{\prime}: K\right]$ variations of $\varphi$ consistent with $[\varphi]$.
Therefore, for an elliptic curve $E / k$ defined over a number field $k$ in one of the class i)-iii) at the beginning of section 4.2 with the condition $\left({ }^{*}\right)$ in Lemma 4.2.4 satisfied by $E / k^{\prime}$ ( $k^{\prime}=L_{f_{z}}$ for $k$ in (iii), and $k^{\prime}=k$ otherwise), computation of the Hasse-Weil $L$-function has now been reduced to computation of the $L$-functions of type $[-1 / 2 ; 1,0]$ Hecke characters of the idele class group of an imaginary quadratic field $K$. The latter-Hecke $L$-functions-is now related to the Mellin transform of Hecke theta functions as follows. We begin with defining the following functions ${ }^{27}$

[^18]Definition 4.2.6. Let $L$ be a number field which has $r_{1}$ real embeddings $\rho_{a}: L \hookrightarrow \mathbb{R}$ $\left(a=1, \cdots, r_{1}\right)$ and $r_{2}$ pairs of imaginary embeddings $\sigma_{b}: L \hookrightarrow \mathbb{C}, \bar{\sigma}_{b}=c c \circ \sigma_{b} ; c c$ stands for the complex conjugation in $\mathbb{C}$. For a sublattice $\Lambda$ of the lattice $\left(\mathcal{O}_{L},\langle-,-\rangle_{L / \mathbb{Q}}\right), x \in L \otimes_{\mathbb{Q}} \mathbb{R} / \Lambda$ and $\mathbf{p} \in(\mathbb{Z} / 2 \mathbb{Z})^{r_{1}} \times \mathbb{Z}^{\oplus r_{2}}$, we set

$$
\begin{align*}
\vartheta_{L}^{\mathrm{p}}(\tau ; \Lambda, x) & :=\sum_{w \in x}[w]_{\mathbf{p}} q^{\frac{\langle w, w\rangle_{L / Q}}{2}},  \tag{72}\\
{[w]_{\mathbf{p}} } & :=\prod_{a=1}^{r_{1}}\left(\rho_{a}(w)\right)^{p_{\rho_{a}}} \prod_{b=1}^{r_{2}}\left\{\begin{array}{ll}
\left(\sigma_{b}(w)\right)^{p_{\sigma_{b}}}, & \text { if } p_{\sigma_{b}} \geq 0 \\
\left(\bar{\sigma}_{b}(w)\right)^{-p_{\sigma_{b}}}, & \text { if } p_{\sigma_{b}}<0
\end{array}\right\} . \tag{73}
\end{align*}
$$

Here, $p_{\rho_{a}}$ 's with $a=1, \cdots, r_{1}$ in $\mathbf{p}=\left(p_{\rho_{1}}, \cdots, p_{\rho_{r_{1}}}, p_{\sigma_{1}}, \cdots, p_{\sigma_{r_{2}}}\right) \in(\mathbb{Z} / 2 \mathbb{Z})^{r_{1}} \times \mathbb{Z}^{\oplus r_{2}}$ are regarded as either 0 or 1 , when they are used in defining a monomial 28

Let us now complete the task of relating the Hasse-Weil $L$-function of an elliptic curve with complex multiplication with the congruent theta functions above. Let $k^{\prime} / K$ be an abelian extension containing $L_{f_{z}} / K$, and $\mathfrak{m}_{f}$ an integral ideal of $K$ so that the ray class field $L_{\mathfrak{m}_{f}}$ contains $k^{\prime}$.

Definition 4.2.7 (Neukirch [20], Lemma VII.7.6). Let $K$ be an imaginary quadratic field, and $\varphi$ its Hecke character of type $[-1 / 2 ; 1,0]$. For such a Hecke character $\varphi$ of $\mathbb{A}_{K}^{\times} / K^{\times}$with the conductor $\mathfrak{c}_{f}$, one can uniquely determine a character $\chi_{f}:\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times} \rightarrow S^{1}$ with respect to the multiplication law in $\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times}$and the group of complex phases $S^{1}$; we will see how $\chi_{f}$ is determined from $\varphi$ in section 4.2.3. We assume that the conductor of $\varphi$ satisfies $\mathfrak{m}_{f} \mid \mathfrak{c}_{f}$. Let us also choose a set of $\mathcal{O}_{K}$ integral ideals $\{\mathfrak{a}(\mathfrak{K})\}_{\mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right)}$ as discussed in section 3.3, with one extra condition that all of $\mathfrak{a}(\mathfrak{K})$ are prime to $\mathfrak{c}_{f}$ (not just prime to $\mathfrak{m}_{f}$ ). Using all these data, we define a Hecke theta function on $\tau \in \mathcal{H}$ by

$$
\begin{equation*}
\vartheta(\tau ; \varphi, \mathfrak{K}):=\frac{1}{\varphi(\mathfrak{a}(\mathfrak{K}))} \sum_{\left.x \in[\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right)\right]_{\mathfrak{a}(\mathfrak{K})} \mathfrak{c}_{f}} \chi_{f}(x) \vartheta_{K}^{1}\left(\frac{\tau}{\mathbb{N a}(\mathfrak{K})} ; \mathfrak{c}_{f} \mathfrak{a}(\mathfrak{K}), x\right) \tag{74}
\end{equation*}
$$

where $\left[[\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right)\right]_{\mathfrak{a}(\mathfrak{K})} \mathfrak{c}_{f}$ is the image of $[\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right)$ under the projection $\mathfrak{a}(\mathfrak{K}) \rightarrow \mathfrak{a}(\mathfrak{K}) / \mathfrak{a}(\mathfrak{K}) \mathfrak{c}_{f}$.
For the Hecke character $\varphi: \mathbb{A}_{K}^{\times} / K^{\times} \rightarrow \mathbb{C}^{\times}$, let $\bar{\varphi}$ be the Hecke character of $\mathbb{A}_{K}^{\times} / K^{\times}$given by $\bar{\varphi}:=c c \circ \varphi ; \bar{\varphi}$ is of type $[-1 / 2 ;-1,0]$, its conductor is the same as $\mathfrak{c}_{f}$, and the character of $\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times}$corresponding to $\bar{\varphi}$ is given by $\bar{\chi}_{f}:=c c \circ \chi_{f}$. A Hecke theta function for $\bar{\varphi}$ is

[^19]defined by
\[

$$
\begin{equation*}
\vartheta(\tau ; \bar{\varphi}, \mathfrak{K}):=\frac{1}{\bar{\varphi}(\mathfrak{a}(\mathfrak{K}))} \sum_{x \in\left[[\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right)\right]_{\mathfrak{a}(\mathfrak{K})} \mathfrak{c}_{f}} \bar{\chi}_{f}(x) \vartheta_{K}^{-1}\left(\frac{\tau}{\mathbb{N a}(\mathfrak{K})} ; \mathfrak{c}_{f} \mathfrak{a}(\mathfrak{K}), x\right) . \tag{75}
\end{equation*}
$$

\]

Now, we are ready to write down the Hecke L-functions $L(s, \varphi)$ in terms of the Mellin transform of the Hecke theta functions.

Theorem 4.2 .8 (e.g., Koblitz [11], II §5; Neukirch [20], VII §7). Let $K$ be an imaginary quadratic field, and $\varphi: \mathbb{A}_{K}^{\times} / K^{\times} \rightarrow \mathbb{C}^{\times}$a Hecke character of type $[-1 / 2 ; 1,0]$. Then the Hecke $L$-function of $\varphi$ is given by

$$
\begin{align*}
L(s, \varphi) & =\sum_{\mathfrak{K} \in \mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right)} L(s, \varphi, \mathfrak{K})  \tag{76}\\
L(s, \varphi, \mathfrak{K}) \frac{\Gamma(s)}{(2 \pi)^{s}} & =\frac{1}{\#\left[\mathcal{O}_{K}^{\times}\right]} \int_{0}^{\infty} \frac{d t}{t} t^{s} \vartheta(i t ; \varphi, \mathfrak{K}) . \tag{77}
\end{align*}
$$

To summarize, the Hasse-Weil $L$-function of an elliptic curve $E / k$ with complex multiplication by an order of an imaginary quadratic field $K$ is given by combining (72, 74, 77, 76, (71) and finally either (68) or (69), if the field of definition $k$ is either an abelian extension containing $L_{f_{z}} / K$ or $F_{f_{z}}^{[E]}$, and the condition $\left(^{*}\right)$ in Lemma 4.2.4 is satisfied (after the base change to $L_{f_{z}}$, if $k=F_{f_{z}}^{[E]}$.

### 4.2.2 Hecke Theta Functions as Modular Forms

There is a more general version of Proposition 3.3.2, which is stated in the form we use in this article:

Theorem 4.2.9 (Iwaniec [8], Cor. 10.7; Miyake [15], Cor. 4.9.4). Here, we use the same notation and assumption as in Definition 4.2.7. Then $\vartheta_{K}^{1}\left(\tau^{\prime} ; \Lambda, x\right)$ for $x \in \Lambda^{*} / \Lambda[$ resp. $\vartheta(\tau ; \varphi, \mathfrak{K})]$ is a cusp form of weight 2 for $\Gamma\left(4 N_{\Lambda}\right) \subset \mathrm{SL}(2 ; \mathbb{Z})\left[\operatorname{resp} .29 \Gamma\left(4 N_{\mathfrak{a}(\mathfrak{K})_{\mathfrak{c}}}\right) \subset \mathrm{SL}(2 ; \mathbb{Z})\right]$. The group $\Gamma\left(4 N_{\Lambda}\right)$ [resp. $\left.\Gamma\left(4 N_{\mathfrak{a}(\mathfrak{K})} \mathfrak{c}_{f}\right)\right]$ acts on $\tau^{\prime} \in \mathcal{H}[$ resp. on the combination $\tau / \mathbb{N a}(\mathfrak{K}) \in \mathcal{H}]$ through the ordinary action of $\operatorname{SL}(2 ; \mathbb{Z})$ on $\mathcal{H}$.

When a non-trivial multiplier system is allowed, $\Gamma\left(4 N_{\Lambda}\right)$ can be replaced by $\Gamma_{1}\left(4 N_{\Lambda}\right)$. If $x \in \mathcal{O}_{K} / \Lambda$, then $(x, x) / 2 \in \mathbb{Z}$, and the multiplier system becomes trivial.
${ }^{29} x \in \mathfrak{a}(\mathfrak{K}) / \mathfrak{a}(\mathfrak{K}) \mathfrak{c}_{f}$ implies $x \in\left(\mathfrak{a}(\mathfrak{K}) \mathfrak{c}_{f}\right)^{*} / \mathfrak{a}(\mathfrak{K}) \mathfrak{c}_{f} ;$ see section 3.4.2,

For many cases we use the theta functions $\vartheta_{K}^{\mathrm{P}}(\tau ; \Lambda, x)$ in this article, $\Lambda$ is not just a general sublattice of $\left(\mathcal{O}_{K},\langle-,-\rangle_{K / \mathbb{Q}}\right)$ (that is, not just an abelian group), but a subring of $\mathcal{O}_{K}$. In such cases, we can think of summing over $\vartheta_{K}^{\mathrm{P}}(\tau ; \Lambda, x)$ over $x$ and seek for an analogue/generalization of Proposition 3.3.1.

Remark 4.2.10 (cf. Miyake [15], Thm. 4.9.3). Let $K$ be an imaginary quadratic field, and $\mathfrak{a}$ an integral ideal of $\mathcal{O}_{K}$. Let $\chi_{f}:\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times} \rightarrow S^{1}$ be a character with respect to the multiplication law in $\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times}$, where $\mathfrak{c}_{f}$ is an integral ideal of $\mathcal{O}_{K}$ relatively prime to $\mathfrak{a}$. Then, for $\gamma=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \Gamma_{0}\left(2 N_{\mathfrak{a c}_{f}}\right)$ and a subgroup $G_{0}$ of $\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times}$,

$$
\begin{align*}
& \sum_{x \in\left[\mathfrak{a}_{G_{0}}\left(\mathfrak{c}_{f}\right)\right] \mathfrak{a}_{f}} \chi_{f}(x) \vartheta_{K}^{\mathbf{P}}\left(\gamma \cdot \tau^{\prime} ; \mathfrak{a c}_{f}, x\right)  \tag{78}\\
= & (c \tau+d)^{1+\operatorname{deg}(\mathbf{P})}\left(\frac{-\operatorname{disc}\left(\mathfrak{a c}_{f}\right)}{d}\right) \sum_{x \in\left[\mathfrak{a}_{G_{0}}\left(\mathfrak{c}_{f}\right)\right] \mathfrak{a}_{f}} \chi_{f}(x) \vartheta_{K}^{\mathbf{P}}\left(\tau^{\prime} ; \mathfrak{a c}_{f}, a \cdot x\right), \tag{79}
\end{align*}
$$

where we used the Legendre (quadratic residue) symbol ( $-/-$ ) in writing down the multiplier system explicitly; $\operatorname{deg}(\mathbf{P})$ is the degree of the monomial $[w]_{\mathbf{P}}$. Therefore, for the subgroup of $\Gamma_{0}\left(2 N_{\mathfrak{a c}_{f}}\right)$ where

$$
\begin{equation*}
(a \cdot):\left[\mathfrak{a}_{G_{0}}\left(\mathfrak{c}_{f}\right)\right]_{\mathfrak{a}_{f}} \ni x \longmapsto a \cdot x \in \mathfrak{a} / \mathfrak{a}_{f} \tag{80}
\end{equation*}
$$

is an isomorphism from $\left[\mathfrak{a}_{G_{0}}\left(\mathfrak{c}_{f}\right)\right]_{\mathfrak{a}_{f}}$ to itself, the sum (78) is a modular form of weight $1+\operatorname{deg}(\mathbf{P})$, with the multiplier system given by $\left(-\operatorname{discr}\left(\mathfrak{a c}_{f}\right) / d\right) \chi_{f}(d)$.

When we deal with the Hecke $L$-functions for elliptic curves $E / H_{K}$ with complex multiplication by $\mathcal{O}_{K}$, we can just choose $G_{0}$ to be the entire $\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times}$. For this case, there is a more definite result (cf. Prop. 3.3.1).

Theorem 4.2.11 (Miyake [15], Thm 4.8.2). The sum $\sum_{\mathfrak{K} \in \mathrm{Cl}_{K}} \vartheta(\varphi, \mathfrak{K})$ of the Hecke theta functions is a cusp form of weight 2 for the group $\Gamma_{0}\left(N_{\chi_{f}}\right)$ acting on $\tau \in \mathcal{H}$, with a multiplier system (homomorphism) $\underline{\chi}: \Gamma_{0}\left(N_{\chi_{f}}\right) \rightarrow \mathbb{C}^{\times}$(for detailed information of $\underline{\chi}$, see [Miyake [15], Thm. 4.8.2]). The level of the group $\Gamma_{0}\left(N_{\chi_{f}}\right)$ is $N_{\chi_{f}}:=\left|D_{K / \mathbb{Q}}\right| \cdot \mathbb{N} \mathfrak{c}_{f}=\mathbb{N}\left(\mathfrak{c}_{f} \mathfrak{d}_{K / \mathbb{Q}}\right)$. $\bullet$

In a broader context, the two Theorems above are concerned about how to extract single component modular forms of a subgroup $\Gamma \subset \operatorname{SL}(2 ; \mathbb{Z})$ of some level from a (multi-component) vector-valued modular form of $\operatorname{SL}(2 ; \mathbb{Z})$, or in the other way around.

### 4.2.3 Choice of Models

We have stated that any $\mathbb{C}$-isomorphism class of elliptic curves with complex multiplication, $[E]_{\mathbb{C}} \in \operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$, has a model $E / k$ over the number field $k=L_{f_{z}}$, over any abelian extension $k / K$ containing $L_{f_{z}} / K$, and over $k=F_{f_{z}}^{[E]}$. It is not that there is just one model $E / k$ for a given $[E]_{\mathbb{C}}$ and $k$, however. To be more precise, one can think of classifying models $E / k$ defined over a number field $k$ for a given $[E]_{\mathbb{C}}$, thinking that two models $E / k$ and $E^{\prime} / k$ are equivalent iff there is an isomorphism between $E$ and $E^{\prime}$ defined over 30 . We will describe here how one can list up the inequivalent models over a given number field $k$ for a given $\mathbb{C}$-isomorphism class $[E]_{\mathbb{C}}$.

We have used a notion of a type of a Hecke character of the idele class group $\mathbb{A}_{L}^{\times} / L^{\times}$for a number field $L$ in section 4.2.1, but did not explain what it is. So, here is

Definition 4.2.12. Let $L$ be a number field which has $r_{1}$ real embeddings $\rho_{a}: L \hookrightarrow \mathbb{C}$ (for $a=1, \cdots, r_{1}$ ) and $r_{2}$ pairs of imaginary embeddings $\sigma_{b}: L \hookrightarrow \mathbb{C}$ and $\bar{\sigma}_{b}=c c \circ \sigma_{b}: L \hookrightarrow \mathbb{C}$ (for $\left.b=1, \cdots, r_{2}\right) ;[L: \mathbb{Q}]=r_{1}+2 r_{2}$. Any Hecke character $\phi: \mathbb{A}_{L}^{\times} / L^{\times} \rightarrow \mathbb{C}^{\times}$can be written in the form of the product of continuous homomorphisms associated with all the inequivalent valuations of $L: \phi=\phi_{f} \cdot \phi_{\infty}$, and

$$
\begin{align*}
\phi_{\mathfrak{p}}: L_{\mathfrak{p}}^{\times} \rightarrow \mathbb{C}^{\times}, & \left(\phi_{f}=\prod_{\mathfrak{p}} \phi_{\mathfrak{p}}\right):\left(\prod_{\mathfrak{p}} L_{\mathfrak{p}}^{\times} \supset \mathbb{A}_{L, f}^{\times}\right) \rightarrow \mathbb{C}^{\times},  \tag{81}\\
\phi_{v \tau}: L_{v \tau}^{\times} \rightarrow \mathbb{C}^{\times} & \left(\phi_{\infty}=\prod_{v \tau} \phi_{v \tau}\right):\left(\mathbb{A}_{L, \infty}^{\times}=\prod_{v \tau} L_{v \tau}^{\times}\right) \rightarrow \mathbb{C}^{\times} . \tag{82}
\end{align*}
$$

Here, $\mathfrak{p}$ runs over the set of all the non-Archimedean valuations, which is equivalent to the set of all the non-zero prime ideals of $\mathcal{O}_{L}$. The index $v \tau$ runs over the set of all the inequivalent Archimedean valuations $\operatorname{Arch}(L)$; let $\Phi_{L}^{\text {real }}=\left\{\rho_{a} \mid a=1, \cdots r_{1}\right\}$, and $\Phi_{L}^{\mathrm{im}} \amalg \bar{\Phi}_{L}^{\mathrm{im}}$ be a mutually exclusive grouping of imaginary embeddings so that no two elements of $\Phi_{L}^{\mathrm{im}}$ are the complex conjugate of the other; then the set $\operatorname{Arch}(L)$ is in one-to-one correspondence with $\Phi_{L}^{\text {real }} \amalg \Phi_{L}^{\mathrm{im}}$. So, we can use $\tau \in \Phi_{L}^{\text {real }} \amalg \Phi_{L}^{\mathrm{im}}$ as a label/index of the product.

In this article, we say ${ }^{31}$ that a Hecke character $\phi: \mathbb{A}_{L}^{\times} / L^{\times} \rightarrow \mathbb{C}^{\times}$is of type $\left[s^{r} ; \mathbf{p}, \mathbf{q}\right]$,

[^20]where
\[

$$
\begin{equation*}
s^{r} \in \mathbb{R}, \quad \mathbf{p}=\left(p_{\rho}, p_{\sigma}\right) \in[\mathbb{Z} /(2 \mathbb{Z})]^{r_{1}} \times \mathbb{Z}^{\oplus r_{2}}, \quad \mathbf{q}=\left(q_{\rho}, q_{\sigma}\right) \in[\mathbb{R} /(2 \pi \mathbb{Z})]^{r_{1}+r_{2}} \tag{83}
\end{equation*}
$$

\]

when $\phi_{\infty}=\prod_{v \tau \in \operatorname{Arch}(L)} \phi_{v \tau}$ is parametrized by

$$
\begin{align*}
& \phi_{v \rho}: L_{v \rho}^{\times} \ni a_{\rho} \mapsto\left(\frac{a_{\rho}}{\left|a_{\rho}\right|_{\mathbb{R}}}\right)^{-p_{\rho}}\left(\left|a_{\rho}\right|_{\mathbb{R}}\right)^{-i q_{\rho}} \times\left(\left|a_{\rho}\right| \mathbb{R}\right)^{s^{r}} \in \mathbb{C}^{\times},  \tag{84}\\
& \phi_{v \sigma}: L_{v \sigma}^{\times} \ni a_{\sigma} \mapsto\left(\frac{a_{\sigma}}{\left|a_{\sigma}\right|_{\mathbb{C}}}\right)^{-p_{\sigma}}\left(\left|a_{\sigma}\right|_{\mathbb{C}}\right)^{-i q_{\rho}} \times\left(\left|a_{\sigma}\right| \mathbb{C}\right)^{2 s^{r}} \in \mathbb{C}^{\times} . \tag{85}
\end{align*}
$$

The type of a given Hecke character $\phi$ looks different depending on how the $2 r_{2}$ imaginary embeddings are grouped into $\Phi_{L}^{\mathrm{im}}$ and $\bar{\Phi}_{L}^{\mathrm{im}}$. When a number field $L$ is an extension of an imaginary quadratic field $K$, and the pair of imaginary embeddings of $K$ is grouped into $\Phi_{K}^{\mathrm{im}}=\{\sigma\}$ and $\bar{\Phi}_{K}^{\mathrm{im}}=\{\bar{\sigma}\}$, we make it a rule to choose a canonical grouping induced from $\Phi_{K}^{\mathrm{im}} \amalg \bar{\Phi}_{K}^{\mathrm{im}}: \Phi_{L}^{\mathrm{im}}$ consists of imaginary embeddings of $L$ that become $\sigma$ upon restriction to $K$, and $\bar{\Phi}_{L}$ of imaginary embeddings of $L$ that become $\bar{\sigma}$ upon restriction to $K$.

The $\phi_{\infty}$ part of a Hecke character is determined completely by the type $\left[s^{r} ; \mathbf{p}, \mathbf{q}\right]$. For example, when $L$ is a totally imaginary field $\left(r_{1}=0\right)$, the type $[-1 / 2 ; \mathbf{1}, \mathbf{0}]$ implies that

$$
\begin{equation*}
\phi_{\infty}: \mathbb{A}_{L, \infty}^{\times} \ni\left(a_{\sigma}\right) \mapsto \frac{1}{\prod_{\sigma \in \Phi_{L}^{\mathrm{im}}} a_{\sigma}} \in \mathbb{C}^{\times} \tag{86}
\end{equation*}
$$

With this definition, we can now write down a result of classification of models $E / k$ modulo isomorphism over $k$, as follows:

Theorem 4.2.13. (See [Shimura [27], Thms. 22.1 and 19.10], [Lang [13], Chap. $10 \S 4$ ] and [Shimura [26], Thm. 5]) Let $K$ be an imaginary quadratic field, and $k / K$ an abelian extension containing a ring class field $L_{f_{z}} / K$ for some $f_{z} \in \mathbb{N}_{>0}$. For a given $[E]_{\mathbb{C}} \in \operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$, its models $E / k$ modulo $k$-isomorphisms are in one-to-one correspondence with Hecke characters $\psi: \mathbb{A}_{k}^{\times} / k^{\times} \rightarrow \mathbb{C}^{\times}$satisfying the two conditions:
(a) It is of type $[-1 / 2 ; \mathbf{1}, \mathbf{0}]$.
(b) For any prime ideal $\mathfrak{P}$ of $\mathcal{O}_{k}$ that is prime to the conductor $\mathfrak{C}_{f}$ of $\psi, \psi_{\mathfrak{F}}\left(\pi_{\mathfrak{F}}\right) \in \mathcal{O}_{f_{z}} \subset$ $K \subset \mathbb{C}$ and $\operatorname{Nm}_{k / K}(\mathfrak{P})=\left(\psi_{\mathfrak{P}}\left(\pi_{\mathfrak{P}}\right)\right)_{\mathcal{O}_{K}}$. Here, $\pi_{\mathfrak{P}}$ is a uniformizer of the ring of $\mathfrak{P}$-adic integers.
$\phi_{\infty}\left(a_{\infty}\right)=\left(\operatorname{Nm}_{L / \mathbb{Q}}\left(a_{\infty}\right)\right)^{s^{r}} / \chi_{\infty}\left(a_{\infty}\right)$ for $a_{\infty} \in \mathbb{A}_{L, \infty}^{\times}$. The parameters $\mathbf{p}$ and $\mathbf{q}$ of the type here are related to $\mathbf{m}$ and $\varphi$ in [Lang [12], XIV] through the relation $p_{\rho}=m_{\rho} \in \mathbb{Z} /(2 \mathbb{Z}), p_{\sigma}=-m_{\sigma} \in \mathbb{Z}, q_{\rho}=-\varphi_{\rho} \in \mathbb{R} /(2 \pi \mathbb{Z})$ and $q_{\sigma}=-2 \varphi_{\sigma} \in \mathbb{R} /(2 \pi \mathbb{Z})$.

There is an isogeny defined over $k$ from a model $E / k$ of $[E]_{\mathbb{C}} \in \operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$ to a model $E^{\prime} / k$ of $\left[E^{\prime}\right]_{\mathbb{C}} \in \operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$ if and only if $\psi_{E / k}=\psi_{E^{\prime} / k}$.

The $L$-function is defined for each one of $k$-isomorphism classes of elliptic curves over $k$ (not necessarily with complex multiplication). The $L$-functions $L(E / k, s)$ and $L\left(E^{\prime} / k, s\right)$ of a pair of models over a common number field $k$ can be the same, even when there is no isomorphism over $k$ between $E / k$ and $E^{\prime} / k$. To be more precise,

Remark 4.2.14. (See [Faltings [4], p.22, Cor. 2]) Let $k$ be a number field and $E$ and $E^{\prime}$ are elliptic curves defined over $k$ (not necessarily with complex multiplication). Then the following are equivalent:

1. $E$ and $E^{\prime}$ are $k$-isogenous,
2. $L_{v}(E, s)=L_{v}\left(E^{\prime}, s\right)$ for almost all places $v$ of $k$.

This statement holds true also for higher-dimensional abelian varieties. As a special case of this theorem, the pair of models over $k$ that are referred to at the end of Thm. 4.2.13 have an identical $L$-function. 32

One might also be interested in the conditions for a model $E / L_{f_{z}}$ defined over the ring class field to be obtained as a base change of a model defined over $F_{f_{z}}^{[E]}$. The answer to this question can also be phrased in terms of the Hecke character $\psi_{E / L_{z}}$ :

Theorem 4.2.15. (See [Shimura [27], Thms. 22.2 and 20.15; [25], Thm. 7.46]) Let $E / L_{f_{z}}$ be an elliptic curve with complex multiplication by $\mathcal{O}_{f_{z}}, \psi_{E / L_{f_{z}}}$ its associated Hecke character of type $[-1 / 2 ; \mathbf{1}, \mathbf{0}]$, and $\mathfrak{C}_{f}$ the conductor of $\psi_{E / L_{f_{z}}}$. There exists a model $E / F_{f_{z}}^{[E]}$ whose base change to $\operatorname{Spec}\left(L_{f_{z}}\right)$ is isomorphic to $E / L_{f_{z}}$ (here, $[E]$ is the $\mathbb{C}$-isomorphism class of $E / L_{f_{z}}$ ), if and only if the following condition 33 is satisfied:

$$
\begin{equation*}
\psi_{E / L_{f_{z}}}\left(\rho_{[E]}(x)\right)=c c \circ \psi_{E / L_{f_{z}}}(x), \quad{ }^{\forall} x \in \mathbb{A}_{L_{f_{z}}}^{\times} . \tag{87}
\end{equation*}
$$

Remark 4.2.16. When an elliptic curve $E$ with complex multiplication by an order $\mathcal{O}_{f_{z}}$ of $K$ is defined over a number field $k$ containing $L_{f_{z}}$, the conductor $N_{E / k}$ of the elliptic curve $E / k$ is given by the conductor $\mathfrak{C}_{f}$ of the associated Hecke character $\psi_{E / k}$ by [Serre-Tate [23], Thm. 12]

$$
\begin{equation*}
N_{E / k}=\mathfrak{C}_{f}^{2} \in \operatorname{Div}\left(\operatorname{Spec}\left(\mathcal{O}_{k}\right)\right) \tag{88}
\end{equation*}
$$

[^21]When an elliptic curve $E$ with complex multiplication by $\mathcal{O}_{K}$ is defined over $F=F_{f_{z}=1}^{[E]}$, , the conductor $N_{E / F}$ of the elliptic curve $E / F$ is related to the conductor $\mathfrak{C}_{f}$ of the associated Hecke character $\psi_{E / L_{f_{z}}}$ by [Gross [5], eq. (10.3.2)]

$$
\begin{equation*}
N_{E / F}=D_{H / F} \cdot \operatorname{Nm}_{H / F}\left(\mathfrak{C}_{f}\right) \in \operatorname{Div}\left(\operatorname{Spec}\left(\mathcal{O}_{F}\right)\right) \tag{89}
\end{equation*}
$$

Note that, in the case $h\left(\mathcal{O}_{K}\right)=1$ (i.e., when $\left.F=\mathbb{Q}\right)$, the ideal $\left(N_{\chi_{f}}\right)_{\mathbb{Z}}$ is equal to the conductor $N_{E / \mathbb{Q}}$. $\bullet$

Presentation so far makes it clear that the Hecke character $\psi_{E / k}$ for the field of definition $k$ is the crucial tool in classifying elliptic curves defined over number fields so far as the $L$-functions are concerned. In the meantime, we have written down a way to express the $L$-functions in terms of Hecke characters for the imaginary quadratic field $K$ of complex multiplication. It is therefore convenient if the $k$-isomorphism classification of models over $k$ (Thm. 4.2.13) is re-stated in terms of Hecke characters $\varphi$ for $K$. We do so in the following, by largely following discussions in [Neukirch [20], VII §6] and [Milne [14], Chap. V].

To start off, we study in detail the structure of the group of Hecke characters for a number field $L$; we intend to use the following discussion for $L=K$.

Let $L$ be a number field, and we follow the notation adopted in Definition 4.2.12. Let $\mathfrak{c}_{f}$ be an integral $\mathcal{O}_{L}$ ideal, and H.Char $\left[C\left(\mathfrak{c}_{f}\right)\right]$ the group of Hecke characters $\phi: \mathbb{A}_{L}^{\times} / L^{\times} \rightarrow \mathbb{C}^{\times}$ with the modulus $\mathfrak{c}_{f}$. It then follows immediately from [Neukirch [20], Prop. VII.6.12] that

$$
\begin{equation*}
0 \rightarrow \text { Char }\left[\mathrm{Cl}_{L}\left(\mathfrak{c}_{f}\right)\right] \rightarrow \text { H.Char }\left[C\left(\mathfrak{c}_{f}\right)\right] \rightarrow \text { Char }\left[\mathbb{A}_{L, \infty}^{\times} / \mathcal{O}_{L, 1}^{\times}\left(\mathfrak{c}_{f}\right)\right] \rightarrow 0 \tag{90}
\end{equation*}
$$

is exact. Here 34

$$
\begin{equation*}
\mathcal{O}_{L, 1}^{\times}\left(\mathfrak{c}_{f}\right):=\left\{\epsilon \in \mathcal{O}_{L}^{\times} \mid \epsilon \in 1+\mathfrak{c}_{f}\right\} . \tag{91}
\end{equation*}
$$

The projection from H.Char $\left[C\left(\mathfrak{c}_{f}\right)\right]$ to $\operatorname{Char}\left[\mathbb{A}_{L, \infty}^{\times} / \mathcal{O}_{L, 1}^{\times}\left(\mathfrak{c}_{f}\right)\right]$ is given by $\phi \mapsto \phi_{\infty}$.
Definition 4.2.17. An integral $\mathcal{O}_{L}$-ideal $\mathfrak{c}_{f}$ as a modulus and a type $[\mathbf{p}, \mathbf{q}]$ are said to be compatible, if a character $\phi_{\infty} \in \operatorname{Char}\left[\mathbb{A}_{L, \infty}^{\times}\right]$of type $\left[s^{r} ; \mathbf{p}, \mathbf{q}\right]$ (with any $s^{r} \in \mathbb{R}$ ) is trivial on $\mathcal{O}_{L, 1}^{\times}\left(\mathfrak{c}_{f}\right)$ (the triviality condition is independent of $s^{r}$ ).

Proposition 4.2.18 (cf. Neukirch [20], VII.6.12-14). Let $\phi: \mathbb{A}_{L}^{\times} / L^{\times} \rightarrow \mathbb{C}^{\times}$be a Hecke character of type $\left[s^{r} ; \mathbf{p}, \mathbf{q}\right]$ with a modulus $\mathfrak{c}_{f}$ that is compatible with the type $[\mathbf{p}, \mathbf{q}]$. Then

[^22]at non-Archimedean valuations for prime ideals $\mathfrak{q}$ outside of the support of $\mathfrak{c}_{f}$ (i.e., $\mathfrak{q} / \mathfrak{c}_{f}$ ), $\phi_{\mathfrak{q}}: L_{\mathfrak{q}}^{\times} \rightarrow \mathbb{C}^{\times}$are given by a unique unitary (i.e., $S^{1}$-valued) character $\chi: I_{L}\left(\mathfrak{c}_{f}\right) \rightarrow S^{1}$ for a given $\phi$ :
\[

$$
\begin{equation*}
\phi_{\mathfrak{q}}: L_{\mathfrak{q}}^{\times} \ni a_{\mathfrak{q}} \mapsto \chi\left(\mathfrak{q}^{\operatorname{ord}_{\mathfrak{q}}\left(a_{\mathfrak{q}}\right)}\right)\left|\pi_{\mathfrak{q}}\right|_{\mathfrak{q}}^{s^{r}}=\chi\left(\mathfrak{q}^{\operatorname{ord}_{\mathfrak{q}}\left(a_{\mathfrak{q}}\right)}\right)(\mathbb{N} \mathfrak{q})^{-s^{r}}, \tag{92}
\end{equation*}
$$

\]

where $I_{L}\left(\mathfrak{c}_{f}\right)$ is the group of fractional ideals of $\mathcal{O}_{L}$ prime to $\mathfrak{c}_{f}$, and $\pi_{\mathfrak{q}}$ is a uniformizer of the ring of $\mathfrak{q}$-adic integers.
$\phi_{\mathfrak{p}}: L_{\mathfrak{p}}^{\times} \rightarrow \mathbb{C}^{\times}$for non-Archimedean valuations in the support of $\mathfrak{c}_{f}$ (i.e., $\mathfrak{p} \mid \mathfrak{c}_{f}$ ) have the following property. For $\alpha \in \mathcal{O}_{L}$ that is prime to $\mathfrak{c}_{f}$, define

$$
\begin{equation*}
\chi_{f}: \alpha \mapsto \chi_{f}(\alpha):=\left(\prod_{\mathfrak{q} / \mathfrak{c}_{f}} \phi_{\mathfrak{q}}\left(\alpha_{\mathfrak{q}}\right)\right) \cdot \phi_{\infty}(\alpha)=\chi\left((\alpha)_{\mathcal{O}_{L}}\right) \cdot \prod_{v \tau}\left(\frac{\alpha_{\tau}}{\left|\alpha_{\tau}\right|_{\mathbb{C}}}\right)^{-p_{\tau}}\left|\alpha_{\tau}\right|_{\mathbb{C}}^{-i q_{\tau}} ; \tag{93}
\end{equation*}
$$

then the triviality of $\phi$ on $L^{\times} \subset \mathbb{A}_{L}^{\times}$(by def) implie 35 that

$$
\begin{equation*}
\left(\prod_{\mathfrak{p} \mid \mathfrak{c}_{f}} \phi_{\mathfrak{p}}\left(\alpha_{\mathfrak{p}}\right)\right)=1 / \chi_{f}(\alpha) . \tag{94}
\end{equation*}
$$

Since $\phi$ has a modulus $\mathfrak{c}_{f}, \chi_{f}$ so defined is a unitary character of $\left[\mathcal{O}_{L} / \mathfrak{c}_{f}\right]^{\times}$. This character $\chi_{f}$ satisfies the condition

$$
\begin{equation*}
\chi_{f}(\epsilon)=\prod_{v \tau \in \operatorname{Arch}(L)}\left(\frac{\epsilon_{\tau}}{\left|\epsilon_{\tau}\right|_{\mathbb{C}}}\right)^{-p_{\tau}}\left|\epsilon_{\tau}\right|_{\mathbb{C}}^{-i q_{\tau}}, \quad{ }^{\forall} \epsilon \in \mathcal{O}_{L}^{\times} / \mathcal{O}_{L, 1}^{\times}\left(\mathfrak{c}_{f}\right) \tag{95}
\end{equation*}
$$

Let H.Char $\left[C\left(\mathfrak{c}_{f}\right)\right]^{\left[s^{r} ; \mathbf{p}, \mathbf{q}\right]}$ be the set of Hecke characters for $L$ with the modulus $\mathfrak{c}_{f}$ and type $\left[s^{r} ; \mathbf{p}, \mathbf{q}\right]$. It is not empty so long as $[\mathbf{p}, \mathbf{q}]$ is compatible with $\mathfrak{c}_{f}$. For a pair of Hecke characters $\phi_{1}$ and $\phi_{2}$ in H.Char $\left[C\left(\mathfrak{c}_{f}\right)\right]^{\left[s^{r} ; \mathbf{p}, \mathbf{q}\right]}, \phi_{1} / \phi_{2}$ should be regarded as an element of $\operatorname{Char}\left[\mathrm{Cl}_{L}\left(\mathfrak{c}_{f}\right)\right]$. This is done by ${ }^{36}$ using $\chi_{1} / \chi_{2}: I_{L}\left(\mathfrak{c}_{f}\right) \rightarrow S^{1}$, which implies that $\chi_{1} / \chi_{2}$ factors through $\mathrm{Cl}_{L}\left(\mathfrak{c}_{f}\right)=I_{L}\left(\mathfrak{c}_{f}\right) / P_{L, 1}\left(\mathfrak{c}_{f}\right)$.

Using an injective homomorphism $c: I_{L}\left(\mathfrak{c}_{f}\right) \hookrightarrow \mathbb{A}_{L}^{\times} / L^{\times}$given by sending a prime ideal $\mathfrak{q} \in I_{L}\left(\mathfrak{c}_{f}\right)$ to $\left(1_{\mathfrak{p}}, \pi_{\mathfrak{q}}, 1_{\tau}\right) \in\left(\prod_{\mathfrak{p} \neq \mathfrak{q}} L_{\mathfrak{p}}^{\times}\right) \times L_{\mathfrak{q}}^{\times} \times \mathbb{A}_{L, \infty}^{\times}$, a character $\phi \circ c: I_{L}\left(\mathfrak{c}_{f}\right) \rightarrow \mathbb{C}^{\times}$is induced; we will abuse the notation and use $\phi$ also for the character $\phi \circ c$ of $I_{L}\left(\mathfrak{c}_{f}\right)$ in this article.

[^23]Proposition 4.2.19 (Neukirch [20], VII $\S 6)$. Let $\phi: I_{L}\left(\mathfrak{c}_{f}\right) \rightarrow \mathbb{C}^{\times}$be a character corresponding to a Hecke character $\phi: \mathbb{A}_{L}^{\times} / L^{\times} \rightarrow \mathbb{C}^{\times}$with a modulus $\mathfrak{c}_{f}$. Then a principal ideal $(\xi)_{\mathcal{O}_{L}}$ prime to $\mathfrak{c}_{f}$ has the following value:

$$
\begin{equation*}
\phi\left((\xi)_{\mathcal{O}_{L}}\right)=\chi\left((\xi)_{\mathcal{O}_{L}}\right) \prod_{\mathfrak{q}} \chi_{\mathfrak{c}_{f}}\left|\xi_{\mathfrak{q}}\right| s_{\mathfrak{q}}^{s^{r}}=\frac{\chi_{f}(\xi)}{\phi_{\infty}(\xi)} \tag{96}
\end{equation*}
$$

This result, combined with (86), determines the expression on the right-hand-side of (74).
We wish to use Prop. 4.2.18 in classifying Hecke characters $\varphi$ for the imaginary quadratic field $K$ of complex multiplication. Remember that a Hecke character $\psi_{E / k^{\prime}}$ of type $[-1 / 2 ; \mathbf{1}, \mathbf{0}]$ of an abelian extension $k^{\prime}$ over $K$ corresponds to a set of $\left[k^{\prime}: K\right.$ ] different Hecke characters of type $[-1 / 2 ; 1,0]$ for $K$, provided that $E / k^{\prime}$ satisfies the condition in Lemma 4.2.4. Therefore, we wish to introduce an equivalence relation where $\chi \sim \chi^{\prime}$ if they are different only by $\operatorname{Char}\left[\operatorname{Gal}\left(k^{\prime} / K\right)\right]$, so that we can classify Hecke characters for $k^{\prime}$ by dealing with characters associated with the imaginary quadratic field $K$.

Let us first develop a result that is useful in dealing with the cases of $k^{\prime}=H_{K}$ (so $f_{z}=1$ ). Now, remember

Lemma 4.2.20 (Milne [14], Thm. V.1.5). The following sequence is exact:

$$
\begin{equation*}
0 \rightarrow \mathcal{O}_{L}^{\times} / \mathcal{O}_{L, 1}^{\times}\left(\mathfrak{c}_{f}\right) \rightarrow\left[\mathcal{O}_{L} / \mathfrak{c}_{f}\right]^{\times} \rightarrow \mathrm{Cl}_{L}\left(\mathfrak{c}_{f}\right) \rightarrow \mathrm{Cl}_{L} \rightarrow 0 \tag{97}
\end{equation*}
$$

Using this fact, we arrive at
Theorem 4.2.21. Let $\mathfrak{c}_{f}$ be an integral $\mathcal{O}_{L}$ ideal to be used as a modulus of Hecke character for a number field $L$, and $[\mathbf{p}, \mathbf{q}]$ a type compatible with $\mathfrak{c}_{f}$. Then the set H.Char $\left[C\left(\mathfrak{c}_{f}\right)\right]^{\left[s^{r} ; \mathbf{p}, \mathbf{q}\right]}$ modulo difference by $\left.\mathrm{Char}^{[ } \mathrm{Cl}_{L}\right]$ is in one-to-one with the set of unitary characters $\chi_{f}$ : $\left[\mathcal{O}_{L} / \mathfrak{c}_{f}\right]^{\times} \rightarrow S^{1}$ that satisfy the condition (95).

We are also interested in cases where the field of definition is $k^{\prime}=L_{f_{z}}$ with $f_{z}>1$, and also in cases where $k^{\prime} / K$ is an abelian extension containing $L_{f_{z}} / K$.

Lemma 4.2.22. Let $k$ be an abelian extension of $L$ that is contained in the ray class field $L_{\mathfrak{c}_{f}}$ of $L$. Then appropriate subgroups of $\left[\mathcal{O}_{L} / \mathfrak{c}_{f}\right]^{\times}$and $\mathcal{O}_{L}^{\times} / \mathcal{O}_{L, 1}^{\times}\left(\mathfrak{c}_{f}\right)$ are determined so that

$$
\begin{equation*}
0 \rightarrow\left[\mathcal{O}_{L}^{\times} / \mathcal{O}_{L, 1}^{\times}\left(\mathfrak{c}_{f}\right)\right]_{k} \rightarrow\left[\mathcal{O}_{L} / \mathfrak{c}_{f}\right]_{k}^{\times} \rightarrow \mathrm{Cl}_{L}\left(\mathfrak{c}_{f}\right) \rightarrow \operatorname{Gal}(k / L) \rightarrow 0 \tag{98}
\end{equation*}
$$

is exact.

Theorem 4.2.23. The notation and assumption being the same as in Thm. 4.2 .21 and Lemma 4.2.22, the set H.Char $\left[C\left(\mathfrak{c}_{f}\right)\right]^{\left[s^{r} ; \mathbf{p}, \mathbf{q}\right]}$ modulo difference by $\operatorname{Char}[\operatorname{Gal}(k / L)]$ is in one-to-one with the set of unitary characters $\chi_{f}:\left[\mathcal{O}_{L} / \mathfrak{c}_{f}\right]^{\times} \rightarrow S^{1}$ satisfying (95) evaluated on $\left[\mathcal{O}_{L} / \mathfrak{c}_{f}\right]_{k}^{\times} \subset\left[\mathcal{O}_{L} / \mathfrak{c}_{f}\right]^{\times}$.

The condition (a) in Thm. 4.2.13 can be implemented in the language of Hecke characters $\varphi$ of $\mathbb{A}_{K}^{\times} / K^{\times}$through Thms. 4.2.21 and 4.2.23 for $L=K$. The condition (b) is implemented as follows:

Theorem 4.2.24. Let $K$ be an imaginary quadratic field, and $k / K$ an abelian extension containing $L_{f_{z}} / K$ for some $f_{z} \in \mathbb{N}_{>0}$. Suppose that an elliptic curve $E / k$ has complex multiplication by $\mathcal{O}_{f_{z}}$, and satisfies the condition $\left(^{*}\right)$ in Lemma 4.2.4. If the conductor $\mathfrak{C}_{f}$ of the Hecke character $\psi_{E / k}$ of $\mathbb{A}_{k}^{\times} / k^{\times}$has the same support as $\pi^{*}\left(\mathfrak{c}_{f}\right)$ for an integral $\mathcal{O}_{K}$ ideal $\mathfrak{c}_{f}$ satisfying $L_{\mathfrak{c}_{f}} \supset k$, where $\pi: \operatorname{Spec}\left(\mathcal{O}_{k}\right) \rightarrow \operatorname{Spec}\left(\mathcal{O}_{K}\right)$, then the $[k: K]$ Hecke characters $\varphi^{\prime}$ s of $\mathbb{A}_{K}^{\times} / K^{\times}$in Lemma 4.2.4 admit $\mathfrak{c}_{f}$ as a modulus,

$$
\begin{equation*}
\mathcal{O}_{K}^{\times} \cdot\left(\left[\mathcal{O}_{f_{z}}\right] \cap\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]_{k}^{\times}\right)=\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]_{k}^{\times}, \tag{99}
\end{equation*}
$$

is of type $[-1 / 2 ; 1,0]$, and satisfy one more condition

$$
\begin{equation*}
\chi_{f}([\alpha]) \in \mathcal{O}_{f_{z}}^{\times} \quad{ }^{\forall}[\alpha] \in\left[\mathcal{O}_{f_{z}}\right] \cap\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]_{k}^{\times} \subset\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times} . \tag{100}
\end{equation*}
$$

Conversely, suppose that $\varphi$ is a Hecke character of $\mathbb{A}_{K}^{\times} / K^{\times}$that admits $\mathfrak{c}_{f}$ as a modulus, which satisfies $k \subset L_{\mathfrak{c}_{f}}$ and the condition (99). Suppose further that $\varphi$ is of type $[-1 / 2 ; 1,0]$ and also satisfies (100). Then $\psi:=\varphi \circ \mathrm{Nm}_{k / K}$ is a Hecke character of $\mathbb{A}_{k}^{\times} / k^{\times}$that admits $\pi^{*}\left(\mathfrak{c}_{f}\right)$ as a modulus, and is of type $[-1 / 2 ; \mathbf{1}, \mathbf{0}]$. The condition (b) of Thm. 4.2.13 is satisfied for prime ideals $\mathfrak{P}$ prime to $\pi^{*}\left(\mathfrak{c}_{f}\right)$. So, if the Hecke character $\psi$ constructed in that way has a conductor $\mathfrak{C}_{f}$ whose support is the same as that of $\pi^{*}\left(\mathfrak{c}_{f}\right)$, then there is a model $E / k$ whose associated Hecke character is this $\psi$.

The condition (87) is also translated as follows.
Theorem 4.2.25. Let $E / L_{f_{z}}$ be a model of $[E] \in \operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$ and $\psi_{E / L_{f_{z}}}$ its Hecke character. Suppose that this model has the property $\left(^{*}\right.$ ) in Lemma 4.2.4. Then the condition (87) is replaced by

$$
\begin{equation*}
\chi_{f}\left(\rho_{[E]}(\alpha)\right)=c c \circ \chi_{f}(\alpha), \quad{ }^{\forall}[\alpha] \in\left[\mathcal{O}_{f_{z}}\right] \cap\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]_{L_{f_{z}}}^{\times} \subset\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times}, \tag{101}
\end{equation*}
$$

where $\mathfrak{c}_{f}$ is the conductor of the Hecke character $\varphi$ of $\mathbb{A}_{K}^{\times} / K^{\times}$. Here, $\rho_{[E]}$ is regarded as an element of $\operatorname{Gal}\left(L_{f_{z}} / F_{f_{z}}^{[E]}\right) \subset \operatorname{Gal}\left(L_{f_{z}} / \mathbb{Q}\right)$, which maps $K \subset L_{f_{z}}$ to itself. It is implicit in the condition here that the integral $\mathcal{O}_{K}$ ideal $\mathfrak{c}_{f}$ is invariant under $\rho_{[E]}$.

Therefore, the conditions ( $\mathrm{a}, \mathrm{b}$ ) in Thm. 4.2.13 and (87) in Thm. 4.2.15 can be implemented by Thms. 4.2.21, 4.2.23, 4.2.24 and 4.2.25. respectively, purely in the language of the imaginary quadratic field $K$ [Koblitz [11], II §5], 37] at the cost of introducing a restriction $\left(^{*}\right)$ on models.

### 4.3 Combining Them Together

### 4.3.1 Models Defined over $H_{K}$ or $F_{K}^{[E]}$

Think of a $\mathbb{C}$-isomorphism class of elliptic curves $\left[E_{z}\right]_{\mathbb{C}} \in \operatorname{Ell}\left(\mathcal{O}_{K}\right)$ for some imaginary quadratic field $K$. It is realized in superstring theory in the form of rational diagonal $T^{2}$ target $\mathcal{N}=(2,2)$ SCFT, parametrized by $f_{\rho} \in \mathbb{N}_{>0}$, which controls the choice of complexified Kähler form on $\left[E_{z}\right]_{\mathbb{C}}$. Now, we observe that the functions $f_{1}\left(\tau_{w s} ; \alpha\right)$ and $\vartheta_{K}^{1}(\tau ; \Lambda, x)$ are almost the same. In fact,

$$
\begin{equation*}
f_{1}\left(\tau_{w s} ; \alpha\right)=\frac{C}{i} \sqrt{\frac{2 a_{z}}{f_{\rho} D_{z}}} \vartheta_{K}^{1}\left(\frac{C^{2} a_{z}}{f_{\rho} D_{z}} \tau_{w s} ; \Omega\left(\Lambda_{\text {winding }}\right), \alpha\right) \tag{102}
\end{equation*}
$$

Now, think of a model $E / H_{K}$ or possibly $E / F_{K}^{[E]}$ of $\left[E_{z}\right]_{\mathbb{C}}$ that corresponds to a multiplicative character $\chi_{f}:\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times} \rightarrow S^{1}$, where $\mathfrak{c}_{f}$ is an integral $\mathcal{O}_{K}$ ideal (Thm. 4.2.21, with $L=K$ ). Focus on string-theory realizations with any $f_{\rho} \in \mathbb{N}_{>0}$ chosen so that

$$
\begin{equation*}
\mathfrak{c}_{f} \cdot \operatorname{LCM}\left(\mathfrak{a}(\mathfrak{K})_{\mathfrak{k} \in \mathrm{Cl}_{K}}\right) \mid \mathfrak{b}_{z} \tag{103}
\end{equation*}
$$

This condition is a little more restrictive than (50) for re-construction of $L\left(H_{e t}^{0}(E), s\right)=\zeta_{k}(s)$ from $f_{0}\left(\tau_{w s} ; \alpha\right)$ 's available in string-theory realizations.

For string-theory realizations of $\left[E_{z}\right]_{\mathbb{C}}$ satisfying the condition (103),

$$
\begin{align*}
f_{1}\left(\tau_{w s} ; \mathfrak{a}(\mathfrak{K}) \mathfrak{c}_{f}, x\right) & :=\sum_{\pi(\alpha)=x} f_{1}\left(\tau_{w s} ; \alpha\right),  \tag{104}\\
\vartheta_{K}^{1}\left(\tau ; \mathfrak{a}(\mathfrak{K}) \mathfrak{c}_{f}, x\right) & =\sum_{\pi(\alpha)=x} \vartheta_{K}^{1}\left(\tau ; \mathfrak{b}_{z}, \alpha\right), \tag{105}
\end{align*}
$$

where $\pi: \mathfrak{a}(\mathfrak{K}) / \mathfrak{b}_{z} \rightarrow \mathfrak{a}(\mathfrak{K}) / \mathfrak{a}(\mathfrak{K}) \mathfrak{c}_{f}$ is the projection. Therefore, the observation (102) implies the main result of this article,

[^24]Theorem 4.3.1. For a $\mathbb{C}$-isomorphism class $\left[E_{z}\right]_{\mathbb{C}} \in \operatorname{Ell}\left(\mathcal{O}_{K}\right)$, think of a model $E / H_{K}$ that satisfies the condition $\left(^{*}\right)$ in Lemma 4.2.4. Then the Hasse-Weil $L$-function of $E / H_{K}$ is obtained from the Boltzmann-weighted sum of $\mathrm{U}(1)$-charges $f_{1}\left(\tau_{w s} ; \alpha\right)$ in (62, 63) in any one of string realizations of $\left[E_{z}\right]_{\mathbb{C}}$ so long as the parameter $f_{\rho}$ satisfies the condition (103). The procedure is to use (102), (104, (105), (74), the Mellin transformation (77), (76), (71), and finally (68).

If $\chi_{f}$ satisfies one more condition, (101), then the model $E / H_{K}$ is obtained as a base change of a model $E / F_{K}^{[E]}$ whose $L$-function is obtained by using (69) instead of (68).

The numbers of vertex operators with given sets of conformal weight and $U(1)$ charge are related to the numbers of solutions of arithmetic models reduced over prime ideals through (102).

From a slightly different perspective, the same result can be stated also as follows:
Theorem 4.3.2. Think of a diagonal rational $T^{2}$-target CFT (or $\mathcal{N}=(2,2) \mathrm{SCFT}$ ) corresponding to a $\mathbb{C}$-isomorphism class of elliptic curves $\left[E_{z}\right]_{\mathbb{C}} \in \operatorname{Ell}\left(\mathcal{O}_{K}\right)$ and a complexified Kähler parameter $\rho=f_{\rho} a_{z} z$. Then linear combinations of the Boltzmann-weighted sum of $\mathrm{U}(1)$ charges of irreducible representations (i.e., $f_{1}\left(\tau_{w s} ; \alpha\right)$ 's) in that CFT yield the HasseWeil $L$-function in the procedure outlined above for models $E / H_{K}$ of $\left[E_{z}\right]_{\mathbb{C}}$ with the property $\left(^{*}\right)$, as long as the conductor $\mathfrak{c}_{f}$ of Hecke characters $\varphi$ of $\mathbb{A}_{K}^{\times} / K^{\times}$in Lemma 4.2.4 satisfies the condition (103). The number of such models of $\left[E_{z}\right]_{\mathbb{C}}$ over $H_{K}$ is finite, once the complex structure (i.e., $\left[E_{z}\right]_{\mathbb{C}} \in \operatorname{Ell}\left(\mathcal{O}_{K}\right)$ ) and the Kähler parameter (i.e., $f_{\rho} \in \mathbb{N}_{>0}$ ) are fixed.

The same procedure also yields the $L$-functions of a finite number of models over $F_{K}^{[E]}$, if there is any such model within the restriction on $\mathfrak{c}_{f}$ set by $f_{\rho}$ through (103).

We have posed a question in Introduction how a $T^{2}$-target CFT, which depends only on the $\mathbb{C}$-isomorphism class $[E]_{\mathbb{C}}$ (and a Kähler parameter $\approx f_{\rho}$ ), can contain information of the $L$-functions of multiple different models of $[E]_{\mathbb{C}}$ defined over some number fields. When $[E]_{\mathbb{C}}$ has complex multiplication, the answer is now clear. Such a CFT has finitely many irreducible representations of the chiral algebra labeled by $\alpha \in i R e p s . \cong \Gamma_{+}^{\vee} / \Gamma_{+} \cong \Lambda_{\text {Cardy }} / \Lambda_{\text {winding }}$. Multiple different linear combinations of $\left\{f_{1}\left(\tau_{w s} ; \alpha\right) \mid \alpha \in\right.$ iReps. $\}$ yield the L-function of multiple different models of $[E]_{\mathbb{C}}$.

The other question posed in Introduction was how the $L$-function of a model $E / k$ become independent of the choice of a Kähler parameter $\left(f_{\rho}\right)$ in a string-theory realization. The procedure summarized in Thm. 4.3.1 allows us to reproduce the $L$-function of $E / k$ from the functions $\left\{f_{1}\left(\tau_{w s} ; \alpha\right) \mid \alpha \in i R e p s.\right\}$ of a string realization with the parameter $f_{\rho}$ that
can be chosen in infinitely many different ways. Although the set $i$ Reps. and the set of functions $\left\{f_{1}(\alpha) \mid \alpha \in i R e p s.\right\}$ depend on the choice of $f_{\rho}$, the $L$-function so computed should remain independent of $f_{\rho}$. The procedure achieves $f_{\rho}$-independence technically, although it is not clear how the independence comes about (without exploiting relations among theta functions).

Note also that the study in this article restricted string realizations of an elliptic curve $[E]_{\mathbb{C}}$ with complex multiplication to be within the class of diagonal rational CFT's; the extra constraint of being diagonal made our problem easier in that $f_{0}$ 's and $f_{1}$ 's are the same regardless of whether we use the left-moving, right-moving or the open string sector. The degeneracy among $f_{0}$ 's and $f_{1}$ 's in those three sectors also obscured, at the same time, how the metric independence has been achieved.

As a side remark, Thms. 4.2 .9 and 4.2 .11 can be used to see the following.
Theorem 4.3.3. Think of any diagonal rational $T^{2}$-target $\mathcal{N}=(2,2)$ SCFT where the parameter $f_{\rho}$ satisfies the condition (103) for some integral $\mathcal{O}_{K}$-ideal $\mathfrak{c}_{f}$ compatible with type $[1,0]$. Suppose that $\mathfrak{c}_{f}$ admits a non-trivial character $\chi_{f}:\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times} \rightarrow S^{1}$ satisfying the conditions (95) and (100). Then individual $f_{1}\left(\tau_{w s} ; x\right)$ 's with $x \in \mathfrak{a}(\mathfrak{K}) / \mathfrak{a}(\mathfrak{K}) \mathfrak{c}_{f}$ are modular forms of weight 2 for $\Gamma\left(4 N_{\mathfrak{a}(\mathfrak{F}) \mathfrak{c}_{f}}\right)$. Because the group $\Gamma\left(4 N_{\mathfrak{a}(\mathfrak{K}) \mathfrak{c}_{f}}\right) \subset \mathrm{SL}(2 ; \mathbb{Z})$ acts ordinarily on the combination $\tau=\tau_{w s} \frac{C^{2} a_{z}}{f_{\rho} D_{z}}$, the common subgroup $\Gamma\left(4 N_{\mathfrak{a}(\tilde{\kappa}) \mathfrak{c}_{f}}\right) \cap \operatorname{SL}(2 ; \mathbb{Z})_{w s}$ is

$$
\begin{equation*}
\operatorname{SL}(2 ; \mathbb{Z})_{w s} \cap\left[\operatorname{diag}\left(\frac{f_{\rho} D_{z}}{C^{2} a_{z}}, 1\right) \cdot \Gamma\left(4 N_{\mathfrak{a}(\mathfrak{\xi}) \mathfrak{c}_{f}}\right) \cdot \operatorname{diag}\left(\frac{C^{2} a_{z}}{f_{\rho} D_{z}}, 1\right)\right] . \tag{106}
\end{equation*}
$$

See also the comment at the end of section 3.4.1.
So long as we deal only with models defined over $F_{K}$ or $H_{K}$, and $[E] \in \operatorname{Ell}\left(\mathcal{O}_{K}\right)$, the following combination,

$$
\begin{equation*}
\sum_{\mathfrak{\beta} \in \mathrm{Cl} 1_{K}} \sum_{\alpha \in \mathfrak{a}(\tilde{\kappa}) / \mathfrak{b}_{z}} \chi_{f}(\pi(\alpha)) f_{1}\left(\tau_{w s} ; \alpha\right), \tag{107}
\end{equation*}
$$

where $\pi: \mathfrak{a}(\mathfrak{K}) / \mathfrak{b}_{z} \rightarrow \mathfrak{a}(\mathfrak{K}) / \mathfrak{a}(\mathfrak{K}) \mathfrak{c}_{f}$ is the projection, is a cusp form of weight 2 for $\Gamma_{0}\left(N_{\chi_{f}}\right)$ for some multiplier system (homomorphism) $\underline{\chi}: \Gamma_{0}\left(N_{\chi_{f}}\right) \rightarrow \mathbb{C}^{\times}$. •

What is interesting is that, in string theory, there is no theoretical motivation to think of a multiplication law among the irreducible representations of the chiral algebra $\mathcal{A}_{-} \times \mathcal{A}_{+}$ or $\mathcal{A}_{+}$. The addition law among $\mathcal{O}_{K} / \Omega\left(\Lambda_{\text {winding }}\right)$, not the multiplication law, corresponds
to the fusion algebra 38 on $i$ Reps. $=\Lambda_{\text {Cardy }} / \Lambda_{\text {winding }}$. Here, however, it is crucial to use the character of the group $\left[\mathcal{O}_{K} / \mathfrak{b}_{z}\right]^{\times}$in multiplication, not in the addition of $\mathcal{O}_{K} / \mathfrak{b}_{z}$, in order to construct the modular forms (107) of weight 2 for $\Gamma_{0}\left(N_{\chi_{f}}\right)$. As a reminder, though, individual $f_{1}\left(\tau_{w s} ; x\right)$ 's are modular forms for a group of the form $\Gamma\left(4 N_{\mathfrak{a}(\tilde{R})} \mathfrak{c}_{f}\right)$ without a sum with the character $\chi_{f}$, if not ${ }^{39}$ for a group of the form of $\Gamma_{0}(M)$ for some $M \in \mathbb{N}$ that scales as $N_{\mathfrak{a}(\mathfrak{R}) \mathfrak{c}_{f}}$.

Here, we illustrate by examples how Theorems 4.3.1 and 4.3.2 work in practice.
Example 4.3.4. For an imaginary quadratic field $K=\mathbb{Q}(\sqrt{-1}),\left[E_{z}\right]$ with $z=\sqrt{-1}=i$ is the only $\mathbb{C}$-isomorphism class of elliptic curves with complex multiplication by $\mathcal{O}_{K}$. It has multiple models defined over the Hilbert class field $H_{K}=K$, and those over $F_{K}=\mathbb{Q}$. A modulus $\mathfrak{c}_{f}$ of a Hecke character $\psi_{E / H_{K}}=\varphi$ for $H_{K}=K$ is compatible with the type [1,0] iff $\mathfrak{c}_{f} \chi(2)_{\mathcal{O}_{K}}$ in this case.

For example, two choices $\mathfrak{c}_{f}=(2+2 i)_{\mathcal{O}_{K}}$ and $\mathfrak{c}_{f}=(4)_{\mathcal{O}_{K}}$ have their own unique primitive multiplicative character $\chi_{f}$ satisfying (95) and (100) for $k=H_{K}=K$. Those two characters $\chi_{f}$ satisfy (101), so the corresponding two models can be defined over $\mathbb{Q}$. It is known that the two models correspond to defining equations $y^{2}=x^{3}-n^{2} x$ with $n=1,2$, respectively [Koblitz [11], Chap. II]. On the other hand, the choice $\mathfrak{c}_{f}=(2+i)_{\mathcal{O}_{K}}$ has a unique multiplicative character satisfying the two conditions (95, 100) for $k=H_{K}=K$, so there exists a model over $K$, but it is not obtained as a base change from a model over $\mathbb{Q}$, because (101) is not satisfied. When we choose $\mathfrak{c}_{f}=(3)_{\mathcal{O}_{K}}$, there are two inequivalent characters $\chi_{f}:\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times} \rightarrow S^{1}$ satisfying (95). The two characters do not satisfy (100), however. So, there is no model of $\left[E_{z=i}\right]_{\mathbb{C}}$ over $k=H_{K}=K$ with the conductor $\mathfrak{C}_{f}=(3)_{\mathcal{O}_{K}}$.

For both of the two models corresponding to $\mathfrak{C}_{f}=(2+2 i)_{\mathcal{O}_{K}}$ and $(4)_{\mathcal{O}_{K}}$, the condition (103) on the parameter $f_{\rho}$ of string realizations is $2 \mid f_{\rho}$; remember that $\Omega\left(\Lambda_{\text {winding }}\right) / f_{\rho}=(2)_{\mathcal{O}_{K}}$ for the case of $\left[E_{z}\right]$ here (see Table [2). In the minimal realization 40 using $f_{\rho}=2$, for example, there are 16 irreducible representations of the chiral algebra $\mathcal{A}_{-} \cong \mathcal{A}_{+}$. Out of the

[^25]Boltzmann-weighted sum of $\mathrm{U}(1)$-charges $f_{1}\left(\tau_{w s} ; \alpha\right)$ of those 16 representations, the following linear combinations yield the $L$-function through the Mellin transformation:

$$
\begin{align*}
& \frac{\left[f_{1}(1)-i f_{1}(i)-f_{1}(-1)+i f_{1}(-i)\right]}{4} \pm i \frac{\left[f_{1}(2+i)-i f_{1}(-1+2 i)-f_{1}(-2-i)+i f_{1}(1-2 i)\right]}{4} \\
& \quad=f_{1}(1) \pm i f_{1}(2+i) \tag{108}
\end{align*}
$$

here, only the second argument $\alpha \in i$ Reps. is retained in the expression above, and a complex number in a unit cell of $\mathbb{C} / \mathfrak{b}_{z}$ is used to refer to an irreducible representation $\alpha$ by exploiting the embedding $\Omega:$ iReps $\cong \Omega\left(\Lambda_{\text {Cardy }}\right) / \mathfrak{b}_{z} \subset K / \mathfrak{b}_{z} \subset \mathbb{C} / \mathfrak{b}_{z}$.

For the model over $K$ corresponding to the multiplicative character with $\mathfrak{C}_{f}=(2+i)_{\mathcal{O}_{K}}$, we find that string realizations with $5 \mid f_{\rho}$ need to be used to obtain the modular forms to be Mellin-transformed.

Example 4.3.5. For an imaginary quadratic field $K=\mathbb{Q}(\sqrt{-3})$, $E l l\left(\mathcal{O}_{K}\right)$ consists of just one $\mathbb{C}$-isomorphism class, $\left[E_{z}\right]_{\mathbb{C}}$ with $z=\zeta_{3}$, where $\zeta_{N}:=e^{2 \pi i / N}$. The three Gepner constructions, ${ }^{41} 1^{\otimes 3} /(\mathbb{Z} / 3), 1^{\otimes 3} /((\mathbb{Z} / 3) \times(\mathbb{Z} / 3))$ and $0^{\otimes 1} 1^{\otimes 1} 4^{\otimes 1} /(\mathbb{Z} / 6)$, all give rise to a single $\mathcal{N}=(2,2)$ SCFT, as one can see by computing and comparing the spectrum for those Gepner constructions; the common spectrum reveals that it is a string-theory realization of this $\left[E_{z}\right]_{\mathbb{C}}$ with $z=\zeta_{3}$, with $f_{\rho}=1$ (i.e., $\rho=\zeta_{3}$ ).

The compatibility condition with the type $[1,0]$ rules out any choice with $\mathfrak{c}_{f} \mid\left(1+\zeta_{6}\right)_{\mathcal{O}_{K}}$ or $\mathfrak{c}_{f} \mid(2)_{\mathcal{O}_{K}}$.

Ref. [22] picks up two models over $\mathbb{Q}$ for this $\mathbb{C}$-isomorphism class $\left[E_{z}\right]_{\mathbb{C}}$. Here is how we understand the two models. First, let us choose the modulus $\mathfrak{c}_{f}=(3)_{\mathcal{O}_{K}}$; there is just one unitary character of $\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times}$satisfying the conditions (95, 100) then. The corresponding Hecke character $\varphi=\psi: \mathbb{A}_{K}^{\times} / K^{\times} \rightarrow \mathbb{C}^{\times}$must be primitive, since any $\mathcal{O}_{K}$-ideals dividing $(3)_{\mathcal{O}_{K}}$ cannot be compatible with the type [1, 0]. This Hecke character $\psi_{E / K}$ therefore has a conductor $\mathfrak{C}_{f}=\mathfrak{c}_{f}=(3)_{\mathcal{O}_{K}}$. The condition (101) is satisfied, and hence this model $E / K$ is obtained as a base change from a model $E / \mathbb{Q}$. The $L$-function of this model $E / \mathbb{Q}$ is $\left[1 / 1^{s}-2 / 4^{s}-1 / 7^{s}+\cdots\right]$ and the conductor of the elliptic curve over $\mathbb{Q}$ is $N_{E / \mathbb{Q}}=(27)_{\mathbb{Z}}$,

[^26]where we used (89). This reproduces the $L$-function and $N_{E / Q}$ of one of the two models over $\mathbb{Q}$ discussed in [22].

The other model over $\mathbb{Q}$ is found by setting $\mathfrak{c}_{f}=\left(4\left(1+\zeta_{6}\right)\right)_{\mathcal{O}_{K}}$. There are four unitary characters of $\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times}$in this case, but one of them is induced from a unitary character for $\mathfrak{c}_{f}=\left(2\left(1+\zeta_{6}\right)\right)_{\mathcal{O}_{K}}$ and two others from two unitary characters for $\mathfrak{c}_{f}=(4)_{\mathcal{O}_{K}}$. There is just one unitary character where the choice of modulus $\mathfrak{c}_{f}=\left(4\left(1+\zeta_{6}\right)\right)_{\mathcal{O}_{K}}$ is primitive. So, there is a unique Hecke character for $K=\mathbb{Q}(\sqrt{-3})$ where the conductor is $\mathfrak{C}_{f}=\left(4\left(1+\zeta_{6}\right)\right)_{\mathcal{O}_{K}}$. It satisfies (101), so the corresponding model is obtained as a base change of a model $E / \mathbb{Q}$. The $L$-function and the conductor $N_{E / \mathbb{Q}}$ of such a model $E / \mathbb{Q}$ are computed to be $\left[1+4 / 7^{s}+\right.$ $\left.2 / 13^{s}-8 / 19^{s}-5 / 25^{s}+\cdots\right]$, and $N_{E / \mathbb{Q}}=(144)_{\mathbb{Z}}$, respectively. So, the other model of $\left[E_{z=\zeta_{3}}\right]_{\mathbb{C}}$ in [22] is reproduced.

In order to obtain the modular forms in the procedure in Theorem4.3.1, we need a string realization with $3 \mid f_{\rho}$, and $4 \mid f_{\rho}$ for the models corresponding to $\mathfrak{C}_{f}=(3)_{\mathcal{O}_{K}}$ and $\left(4\left(1+\zeta_{6}\right)\right)_{\mathcal{O}_{K}}$, respectively. •

There are nine imaginary quadratic fields $K=\mathbb{Q}\left(\sqrt{-d_{0}}\right)$ where $h\left(\mathcal{O}_{K}\right)=1\left(d_{0}=\right.$ $1,2,3,7,11,19,43,67,163)$. That is when just one linear combination of $f_{1}\left(\tau_{w s} ; \alpha\right)$ 's (for $\vartheta(\tau ; \varphi, \mathfrak{K})$ with $\left.\{\mathfrak{K}\}=\mathrm{Cl}_{K}\right)$ is enough in constructing the $L$-functions of models over $\mathbb{Q}$ or over $K$. The two examples above, where $K=\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, are both in this category. Let us also take a look at an example of imaginary quadratic fields other than these nine special ones.

Example 4.3.6. For an imaginary quadratic field $K=\mathbb{Q}(\sqrt{-5})$, a modulus $\mathfrak{c}_{f}$ of a Hecke character of $\mathbb{A}_{K}^{\times} / K^{\times}$is compatible with the type $[1,0]$ iff $\mathfrak{c}_{f} /(2)_{\mathcal{O}_{K}}$.

If we choose $\mathfrak{c}_{f}=(3,+1+\sqrt{5} i)_{\mathcal{O}_{K}}$, or $\mathfrak{c}_{f}=(3,-1+\sqrt{5} i)_{\mathcal{O}_{K}}$, there is one unique character $\chi_{f}$ satisfying (95, 100) for each case. $\pi^{*}\left(\mathfrak{c}_{f}\right)$ is a prime ideal of $\mathcal{O}_{H_{K}}$ for any one of the two choices of $\mathfrak{c}_{f}$ above, and hence $\psi=\varphi \circ \operatorname{Nm}_{H_{K} / K}$ is primitive; $\mathfrak{C}_{f}=\pi^{*}\left(\mathfrak{c}_{f}\right)$. Thus, there is one model over $H_{K}$ for each choice of $\mathfrak{c}_{f}$ and for each one of $\operatorname{Ell}\left(\mathcal{O}_{K}\right)$. The condition (103) implies that we need to employ string realizations with $3 \mid f_{\rho}$ in order to write down the inverse Mellin transform of the Hecke $L$-functions in terms of $f_{1}$ 's. The four models over $H_{K}$ here, corresponding to two possible $\mathfrak{c}_{f}$ 's and two in $\operatorname{Ell}\left(\mathcal{O}_{K}\right)$, cannot be obtained by the base change of a model over $F_{K}=\mathbb{Q}(\sqrt{5})$, because the two $\chi_{f}$ 's do not satisfy (101).

### 4.3.2 Models Defined over a Ramified Extension of $K$

Even for arithmetic models of elliptic curves with complex multiplication by a non-maximal order in general, or for models of elliptic curves with complex multiplication whose field of definition $k$ is an abelian extension over $K$ containing the ring class field as a proper subfield, we can use Thms. 4.2 .23 and 4.2 .24 to list up inequivalent models over a number field $k$ in the class we consider in this article. For those models, general algorithm for the computation of the $L$-functions reviewed in section 4.2.1 still works (as stated there); the relation (102) also holds for these models; therefore, the functions $f_{1}$ 's obtained from string realizations of these arithmetic models can be organized by using the character $\chi_{f}$ as linear combination coefficients so that the Mellin transform of the linear combinations become Hecke $L$-functions to be used in rebuilding the Hasse-Weil $L$-function.

The only one change we have to make is to replace the condition (103) on the parameter $f_{\rho}$ of string realizations by

$$
\begin{equation*}
\mathfrak{c}_{f} \cdot \operatorname{LCM}\left(\mathfrak{a}(\mathfrak{K})_{\mathfrak{k} \in \mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right)}\right) \supset \mathfrak{b}_{z}, \tag{109}
\end{equation*}
$$

because $\mathfrak{b}_{z}$ is a proper $\mathcal{O}_{f_{z}}$-ideal for elliptic curves in $\operatorname{Ell}\left(\mathcal{O}_{f_{z}}\right)$ and is not necessarily an $\mathcal{O}_{K}$ ideal, when $f_{z}>1$. We do not reiterate Thms. 4.3.1, 4.3.2 and the first half of 4.3 .3 modified for these general cases, because the necessary modification is obvious.

Example 4.3.7. For an imaginary quadratic field $K=\mathbb{Q}(\sqrt{-1}), k=K(\sqrt{3})$ is an abelian extension of $K=L_{f_{z}=1}=L_{f_{z}=2}$ as well as $L_{f_{z}=3}=K(\sqrt{3})$ (see Table (1). It is also isomorphic to the ray class field $L_{\mathfrak{m}_{f}}$ for $\mathfrak{m}_{f}=(3)_{\mathcal{O}_{K}}$.

Here, we look at models over $k=K(\sqrt{3})$ of elliptic curves in $\operatorname{Ell}\left(\mathcal{O}_{K}\right)=\left\{\left[E_{z=i}\right]\right\}$ and $\operatorname{Ell}\left(\mathcal{O}_{f_{z}=3}\right)=\left\{\left[E_{z=3 w_{K}}\right],\left[E_{z=\left(1+3 w_{K}\right) / 2}\right]\right\}$ where the Hecke characters $\varphi$ of $\mathbb{A}_{K}^{\times} / K^{\times}$in Lemma 4.2.4 have a conductor $\mathfrak{c}_{f}=(3)_{\mathcal{O}_{K}}$. We have already seen in Example 4.3.4 that $\mathfrak{c}_{f}=(3)_{\mathcal{O}_{K}}$ as a modulus is compatible with the type [1, 0]. Moreover, the condition (99) is satisfied for $f_{z}=3$; trivial for $f_{z}=1$.

There are two inequivalent characters $\chi_{f}:\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times} \rightarrow S^{1}$ satisfying (95)):

$$
\begin{array}{r}
\chi_{f}(1)=1, \quad \chi_{f}(i)=-i, \quad \chi_{f}(2)=-1, \quad \chi_{f}(2 i)=i, \\
\chi_{f}(1+i)=a, \quad \chi_{f}(2+i)=-i a, \quad \chi_{f}(2+2 i)=-a, \quad \chi_{f}(1+2 i)=i a \tag{111}
\end{array}
$$

where $a=e^{\pi i / 4}$ or $-e^{\pi i / 4}$. The two $\chi_{f}$ 's become the same, when they are restricted to $\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]_{k}^{\times}=\{[1],[2],[i],[2 i]\}$, however, and the value of $\chi_{f}$ remains within $\mathcal{O}_{K}^{\times}$(the condition (100) for $k)$; because $\pi^{*}\left((3)_{\mathcal{O}_{K}}\right)=\mathfrak{P}_{3}^{2}$ for a norm-3 prime ideal $\mathfrak{P}_{3}$ of $\mathcal{O}_{k}$, the support of
the conductor of $\psi=\varphi \circ \mathrm{Nm}_{k / K}$ cannot be different from that of $\pi^{*}\left(\mathfrak{c}_{f}\right)$; this means that there is just one model over $k$ for $\left[E_{z=i}\right]_{\mathbb{C}}$ (Thms. 4.2.23 and 4.2.24). There is also just one model over $k$ for each one of $\left[E_{z=3 w_{K}}\right]_{\mathbb{C}}$ and $\left[E_{\left(1+3 w_{K}\right) / 2}\right]_{\mathbb{C}}$, since the value of $\chi_{f}$ remains to be within $\mathcal{O}_{f_{z}}^{\times}=\{ \pm 1\}$ for $\left[\mathcal{O}_{f_{z}}\right] \cap\left[\mathcal{O}_{K} / \mathfrak{c}_{f}\right]^{\times}=\{[1],[2]\}$. The Hecke $L$-function of $\psi_{E / k}$ of $\mathbb{A}_{k}^{\times} / k^{\times}$ is given by $L\left(s, \psi_{E / k}\right)=\prod_{a \in\left\{ \pm e^{\pi i / 4}\right\}} L\left(s, \varphi_{a}\right)$ for all the three models over $k$; the two $\varphi_{a}$ 's are the Hecke characters of $\mathbb{A}_{K}^{\times} / K^{\times}$corresponding to the two $\chi_{f}$ 's. The Hecke $L$-function $L\left(s, \varphi_{a}\right)$ is given by the Mellin transform of appropriate linear combinations of $f_{1}$ 's of string realizations of the three models, if $3\left|f_{\rho}, 3\right| f_{\rho}$ and $3 \mid f_{\rho}$, respectively.

## 5 The $L$-function for $H_{e t}^{2}(E)$

The $L$-function is defined for an elliptic curve $E / k$ defined over a number field $k$ also in association with $H_{e t}^{2}(E)$. Because $L\left(H_{e t}^{2}(E), s\right)=\zeta_{k}(s-1)$, this $L$-function does not carry any information not contained in $L\left(H_{e t}^{0}(E), s\right)=\zeta_{k}(s)$.

If one is interested in reconstructing $L\left(H_{e t}^{2}(E), s\right)$ directly from some data available in string-theory realizations of $E / k$, than shifting the argument of $L\left(H_{e t}^{0}(E), s\right)$, then we can use

$$
\begin{equation*}
f_{2}\left(i t_{w s} ; \alpha\right):=(-i) \operatorname{Tr}_{V_{\alpha}^{o} ; R}\left[F e^{\pi i F} \Omega^{\prime} \bar{\Omega}^{\prime} q^{L_{0}-c / 24}\right], \quad q=e^{-2 \pi t_{w s}} \tag{112}
\end{equation*}
$$

for $\alpha \in i$ Reps., instead of $f_{0}$ 's. By replacing $f_{0}$ 's that appear in section 3 by $f_{2}$ 's, we simply ${ }^{42}$ obtain $\zeta_{k}(s-1)$.

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[^0]:    ${ }^{1}$ The second named author has been led to this question through the application of arithmetic ideas to the cosmological constant / gravitino mass problem in the context of flux compactification [18, 3, 1, 9].
    ${ }^{2}$ a caveat: topological string theory
    ${ }^{3}$ In a more mathematical language, elliptic curves used as a target space in string theory are objects in the category of Kähler manifolds (if duality is ignored). On the other hand, the $L$-function is defined for individual objects in categories of algebraic varieties defined over some number fields. There is no canonical functor in one way or the other. There is just correspondence between them, where infinitely many objects on one side correspond to infinitely many objects on the other.

[^1]:    ${ }^{4} D_{K}$ is the discriminant of an imaginary quadratic field $K=\mathbb{Q}\left(\sqrt{-d_{0}}\right)$. When $d_{0}$ satisfies $d_{0} \equiv 3 \bmod 4$, $-D_{K}=d_{0} ;$ when $d_{0} \equiv 1,2 \bmod 4$, however, $-D_{K}=4 d_{0}$.
    ${ }^{5}$ It is a quotient of the ray ideal class group $\mathrm{Cl}_{K}\left(\mathfrak{m}_{f}\right)$ with the modulus $\mathfrak{m}_{f}=\left(f_{z}\right)_{\mathcal{O}_{K}}$. The ray class field corresponding to the modulus $\mathfrak{m}_{f}=\left(f_{z}\right)_{\mathcal{O}_{K}}$ is denoted by $L_{\mathfrak{m}_{f}}=L_{\left(f_{z}\right)_{\mathcal{O}_{K}}}$ in this article, whereas $L_{f_{z}}$ stands for the abelian extension of $K$ satisfying $\operatorname{Gal}\left(L_{f_{z}} / K\right) \cong \mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}}\right) . L_{f_{z}}$ is a subfield of $L_{\left(f_{z}\right) \mathcal{O}_{K}}$. See [13, 19] for class field theory associated with elliptic curves.
    ${ }^{6}$ A string-theorist-friendly review on global class field theory is found in [16, 18. Systematic expositions on global class field theory are also found in such textbooks as [20, 12, 14, and a little more introductory ones in [13, 19].

[^2]:    ${ }^{7}$ If we do not impose this condition, any $[\rho] \in \mathcal{H} / \mathrm{SL}(2 ; \mathbb{Z})$ is fine.

[^3]:    ${ }^{8}$ Due to the fact that the pair of primitive sublattices $\Gamma_{-}$and $\Gamma_{+}$fit into the even unimodular lattice $I_{2,2}$, one can see that the discriminant groups and forms of $\Gamma_{-}[-1]$ and $\Gamma_{+}$are identical. This condition is still weaker than the existence of a lattice isometry $\Gamma_{-}[-1] \cong \Gamma_{+}$. See [7]. Furthermore, the existence of a lattice isometry $\Gamma_{-}[-1] \cong \Gamma_{+}$is still a weaker condition than the one for a rational $T^{2}$-target CFT to be diagonal.

[^4]:    ${ }^{9}$ With an abuse of notation, we also use $\varphi_{0}$ for this isomorphism.

[^5]:    ${ }^{10}$ Cardy states are certain class of boundary conditions to be imposed on string theories on a worldsheet with a boundary. It may sound strange to refer to a boundary condition as a "state," but it is legitimate, because choice of such a boundary condition can be regarded as a choice of a state in a vector space on which linear operators act, in a certain perspective in string theory. Because we do not need to exploit this perspective in this note, Cardy states can be understood as a certain class of choices of boundary conditions imposed on a string theory on a worldsheet with a boundary.
    In order to define Cardy states, one needs to specify the linear combination of operators from $\mathcal{A}_{-}$and $\mathcal{A}_{+}$ whose kernel the boundary states are in. The prefix " $\varphi_{0^{-}}$" specifies the linear combination.

[^6]:    ${ }^{11}$ In a more colloquial language, $X$ is a subvariety of a projective space over a number field $k$, given by a set of defining equations whose coefficients are in $k$.
    ${ }^{12}$ In a more colloquial language, this is an operation to include all the points where the affine coordinates take value in $k^{\prime}$ (although defining equations have coefficients in $k$ ).

[^7]:    ${ }^{13}$ The definition of the $L$-function is written down in [Shimura [27, 19.1-6] for the case $X$ is an abelian variety, and $i=1$; for more general cases, we need to refer to etale cohomology groups to define $L\left(H_{e t}^{i}(X), s\right)$. When $X$ is an elliptic curve, however, the $L$-function for $i=1$ has a more intuitive characterization in terms of the number of $\mathbb{F}_{\mathfrak{p}}$ (residue field)-points of the reduction of $X$ at a non-zero prime ideal $\mathfrak{p}$ of $k$.

[^8]:    ${ }^{14}$ Here, we think of a worldsheet made of a strip with width $\pi$ and length $2 \pi t_{w s}$; it is made periodic to be a cylinder in the " $2 \pi t_{w s}$ " direction.

[^9]:    ${ }^{15}$ The subscript $f$ here is a reminder that we are referring to the finite part of a modulus. Almost all the subscripts ${ }_{f}$ in this article, such as those in $\mathfrak{c}_{f}, \mathfrak{C}_{f}, \phi_{f}$, and $\chi_{f}$, are reminders for the finite part. When the same letter " f " is used in the form of $f_{z} \in \mathbb{N}$, and $L_{f_{z}}$, however, it originates from a German word meaning conductor.

[^10]:    ${ }^{16}$ Note also that there are only finitely many units in $\mathcal{O}_{K}$ when $K$ is an imaginary quadratic field. That is, $\#\left[\mathcal{O}_{K}^{\times}\right]<\infty$.

[^11]:    ${ }^{17}$ The level $N_{\Lambda}$ of an integral lattice $\Lambda$ is $N_{\Lambda}:=\operatorname{Min}\left\{N \in \mathbb{N}_{>0} \mid \Lambda^{*}[N]\right.$ is integral $\}$. Here, $\Lambda^{*}$ stands for the dual lattice of $\Lambda$.

[^12]:    ${ }^{18}[\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right)$ is the $\mathfrak{a}=\mathfrak{a}(\mathfrak{K})$ case of (46).
    ${ }^{19}$ Note that $[\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right) \subset \mathfrak{a}(\mathfrak{K})$ has a periodicity $\mathfrak{a}(\mathfrak{K}) \mathfrak{m}_{f}$ and hence $\mathfrak{a}(\mathfrak{K}) \mathfrak{m}_{f}$ or any abelian subgroup $\mathfrak{m}$ of it acts on $[\mathfrak{a}(\mathfrak{K})]_{1}\left(\mathfrak{m}_{f}\right)$ by the addition law.

[^13]:    ${ }^{20}$ In fact, this is a special case of a more general statement. For the $\mathbb{C}$-isomorphism class of elliptic curves that correspond to the trivial element in $\operatorname{Ell}\left(\mathcal{O}_{K}\right) \cong \mathrm{Cl}_{K}$ (that is not necessarily trivial), one can verify that $\mathfrak{b}_{z} / f_{\rho}=\mathfrak{d}_{K / \mathbb{Q}}$, the different of the extension $K / \mathbb{Q}$. The different $\mathfrak{d}_{E / F}$ associated with extension of number fields $E / F$ is always an integral ideal of $\mathcal{O}_{E}$. So, $\left(\mathfrak{a}([0])=\mathcal{O}_{K}\right) \mid \mathfrak{d}_{K / \mathbb{Q}}=\mathfrak{b}_{z} / f_{\rho}$.

[^14]:    ${ }^{21}$ For $\mathfrak{a}([0])=\mathcal{O}_{K}, N_{\mathfrak{a}(\mathfrak{\kappa})}=\left|D_{K / \mathbb{Q}}\right|$.

[^15]:    ${ }^{22}$ Note that $\mathrm{Cl}_{K}\left(\mathcal{O}_{f_{z}=3}\right) \cong \mathrm{Cl}_{K}\left(\left(f_{z}=3\right)_{\mathcal{O}_{K}}\right)$ in this example.
    ${ }^{23}$ See the explanation below (47) for the notation $\mathfrak{a}_{\mathbb{Z}}\left((f)_{\mathcal{O}_{K}}\right)$; here, we use it for $\mathfrak{a}=\mathfrak{a}(\mathfrak{K})$.

[^16]:    ${ }^{24} \mathrm{cf}$ footnote 5

[^17]:    ${ }^{25}$ also [Silverman [29, Thm.II.9.1. and II.9.2].
    ${ }^{26}$ also [Silverman [29, Thm. II.10.5].

[^18]:    ${ }^{27}$ If we wish to express $L(s, \psi)$ for a Hecke character $\psi$ of $\mathbb{A}_{k}^{\times}$of a number field $k$ with $[k: \mathbb{Q}]>2$ directly as the Mellin transform of a modular form, rather than through decomposing it as in (71), more general form of theta functions (for $L=k$ ) needs to be introduced; see [Neukirch [20], VII §8]. The general version of the theta functions, however, does not fit into the observation (102). For that reason, we do not exploit the general version of the theta functions and rely on Lemma 4.2.4 and Prop. 4.2.5 instead in this article.

[^19]:    ${ }^{28}$ Here, monomials $[w]_{\mathbf{p}}$ labeled by $\mathbf{p} \in(\mathbb{Z} / 2 \mathbb{Z})^{r_{1}} \times \mathbb{Z}^{\oplus r_{2}}$ are called spherical functions. The Poisson resummation (Fricke involution) formula is not messed up by insertion of such a monomial [8, 15, 20].

[^20]:    ${ }^{30}$ In a colloquial language, that is whether one can find a map between coordinates of $E$ and $E^{\prime}$ where the map in the form of polynomials in the coordinates have coefficients in the field $k$, not just in $\mathbb{C}$.
    ${ }^{31}$ It seems that there is no standard jargon / parametrization in referring to the notion we call "type" of a Hecke character. The parametrization in terms of $\mathbf{p}$ and $\mathbf{q}$ are chosen so that $\phi$ with $s^{r}=0$ reproduces the Hecke character / Grössen-character in [Neukirch [20, VII §6]; $\phi_{\infty}$ here and $\chi_{\infty}$ in [20] are in the relation

[^21]:    ${ }^{32}$ That is consistent with the fact that the $L$-function is determined from the associated Hecke character, as in (68).
    ${ }^{33}$ It is implicit in this condition that the ideal $\mathfrak{C}_{f}$ of $\mathcal{O}_{L_{f_{z}}}$ is invariant under $\rho_{[E]}$.

[^22]:    ${ }^{34}$ Notation: the group $\mathcal{O}_{L, 1}^{\times}\left(\mathfrak{c}_{f}\right)$ in this article corresponds to $\mathcal{O}^{\mathfrak{m}}$ in [20].

[^23]:    ${ }^{35}$ This property specifies $\phi_{\mathfrak{p}}$ 's with $\mathfrak{p} \mid \mathfrak{c}_{f}$ for $a_{\mathfrak{p}} \in L_{\mathfrak{p}}^{\times}$with $\operatorname{ord}_{\mathfrak{p}}\left(a_{\mathfrak{p}}\right)=0$, but $\phi_{\mathfrak{p}}\left(a_{\mathfrak{p}}\right)$ remains unspecified for $a_{\mathfrak{p}}$ with $\operatorname{ord}_{\mathfrak{p}}\left(a_{\mathfrak{p}}\right) \neq 0$. One should exploit the triviality of $\phi$ for $\alpha \in \mathcal{O}_{L}$ that is not prime to $\mathfrak{c}_{f}$ to determine $\phi_{\mathfrak{p}}$ completely.
    ${ }^{36}$ The ratio $\chi_{f, 1} / \chi_{f, 2}:\left[\mathcal{O}_{L} / \mathfrak{c}_{f}\right] \times \rightarrow S^{1}$ cannot retain more detailed information of $\phi_{1} / \phi_{2}$ than $\chi_{1} / \chi_{2}$ does, because $\chi_{f}$ is completely determined by $\chi$ (after a type is specified).

[^24]:    ${ }^{37}$ memo: $\chi^{\prime}$ in [11] corresponds to $\chi_{f}$ in this article, and $\tilde{\chi}$ in [11] to $\varphi$ or $\psi$ in this article.

[^25]:    ${ }^{38}$ Generally in a rational CFT, the fusion algebra introduces a structure of algebra into $\mathbb{Z}[i R e p s$.$] . In$ the case of a $T^{2}$-target rational CFT, however, the multiplication law on $\mathbb{Z}[i R e p s$.$] can be induced from an$ abelian group law on $i$ Reps.. The addition law referred to in the main text corresponds to this abelian group law on $i$ Reps..
    ${ }^{39}$ They are for a group $\Gamma_{0}(M)$ for an $M$ that scales as $N_{\mathfrak{a}(\mathfrak{K}) \mathfrak{c}_{f}}^{2}$.
    ${ }^{40}$ The $L$-function written down in $\left[22\right.$ for this $\mathbb{C}$-isomorphism class of elliptic curves, $\left[E_{z}\right]_{\mathbb{C}}$ with $z=i$, is the one for $\mathfrak{C}_{f}=(4)_{\mathcal{O}_{K}}$. The conductor of the elliptic curve over $\mathbb{Q}$ is $N_{E / \mathbb{Q}}=N_{\chi_{f}}=4 \times \operatorname{Nm}_{K / \mathbb{Q}}\left(\mathfrak{c}_{f}\right)=(64)_{\mathbb{Z}}$, as in [22]. The modular forms (108) -we know that they are (due to Theorem 4.3.3) - are obtained from string realizations with $f_{\rho}=2$ or any $2 \mid f_{\rho}$, not from the one with $f_{\rho}=1$.
    It should be noted, on the other hand, that the two Gepner constructions $0^{\otimes 1} 2^{\otimes 2} /(\mathbb{Z} / 4)$ and $0^{\otimes 1} 2^{\otimes 2} /((\mathbb{Z} / 4) \times$ $(\mathbb{Z} / 4 \mathbb{Z}))$ yield the same $\mathcal{N}=(2,2)$ SCFT, as one can see by computing the spectrum. This string realization corresponds to $z=\rho=i$, so $f_{\rho}=1$. See also footnote 41 .

[^26]:    ${ }^{41}$ An orbifold construction is a procedure of using a modular invariant SCFT to build a modular invariant SCFT that is different from the original one, as well as the SCFT so constructed. Gepner construction is a special version of that. In this article, we study the relation between the $L$-functions of a variety $X$ and the spectrum (i.e., $f_{0}$ 's and $f_{1}$ 's) of the $X$-target SCFT, for natural reason, instead of the spectrum of the SCFT (such as $N=2$ minimal models) which one may use to construct the $X$-target SCFT. The fact that multiple different Gepner constructions may give rise to the same geometry-target SCFT motivates the way we set the problem in this article.

[^27]:    ${ }^{42}$ The corresponding theta functions, however, do not fall into the category of congruent theta functions introduced in (72); the Poisson resummation formula is messed up by insertion of a monomial $|w|_{\mathbb{C}}^{2}$.

