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AN ABSTRACT APPROACH TO BOHR'S PHENOMENON

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Abstract. In 1914 Bohr discovered that there exists $r \in (0,1)$ such that if a power series converges in the unit disk and its sum has modulus less than 1, then for $|z| < r$ the sum of absolute values of its terms is again less than 1. Recently analogous results were obtained for functions of several variables. Our aim here is to present an abstract approach to the problem and show that Bohr's phenomenon occurs under very general conditions.

1. Introduction

The classical (improved) result of H. Bohr [4], which was put in final form by M. Riesz, I. Schur and F. Wiener reads as follows:

Theorem 1. *If a power series*

$$\sum_{k=0}^{\infty} c_k z^k \tag{1}$$

converges in the unit disk and its sum has modulus less than 1*, then*

$$\sum_{k=0}^{\infty} | c_k z^k | < 1 \tag{2}$$

in the disk $\{z : \ |z| < 1/3\}$ *and the constant* $1/3$ *cannot be improved.*

For holomorphic functions with positive real part the following analogous statement holds.

Theorem 2. *If the function*

$$f(z) \ = \ \sum_{k=0}^{\infty} c_k z^k, \qquad |z| < 1,$$

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has a positive real part and $f(0) > 0$, *then*

$$\sum_{k=0}^{\infty} | c_k z^k | < 2f(0)$$

in the disk $\{z : \ |z| < 1/3\}$ *and the constant* $1/3$ *cannot be improved.*

Proof. Obviously, it is enough to prove the statement in the case $f(0) = 1$. Then by Carathéodory's inequality [5] (see also [6], sect. 2.5) we have $|c_k| \le 2$, so for $|z| < 1/3$ it follows that

$$\sum_{k=0}^{\infty} | c_k z^k | < 1 + \sum_{k=1}^{\infty} 2(1/3)^k \ = \ 2.$$

On the other hand the Möbius function

$$f(z) = \frac{1+z}{1-z} \ = \ 1 + \sum_{1}^{\infty} 2z^k$$

has positive real part, and the sum of the moduli of the terms of its expansion equals 2 for $|z| = 1/3$, so the constant $1/3$ is the best possible.

Multidimensional analogues of Theorem 1 for Taylor expansions of functions on complete Reinhardt domains were considered in [3] and [1]. The authors showed in [2] that an analogous phenomenon occurs for a complex manifold M and expansions with respect to a certain basis in the space of analytic functions on M, provided the basis we consider not only exists, but has some additional properties. Our aim here is to obtain multidimensional generalizations of Theorems 1 and 2 in a more general setting and in the spirit of Functional Analysis.

2. General setting of the problems

Suppose M is a complex manifold. We denote by $H(M)$ the space of holomorphic functions on M and for any compact subset $K \subset M$ we set

$$|f|_K \ = \ \sup_K |f(z)|, \quad f \in H(M).$$

The system of seminorms $|f|_K, \ K \subset\subset M$, defines the topology of uniform convergence on compact subsets of M. Equipped with this norm system $H(M)$ is a nuclear Fréchet space (e.g. [7]).

Let $\|.\|_r, \ r \in (0,1)$, be one-parameter family seminorms in $H(M)$, that are continuous with respect to the topology of uniform convergence on compact subsets of M. We always assume in the following that

$$\|.\|_{r_1} \le \|.\|_{r_2} \quad \text{if} \quad r_1 \le r_2. \tag{3}$$

Consider the following problem:

Problem B_1. Is there an $r \in (0,1)$ and a $K \subset\subset M$ such that

$$\|f\|_r \le |f|_K \quad \forall\, f \in H(M)?$$

Remark. Obviously, one may state Problem B_1 for a sequence of seminorms.

Suppose $(\varphi_n)_{n=0}^{\infty}$ is a basis in the space $H(M)$. For each $K \subset\subset M$ and $f \in H(M)$, $f \ = \ \sum f_n \varphi_n$, we can consider the norms

$$\|f\|_K := \sum |f_n| |\varphi_n|_K . \tag{4}$$

Fix a point $z_0 \in M$; let $K_r \downarrow z_0$ as $r \to 0$ be a system of compact subsets of M shrinking to the point z_0. Theorem 5 in [2] says that Problem B_1 has a positive solution for the system of norms $\|f\|_{K_r}$, if, in addition, $\varphi_0 = 1$ and $\varphi_n(z_0) = 0$, $n \geq 1$.

We present here a generalization of this result. Its proof, as the proof of Theorem 5 in [2], depends on the following lemma, which gives an upper bound for the modulus of a function on a compact subset from the bound of its real part on a larger compact subset. It generalizes the theorem of Borel and Carathéodory (see [8]) for the disk case to arbitrary domains.

Lemma 3. *If G is an open domain on a complex manifold, then for any $z_0 \in G$ and $K \subset\subset G$ there exists a constant $C > 0$ such that whenever $f \in H(G)$ and $f(z_0) = 0$ we have*

$$\sup_K |f(z)| \ \leq C \sup_G Re f(z).$$

Proof. The set of holomorphic functions

$$\Phi \ = \ \{\varphi \in H(G) \ : \quad \sup_G |\varphi(z)| \leq 1, \ \varphi(z_0) = 0\}$$

is compact. Set

$$V_m \ = \ \{\varphi \in H(G) \ : \quad |\varphi|_K < m/(m+1)\}, \quad m = 1, 2, \ldots .$$

The sets V_m are open, $V_m \subset V_{m+1}$, and (since there is no non-zero constant function in Φ)

$$\Phi \subset \bigcup_m V_m.$$

Thus there exists m_0 such that $\Phi \subset V_{m_0}$.

Fix any $f \in H(G)$ such that $f(z_0) = 0$ and $\sup_G Re f(z) = 1$ (obviously, it is enough to prove the theorem for such functions). Consider the function $\varphi(z) = f(z)/(2 - f(z))$; then $\varphi \in \Phi$, so we have

$$|\varphi(z)| \ \leq \ m_0/(m_0 + 1) \quad \forall z \in K.$$

Therefore

$$(m_0 + 1)|f(z)| \ \leq \ m_0(2 + |f(z)|) \quad \forall z \in K;$$

hence $|f(z)| \leq 2m_0 \quad \forall z \in K$, which proves the statement.

Theorem 4. *Let M be a complex manifold, $z_0 \in M$ and $\|.\|_r$, $r \in (0, 1)$, be a one-parameter family of continuous seminorms in $H(M)$ such that* (3) *and the following conditions hold:*

(a) $\|f\|_r \to |f(z_0)| \quad$ *as* $\quad r \to 0$;
(b) $\|1\|_r = 1 \qquad \forall\, r \in (0, 1)$.

Then Problem B_1 has a positive solution.

Proof. Fix any $r_1 \in (0, 1)$. Since $\|.\|_{r_1}$ is a continuous seminorm, there exist a compact subset $K_1 \subset\subset M$ and $C_1 > 0$ such that

$$\|f\|_{r_1} \ \leq \ C_1 |f|_{K_1} \qquad \forall f \in H(M).$$

By Lemma 3 there exist a compact subset $K_2 \subset\subset M$ and $C_2 > 0$ such that

$$|f - f(z_0)|_{K_1} \ \leq \ C_2 \sup_{K_2} Re(f(z) - f(z_0)) \qquad \forall\, f \in H(M). \tag{5}$$

Fix $f \in H(M)$ such that $f \neq$ constant and $f(z_0) \geq 0$ (obviously, it is enough to prove the theorem for such functions). Then we have by (5)

$$\|f\|_r \leq f(z_0) + \|f - f(z_0)\|_r$$

$$\leq f(z_0) + \frac{\|f - f(z_0)\|_r}{|f - f(z_0)|_{K_1}} \cdot C_2(\sup_{K_2} |f(z)| - f(z_0)).$$

Obviously the statement will be proved, if we show that $\frac{\|f-f(z_0)\|_r}{|f-f(z_0)|_{K_1}} \to 0$ uniformly for $f \in H(M),\ f \neq$ constant, as $r \to 0$. Fix $\varepsilon > 0$ and a bounded domain G on M such that $K_1 \subset\subset G \subset\subset M$, and consider the sets

$$\Phi = \{\varphi \in H(G) : \quad \sup_G |\varphi(z)| = 1, \ \ \varphi(z_0) = 0\}$$

and

$$U_r = \{\varphi \in H(G) \ : \quad \|\varphi\|_r < \varepsilon |\varphi|_{K_1}\}.$$

It is easy to see that Φ is a compact subset of $H(G)$ and $U_r,\ r \in (0,1)$, is an open cover of Φ, due to (a). Since $U_{r_1} \supset U_{r_2}$ if $r_1 < r_2$ there exist r_0 such that $\Phi \subset U_{r_0}$. But then we have

$$\|\varphi\|_{r_0} < \varepsilon |\varphi|_{K_1} \qquad \forall\, \varphi \in \Phi,$$

which proves the statement.

Remark. In order to prove a weaker property, say $B_{1+\varepsilon}$:

$$\exists\, r \in (0,1),\ K \subset\subset M \ : \quad \|f\|_r \leq (1+\varepsilon)|f|_K,$$

one does not need Lemma 3. It is enough to use the triangle inequality and the fact that $\frac{\|f-f(z_0)\|_r}{|f-f(z_0)|_{K_1}} \to 0$ uniformly for $f \in H(M),\ f \neq$ constant, as $r \to 0$.

As we see, Problem B_1 has a positive solution for each system of seminorms satisfying the conditions (a) and (b). In view of the previous result we can consider the following problem:

Problem B_2. Let M be a complex manifold; find

$$\sup\{r : \quad \|f\|_r \leq \sup_M |f(z)| \quad \forall\, f \in H(M) \ \text{ and bounded}\}.$$

We call the finite solution of Problem B_2 the *Bohr radius*. Theorem 4 proves that the Bohr radius exists for any family of continuous seminorms $\|.\|_r, r \in (0,1)$, satisfying conditions (a) and (b).

Let us mention some known results from this point of view (with $z_0 = 0$):

1. Theorem 1 says that in case M coincides with the unit disk and

$$\|f\|_r = \sup_{|z|\leq r} \sum_n |c_n z^n|, \tag{6}$$

where $f(z) = \sum_n c_n z^n$ is the Taylor expansion of f, Problem B_2 has solution $r = 1/3$, i.e. in this case the Bohr radius equals 1/3.

2. Boas and Khavinson [3] considered Problem B_2 for the polydisk $U_1 = \{z \in C^n \ : |z_k| < 1,\ k = 1, \ldots, n\}$ and the system of norms

$$\|f\|_r = \sup_{r \cdot U_1} \sum_\alpha |c_\alpha z^\alpha|, \qquad f \in H(U_1), \tag{7}$$

where $r \cdot U_1$ is the homothetic transformation of U_1 with coefficient r, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $z^\alpha = {z_1}^{\alpha_1} \dots {z_n}^{\alpha_n}$ and $f(z) = \sum_\alpha c_\alpha z^\alpha$ is the Taylor expansion of f. Let $K_n(U_1)$ be the corresponding Bohr radius. Boas and Khavinson proved that for $n > 1$

$$\frac{1}{3\sqrt{n}} < K_n(U_1) < \frac{2\sqrt{\log n}}{\sqrt{n}} \tag{8}$$

and generalized the left-hand side of (8) for complete Reinhardt domains.

Let us note that the above estimates depend on n. In [1] for the unit hypercone $D^\circ = \{z : |z_1| + ... + |z_n| < 1\}$ estimates that do not depend on n were obtained. Namely, if $K_n(D^\circ)$ is the corresponding Bohr radius, then we have

$$\frac{1}{3e^{1/3}} < K_n(D^\circ) \leq 1/3\ . \tag{9}$$

3. Other multidimensional extensions of Problem B_2 were considered in [1]. Let D be a complete Reinhardt domain; consider the system of norms

$$\|f\|_r = \sum_\alpha \sup_{D_r} |c_\alpha z^\alpha| \qquad f \in H(D), \tag{10}$$

where $D_r = r \cdot D$ is the homothetic transformation of D. Let $B_n(D)$ denote the corresponding Bohr radius. In [1] it is proved that the inequality

$$1 - \sqrt[n]{\frac{2}{3}} < B_n(D) \tag{11}$$

holds for any complete bounded n-circular domain D; these estimates can be improved for concrete domains (unit ball, unit hypercone, complete Cartan's circular domains), and, moreover, one can also obtain estimates from above in these cases.

3. Bohr's phenomenon for holomorphic functions with positive real part

Analogous questions may be posed for the class of holomorphic functions with positive real part. Suppose again M is a complex manifold, and $z_0 \in M$. Let

$$P \ = \ \{f \in H(G) \ : \quad Ref(z) > 0, \ \ f(z_0) > 0\}. \tag{12}$$

For a given system of continuous seminorms in $H(M)$, $\|.\|_r$, $r \in (0,1)$, we consider the following problems:

Problem PB_1. Does there exist an $r \in (0,1)$ such that

$$\|f\|_r < 2f(z_0) \qquad \forall\, f \in P? \tag{13}$$

Problem PB_2. In case Problem PB_1 has a positive solution find

$$\sup\{r : \ \ \|f\|_r < 2f(z_0) \quad \forall\, f \in P\}. \tag{14}$$

We will again call the solution of Problem PB_2 the *Bohr radius*.

Problem PB_1 has a positive solution under a weaker condition on the norms compared to the case of Problem B_1 of the previous section.

Theorem 5. *Let M be a complex manifold, $z_0 \in M$ and $\|.\|_r$, $r \in (0,1)$, be a system of continuous seminorms in $H(M)$, satisfying* (3) *and the condition*

$$(\tilde{a}) \qquad \exists\, c \in (0,2) : \quad \|f\|_r \to cf(z_0) \quad as \quad r \to 0 \quad \forall\, f \in H.$$

Then there exists an $r \in (0,1)$ such that (13) *holds, i.e. Problem PB_1 has a positive solution.*

Proof. Obviously, it is enough to prove the theorem for functions belonging to the set

$$P_1 \;=\; \{f \in P\ : f(z_0) = 1\}.$$

The set P_1 is compact. Indeed, let (f_n) be a sequence in P_1. Consider the functions

$$g_n \;=\; \frac{1-f_n}{1+f_n}, \qquad n = 1, 2, \dots .$$

Then we have $|g_n(z)| < 1$ for all $z \in M$, so the sequence (g_n) has a convergent in $H(M)$ subsequence $g_{n_k} \to g$. Since $g_n(z_0) = 0\ \forall\, n$ we have $g(z_0) = 0$, therefore $|g(z)| < 1$ for all $z \in M$. Hence

$$f_{n_k} = \frac{1-g_{n_k}}{1+g_{n_k}} \to \frac{1-g}{1+g}\ \in P_1.$$

The family of sets

$$V_r \;=\; \{f \in H(M)\ :\quad \|f\|_r < 2\}, \quad r \in (0,1),$$

is an open cover of P_1 (due to $(\tilde{a})$), and moreover, $V_{r_1} \supset V_{r_2}$ if $r_1 < r_2$. Therefore there exists r such that $P_1 \subset V_r$, which proves the theorem.

Theorems 1 and 2 show that in the case when M is the unit disk, $z_0 = 0$ and the system of norms as in (6), Bohr radii of Problems B_2 and PB_2 coincide and is 1/3.

Our next results shows that Bohr radii of Problems B_2 and PB_2 are equal in more general situations.

Theorem 6. *Let M be a complex manifold, $z_0 \in M$ and $\|.\|$ be a continuous semi-norm in $H(M)$ such that for all $f, g \in H(M)$,*

(i) $\|f\| \;=\; |f(z_0)| \;+\; \|f - f(z_0)\|$;
(ii) $\|f \cdot g\| \;\le\; \|f\| \cdot \|g\|$.

Then the following statements are equivalent:

(PB) $\|f\| \le 2f(z_0)$ *if* $Ref(z) > 0 \quad \forall\, z \in M$ *and* $f(z_0) > 0$;
(B) $\|f\| \le \sup_M |f(z)| \qquad \forall\, f \in H(M)$.

Proof. First we show that $(B) \Rightarrow (PB)$. Fix $g \in H(M)$ such that $Reg(z) > 0$ and $g(z_0) = 1$ (obviously, it is enough to prove (PB) for such functions). Set

$$f_\varepsilon \;=\; \frac{1-\varepsilon g}{1+\varepsilon g}, \qquad \varepsilon \in (0,1).$$

Then $|f_\varepsilon(z)| < 1\ \forall\, z \in M$; thus we have (by (i) and (B)) $\|f_\varepsilon\| = f_\varepsilon(z_0) + \|f_\varepsilon - f_\varepsilon(z_0)\| \le 1$, so we have $\|f_\varepsilon - f_\varepsilon(z_0)\| \le 1 - f_\varepsilon(z_0)$. Moreover, since $f_\varepsilon(z_0) \;=\; (1-\varepsilon)/(1+\varepsilon)$ it follows that

$$\|1 - f_\varepsilon\| = 1 - f_\varepsilon(z_0) + \|f_\varepsilon(z_0) - f_\varepsilon\| \le 2(1 - f_\varepsilon(z_0)) \;=\; \frac{4\varepsilon}{1+\varepsilon}.$$

On the other hand

$$\varepsilon g \;=\; \frac{1-f_\varepsilon}{1+f_\varepsilon} \;=\; \sum_{n=1}^{\infty} \left(\frac{1-f_\varepsilon}{2}\right)^n;$$

hence by (ii)

$$\|\varepsilon g\| \;\le\; \sum_{n=1}^{\infty} \left\|\frac{1-f_\varepsilon}{2}\right\|^n \;\le\; \sum_{n=1}^{\infty} \left(\frac{2\varepsilon}{1+\varepsilon}\right)^n \;=\; \frac{2\varepsilon}{1-\varepsilon}.$$

From here it follows that

$$\|g\| \ \le\ \frac{2}{1-\varepsilon} \qquad \forall\, \varepsilon \in (0,1).$$

Letting $\varepsilon \to 0$ we obtain $\|g\| \le 2$.

Next we prove the implication $(PB) \Rightarrow (B)$. Fix $f \in H(M)$ such that $\sup\{|f(z)|, z \in M\} = 1$ and $0 \le f(z_0) < 1$ (it is enough to prove the statement for such functions). Then the function $g = (1-f)/(1+f)$ has positive real part and, in addition, by (i) and (PB) we have

$$\|1-g\| = 1 - g(z_0) + \|g(z_0) - g\| \ \le\ 1 - g(z_0) + g(z_0) \ =\ 1.$$

On the other hand

$$f \ =\ \frac{1-g}{1+g} \ =\ \sum_{n=1}^{\infty} \left(\frac{1-g}{2}\right)^n;$$

hence from (ii) it follows that

$$\|f\| \ \le\ \sum_{n=1}^{\infty} \left\|\frac{1-g}{2}\right\|^n \ =\ \frac{\|1-g\|}{2-\|1-g\|} \ \le\ 1.$$

As a corollary we immediately obtain the following theorem.

Theorem 7. *Let M be a complex manifold, $z_0 \in M$ and $\|.\|_r$, $r \in (0,1)$, be a family of continuous seminorms in $H(M)$ such that conditions*

(a) $\|f\|_r \to |f(z_0)|$ *as* $r \to 0$;
(b) $\|f\|_r \ =\ |f(z_0)| + \|f - f(z_0)\|_r$;
(c) $\|f \cdot g\|_r \ \le\ \|f\|_r \cdot \|g\|_r \quad \forall\, r \in (0,1)$

hold. Then the Bohr radii corresponding to Problems B_2 and PB_2 are equal.

Of course, in general, one should expect that Bohr radii corresponding to Problems B and PB are different. The next example confirms this.

Example 1. Let $\Delta = \{z : |z| < 1\}$ be the unit disk. Consider in the space $H(\Delta)$ the system of norms

$$\|f\|_r \ =\ \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta\right)^{1/2}, \qquad r \in (0,1).$$

Obviously for each $r \in (0,1)$ we have

$$\|f\|_r \ \le \sup_{\Delta} |f(z)| \quad \forall\, f \in H(\Delta).$$

Thus the Bohr radius corresponding to Problem B equals 1.

Next we compute the Bohr radius corresponding to Problem PB. For any holomorphic function $f(z) = c_0 + \sum_1^\infty c_n z^n$ with positive real part and $c_0 > 0$ we have by Carathéodory inequality $|c_n| \le 2c_0$, whence it follows that

$$\|f\|_r^2 \ \le\ c_0^2 \left(1 + 4\sum_{n=1}^{\infty} r^{2n}\right) \ =\ c_0^2 \left(1 + \frac{4r^2}{1-r^2}\right).$$

Since the expression in the parentheses is less than 4 for $r < \sqrt{3/7}$ the corresponding Bohr radius is larger than or equal to $\sqrt{3/7}$.

On the hand, for the function

$$g(z) \ = \ (1+z)/(1-z) \ = \ 1 + 2\sum_1^\infty z^n$$

we obtain $\|g\|_r^2 = 1 + 4r^2/(1-r^2) > 4$ if $r > \sqrt{3/7}$. Hence the Bohr radius of Problem PB_2 equals $\sqrt{3/7}$.

The following examples show that both Problems B_1 and PB_1 may have negative solutions, or Problem B_1 may have negative solution and simultaneously Problem PB_1 may have positive solution.

Example 2. It is easy to see that the system of functions

$$\{1, \ \frac{z-1}{2}, \ z^2, \ z^3, \dots \}$$

is a basis in the space $H(\Delta)$. Consider the family of norms that corresponds to this basis by (7), $\|f\|_{K_r}, r \in (0,1)$, where $K_r = \{z : |z| \le r\}$. Then for this family both Problems B_1 and PB_1 have negative solutions.

Indeed, we have $(z+1)/2 = 1 + (z-1)/2$, thus for every $r \in (0,1)$

$$\|(z+1)/2\|_{K_r} = 1 + |(z-1)/2|_{K_r} > 1.$$

Of course, in this example condition $(\tilde{a})$ of Theorem 5 (as well condition (a) of Theorem 4) is violated.

Example 3. Consider the following basis in $H(\Delta)$:

$$\{\frac{z+1}{2}, \ z, \ z^2, \ z^3, \dots \}.$$

Then for the family of norms $\|f\|_{K_r}, \ r \in (0,1)$, corresponding by (7) to this basis, Problem B_1 has negative solution, but Problem PB_1 has positive solution.

Indeed, we have $1 = 2 \cdot (z+1)/2 - z$, therefore $\|1\|_{K_r} = 1 + 2r > 1$ for any $r \in (0,1)$.

On the other hand, if $Ref(z) > 0$ and $f(z) = \sum_0^\infty c_n z^n$ is the Taylor expansion of the function f, then

$$f \ = \ 2c_0 \frac{z+1}{2} + (c_1 - c_0)z + \sum_2^\infty c_n z^n$$

is the expansion of f with respect to the basis we consider. From here, using Carathèodory inequality $|c_k| \le 2c_0$ we obtain

$$\|f\|_{K_r} = c_0(1+r) + |c_1 - c_0| r + \sum_2^\infty |c_n| r^n \le c_0(1 + 4r + \frac{r^2}{1-r}).$$

Since the expression in the parentheses is less than 2 for $r < (5 - \sqrt{13})/6$ the problem PB_1 has a positive solution, and moreover the corresponding Bohr radius is larger than or equal to $(5 - \sqrt{13})/6$.

Finally, since the class P contains unbounded functions, we can easily give an example, where Problem B_1 has a positive solution, but Problem PB_1 has negative solution.

Example 4. Consider in the space $H(\Delta)$ the system of norms

$$\|f\|_r = \sup\{|f(z)| \ : \quad |\, z - 3/4\,| \le r/4\,\}, \quad r \in (0,1).$$

Then, obviously Problem B_1 has a positive solution and the corresponding Bohr radius equals 1. On the other hand, Problem PB_1 has negative solution, because for the function $g = (1+z)/(1-z)$ we have $\|g\|_r > 7$ for any $r \in (0,1)$. This example also shows that one cannot replace (i) of Theorem 6 by the weaker condition (b) of Theorem 4.

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References

[1] L. Aizenberg, Multidimensional analogues of Bohr's theorem on power series, Proc. Amer. Math. Soc., to appear. CMP 98:16
[2] L. Aizenberg, A. Aytuna, P. Djakov, Generalization of Bohr's theorem for arbitrary bases in spaces of holomorphic functions of several variables, preprint METU, Mathematics 98/168.
[3] H. P. Boas, D. Khavinson, Bohr's power series theorem in several variables, Proc. Amer. Math. Soc. **125** (1997), 2975-2979. MR **98i:**32002
[4] H. Bohr, A theorem concerning power series, Proc. London Math. Soc. (2) **13** (1914) 1-5.
[5] C. Carathéodory, Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene werte nicht annemen, Math. Ann. **64** (1907), 95-115.
[6] P. L. Duren, Univalent Functions, Springer, 1983. MR **85j:**30034
[7] R. Meise, D. Vogt, Introduction to Functional Analysis, Clarendon Press, Oxford 1997. MR **98g:**46001
[8] E. C. Titchmarsh, The theory of functions, Oxford University Press, 1939.

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