Monopole Equations on 8-Manifolds with Spin(7) Holonomy

Ayşe Hümeyra Bilge
Department of Mathematics
İstanbul Technical University
İstanbul, Turkey
E.mail : bilge@hidiv.cc.itu.edu.tr
Tekin Dereli
Department of Physics
Middle East Technical University
Ankara, Turkey
E.mail : tekin@dereli.physics.metu.edu.tr
Şahin Koçak
Department of Mathematics
Anadolu University
Eskişehir, Turkey
E.mail : skocak@vm.baum.anadolu.edu.tr

Abstract

We construct a consistent set of monopole equations on eight-manifolds with Spin(7) holonomy. These equations are elliptic and admit non-trivial solutions including all the 4-dimensional Seiberg-Witten solutions as a special case.

1. Introduction

In a remarkable paper [1] Seiberg and Witten have shown that diffeomorphism invariants of 4-manifolds can be found essentially by counting the number of solutions of a set of massless, Abelian monopole equations [2],[3]. It is later noted that topological quantum field theories which are extensively studied in this context in 2, 3 and 4 dimensions also exist in higher dimensions [4],[5],[6],[7]. Therefore it is of interest to consider monopole equations in higher dimensions and thus generalizing the 4-dimensional Seiberg-Witten theory.

In fact Seiberg-Witten equations can be constructed on any even dimensional manifold (D=2n) with a spin^c-structure [8]. But there are problems. The self-duality of 2-forms plays an eminent role in 4-dimensional theory and we encounter projection maps $\rho^+(F_A) = \rho^+(F_A^+) = \rho(F_A^+)$ (see the next section). The first projection $\rho^+(F_A)$ is meaningful in any dimension $2n \geq 4$. However, a straightforward generalization of the Seiberg-Witten equations using this projection yields an over determined set of equations having no non-trivial solutions even locally [9]. To use the other projections, one needs an appropriately generalized notion of self-dual 2-forms. On the other hand there is no unique definition of self-duality in higher than four dimensions. In a previous paper [10] we reviewed the existing definitions of self-duality and gave an eigenvalue criterion for specifying self-dual 2-forms on any even dimensional manifold. In particular, in $D = 8$ dimensions, there is a linear notion of self-duality defined on 8-manifolds with Spin(7) holonomy [11],[12]. This corresponds to a specific choice of a maximal linear subspace in the set of (non-linear) self-dual 2-forms as defined by our eigenvalue criterion [13]. Eight dimensions is special because in this particular case the set of linear Spin(7) self-duality equations can be solved by making use of octonions [14]. The existence of *octonionic instantons* which realise the last Hopf fibration $S^{15} \rightarrow S^8$ is closely related with the properties of the octonion algebra [15],[16],[17].

Here we use this linear notion of self-duality to construct a consistent set of Abelian monopole equations on 8-manifolds with Spin(7) holonomy. These equations turn out to be elliptic and locally they admit non-trivial solutions which include all 4-dimensional Seiberg-Witten solutions as a special case. But before giving our 8-dimensional monopole equations, we first wish in the next section to give the set up and generalizations of 4-dimensional Seiberg-Witten equations to arbitrary even dimensional manifolds with spin^c-structure as proposed by Salamon [8]. This is going to help us put our monopole equations into their proper context. We also wish to note that any 8-manifold with Spin(7) holonomy is automatically a spin manifold [18],[19] and thus carries a spin^c-structure; making the application of the general approach possible. In fact our monopole equations can always be expressed purely in the real realm, but in order to relate them to the 4-dimensional Seiberg-Witten equations, it is preferable to use the spin^c-structure and complex spinors.

2. Definitions and notation

A spin^c-structure on a $2n$-dimensional real inner-product space V is a pair (W, Γ), where W is a 2^n-dimensional complex Hermitian space and $\Gamma : V \to End(W)$ is a linear map satisfying

$$\Gamma(v)^* = -\Gamma(v), \qquad \Gamma(v)^2 = -\|v\|^2$$

for $v \in V$. Globalizing this defines the notion of a spin^c-structure $\Gamma : TX \to End(W)$ on a $2n$-dimensional (oriented) manifold X, W being a 2^n-dimensional complex Hermitian vector bundle on X. Such a structure exists if and only if $w_2(X)$ has an integral lift. Γ extends to an isomorphism between the complex Clifford algebra bundle $C^c(TX)$ and $End(W)$. There is a natural splitting $W = W^+ \oplus W^-$ into the $\pm i^n$ eigenspaces of $\Gamma(e_{2n}e_{2n-1}\cdots e_1)$ where $e_1, e_2, \cdots, e_{2n}$ is any positively oriented local orthonormal frame of TX.

The extension of Γ to $C_2(X)$ gives, via the identification of $\Lambda^2(T^*X)$ with $C_2(X)$, a map

$$\rho : \Lambda^2(T^*X) \to End(W)$$

given by

$$\rho(\sum_{i<j} \eta_{ij} e_i^* \wedge e_j^*) = \sum_{i<j} \eta_{ij} \Gamma(e_i)\Gamma(e_j).$$

The bundles $W^\pm$ are invariant under $\rho(\eta)$ for $\eta \in \Lambda^2(T^*X)$. Denote $\rho^\pm(\eta) = \rho(\eta)|_{W^\pm}$. The map ρ (and $\rho^\pm$) extends to

$$\rho : \Lambda^2(T^*X) \otimes \mathbf{C} \to End(W).$$

(If $\eta \in \Lambda^2(T^*X) \otimes \mathbf{C}$ is real-valued then $\rho(\eta)$ is skew-Hermitian and if η is imaginary-valued then $\rho(\eta)$ is Hermitian.) A Hermitian connection ∇ on W is called a spin^c connection (compatible with the Levi-Civita connection) if

$$\nabla_v(\Gamma(w)\Phi) = \Gamma(w)\nabla_v\Phi + \Gamma(\nabla_v w)\Phi$$

where Φ is a spinor (section of W), v and w are vector fields on X and $\nabla_v w$ is the Levi-Civita connection on X. ∇ preserves the subbundles $W^{\pm}$. There is a principal $\mathrm{Spin}^c(2n) = \{e^{i\theta}x | \theta \in \mathbf{R}, x \in Spin(2n)\} \subset C^c(\mathbf{R}^{2n})$ bundle P on X such that W and TX can be recovered as the associated bundles

$$W = P \times_{Spin^c(2n)} \mathbf{C}^{2^n}, \qquad TX = P \times_{Ad} \mathbf{R}^{2n},$$

Ad being the adjoint action of $\mathrm{Spin}^c(2n)$ on $\mathbf{R}^{2n}$. We get then a complex line bundle $L_\Gamma = P \times_\delta \mathbf{C}$ using the map $\delta : Spin^c(2n) \to S^1$ given by $\delta(e^{i\theta}x) = e^{2i\theta}$.

There is a one-to-one correspondence between spin^c connections on W and spin^c(2n)=Lie(Spin^c(2n)=spin(2n)$\oplus i\mathbf{R}$ -valued connection 1-forms $\hat{A} \in \mathbf{A}(P) \subset \Omega^1(P,$spin^c(2n)) on P. Now consider the trace-part A of $\hat{A}$: $A = \frac{1}{2^n} trace(\hat{A})$. This is an imaginary valued 1-form $A \in \Omega^1(P, i\mathbf{R})$ which is equivariant and satisfies

$$A_p(p \cdot \xi) = \frac{1}{2^n} trace(\xi)$$

for $v \in T_pP, g \in$ Spin^c(2n),$\xi \in$ spin^c(2n) (where $p \cdot \xi$ is the infinitesimal action). Denote the set of imaginary valued 1-forms on P satisfying these two properties by $\mathbf{A}(\Gamma)$. There is a one-to-one correspondence between these 1-forms and spin^c connections on W. Denote the connection corresponding to A by ∇_A. $\mathbf{A}(\Gamma)$ is an affine space with parallel vector space $\Omega^1(X, i\mathbf{R})$. For $A \in \mathbf{A}(\Gamma)$, the 1-form $2A \in \Omega^1(P, i\mathbf{R})$ represents a connection on the line bundle L_Γ. Because of this reason A is called a *virtual connection* on the *virtual line bundle* $L_\Gamma^{1/2}$. Let $F_A \in \Omega^2(X, i\mathbf{R})$ denote the curvature of the 1-form A. Finally, let D_A denote the Dirac operator corresponding to $A \in \mathbf{A}(\Gamma)$,

$$D_A : C^\infty(X, W^+) \to C^\infty(X, W^-)$$

defined by

$$D_A(\Phi) = \sum_{i=1}^{2n} \Gamma(e_i) \nabla_{A,e_i}(\Phi)$$

where $\Phi \in C^\infty(X, W^+)$ and $e_1, e_2, \cdots, e_{2n}$ is any local orthonormal frame.

The Seiberg-Witten equations can now be expressed as follows. Fix a spin^c-structure $\Gamma : TX \to End(W)$ on X and consider the pair $(A, \Phi) \in \mathbf{A}(\Gamma) \times C^\infty(X, W^+)$. The Seiberg- Witten equations read

$$D_A(\Phi) = 0 \ , \qquad \rho^+(F_A) = (\Phi\Phi^*)_0$$

where $(\Phi\Phi^*)_0 \in C^\infty(X, End(W^+))$ is defined by $(\Phi\Phi^*)(\tau) = <\Phi, \tau> \Phi$ for $\tau \in C^\infty(X, W^+)$ and $(\Phi\Phi^*)_0$ is the traceless part of $(\Phi\Phi^*)$.

3. Seiberg-Witten equations on 4-manifolds

Before going over to 8-manifolds, we first show that the Seiberg-Witten equations on 4-manifolds (Ref.[8], p.232) can be rewritten in a different form. The Dirac equation

$$D_A(\Phi) = 0 \tag{1}$$

can be explicitly written as

$$\nabla_1\Phi = I\nabla_2\Phi + J\nabla_3\Phi + K\nabla_4\Phi, \tag{2}$$

and

$$\rho^+(F_A) = (\Phi\Phi^*)_0 \tag{3}$$

is equivalent to the set

$$\begin{aligned} F_{12} + F_{34} &= -1/2\Phi^* I\Phi, \\ F_{13} - F_{24} &= -1/2\Phi^* J\Phi, \\ F_{14} + F_{23} &= -1/2\Phi^* K\Phi, \end{aligned} \tag{4}$$

where $\Phi : \mathbf{R}^4 \to \mathbf{C}^2$, $\nabla_i\Phi = \frac{\partial\Phi}{\partial x_i} + A_i\Phi$,
$A = \sum_{i=1}^4 A_i dx_i \in \Omega^1(\mathbf{R}^4, i\mathbf{R})$, $F_A = \sum_{i<j} F_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbf{R}^4, i\mathbf{R})$,
and

$$I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, K = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

In the most explicit form, these equations can be written as

$$\begin{aligned} \frac{\partial\phi_1}{\partial x_1} + A_1\phi_1 &= i(\frac{\partial\phi_1}{\partial x_2} + A_2\phi_1) + \frac{\partial\phi_2}{\partial x_3} + A_3\phi_2 + i(\frac{\partial\phi_2}{\partial x_4} + A_4\phi_2), \\ \frac{\partial\phi_2}{\partial x_1} + A_1\phi_2 &= -i(\frac{\partial\phi_2}{\partial x_2} + A_2\phi_2) - (\frac{\partial\phi_1}{\partial x_3} + A_3\phi_1) + i(\frac{\partial\phi_1}{\partial x_4} + A_4\phi_1) \end{aligned} \tag{5}$$

(for $D_A(\Phi) = 0$) and

$$\begin{aligned} F_{12} + F_{34} &= -i/2(\phi_1\bar{\phi}_1 - \phi_2\bar{\phi}_2), \\ F_{13} - F_{24} &= 1/2(\phi_1\bar{\phi}_2 - \phi_2\bar{\phi}_1), \\ F_{14} + F_{23} &= -i/2(\phi_1\bar{\phi}_2 + \phi_2\bar{\phi}_1) \end{aligned} \tag{6}$$

(for $\rho^+(F_A) = (\Phi\Phi^*)_0$).

We will reinterpret the second part of these equations in the following way: The 6-dimensional bundle of real-valued 2-forms on $\mathbf{R}^4$ has a 3-dimensional subbundle of self-dual forms with orthogonal basis

$$\begin{aligned} f_1 &= dx_1 \wedge dx_2 + dx_3 \wedge dx_4, \\ f_2 &= dx_1 \wedge dx_3 - dx_2 \wedge dx_4, \\ f_3 &= dx_1 \wedge dx_4 + dx_2 \wedge dx_3, \end{aligned} \tag{7}$$

in each fiber with respect to the usual metric. These forms span a 3-dimensional complex subbundle of the bundle of complex-valued 2-forms. The projection of a (global) 2-form $F = \sum F_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbf{R}^4, i\mathbf{R})$ onto this complex subbundle is given by

$$F^+ = 1/2(F_{12} + F_{34}) f_1 + 1/2(F_{13} - F_{24}) f_2 + 1/2(F_{14} + F_{23}) f_3. \tag{8}$$

We have $\rho^+(f_1) = 2I, \rho^+(f_2) = 2J, \rho^+(f_3) = 2K$, so that,

$$\rho^+(F^+) = (F_{12} + F_{34})I + (F_{13} - F_{24})J + (F_{14} + F_{23})K. \tag{9}$$

On the other hand, the orthogonal projection $(\Phi\Phi^*)^+$ of $\Phi\Phi^*$ onto the subbundle of the positive spinor bundle generated by the (Hermitian-) orthogonal basis $(\rho^+(f_1), \rho^+(f_2), \rho^+(f_3))$ is given by

$$\begin{aligned} &< 2I, \Phi\Phi^* > 2I/|2I|^2 + < 2J, \Phi\Phi^* > 2J/|2J|^2 + < 2K, \Phi\Phi^* > 2K/|2K|^2 \\ &= \frac{1}{2} < I, \Phi\Phi^* > I + \frac{1}{2} < J, \Phi\Phi^* > J + \frac{1}{2} < K, \Phi\Phi^* > K. \end{aligned} \tag{10}$$

Since

$$< I, \Phi\Phi^* >= -\Phi^* I \Phi, \quad < J, \Phi\Phi^* >= -\Phi^* J \Phi, \quad < K, \Phi\Phi^* >= -\Phi^* K \Phi, \tag{11}$$

this shows that the second part of the Seiberg-Witten equations can be expressed as follows: Given any (global, imaginary-valued) 2-form F, the image under the map ρ^+ of its self-dual part F^+ coincides with the orthogonal projection of $\Phi\Phi^*$ onto the subbundle of the positive spinor bundle which is the image bundle of the complexified subbundle of self-dual 2-forms under the map ρ^+, that is,

$$\rho^+(F^+) = (\Phi\Phi^*)^+. \tag{12}$$

Indeed, in the present case $(\Phi\Phi^*)^+$ is nothing else than $(\Phi\Phi^*)_0$. In this modified form the Seiberg-Witten equations allow a tempting generalisation.

Suppose we are given a subbundle $S \subset \Lambda^2(T^*X)$. Denote the complexification of S by S^*, the projection of an imaginary valued 2-form field F onto S^* by F^+ and the projection of $\phi\phi^*$ onto $\rho^+(S^*)$ by $(\phi\phi^*)^+$. Then the equation $\rho^+(F^+) = (\phi\phi^*)^+$ can be taken as a substitute of the 4-dimensional equation (3) in 2n-dimensions. An arbitrary choice of S wouldn't probably give anything interesting, but stable subbundles with respect to certain structures on X are likely to give useful equations.

4. Monopole equations on 8-manifolds

We now consider 8-manifolds with Spin(7) holonomy. In this case there are two natural choices of S which have already found applications in the existing literature. In the 28-dimensional space of 2-forms $\Omega^2(\mathbf{R}^8, \mathbf{R})$, there are two orthogonal subspaces S_1 and S_2 (7 and 21 dimensional, respectively) which are Spin(7) $\subset$ SO(8) invariant [11],[12]. On an 8-manifold X with Spin(7) holonomy (so that the structure group is reducible to Spin(7)) they give rise to global subbundles (denoted by the same letters) $S_1, S_2 \subset \Lambda^2(T^*X)$ which can play the above mentioned role. We will concentrate on the 7-dimensional subbundle S_1 and show that the resulting equations are elliptic, exemplify the local existence of non-trivial solutions and show that they are related to solutions of the 4-dimensional Seiberg-Witten equations. We would like to point out that instead of the widely known CDFN 7-plane, we are working with another 7-plane in $\Omega^2(\mathbf{R}^8, \mathbf{R})$, which is conjugated to the CDFN 7-plane and thus invariant under a conjugated Spin(7) embedding in SO(8). This has the advantage that the 2-forms in this 7-plane can be expressed in an elegant way in terms of 4-dimensional self-dual and anti-self-dual 2-forms. (For a general account we refer to our previous work, Ref.[10].) We will define this 7-plane below, but before that, for the sake of clarity, we first wish to present the global monopole equations. Let X be an 8-manifold with Spin(7) holonomy and S be any stable subbundle of $\Lambda^2(T^*X)$ and S^* its complexification. Given an imaginary valued global 2-form F, let us denote its projection onto S^* by F^+ and the projection of any global spinor ϕ onto the subbundle $\rho^+(S^*) \subset End(W^+)$ by ϕ^+. Then the monopole equations read

$$D_A(\phi) = 0, \tag{13}$$

$$\rho^+(F_A^+) = (\phi\phi^*)^+. \tag{14}$$

Now, we define $S_1 \subset \Omega^2(\mathbf{R}^8, \mathbf{R})$ to be the linear space of 2-forms

$$\omega = \sum_{i<j} \omega_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbf{R}^8, \mathbf{R}),$$

which can be expressed in matrix form as

$$\omega = \omega_{12} f + \begin{pmatrix} \omega' & \omega'' \\ \omega'' & -\omega' \end{pmatrix}, \tag{15}$$

where ω_{12} is a real function, ω' is the matrix of a 4-dimensional self-dual 2-form, ω'' is the matrix of a 4-dimensional anti-self-dual 2-form and we let $f = -J \otimes id_4$. These 2-forms span a 7-dimensional linear subspace S_1 in the 28-dimensional space of 2-forms and the square of any element in this subspace is a scalar matrix. S_1 is maximal with respect to this property. We choose the following orhogonal basis for this maximal linear subspace of self-dual 2-forms:

$$\begin{aligned}
f_1 &= dx_1 \wedge dx_5 + dx_2 \wedge dx_6 + dx_3 \wedge dx_7 + dx_4 \wedge dx_8, \\
f_2 &= dx_1 \wedge dx_2 + dx_3 \wedge dx_4 - dx_5 \wedge dx_6 - dx_7 \wedge dx_8, \\
f_3 &= dx_1 \wedge dx_6 - dx_2 \wedge dx_5 - dx_3 \wedge dx_8 + dx_4 \wedge dx_7, \\
f_4 &= dx_1 \wedge dx_3 - dx_2 \wedge dx_4 - dx_5 \wedge dx_7 + dx_6 \wedge dx_8, \\
f_5 &= dx_1 \wedge dx_7 + dx_2 \wedge dx_8 - dx_3 \wedge dx_5 - dx_4 \wedge dx_6, \\
f_6 &= dx_1 \wedge dx_4 + dx_2 \wedge dx_3 - dx_5 \wedge dx_8 - dx_6 \wedge dx_7, \\
f_7 &= dx_1 \wedge dx_8 - dx_2 \wedge dx_7 + dx_3 \wedge dx_6 - dx_4 \wedge dx_5.
\end{aligned} \tag{16}$$

In matrix notation we set $f_1 = f$, and take

$$\begin{aligned}
f_2 &= -iI \otimes a_1, & f_3 &= -iK \otimes b_1 \\
f_4 &= -iI \otimes a_2, & f_5 &= -iK \otimes b_2 \\
f_6 &= -iI \otimes a_3, & f_7 &= -iK \otimes b_3.
\end{aligned} \tag{17}$$

where (I, J, K) are as given as before and we have

$$a_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, a_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, a_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$b_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, b_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

At this point it will be instructive to show that the above basis corresponds to a representation of the Clifford algebra Cl_7 induced by right multiplications in the algebra of octonions. We adopt the Cayley-Dickson approach and describe a quaternion by a pair of complex numbers so that $a = (x + iy) + j(u + iv)$ where $(i, j, ij = k)$ are the imaginary unit quaternions. In a similar way an octonion is described by a pair of quaternions (a, b). Then the octonionic multiplication rule is

$$(a, b) \cdot (c, d) = (ac - \bar{d}b, da + b\bar{c}). \tag{18}$$

If we now represent an octonion (a, b) by a vector in $\mathbf{R}^8$, its right multiplication by imaginary unit octonions correspond to linear transformations on $\mathbf{R}^8$. We thus obtain the following correspondences:

$$\begin{aligned} (0,1) \to f_1, (i,0) \to f_2, (j,0) \to f_3, (k,0) \to f_4, \\ (0,i) \to f_5, (0,j) \to f_6, (0,k) \to f_7. \end{aligned} \tag{19}$$

The projection F^+ of a 2-form $F_= \sum_{i<j} F_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbf{R}^8, i\mathbf{R})$ onto the complexification of the above self-dual subspace is given by

$$\begin{aligned} F^+ = & \ 1/4(F_{15} + F_{26} + F_{37} + F_{48}) f_1 \\ & +1/4(F_{12} + F_{34} - F_{56} - F_{78}) f_2 \\ & +1/4(F_{16} - F_{25} - F_{38} + F_{47}) f_3 \\ & +1/4(F_{13} - F_{24} - F_{57} + F_{68}) f_4 \\ & +1/4(F_{17} + F_{28} - F_{35} - F_{46}) f_5 \\ & +1/4(F_{14} + F_{23} - F_{58} - F_{67}) f_6 \\ & +1/4(F_{18} - F_{27} + F_{36} - F_{45}) f_7. \end{aligned}$$

We now fix the constant spin^c-structure $\Gamma : \mathbf{R^8} \longrightarrow \mathbf{C^{16\times 16}}$ given by

$$\Gamma(e_i) = \begin{pmatrix} 0 & \gamma(e_i) \\ -\gamma(e_i)^* & 0 \end{pmatrix} \tag{20}$$

where $e_i, i = 1, 2, ..., 8$ is the standard basis for $\mathbf{R^8}$ and $\gamma(e_1) = Id, \quad \gamma(e_i) = f_{i-1}$ for $i = 2, 3, ..., 8$. We note that this choice is specific to 8 dimensions , because $2n = 2^{n-1}$ only for $n = 4$. We have $X = \mathbf{R}^8, W = \mathbf{R}^8 \times \mathbf{C}^{16}, W^{\pm} = \mathbf{R}^8 \times \mathbf{C}^8$ and $L_\Gamma = {L_\Gamma}^{1/2} = \mathbf{R}^8 \times \mathbf{C}$. Consider the connection 1-form

$$A = \sum_{i=1}^{8} A_i dx_i \in \Omega^1(\mathbf{R}^8, i\mathbf{R}) \tag{21}$$

on the line bundle $\mathbf{R}^8 \times \mathbf{C}$. Its curvature is given by

$$F_A = \sum_{i<j} F_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbf{R}^8, i\mathbf{R}) \tag{22}$$

where $F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$. The spin^c connection $\nabla = \nabla_A$ on W^+ is given by

$$\nabla_i \Phi = \frac{\partial \Phi}{\partial x_i} + A_i \Phi \tag{23}$$

$(i = 1, ..., 8)$ where $\Phi : \mathbf{R}^8 \to \mathbf{C}^8$. Therefore the map

$$\rho^+ : \Lambda^2(T^*X) \otimes \mathbf{C} \to End(W^+)$$

can be computed for our generators f_i to give

$$\begin{aligned}
\rho^+(f_1) &= \gamma(e_1)\gamma(e_5) + \gamma(e_2)\gamma(e_6) + \gamma(e_3)\gamma(e_7) + \gamma(e_4)\gamma(e_8) \\
\rho^+(f_2) &= \gamma(e_1)\gamma(e_2) + \gamma(e_3)\gamma(e_4) - \gamma(e_5)\gamma(e_6) - \gamma(e_7)\gamma(e_8) \\
\rho^+(f_3) &= \gamma(e_1)\gamma(e_6) - \gamma(e_2)\gamma(e_5) + \gamma(e_3)\gamma(e_8) + \gamma(e_4)\gamma(e_7) \\
\rho^+(f_4) &= \gamma(e_1)\gamma(e_3) - \gamma(e_2)\gamma(e_4) - \gamma(e_5)\gamma(e_7) + \gamma(e_6)\gamma(e_8) \\
\rho^+(f_5) &= \gamma(e_1)\gamma(e_7) + \gamma(e_2)\gamma(e_8) - \gamma(e_3)\gamma(e_5) - \gamma(e_4)\gamma(e_6) \\
\rho^+(f_6) &= \gamma(e_1)\gamma(e_4) + \gamma(e_2)\gamma(e_3) - \gamma(e_5)\gamma(e_8) - \gamma(e_6)\gamma(e_7) \\
\rho^+(f_7) &= \gamma(e_1)\gamma(e_8) - \gamma(e_2)\gamma(e_7) + \gamma(e_3)\gamma(e_6) - \gamma(e_4)\gamma(e_5).
\end{aligned}$$

Then for a connection $A = \sum_{i=1}^{8} A_i dx_i \in \Omega^1(\mathbf{R}^8, i\mathbf{R})$ and a given complex 8-spinor $\Psi = (\psi_1, \psi_2, ..., \psi_8) \in C^\infty(X, W^+) = C^\infty(\mathbf{R}^8, \mathbf{R}^8 \times \mathbf{C}^8)$ we state our 8-dimensional monopole equations as follows:

$$D_A(\Psi) = 0 \quad , \qquad \rho^+({F_A}^+) = (\Psi\Psi^*)^+. \tag{24}$$

Here $(\Psi\Psi^*)^+$ is the orthogonal projection of $\Psi\Psi^*$ onto the spinor subbundle spanned by $\rho^+(f_i), i = 1, 2, ..., 7$. More explicitly, $D_A(\Psi) = 0$ can be expressed as

$$\nabla_1\Psi = \gamma(e_2)\nabla_2\Psi + \gamma(e_3)\nabla_3\Psi + ... + \gamma(e_8)\nabla_8\Psi \tag{25}$$

and $\rho^+(F_A{}^+) = (\Psi\Psi^*)^+$ is equivalent to the equation

$$\rho^+(F_A{}^+) = \sum_{i=2}^{8} < \rho^+(f_i), \Psi\Psi^* > \rho^+(f_i)/|\rho^+(f_i)|^2. \tag{26}$$

(26) is equivalent to the set of equations

$$\begin{aligned}
F_{15} + F_{26} + F_{37} + F_{48} &= 1/8 < \rho^+(f_1), \Psi\Psi^* >,\\
F_{12} + F_{34} - F_{56} - F_{78} &= 1/8 < \rho^+(f_2), \Psi\Psi^* >,\\
F_{16} - F_{25} - F_{38} + F_{47} &= 1/8 < \rho^+(f_3), \Psi\Psi^* >,\\
F_{13} - F_{24} - F_{57} + F_{68} &= 1/8 < \rho^+(f_4), \Psi\Psi^* >,\\
F_{17} + F_{28} - F_{35} - F_{46} &= 1/8 < \rho^+(f_5), \Psi\Psi^* >,\\
F_{14} + F_{23} - F_{58} - F_{67} &= 1/8 < \rho^+(f_6), \Psi\Psi^* >,\\
F_{18} - F_{27} + F_{36} - F_{45} &= 1/8 < \rho^+(f_7), \Psi\Psi^* > .
\end{aligned}$$

or still more explicitly to the equations

$$\begin{aligned}
F_{15}+F_{26}+F_{37}+F_{48} &= 1/4(\psi_1\bar\psi_3-\psi_3\bar\psi_1-\psi_2\bar\psi_4+\psi_4\bar\psi_2-\psi_5\bar\psi_7+\psi_7\bar\psi_5-\psi_6\bar\psi_8+\psi_8\bar\psi_6),\\
F_{12}+F_{34}-F_{56}-F_{78} &= 1/4(\psi_1\bar\psi_5-\psi_5\bar\psi_1-\psi_2\bar\psi_6+\psi_6\bar\psi_2+\psi_3\bar\psi_7-\psi_7\bar\psi_3+\psi_4\bar\psi_8-\psi_8\bar\psi_4),\\
F_{16}-F_{25}-F_{38}+F_{47} &= 1/4(\psi_1\bar\psi_7-\psi_7\bar\psi_1+\psi_2\bar\psi_8-\psi_8\bar\psi_2-\psi_3\bar\psi_5+\psi_5\bar\psi_3+\psi_4\bar\psi_6-\psi_6\bar\psi_4),\\
F_{13}-F_{24}-F_{57}+F_{68} &= 1/4(\psi_1\bar\psi_2-\psi_2\bar\psi_1+\psi_3\bar\psi_4-\psi_4\bar\psi_3+\psi_5\bar\psi_6-\psi_6\bar\psi_5-\psi_7\bar\psi_8+\psi_8\bar\psi_7),\\
F_{17}+F_{28}-F_{35}-F_{46} &= 1/4(\psi_1\bar\psi_4-\psi_4\bar\psi_1+\psi_2\bar\psi_3-\psi_3\bar\psi_2-\psi_5\bar\psi_8+\psi_8\bar\psi_5+\psi_6\bar\psi_7-\psi_7\bar\psi_6),\\
F_{14}+F_{23}-F_{58}-F_{67} &= 1/4(-\psi_1\bar\psi_6+\psi_6\bar\psi_1-\psi_2\bar\psi_5+\psi_5\bar\psi_2-\psi_3\bar\psi_8+\psi_8\bar\psi_3+\psi_4\bar\psi_7-\psi_7\bar\psi_4),\\
F_{18}-F_{27}+F_{36}-F_{45} &= 1/4(\psi_1\bar\psi_8-\psi_8\bar\psi_1-\psi_2\bar\psi_7+\psi_7\bar\psi_2-\psi_3\bar\psi_6+\psi_6\bar\psi_3-\psi_4\bar\psi_5+\psi_5\bar\psi_4).
\end{aligned}$$

5. Conclusion

We will now show that the system of monopole equations (25)-(26) form an elliptic system. These equations can be written compactly in the form

$$\langle F, f_i \rangle = 1/8 \langle \rho^+(f_i), \Psi\Psi^* \rangle, \quad i = 1 \ldots 7, \quad D_A(\Psi) = 0.$$

If in addition we impose the Coulomb gauge condition

$$\sum_{i=1}^{8} \partial_i A_i = 0,$$

we obtain a system of first order partial differential equations consisting of eight equations for the components of the spinor Ψ and eight equations for the components of the connection 1-form A. The characteristic determinant of this system [20] is the product of the characteristic determinants of the equations for Ψ and A. As the Dirac operator is elliptic[19], the ellipticity of the present system depends on the characteristic determinant of the system consisting of $\langle F, f_i \rangle = 1/8 \langle \rho^+(f_i), \Psi\Psi^* \rangle, \quad i = 1 \ldots 7$ and the Coulomb gauge condition. In the computation of the characteristic determinant, the fifth row, for instance, is obtained from

$$F_{15}+F_{26}+F_{37}+F_{48} = \partial_1 A_5 - \partial_5 A_1 + \partial_2 A_6 - \partial_6 A_2 + \partial_3 A_7 - \partial_7 A_3 + \partial_4 A_8 - \partial_8 A_4$$

by replacing ∂_i by ξ_i. Thus after a rearrangement of the order of the equations, the characteristic determinant can be obtained as

$$det \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_8 \\ -\xi_2 & \xi_1 & -\xi_4 & \xi_3 & \xi_6 & -\xi_5 & \xi_8 & -\xi_7 \\ -\xi_3 & \xi_4 & \xi_1 & -\xi_2 & \xi_7 & -\xi_8 & -\xi_5 & \xi_6 \\ -\xi_4 & -\xi_3 & \xi_2 & \xi_1 & \xi_8 & \xi_7 & -\xi_6 & -\xi_5 \\ -\xi_5 & -\xi_6 & -\xi_7 & -\xi_8 & \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ -\xi_6 & \xi_5 & \xi_8 & -\xi_7 & -\xi_2 & \xi_1 & \xi_4 & -\xi_3 \\ -\xi_7 & -\xi_8 & \xi_5 & \xi_6 & -\xi_3 & -\xi_4 & \xi_1 & \xi_2 \\ -\xi_8 & \xi_7 & -\xi_6 & \xi_5 & -\xi_4 & \xi_3 & -\xi_2 & \xi_1 \end{pmatrix}.$$

It is equal to

$$(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 + \xi_6^2 + \xi_7^2 + \xi_8^2)^4.$$

and this proves ellipticity.

Finally we point out that the monopole equations (25)-(26) admit non-trivial solutions. For example, if the pair (A, Φ) with

$$A = \sum_{i=1}^{4} A_i(x_1, x_2, x_3, x_4) dx_i$$

and

$$\Phi = (\phi_1(x_1, x_2, x_3, x_4), \phi_2(x_1, x_2, x_3, x_4))$$

is a solution of the 4-dimensional Seiberg-Witten equations, then the pair (B, Ψ) with

$$B = \sum_{i=1}^{4} A_i(x_1, x_2, x_3, x_4) dx_i$$

(i.e. the first four components B_i of B coincide with A_i, thus not depending on x_5, x_6, x_7, x_8 and the last four components of B vanish) and

$$\Psi = (0, 0, \phi_1, \phi_2, 0, 0, i\phi_1, -i\phi_2),$$

where ϕ_1 and ϕ_2 depend only on x_1, x_2, x_3, x_4, is a solution of these new 8-dimensional monopole equations. It can directly be verified that Ψ is harmonic with respect to B and the second part of the equations is also satisfied.

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