

FINITE ACTION YANG-MILLS SOLUTIONS ON THE GROUP MANIFOLD

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Abstract

We demonstrate that the left (and right) invariant Maurer-Cartan forms for any semi-simple Lie group enable one to construct solutions of the Yang-Mills equations on the group manifold equipped with the natural Cartan-Killing metric. For the unitary unimodular groups the Yang-Mills action integral is finite for such solutions. This is explicitly exhibited for the case of $SU(3)$.

I. INTRODUCTION

Classical solutions of the Yang-Mills field equations have proved to be of immense value in the development of both mathematical and physical theory. Instanton and monopole solutions have provided a fertile source of approximation techniques in quantum field theory

and recent developments in duality theory have led to exciting new computational schemes in differential geometry based on supersymmetric Yang-Mills actions. Although self-duality has featured prominently in the construction of finite-action solutions not all such field configurations fall into this category. One of the earliest solutions found in $SU(2)$ Yang-Mills theory was the “meron” [1–7]. Its generalisations gave rise to an interesting proposal for the confinement of quarks [8,9]. Little attention has been devoted to the exploration of finite action solutions associated with higher rank Lie groups in four [10] or higher dimensions [21].

It is the purpose of this note to point out that solutions of “meron” type can occur in a much broader context. We demonstrate below that there exist similar finite-action solutions of the Yang-Mills field equations associated with *any compact semi-simple Lie group* if the action functional for the field equations is constructed over the group manifold itself in terms of a group invariant metric on the Lie algebra. Of particular interest for physics are the unitary unimodular groups. Such a broad class of solutions may have some relevance to one or more of the many compactification schemes that feature in the dimensional reduction of higher dimensional descriptions of the fundamental interactions. The explicit construction of any solution in this class depends upon the manner in which the group manifold is coordinated. We content ourselves here with the explicit construction of a finite-action solution associated with $SU(3)$.

II. YANG-MILLS SOLUTIONS

Suppose G is a semi-simple Lie group. Let $\{\omega^\alpha\}$ be the set of left invariant basis 1-forms on G that satisfy the Maurer-Cartan structure equations

$$d\omega^\alpha = -\frac{1}{2}C_{\beta\gamma}{}^\alpha\omega^\beta \wedge \omega^\gamma \tag{1}$$

where $C_{\beta\gamma}{}^\alpha$ are the structure constants of the Lie algebra of G . If we define the dual basis of vectors $\{X_\alpha\}$ so that $\omega^\alpha(X_\beta) = \delta^\alpha_\beta$ then

$$[X_\alpha, X_\beta] = C_{\alpha\beta}{}^\gamma X_\gamma. \quad (2)$$

Let g denote the Cartan-Killing metric on the group manifold G so that

$$g_{\alpha\beta} \equiv g(X_\alpha, X_\beta) = C_{\alpha\gamma}{}^\delta C_{\beta\delta}{}^\gamma. \quad (3)$$

We also introduce an irreducible representation¹ of the Lie algebra of G by $N \times N$ matrices $\{\lambda_\alpha\}$ that satisfy

$$[\lambda_\alpha, \lambda_\beta] = C_{\alpha\beta}{}^\gamma \lambda_\gamma. \quad (4)$$

Thus the matrices are scaled in such a way that

$$\text{tr}(\text{ad } \lambda_\alpha \text{ ad } \lambda_\beta) = g_{\alpha\beta}. \quad (5)$$

Define a Yang-Mills potential 1-form

$$A = c \omega^\alpha \lambda_\alpha \quad (6)$$

where c is a constant to be determined. The corresponding gauge field 2-form is

$$F \equiv dA + \frac{1}{2}[A, A] = -c(c-1)d\omega^\alpha \lambda_\alpha. \quad (7)$$

Now consider the Hodge dual $*F$ with respect to the Cartan-Killing metric and substitute it in the Yang-Mills equation

$$d * F + A \wedge * F - * F \wedge A = 0. \quad (8)$$

We find

$$d * F = -c(c-1) d * d\omega^\alpha \lambda_\alpha \quad (9)$$

and

¹ For $G = SU(N)$ these matrices would be a set of anti-Hermitian $N \times N$ traceless matrices corresponding to the fundamental representation.

$$A \wedge *F - *F \wedge A = c^2(c-1) C_{\beta\gamma}{}^\alpha \omega^\beta \wedge *d\omega^\gamma \lambda_\alpha. \quad (10)$$

Excluding the trivial solutions $c = 0, 1$ the Yang-Mills equation is satisfied provided we have

$$d * d\omega^\alpha = -c C_{\beta\gamma}{}^\alpha \omega^\beta \wedge *d\omega^\gamma. \quad (11)$$

From Eq. (1) the left-hand side is

$$\begin{aligned} d * d\omega^\alpha &= -\frac{1}{2} C_{\beta\gamma}{}^\alpha d * (\omega^\beta \wedge \omega^\gamma) \\ &= -\frac{1}{2} C_{\beta\gamma}{}^\alpha d \left(\frac{1}{(N-2)!} \epsilon^{\beta\gamma}{}_{\delta\rho\dots} \omega^\delta \wedge \omega^\rho \wedge \dots \right), \end{aligned} \quad (12)$$

where $\epsilon^{\beta\gamma}{}_{\delta\rho\dots}$ is the Levi-Civita alternating density associated with the Cartan-Killing metric.

With Eqs. (1) and (3) it follows that

$$\begin{aligned} d * d\omega^\alpha &= -\frac{1}{2} C_{\beta\gamma}{}^\alpha d\omega^\delta \wedge *(\omega^\beta \wedge \omega^\gamma \wedge \omega_\delta) \\ &= \frac{1}{4} C_{\beta\gamma}{}^\alpha C_{\rho\sigma}{}^\delta \omega^\rho \wedge \omega^\sigma \wedge *(\omega^\beta \wedge \omega^\gamma \wedge \omega_\delta). \end{aligned} \quad (13)$$

With the aid of the identities

$$\omega^\alpha \wedge *(\omega^\beta \wedge \omega^\gamma \wedge \omega^\delta) = g^{\alpha\beta} *(\omega^\gamma \wedge \omega^\delta) - g^{\alpha\gamma} *(\omega^\beta \wedge \omega^\delta) + g^{\alpha\delta} *(\omega^\beta \wedge \omega^\gamma), \quad (14)$$

$$\omega^\alpha \wedge *(\omega^\beta \wedge \omega^\gamma) = -g^{\alpha\beta} * \omega^\gamma + g^{\alpha\gamma} * \omega^\beta \quad (15)$$

this gives

$$d * d\omega^\alpha = \frac{1}{2} g^{\beta\rho} g^{\gamma\sigma} C_{\beta\gamma}{}^\alpha C_{\rho\sigma}{}^\delta * \omega_\delta. \quad (16)$$

On the other hand the right-hand side of Eq. (11) may be written

$$\begin{aligned} -c C_{\beta\gamma}{}^\alpha \omega^\beta \wedge *d\omega^\gamma &= \frac{c}{2} C_{\beta\gamma}{}^\alpha \omega^\beta \wedge C_{\rho\sigma}{}^\gamma *(\omega^\rho \wedge \omega^\sigma) \quad (\text{from Eq. (1)}) \\ &= -c C_{\beta\gamma}{}^\alpha C_{\rho\sigma}{}^\gamma g^{\beta\rho} * \omega^\sigma \quad (\text{by Eq. (15)}), \end{aligned} \quad (17)$$

with $g^{\alpha\beta}$ being the inverse of the Cartan-Killing metric. Hence the Yang-Mills equation is satisfied if

$$\frac{1}{2} C_{\beta\gamma}{}^\alpha C^{\beta\gamma\delta} = -c C_{\beta\gamma}{}^\alpha C^{\beta\delta\gamma}, \quad (18)$$

where

$$C_{\alpha\beta\gamma} \equiv C_{\alpha\beta}{}^{\delta} g_{\delta\gamma} \quad (19)$$

are totally skew-symmetric. This implies $c = \frac{1}{2}$.

If we now restrict to the compact unimodular unitary groups, the group manifold has a finite volume $\int_G *1$ and the above solution yields a finite Yang-Mills action:

$$I[A] = \int_G \text{tr} (F \wedge *F) = \frac{C_{\alpha\beta\gamma} C^{\alpha\beta\gamma}}{32} \int_G *1, \quad (20)$$

where we have used Eq. (15) and

$$\omega^\alpha \wedge *\omega^\beta = g^{\alpha\beta} *1. \quad (21)$$

The construction of group volumes is not trivial in general [11–17] and the precise values depend on the choice of normalisation of the group generators. We illustrate this by computing the above action for $G = SU(3)$.

If we denote the anti-Hermitian generators of the Lie algebra of $SU(3)$ by

$$\begin{aligned} \lambda_1 &= \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \frac{\sqrt{2}}{2} \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 &= \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_5 = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \lambda_6 = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \\ \lambda_7 &= \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \lambda_8 = \frac{\sqrt{6}}{6} \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix} \end{aligned} \quad (22)$$

which are just $\frac{i}{\sqrt{2}}$ times the Gell-Mann “ λ -matrices” normalised such that $\text{tr} (\lambda_i \lambda_j) = -\delta_{ij}$, then the group parametrisation of $SU(3)$ given by Holland [17] reads

$$\mathcal{G}(\boldsymbol{\theta}) = e^{-\sqrt{2}\theta^1\lambda_3} e^{-\sqrt{2}\theta^2\lambda_2} e^{-\sqrt{2}\theta^3\lambda_3} e^{-\sqrt{6}\theta^4\lambda_8} e^{-\sqrt{2}\theta^5\lambda_4} e^{-\sqrt{2}\theta^6\lambda_3} e^{-\sqrt{2}\theta^7\lambda_2} e^{-\sqrt{2}\theta^8\lambda_3} \quad (23)$$

The parameter ranges for the eight angles are $0 \leq \theta^1, \theta^3, \theta^6, \theta^8 \leq 2\pi$, $0 \leq \theta^4 \leq \pi$, and $0 \leq \theta^2, \theta^5, \theta^7 \leq \frac{1}{2}\pi$.

The Maurer-Cartan 1-forms $\omega^1, \omega^2, \dots, \omega^8$ are defined by

$$\omega^i = -\text{tr}(\lambda_i \mathcal{G}^{-1} d\mathcal{G}). \quad (24)$$

The Cartan-Killing metric may be written in this parametrisation as

$$-g_{CK} = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \dots + \omega^8 \otimes \omega^8 \quad (25)$$

and the wedge product of the Cartan-Killing 1-forms, $*1$, is calculated to be

$$*1 = \omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^8 = 8\sqrt{3} \sin(2\theta^2) \sin(2\theta^5) \sin^2(\theta^5) \sin(2\theta^7) d\theta^1 \wedge d\theta^2 \wedge \dots \wedge d\theta^8. \quad (26)$$

We may integrate this to obtain the group volume $V_{SU(3)}$ of $SU(3)$ with respect to the Cartan-Killing metric volume element:

$$\begin{aligned} V_{SU(3)} &= \int_{SU(3)} *1 \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 8\sqrt{3} \sin(2\theta^2) \sin(2\theta^5) \sin^2(\theta^5) \sin(2\theta^7) \\ &\quad \times d\theta^1 d\theta^2 d\theta^3 d\theta^4 d\theta^5 d\theta^6 d\theta^7 d\theta^8 \\ &= 8\sqrt{3} \left[(2\pi) \times 1 \times (2\pi) \times \pi \times \frac{1}{2} \times (2\pi) \times 1 \times (2\pi) \right] = 64\sqrt{3} \pi^5 \end{aligned} \quad (27)$$

Eqs. (20), and (27) give explicit formulae for the Yang-Mills action associated with the “meron” type solution discussed above. For $SU(3)$, this solution has both Pontryagin numbers zero.

III. CONCLUSION

We have shown that for any semi-simple Lie group with general element \mathcal{G} the potential form $A = c\mathcal{G}^{-1}d\mathcal{G}$ is a solution of the Yang-Mills equation on a group manifold with the natural Cartan-Killing metric, if c takes the value $\frac{1}{2}$. By a simple local gauge transformation $A = (1-c)\mathcal{G}d\mathcal{G}^{-1}$ is an equivalent solution. If the group is unimodular, both left and right

invariant group volume elements can be chosen to coincide, so for compact groups the Yang-Mills action is finite for such solutions. For $SU(3)$ the result is

$$|I[A]| = \mu^2 96 \sqrt{3} \pi^5$$

for the metric $g = \mu g_{CK}$ where μ is any real constant.

We have also briefly examined the more general ansatz

$$A = \sum_{i=1}^8 c_i \omega^i \lambda_i.$$

For a constant vector $\{c_i\}$, $\sum_{i=1}^8 c_i \omega^i$ is the most general left invariant 1-form on the group. One finds that A generates a solution provided the components $\{c_i\}$ satisfy a set of coupled quartic equations. For $SU(3)$ we demand that these equations have real solutions. Such solutions exist but may be shown to be related to the solution discussed above by constant gauge transformations (automorphisms of the Lie algebra). The remaining complex solutions give rise to finite but complex Yang-Mills actions.

There have been a number of papers [18–20] devoted to the generalisations of the Yang-Mills solutions to higher dimensions. In their search for the higher dimensional analogues of instantons and merons these papers have modified the form of the original Yang-Mills action in an attempt to preserve the conformal properties that the theory exhibits in four dimensions. In this note we have given a simple proof of the existence of finite action solutions to the original Yang-Mills equation expressed in terms of the natural group invariant metric provided by the group manifold itself.

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