

Self-Dual Yang-Mills Fields in Eight Dimensions

Ayşe Hümeýra Bilge*, Tekin Dereli[†], Şahin Koçak*

* Department of Mathematics,
Anadolu University,
Eskişehir, Turkey
e.mail: bilge@yunus.mam.tubitak.gov.tr

[†] Department of Mathematics,
Middle East Technical University,
Ankara, Turkey
e.mail: tdereli@rorqual.cc.metu.edu.tr

Abstract

Strongly self-dual Yang-Mills fields in even dimensional spaces are characterised by a set of constraints on the eigenvalues of the Yang-Mills fields $F_{\mu\nu}$. We derive a topological bound on \mathbf{R}^8 , $\int_M (F, F)^2 \geq k \int_M p_1^2$ where p_1 is the first Pontrjagin class of the $\text{SO}(n)$ Yang-Mills bundle and k is a constant. Strongly self-dual Yang-Mills fields realise the lower bound.

1. Introduction.

Self-dual Yang-Mills fields in four dimensions have many remarkable properties. But the notion of self-duality cannot be easily generalised to higher dimensions. Let us first highlight the main points concerning the self dual Yang-Mills theory in four dimensions before we discuss a generalisation to higher dimensions. Our notation and conventions are given in the Appendix.

1. In four dimensions the Yang-Mills functional is the L_2 norm of the curvature 2-form F

$$\|F\|^2 = \int_M \text{tr}(F \wedge *F) \quad (1)$$

where $*$ denotes the Hodge dual of F . The Yang-Mills equations are the critical points of the Yang-Mills functional. They are of the form

$$d_E F = 0, \quad *d_E^* F = 0, \quad (2)$$

where d_E is the bundle covariant derivative and $-*d_E^*$ is its formal adjoint.

2. F is self-dual or anti-self-dual provided

$$*F = \pm F. \quad (3)$$

Self-dual or anti-self-dual solutions of (2) are the global extrema of the Yang-Mills functional (1). This can be seen as follows: The Yang-Mills functional has a topological bound

$$\|F\|^2 \geq \int_M \text{tr}(F \wedge F). \quad (4)$$

The term $\text{tr}(F \wedge F)$ is related to the Chern classes of the bundle ^[1]. Actually if E is a complex 2-plane bundle with $c_1(E) = 0$, then the topological bound is proportional to $c_2(E)$ and this lower bound is realised by a (anti)self-dual connection. Furthermore, $SU(2)$ bundles over a four manifold are classified by $\int c_2(E)$, hence self-dual connections are minimal representatives of the connections in each equivalence class of $SU(2)$ bundles ^[2]. This is a generalisation of the fact that an $SU(2)$ bundle admits a flat connection if and only if it is trivial.

3. Self-dual Yang-Mills equations form an elliptic system in the Coulomb gauge $\text{div} A = 0$. The importance of this property is that, in gauge theory, elliptic equations lead to finite dimensional moduli spaces ^[3].

4. Hodge duality can equivalently be defined by requiring that F is an eigen bi-vector of the completely antisymmetric ϵ_{ijkl} tensor. In four dimensions such a tensor is unique and have the same constant coefficients in all coordinate systems. Thus self-dual Yang-Mills equations are linear equations that have the same constant coefficients in all coordinate systems that leave the Euclidean metric invariant .

5. Self-dual Yang-Mills equations can be written in terms of the commutator of two linear operators depending on a complex parameter. Then their solution can be reduced to the solution of a Riemann problem and this case is considered as completely solvable in integrable system theory .

The earliest attempt at a generalisation of self-duality to higher dimensions is due to Tchrakian^[4]. His approach is pursued further for physical applications^[5]. In a similar vein, Corrigan,Devchand,Fairlie and Nuyts^[6] studied the first-order equations satisfied by Yang-Mills fields in spaces of dimension greater than four (property 4 above) and derived $SO(7)$ self-duality equations in \mathbf{R}^8 . This construction was followed by an investigation of completely solvable Yang-Mills equations (property 5 above) in higher dimensions by Ward ^[7]. A different approach is taken by Feza Gürsey and his coworkers ^[8] who showed that the notion of self-duality can be carried over to eight dimensions by making use of octonions. This observation proved fruitful as $SO(7)$ self-duality equations in \mathbf{R}^8 can be solved in terms of octonions ^[9]. The existence of the so called *octonionic instantons* is closely tied to the properties of the Cayley algebra. These turned out to be relevant to superstring models. In fact soliton solutions of the string equations of motion to the lowest order in the string tension parameter, that preserve at least one supersymmetry, are given by octonionic instantons ^[10,11]. A further connection between strings and octonions is provided by the existence of supersymmetric Yang-Mills theories and superstring actions only in dimensions 3,4,6,10. The number of transverse dimensions 1,2,4,8 coincide with the dimensions of the division algebras $\mathbf{R}, \mathbf{C}, \mathbf{H}$ and \mathbf{O} ^[12]. These are related with the four Hopf fibrations, the last of which $S^{15} \rightarrow S^8$ is realised by the octonionic instantons ^[13].

Here we present a characterisation of strongly self-dual Yang-Mills fields by an eigenvalue criterion in even dimensional spaces. In particular, the self-dual Yang-Mills equations in four dimensions and the self-duality equations of Corrigan et al in eight dimensions are consistent with our strong self-duality criteria.

2. Self-duality of a 2-form as an eigenvalue criterion.

Any 2-form in $2m$ dimensions can be represented by a $2m \times 2m$ skew symmetric real matrix. Such a matrix Ω has pure imaginary eigenvalues that are pairwise conjugate. Let λ_k , $k = 1, \dots, m$ be these eigenvalues. This is equivalent to the existence of a basis $\{X_k, Y_k\}$ such that

$$\Omega X_k = -\lambda_k Y_k, \quad \Omega Y_k = \lambda_k X_k, \quad (5)$$

for each k . In this basis the matrix Ω takes the block diagonal form:

$$\Omega = \begin{pmatrix} 0 & \lambda_1 & & & & \\ -\lambda_1 & 0 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 0 & \lambda_m \\ & & & & -\lambda_m & 0 \end{pmatrix} \quad (6)$$

We first consider the case $m = 2$. In four dimensions self-duality can be described by the following eigenvalue criterion which allows an immediate generalisation to higher dimensions. Let Ω be the skew symmetric matrix representing a 2-form F in four dimensions. Then it can be seen that the eigenvalues of the matrix Ω satisfy

$$\lambda_1 \mp \lambda_2 = \sqrt{(\Omega_{12} \mp \Omega_{34})^2 + (\Omega_{13} \pm \Omega_{24})^2 + (\Omega_{14} \mp \Omega_{23})^2} \quad (7)$$

Thus for self-duality

$$\lambda_1 = \lambda_2, \quad (8)$$

while for anti-self-duality

$$\lambda_1 = -\lambda_2. \quad (9)$$

In both cases the absolute values of the eigenvalues are equal. Two cases are distinguished by the sign of

$$\Omega_{12}\Omega_{34} - \Omega_{13}\Omega_{24} + \Omega_{14}\Omega_{23}. \quad (10)$$

In higher dimensions we propose the following

Definition Let F be a 2-form in $2m$ dimensional space and $\pm i\lambda_k$, $k = 1, \dots, m$ be its eigenvalues. Then F is *strongly self-dual* if it can be brought to block diagonal form (6) with respect to an oriented basis such that $\lambda_1 = \lambda_2 = \dots = \lambda_m$.

The strong self-duality or strong anti-self-duality can be characterised by requiring the equality of the absolute values of the eigenvalues:

$$|\lambda_1| = |\lambda_2| = \dots = |\lambda_m|. \quad (11)$$

There are 2^m possible ways to satisfy the above set of equalities. The half of these correspond to strongly self-dual 2-forms, while the remaining half correspond to strongly anti-self-dual 2-forms.

Now consider a 2-form F in eight dimensions and let the corresponding matrix be Ω . Then

$$\det(I + t\Omega) = 1 + \sigma_2 t^2 + \sigma_4 t^4 + \sigma_6 t^6 + \sigma_8 t^8 \quad (12)$$

where, provided the eigenvalues of Ω are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we have

$$\begin{aligned} \sigma_2 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = \text{tr}\Omega^2 \\ \sigma_4 &= \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_1^2\lambda_4^2 + \lambda_2^2\lambda_3^2 + \lambda_2^2\lambda_4^2 + \lambda_3^2\lambda_4^2 \\ \sigma_6 &= \lambda_1^2\lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_2^2\lambda_4^2 + \lambda_1^2\lambda_3^2\lambda_4^2 + \lambda_2^2\lambda_3^2\lambda_4^2 \\ \sigma_8 &= \lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 = \text{Det}\Omega. \end{aligned} \quad (13)$$

The sign of square root of the determinant

$$\sqrt{\sigma_8} = \lambda_1\lambda_2\lambda_3\lambda_4 \quad (14)$$

distinguishes between the self-dual and anti-self-dual cases.

In the next section we will need various estimates involving the norm of F . To this end we prove the following

Lemma 1. *Let $F \in \Lambda^2(\mathbf{R}^8)$. Then $3(F, F)^2 - 2(F^2, F^2) \geq 0$*

Proof. From above it can be seen that

$$\begin{aligned} 3(F, F)^2 - 2(F^2, F^2) &= 3\sigma_2^2 - 8\sigma_4 \\ &= 3(\lambda_1^4 + \lambda_2^4 + \lambda_3^4 + \lambda_4^4) \\ &\quad - 2(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_1^2\lambda_4^2 + \lambda_2^2\lambda_3^2 + \lambda_2^2\lambda_4^2 + \lambda_3^2\lambda_4^2) \\ &= (\lambda_1^2 - \lambda_2^2)^2 + (\lambda_1^2 - \lambda_3^2)^2 + (\lambda_1^2 - \lambda_4^2)^2 \\ &\quad + (\lambda_2^2 - \lambda_3^2)^2 + (\lambda_2^2 - \lambda_4^2)^2 + (\lambda_3^2 - \lambda_4^2)^2 \\ &\geq 0. \quad \square \end{aligned}$$

Lemma 2. *Let $F \in \Lambda^2(\mathbf{R}^8)$. Then $(F^2, F^2) - *F^4 \geq 0$*

Proof. From above we have

$$\begin{aligned} \frac{1}{4}((F^2, F^2) - *F^4) &= (\sigma_4 - 6\sqrt{\sigma_8}) \\ &= (\lambda_1\lambda_2 - \lambda_3\lambda_4)^2 + (\lambda_1\lambda_3 - \lambda_2\lambda_4)^2 + (\lambda_1\lambda_4 - \lambda_2\lambda_3)^2 \\ &\geq 0 \quad . \quad \square \end{aligned}$$

Equality holds in both the above lemmas if and only if the absolute values of all eigenvalues are equal.

The computation above allows us also to prove the following

Proposition. *Let $F \in \Lambda^2(\mathbf{R}^8)$ and $F_{\mu\nu}$ be its coordinate components. If $\pm\lambda_k$, $k = 1, \dots, 4$ are the corresponding eigenvalues, then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ if the following equations are satisfied:*

$$\begin{aligned} F_{12} - F_{34} = 0 \quad F_{12} - F_{56} = 0 \quad F_{12} - F_{78} = 0 \\ F_{13} + F_{24} = 0 \quad F_{13} - F_{57} = 0 \quad F_{13} + F_{68} = 0 \\ F_{14} - F_{23} = 0 \quad F_{14} + F_{67} = 0 \quad F_{14} + F_{58} = 0 \\ F_{15} + F_{26} = 0 \quad F_{15} + F_{37} = 0 \quad F_{15} - F_{48} = 0 \\ F_{16} - F_{25} = 0 \quad F_{16} - F_{38} = 0 \quad F_{16} - F_{47} = 0 \\ F_{17} + F_{28} = 0 \quad F_{17} - F_{35} = 0 \quad F_{17} + F_{46} = 0 \\ F_{18} - F_{27} = 0 \quad F_{18} + F_{36} = 0 \quad F_{18} + F_{45} = 0 \end{aligned} \tag{15}$$

Proof. From the proof of Lemma 1, $3\sigma_2^2 - 8\sigma_4 = 0$ if and only if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$. On the other hand, by direct computation with the EXCALC package in REDUCE it is shown that this condition holds when the above equations are satisfied. \square

These are precisely the self-duality equations of Corrigan, Devchand, Fairlie and Nuyts ^[6].

3. A topological lower bound

How to choose the action density and its topological lower bound? The idea underlying the self-dual Yang-Mills theory is that the Yang-Mills action $\|F\|^2$ (that is a second order curvature invariant) is bounded below by a topological invariant of the bundle. In the case of an $SU(2)$ bundle over a

four manifold, the bundle is classified by the second Chern number $\int c_2(E)$. This lower bound is achieved for self-dual fields, hence a self-dual connection on an $SU(2)$ bundle is the connection whose norm is minimal among all connections on that bundle.

In higher dimensions, we want similarly to find a lower bound for the norm of a curvature invariant in terms of topological invariants of the bundle. If the base manifold is eight dimensional, we will need to integrate an 8-form, hence assuming that the first Chern class of the bundle vanishes $c_1(E) = 0$, an appropriate lower bound can be given by a linear combination of the only available Pontryagin classes :

$$\beta p_1(E)^2 + \gamma p_2(E). \quad (16)$$

But then the topological lower bound would be a fourth order expression in F , so that it is natural to seek a lower bound to some fourth order curvature invariant. The action density should be written in terms of the local curvature 2-form matrix in a way independent of the local trivialization of the bundle. Hence it should involve invariant polynomials of the local curvature matrix. We want to express the action as an inner product in the space of k -forms, which gives a quantity independent of the local coordinates. This leads to the following generic action density

$$a(F, F)^2 + b(F^2, F^2) + c(\text{tr} F^2, \text{tr} F^2). \quad (17)$$

The action density proposed by Grossman, Kephardt and Stasheff^[13] corresponds to the choice $a = c = 0$. Then the self-duality (in the Hodge sense) of F^2 gives global minima of this action involving the characteristic number $\int p_2(E)$.

We derive an alternative lower bound involving the other characteristic number $\int p_1(E)^2$. In analogy with the fact that in four dimensions the L_2 norm of the curvature is bounded below by a topological number, we give a lower bound for the expression $\int (F, F)^2$ for an $SO(n)$ bundle. However, from the Schwarz inequality, it can be seen that for a compact manifold M ,

$$\int_M (F, F) \leq \left(\int (F, F)^2 \right)^{1/2} \text{vol}M, \quad (18)$$

hence lower bounds on the $\int (F, F)^2$ cannot be related to the L_2 norm of F .

Consider an $SO(n)$ valued curvature 2-form matrix $F = (F_{ab})$. Then

$$(F, F) = 2 \sum_{a=1}^n \sum_{b>a}^n (F_{ab}, F_{ab}) \quad (19)$$

and

$$\begin{aligned} (F, F)^2 &= 4 \sum_{a=1}^n \sum_{b>a}^n (F_{ab}, F_{ab})^2 \\ &+ 8 \sum_{a=1}^n \sum_{b>a}^n \sum_{c>a}^n \sum_{d>c}^n (F_{ab}, F_{ab})(F_{cd}, F_{cd}) \end{aligned} \quad (20)$$

On the other hand

$$\text{tr} F^2 = -2 \sum_{a=1}^n \sum_{b>a}^n F_{ab}^2 \quad (21)$$

hence

$$\begin{aligned} (\text{tr} F^2, \text{tr} F^2) &= 4 \sum_{a=1}^n \sum_{b>a}^n (F_{ab}^2, F_{ab}^2) \\ &+ 8 \sum_{a=1}^n \sum_{b>a}^n \sum_{c>a}^n \sum_{d>c}^n (F_{ab}^2, F_{cd}^2). \end{aligned} \quad (22)$$

From the Schwarz inequality we have

$$\begin{aligned} (\text{tr} F^2, \text{tr} F^2) &= 4 \sum_{a=1}^n \sum_{b>a}^n (F_{ab}^2, F_{ab}^2) \\ &+ 8 \sum_{a=1}^n \sum_{b>a}^n \sum_{c>a}^n \sum_{d>c}^n (F_{ab}^2, F_{ab}^2)^{1/2} (F_{cd}^2, F_{cd}^2)^{1/2}. \end{aligned} \quad (23)$$

Then from Lemma 1 we obtain

$$(F, F)^2 \geq \frac{2}{3} (\text{tr} F^2, \text{tr} F^2), \quad (24)$$

while from Lemma 2

$$(\text{tr} F^2, \text{tr} F^2) \geq * \text{tr} F^4. \quad (25)$$

In both (24) and (25) the lower bound is realised by strongly (anti)self-dual fields.

4. Conclusion

In this paper we have characterised strongly (anti)self-dual Yang-Mills fields in even dimensional spaces by putting constraints on the eigenvalues of F . The previously known cases of self-dual Yang-Mills fields in four and eight dimensions are consistent with our characterisation. We believe this new approach to self-duality in higher dimensions deserves further study.

We have also derived a topological bound

$$\int (F, F)^2 \geq k \int_M p_1(E)^2 \quad (26)$$

that is achieved by strongly (anti)self-dual Yang-Mills fields. This is to be compared with a topological bound obtained earlier ^[13]

$$\int_M (F^2, F^2) \geq k' \int_M p_2(E). \quad (27)$$

As already noted, the self-duality of F^2 in the Hodge sense attains the lower bound in (27). But as opposed to the self duality of F in four dimensions, this condition is too restrictive since it yields an overdetermined system for the connection, namely 21 nonlinear equations for 8 variables.

The new topological bound we found here will also be important for applications because, for instance, there are bundles such that $\int p_2(E) = 0$ while $\int p_1(E)^2 \neq 0$.^[14] Given any sequence of numbers p_1, p_2, \dots, p_n , they possibly cannot be realised as Pontryagin numbers of a $4n$ -manifold. But it can be shown that for an appropriate k , the multiples kp_1, kp_2, \dots, kp_n can be realised. As a corollary, there are infinitely many distinct $8n$ -manifolds with $p_2 = 0$ but $p_1 \neq 0$.

Appendix: Notation and Conventions

Let M be an m dimensional differentiable manifold, and E be a vector bundle over M with standard fiber R^n and structure group G . A connection on E can be represented by \mathcal{G} valued connection 1-form A , where \mathcal{G} is a linear representation of the Lie algebra of the structure group G . Then the curvature F of the connection A can be represented locally by a \mathcal{G} valued 2-form given by

$$F = dA - A \wedge A. \quad (28)$$

We omit the wedge symbols in what follows. The local curvature 2-form depends on a trivialization of the bundle, but its invariant polynomials σ_k defined by

$$\det(I + tF) = \sum_{k=0}^n \sigma_k t^k \quad (29)$$

are invariants of the local trivialization. We recall that σ_k 's are closed $2k$ -forms, hence they define the deRham cohomology classes in H^{2k} . Furthermore these cohomology classes depend only on the bundle. They are the Chern classes of the bundle E up to some multiplicative constants [10]. It is also known that the $2k$ -form σ_k can be obtained as linear combinations of $tr F^k$ where F^k means the product of the matrix F with itself k times, with the wedge multiplication of the entries.

(A, B) denotes the inner product in the fibre where A, B are Lie algebra valued forms. In particular if α, β are scalar valued p -forms on M then $(\alpha, \beta) = \alpha \wedge * \beta$. The L_2 norm squared of A on the manifold $\|A\|^2 = \int_M (A, A) d\text{vol}$.

The indices $(\mu, \nu \dots)$ will be used as space-time indices while the indices (a, b, \dots) for Lie algebra basis. Let $e^\mu, \mu = 1, \dots, 8$ be any orthonormal basis of 1-forms $\Lambda^1(R^8)$. Then $e^{\mu\nu} = e^\mu e^\nu, \mu < \nu$ will denote the basis of 2-forms. Similarly, $e^{\mu\nu\alpha\beta}, \mu > \nu > \alpha > \beta$ will be a basis of 4-forms. Let E_a be a basis for the Lie algebra of the structure group. Then we can write F as

$$F = \sum_{\mu > \nu} F_{\mu\nu} e^{\mu\nu} = \sum_{a > b} F^{ab} E_{ab} = \sum_{a > b} \sum_{\mu > \nu} F_{\mu\nu}^{ab} E_{ab} \otimes e^{\mu\nu}. \quad (30)$$

Note that $F_{\mu\nu}$ is an element of the Lie algebra, F^{ab} are 2-forms and $F_{\mu\nu}^{ab}$ are functions on M .

References

- [1] J.M.Milnor and J.D. Stasheff, **Characteristic Classes** (Princeton U.P., 1974)
- [2] C.H.Taubes, Contemp. Math. **35**(1984)493 and references therein.
- [3] S.K.Donaldson and P.B.Kronheimer **The Geometry of Four Manifolds** (Clarendon , Oxford, 1990)
- [4] D. H. Tchrakian, J. Math. Phys. **21**(1980)166
- [5] T.N.Sherry, D.H.Tchrakian, Phys.Lett.**B147**(1984)121
D.H.Tchrakian, Phys.Lett.**B150**(1985)360
C. Saçhoğlu, Nucl.Phys.**B277**(1986)487
- [6] E.Corrigan, C.Devchand, D.B.Fairlie, J.Nuyts, Nucl.Phys.**B214** 452(1983)452
- [7] R.S.Ward , Nucl.Phys.**B236**(1984)381
- [8] R.Dündarer,F.Gürsey,C.-H.Tze, J. Math.Phys.**25**(1984)1496
- [9] D.B.Fairlie,J.Nuyts, J. Phys.**A17**(1984)2867
S.Fubini,H.Nicolai,Phys. Lett.**155B**(1985)369
R.Dündarer,F.Gürsey,C.-H.Tze, Nucl. Phys.**B266**(1986)440
- [10] J.A.Harvey,A.Strominger, Phys. Rev. Lett.**66**(1991)549
- [11] T.A.Ivanova, Phys. Lett. **B315**(1993)277
- [12] J. M. Evans, Nucl. Phys. **B298**(1988)92
- [13] B.Grossman, T.W.Kephart, J.D.Stasheff, Commun. Math. Phys.**96**(1984)431
Err. ,ibid **100**(1985) 311
- [14] F. Hirzebruch **Topological Methods in Algebraic Geometry**
(Springer, 1966)