PRIME GRAPHS OF SOLVABLE GROUPS

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ABSTRACT

PRIME GRAPHS OF SOLVABLE GROUPS

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If G is a finite group, its prime graph Γ_G is constructed as follows: the vertices are the primes dividing the order of G, two vertices p and q are joined by an edge if and only if G contains an element of order pq. This thesis is mainly a survey that gives some important results on the prime graphs of solvable groups by presenting their proofs in full detail.

Keywords: prime graph, solvable group, Frobenius group, Fitting length

ÖZ

ÇÖZÜLEBİLİR GRUPLARIN ASAL ÇİZGELERİ

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G sonlu bir grup olsun. G'nin asal bölen çizgesi, köşe kümesi G'nin mertebesini bölen asal sayıların kümesi olacak şekilde; ve herhangi p ve q köşeleri ancak ve ancak G grubunun mertebesi pq olan bir elemanı varsa birleştirilerek elde edilen çizgedir. Bu tez, çözülebilir grupların asal çizgelerini karakterize eden bazı önemli sonuçları detaylarıyla ortaya koyan bir incelemedir.

Anahtar Kelimeler: asal çizge, çözülebilir grup, Frobenius grup, Fitting uzunluk

To my family

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LIST OF SYMBOLS

SYMBOLS

$o(g)$	the order of g
$H \leq G$	H is a subgroup of G
$H < G$	H is a proper subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
$H\ char\ G$	H is a characteristic subgroup of G
$\|G : H\|$	the index of H in G
H^g	$g^{-1}Hg$
$\langle X \rangle$	the subgroup generated by X
$C_G(H)$	the centralizer of H in G
$N_G(H)$	the normalizer of H in G
G_α	the stabilizer of α in G
$Z(G)$	the center of G
$F(G)$	the Fitting subgroup of G
$O_\pi(G)$	the largest normal π-subgroup of G
$\Phi(G)$	the Frattini subgroup of G
$Syl_p(G)$	the set of Sylow p-subgroups of G
$\ker(\alpha)$	the kernel of α
$Aut(G)$	the automorphism group of G
$Inn(G)$	the inner automorphism group of G
$Out(G)$	the outer automorphism group of G
$[H, K]$	the commutator of H and K

G'	the commutator subgroup of G
$G^{(i)}$	the i-th term in the derived series of G
G^i	the i-th term in the lower central series of G
$\|G\|$	the order of G
$\pi(G)$	the set of primes dividing $\|G\|$
Γ_G	the prime graph of G
$V(\Gamma)$	the vertex set of Γ
$E(\Gamma)$	the edge set of Γ
$\overline{\Gamma}$	the complementary graph of Γ
$\chi(\Gamma)$	the chromatic number of Γ
$\Gamma[\pi]$	the induced π-subgraph of Γ
$\Gamma \setminus u$	the induced subgraph $\Gamma[V(\Gamma) \setminus \{u\}]$ of Γ
$u \to v$	the directed edge uv
$N^1(u)$	the set of vertices connected to u
$N^1_{\uparrow}(v)$	the set of vertices u such that $u \to v$
$N^2_{\uparrow}(v)$	the set of vertices u such that $u \to x$ for some $x \in N^1_{\uparrow}(v)$
$N^1_{\downarrow}(v)$	the set of vertices u such that $v \to u$
$N^2_{\downarrow}(v)$	the set of vertices u such that $x \to u$ for some $x \in N^1_{\downarrow}(v)$
$diam(\Gamma)$	the diameter of Γ
$d(u, v)$	the distance between u and v
$\ell_F(G)$	the Fitting length of G
$\mathbb{Z}_n$	the additive group of integers modulo n
$\mathbb{Z}_n^*$	the multiplicative group of integers modulo n
$\mathbb{F}_q$	the field of q elements
$\sigma(n)$	the number of prime divisors of n
$\mathbb{P}$	the set of prime numbers

CHAPTER 1

INTRODUCTION

There has been a lot of research studying the relationship between group and graph theories. One possible reason is that one approach might help us to see things which are not obvious in the other. For example, some combinatorial properties of graphs can be employed to understand the structure of groups. On the other hand group theory can be employed to understand the symmetries, namely the automorphism groups, of graphs.

Given a group G, there are several ways to associate a graph with G starting with Cayley graphs in 1878. Some examples of other graphs associated with a group which have been extensively studied are commuting graph, non-commuting graph, intersection graph, conjugacy class graph, power graph and prime graph (or Gruenberg-Kegel graph) (See [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]). In general, such graphs are all regarded as geometric objects which encode certain information about the structure of a group G.

If G is a finite group its prime graph Γ_G is constructed as follows: the vertices are the primes dividing the order of the group, two vertices p and q are joined by an edge if and only if G contains an element of order pq. This was introduced by Gruenberg in 1970s as a concept arising from cohomological questions associated with integral representation of finite groups (See Gruenberg [14], Gruenberg- Roggenkamp [15]). The study of prime graphs of both solvable and non-solvable groups has always become an attractive research area over the years as they provide rich information about the group structure. One of the first results concerning prime graphs of groups was

the unpublished Gruenberg-Kegel Theorem, which was published later in a paper of Williams [3]. This result implies the complete description of a solvable group G with disconnected prime graph, namely, G is either Frobenius or a 2-Frobenius. Moreover, if G is a Frobenius group or 2-Frobenius group, then its prime graph Γ_G has exactly two connected components (see [4], Lemma 1); and if G is solvable, then each connected component of Γ_G is a clique, that is a subset of vertices whose induced subgraph is complete. In 2005 Vasil'ev ([12], Propositions 2, 3) obtained a generalization of the Gruenberg-Kegel Theorem. It is worth mentioning that Williams and Kondrat'ev described connected components of the prime graphs of all simple nonabelian groups in [3] and [10]. In particular, they proved that the number of connected components of the prime graph of a simple nonabelian group is at most 6. An adjacency criterion of the prime graphs of simple groups and cocliques of maximal sizes in the prime graphs of simple groups are known by [12] and [13]. In 1999 Lucido ([4], Theorem 1) proved that if G is a solvable group then the complement of its prime graph is triangle free, which also follows by some earlier results of Hall and Higman. As an immediate consequence, the diameter of Γ_G is at most 3 when G is solvable. Lucido also studied those solvable groups G for which Γ_G has diameter 3 , and classified groups whose prime graph is a tree in [4], and with Moghaddamfar also characterized the finite simple groups with a complete prime graph in [8].

This thesis is mainly a survey that gives some important results on the prime graphs of solvable groups by presenting their proofs in full detail. The focus of our attention is included in the main theorems of [1], [4], and [15] which are listed below.

Theorem 1. *Let G be a solvable group. If p, q, r are three distinct primes dividing the order of G, then G contains an element of order the product of two of these primes. Equivalently, the complement of the prime graph of a solvable group contains no triangles.*

Theorem 2. *If the prime graph of a solvable group G is disconnected, then G is either Frobenius or 2-Frobenius. Furthermore, if G is 2-Frobenius, its prime graph*

consists of exactly two components.

Theorem 3. *Let G be a solvable group such that the diameter of its prime graph is 3. Then either $\ell_F(G) \leq 3$ or $\ell_F(G) = 4$ and G has a normal section isomorphic to $2O$. Here $\ell_F(G)$ denotes the Fitting length of G.*

Theorem 4. *A graph is isomorphic to the prime graph of some solvable group if and only if its complement is 3-colorable and triangle free.*

Theorem 5. *Let G be a solvable group having a minimal prime graph. Then $3 \leq \ell_F(G) \leq 4$.*

Theorem 1, which is of independent interest too, serves as a key result pertaining to most of the interesting conclusions on the prime graph of solvable groups. Theorem 2 is a special case of Gruenberg-Kegel theorem, which characterizes solvable groups with disconnected prime graphs. Theorem 3 provides a nice description of the structure of a solvable group whose prime graph has diameter 3, to which we appeal in proving Theorem 5.

Among these, Theorem 4 is of special importance as it provides a very nice characterization of the prime graphs of solvable groups. By Theorem 2 it is known that a solvable group with a disconnected prime graph is either Frobenius or 2-Frobenius. In proving Theorem 4, the main idea is to apply Theorem 3 to disconnected induced subgraphs of the prime graph by introducing the concept of a Frobenius digraph, namely a special orientation of the complement of the prime graph. We then take into account fixed point free actions of the Sylow subgroups of the Hall $\{p, q\}$-subgroups for primes p, q dividing the order of G. The last result Theorem 5 measures how far a solvable group G with a minimal prime graph is from being nilpotent. It is shown by examples that the bounds given in Theorem 5 are the best possible.

Apart from these five theorems we present a partial answer to a conjecture which

states that in a finite solvable group G, there exists an element x such that the number of primes dividing $o(x)$ is at least one third of the number of primes dividing the order of G.

The outline of the thesis is as follows:

Chapter 2 gives the necessary preparation from graph theory and group theory. Most of them are not elementary results which will be referred throughout the thesis.

Chapter 3 provides a detailed description of the structures of Frobenius and 2-Frobenius groups which are indispensable in the theory of prime graphs.

Chapter 4 is essentially devoted to the proofs of Theorem 1, Theorem 2, Theorem 3, and Theorem 4 after introducing the concept of Frobenius digraph.

Finally, Chapter 5 contains a proof of Theorem 5 and the examples which show that the bounds of Theorem 5 are the best possible.

CHAPTER 2

PRELIMINARIES

2.1 Graph Theoretical Part

A **graph** is a pair of sets (V, E) where E consists of elements in the form $\{u, v\}$ for some pairs $u, v \in V$.

Let $\Gamma = (V, E)$ be a graph. Any element in $V = V(\Gamma)$ is called a vertex of Γ, and any element in $E = E(\Gamma)$ is called an edge of Γ. Let v_i in V for $i = 0, 1, \ldots, n$. A graph Ω is called a **subgraph** of Γ if $V(\Omega) \subseteq V$ and $E(\Omega) \subseteq E$. We say Ω is an **induced subgraph** if Ω contains each edge of Γ which is between the vertices in $V(\Omega)$. The induced prime graph of Γ with the set of vertices π is denoted by $\Gamma[\pi]$.

If e_i is the edge in the form $\{v_{i-1}, v_i\}$ for $i = 1, 2, \ldots, n$, we call the sequence $v_0 e_1 v_1 \ldots v_{n-1} e_n v_n$ a **walk**, where n is said to be the **length** of this walk. The walk is called **closed** if $v_0 = v_n$. If no vertex is repeated in the sequence $v_0 v_1 \ldots v_n$, then it is called an n**-path**. If no vertex is repeated in the sequence $v_0 v_1 \ldots v_{n-1}$, then the closed walk $v_0 e_1 v_1 \ldots v_{n-1} e_n v_n$ is called an n**-cycle**. The **girth** of Γ is defined to be the minimum of the lengths of all cycles in Γ.

Lemma 2.1. *Every closed walk w of odd length contains an odd cycle.*

Proof. We proceed by induction on the length of w. Let $w = v_1 v_2 \ldots v_{2k+1} v_1$ for a positive integer k. If there is no repeated vertex in w, then w is an odd cycle. So we can assume that $v_i = v_j$ for some i, j with $1 \leq i < j \leq 2k + 1$. We observe that $v_1 v_2 \ldots v_i v_{j+1} \ldots v_{2k+1} v_1$ and $v_i v_{i+1} \ldots v_{j-1} v_i$ are two closed walks whose union

gives w. It follows that one of them has odd length, and hence contains an odd cycle by induction hypothesis. This completes the proof. $\square$

We say Γ is **connected** if there is a path between any pair of vertices in Γ. The **distance** between two vertices a and b in a connected graph is the length of the shortest path between them, and is denoted by $d(a, b)$. The **diameter** of a connected graph is defined to be the largest distance amongst all pair of vertices, and is denoted by $diam(\Gamma)$.

Lemma 2.2. *A connected finite graph Γ in which every vertex has degree at most 2 is either a path or a cycle.*

Proof. Let a and b be vertices such that $d(a, b) = k = diam(\Gamma)$. Then there exists a path w of length k between a and b. Clearly all vertices on w for a and b do not have any neighbors from outside w. If a and b are both of degree 1, then Γ is a path. If a has a neighbor c outside w, then there must be a path w' of length at most k between c and b. Since w' cannot go over a, the union of w, w' and ac forms a cycle. This completes the proof. $\square$

A set of vertices in Γ is called an **independent set** if there is no edge between any pair of vertices lying in this set. Γ is said to be **bipartite** if $V(\Gamma)$ can be decomposed into two disjoint subsets so that there is no edge between any pair of vertices lying in the same subset, in other words, $V(\Gamma)$ is a disjoint union of two independent sets.

Lemma 2.3. *Let Γ be a graph. Then Γ does not contain an odd cycle if and only if Γ is bipartite.*

Proof. Assume that that Γ contains an odd cycle and Γ is bipartite. Let $v_1 v_2 \dots v_n$ be the ordered sequence of vertices in an odd cycle of length n in Γ. Since Γ is bipartite, $V(\Gamma)$ is a disjoint union of two subsets A and B such that there is no edge between any pair of vertices lying in the same subset. Let $v_1 \in A$. Then $v_2 \in B$ as it is adjacent to v_1. Repeating the same argument for each $k = 1, \dots, n$, we have $v_k \in A$

if k is odd and $v_k \in B$ if k is even. Hence both v_1 and v_n are in A, but they are also adjacent, which is a contradiction.

Next assume that Γ does not contain an odd cycle. We may assume that Γ is connected. Fix a vertex v_0 and partition $V(\Gamma)$ into two sets $A = \{v \in V(\Gamma) \mid d(v, v_0) \text{ is odd}\}$ and $B = \{v \in \Gamma \mid d(v, v_0) \text{ is even}\}$. Assume that a_1 and a_2 are two connected vertices in A. Then the closed walk consisting of the shortest paths between v_0 to a_1, a_1 to a_2, and a_2 to v_0 contains an odd cycle by Lemma 2.1, which is a contradiction. It follows that A is an independent set, and similarly B is also an independent set. Thus Γ is bipartite. $\square$

An n**-coloring** of a graph Γ is a labeling of $V(\Gamma)$ using n colors such that no two vertices sharing the same edge are labeled with the same color. If Γ has a coloring using at most n colors, then Γ is said to be n**-colorable**. The **chromatic number** $\chi(\Gamma)$ of Γ is defined to be the smallest positive integer n such that Γ is n-colorable.

A **directed** graph is a pair of sets (V, D) where $D \subseteq V \times V$. Any element $(u, v) \in D$ is a **directed edge** from u to v. A directed graph $\overrightarrow{\Gamma} = (V, D)$ is called an **orientation** of a graph $\Gamma = (V, E)$ if exactly one of (u, v) and (v, u) is in D for every $\{u, v\} \in E$. Let $v_0 e_1 v_1 \ldots v_{n-1} e_n v_n$ be a walk in Γ. If there are directed edges $d_i = (v_{i-1}, v_i)$ in D for $i = 1, \ldots, n$, then $v_0 d_1 v_1 \ldots v_{n-1} d_n v_n$ is called a **directed walk** in $\overrightarrow{\Gamma}$. If no vertex is repeated in the sequence $v_0 v_1 \ldots v_n$, then $v_0 d_1 v_1 \ldots v_{n-1} d_n v_n$ is called a **directed path** of length n. We call $v_0 d_1 v_1 \ldots v_{n-1} d_n v_n$ a **directed cycle** if the only repeated vertices in $v_0 v_1 \ldots v_n$ are v_0 and v_n. The directed graph $\overrightarrow{\Gamma}$ is said to be an **acyclic orientation** of Γ if $\overrightarrow{\Gamma}$ does not contain any directed cycle.

Theorem 2.4. *(Gallai-Roy) Let* $n = min\{\ell \mid \ell$ *is the length of a longest directed path in an orientation of* $\Gamma\}$. *Then* $\chi(\Gamma) = n + 1$.

Proof. Let $\overrightarrow{\Gamma}$ be an acyclic orientation of Γ for which n is attained. We decompose $V(\overrightarrow{\Gamma})$ into s disjoint subsets A_i, for $i = 1, \ldots, s$, using the following procedure: Let A_1 consist of the vertices with no incoming edge; A_2 consist of the vertices with only incoming edges coming from A_1 , ... Mainly once A_i has been constructed

A_{i+1} is defined to be the set of all vertices whose all incoming edges are coming from $\cup_{j=1}^{i} A_j$. We should note that for each $v \in A_{i+1}$, for $i = 1, \ldots, s-1$, there always exists a directed edge (u, v) such that $u \in A_i$. This gives us a directed path of length $s-1$. Let $v_0 d_1 v_1 \ldots v_{n-1} d_n v_n$ be a longest directed path in $\overrightarrow{\Gamma}$. It is clear from the above construction that $v_{i-1} \in \cup_{t=1}^{j-1} A_t$. Thus v_{i-1} and v_i lie in A_k and A_l respectively, with $k < l$. Consequently, we have $n+1 \leq s$, and hence the length of a longest directed path in this orientation of Γ is $n = s-1$. As no two vertices in A_i are connected for $i = 0, 1, \ldots, s$, we can color the vertices in A_i with color i, and get an s-coloring of Γ. Then we have $\chi(\Gamma) \leq s = n+1$.

Let Γ have a k-coloring. Name these colors by numbers $1, 2, \ldots, k$. If there exist a vertex v with color $a > 1$ and a color $b < a$, such that v has no neighbor with color b, change the color of v from a to b. Repeating this procedure, we reduce the total sum of colors of all vertices, and finally get a coloring for which each vertex colored with a has at least one neighbor colored with b for each color $b < a$. This enables us to construct an acyclic orientation of Γ by directing edges from the vertices with smaller color to the vertices with greater color. Let A_i's be the pairwise disjoint subsets of $V(\Gamma)$ constructed as in the first paragraph by using this orientation. Notice that A_i consists of all vertices colored by i. In this orientation the length of a longest directed path is less than or equal to $k-1$, and hence $k \geq n+1$. This shows that Γ is not n-colorable, and hence we have $\chi(\Gamma) = n+1$. $\square$

2.2 Group Theoretical Part

2.2.1 Automorphism Groups

For any group G, the set $Aut(G)$ consisting of all automorphisms of G forms a group under the composition of mappings which is called the **automorphism group** of G. For each $g \in G$, the map $\tau_g : G \to G$ defined by $\tau_g(x) = g^{-1}xg$ is an automorphism of G called the **inner automorphism** of G induced by g. The set of all inner automorphisms is a normal subgroup of $Aut(G)$ which is denoted by $Inn(G)$ and is

called the **inner automorphism group** of G. The quotient group $Aut(G)/Inn(G)$ is called the **outer automorphism group** of G, and is denoted by $Out(G)$.

We shall present some results which will be used in the proofs of main theorems of the thesis.

Lemma 2.5. *The automorphism group of a cyclic group of order n is isomorphic to $\mathbb{Z}_n^*$, and hence it is abelian.*

Proof. Let c be a generator of a cyclic group C of order n. Then any automorphism of C takes c to another generator c^m . Conversely for any positive integer m with $(m, n) = 1$, the map $\alpha_m : C \to C$ defined by $\alpha_m(k) = k^m$ is an automorphism of C. It easily follows that $Aut(C) = \{\alpha_m \mid 1 \leq m < n \text{ and } (m, n) = 1\}$. Furthermore, the map $\beta : \alpha_m \mapsto m$ defines an isomorphism from $Aut(C)$ to $\mathbb{Z}_n^*$. □

Lemma 2.6. *For any subgroup H of G, the group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$.*

Proof. Consider the map $\alpha : N_G(H) \to Aut(H)$, defined by $\alpha(g) = \tau_g$. Since $\ker(\alpha) = C_G(H)$, the result follows. □

Lemma 2.7. *Let $G = A_1 \times A_2 \times \cdots \times A_n$, where A_i is a characteristic subgroup of G for $i = 1, \ldots, n$. Then*

(i) $Aut(G) \cong Aut(A_1) \times Aut(A_2) \times \cdots \times Aut(A_n)$.

(ii) $Inn(G) \cong Inn(A_1) \times Inn(A_2) \times \cdots \times Inn(A_n)$.

(iii) $Out(G) \cong Out(A_1) \times Out(A_2) \times \cdots \times Out(A_n)$.

Proof. (i) Let α be an automorphism of G. Notice that A_i is the unique subgroup of G of order $|A_i|$ for each $i = 1, \cdots, n$. Then α fixes each A_i, and hence $\alpha|_{A_i} \in Aut(A_i)$. Define $\theta : Aut(G) \to Aut(A_1) \times \cdots \times Aut(A_n)$ by $\theta(\alpha) = (\alpha|_{A_1}, \ldots, \alpha|_{A_n})$ where $\alpha \in Aut(G)$. One can see that θ is a group isomorphism.

(ii) Let θ be defined as in part (i), and let $g = (g_1, \ldots, g_n) \in G$. Then $\theta(\tau_g) = (\tau_g|_{A_1}, \ldots, \tau_g|_{A_n})$. Clearly $\tau_g|_{A_i} = \tau_{g_i}$ since g_j commute with A_i for $i \neq j$. It is clear that the image of $\theta|_{Inn(G)}$ is $Inn(A_1) \times \cdots \times Inn(A_n)$, and hence $Inn(G) \cong Inn(A_1) \times \cdots \times Inn(A_n)$.

(iii) Combining first two parts, we get

$$Out(G) = Aut(G)/Inn(G) \cong (Aut(A_1) \times \cdots \times Aut(A_n))/(Inn(A_1) \times \cdots \times Inn(A_n)) \cong$$

$$(Aut(A_1)/Inn(A_1)) \times \cdots \times (Aut(A_n)/Inn(A_n)) = Out(A_1) \times \cdots \times Out(A_n)$$

as claimed. □

2.2.2 Commutators, Solvable Groups and Hall Subgroups

Let G be a group. For any $h, k \in G$, and $[h, k]$ is defined by $h^{-1}k^{-1}hk$ and read as the **commutator** of h and k. Let H and K be two subgroups of G. The subgroup generated by all commutators $[h, k]$ where $h \in H$ and $k \in K$, is denoted by $[H, K]$. Clearly H and K commute if and only if $[H, K] = 1$.

Lemma 2.8. *Let G be a group, and H and K be subgroups of G. Then*

(i) $[H, K] \trianglelefteq \langle H, K \rangle$.

(ii) $[H, K] \leq H$ *if and only if* $K \leq N_G(H)$.

Proof. (i) Note that $[H, K] = [K, H]$. It is enough to show that $[h, k]^g \in [H, K]$ for any $g \in H$, and a generator $[h, k] \in [H, K]$. Notice that $[h, k]^g = [hg, k][g, k]^{-1} \in [H, K]$. This completes the proof.

(ii) Observe that K normalizes H if and only if $h^{-1}h^k = [h, k] \in H$, for all $h \in H$, $k \in K$ if and only if $[H, K] \leq H$. □

We call $[G, G]$ the **commutator subgroup** of G and denote it by G'. Notice that $G' = 1$ if and only if G is abelian.

Set $G^{(0)} = G$, and define $G^{(n)} = (G^{(n-1)})'$ for any integer $n \geq 1$. Clearly $G^{(n)}$ is a characteristic subgroup of G for $n \geq 0$. Also it can be easily verified by induction that $(G/N)^{(i)} = G^{(i)}N/N$ for any integer $i \geq 0$. In particular G/G' is abelian. Furthermore, for any normal subgroup N of G with G/N is abelian, we have $G' \leq N$ because $(G/N)' = G'N/N = 1$. We can also observe by induction that $(G^{(i)})^{(j)} = G^{(i+j)}$ for any pair i, j of nonnegative integers.

The chain $G \geq G' \geq G^{(2)} \geq \ldots$ is called the **derived series** of G. We say G is **solvable** if the derived series of G terminates at 1 after a finite number of steps, that is, $G^{(n)} = 1$ for some positive integer n.

Theorem 2.9. *Let G be a group.*

(a) If G is solvable, then every subgroup of G is solvable.

(b) If G is solvable, then every homomorphic image of G is solvable.

(c) Let $N \trianglelefteq G$. Suppose that N and G/N are both solvable. Then G is solvable.

Proof. (a) Let H be a subgroup of G. Then we have $H' \leq G'$. It easily follows by induction $H^{(k)} \leq G^{(k)}$ for any positive integer k. Since G is solvable, we have $G^{(m)} = 1$ for some integer $n \geq 0$. Thus $H^{(m)} = 1$, and H is also solvable.

(b) Let $G^{(m)} = 1$ for some integer $m \geq 0$ and $\varphi : G \to A$ be a surjective homomorphism. By induction $\varphi(G^{(i)}) = A^{(i)}$ for all $i \geq 0$. Therefore $A^{(m)} = 1$, and hence A is solvable.

(c) Let $N^{(n)} = 1$, and $(G/N)^{(m)} = N$ for some integers $n, m \geq 0$. Therefore we get $G^{(m+n)} = (G^{(m)})^{(n)} \leq N^{(n)} = 1$, and hence G is solvable. $\square$

In a group G, a nontrivial normal subgroup M is called a **minimal normal** subgroup if no nontrivial proper subgroup of M is normal in G.

Lemma 2.10. *Let M be a finite solvable minimal normal subgroup of a group G. Then M is an elementary abelian p-group for some prime p.*

Proof. Since M is solvable and nontrivial, we have that $M > M'$. Clearly M' is normal in G since M' is characteristic in M and M is normal in G, which forces that $M' = 1$ by the minimality of M. Thus M is abelian. Now let p be a prime divisor of $|M|$, and consider the characteristic subgroup $A = \{x \in M \mid x^p = 1\}$ of M. By Cauchy's Theorem, A is not trivial. Then $A = M$ by the minimality of M, that is, M is an elementary abelian p-group. □

Let π be a set of primes. We denote the complement of π in the set of all prime numbers by π'. A positive integer is called a **π-number** if its all prime divisors lie in π. A finite group is said to be a **π-group** if its order is a π-number.

For any finite group G, a subgroup H of G is called a **Hall π-subgroup** if H is a π-group and $|G : H|$ is a π'-number. It is clear that H is a Sylow p-subgroup of G in case where $\pi = \{p\}$. Recall that the set of all Sylow p-subgroups of G forms a single conjugacy class.

We should mention here a well-known theorem in finite group theory which will be used in the proofs of some results.

Theorem 2.11. *(Schur-Zassenhaus) Let $N \trianglelefteq G$, where G is a finite group, and assume that $(|N|, |G : N|) = 1$. Then N has a complement in G, in other words there exists a subgroup H of order $|G : N|$ of G. Furthermore, if N or G/N is solvable, then all complements of N in G are conjugate.*

Proof. See section 3B in [2]. □

As a consequence of the above theorem, we can make the following important observations related to finite solvable groups.

Theorem 2.12. *Let π be a set of prime numbers. Every finite solvable group G has a Hall π-subgroup, and any two Hall π-subgroups of G are conjugate.*

Proof. The case $|G| = 1$ is trivial. Assume that $G > 1$ and proceed by induction on $|G|$. Let M be a minimal normal subgroup of G. Then M is solvable and hence is

an elementary abelian p-group for some prime p, by Theorem 2.9 and Lemma 2.10. It follows by Theorem 2.9 that G/M is solvable. Then G/M has a Hall π-subgroup, say H/M, by induction. Now H/M is a π-group and $|G : M|/|H : M| = |G : H|$ is a π'-number. If $p \in \pi$, we have $|H| = |H : M||M|$, which is a π-number, and hence H is a Hall π-subgroup of G. If $p \notin \pi$, then by Schur-Zassenhaus Theorem, M has a complement K in H, namely $(|K|, |M|) = 1$ and $KM = H$. Then K is a π-group and

$$|G : K| = |G : H||H : K| = |G : H||M|$$

is a π'-number. Thus K is a Hall π-subgroup of G.

Also for any Hall subgroup L of G, by the induction hypothesis, LM/M is conjugate to HM/M. Then $HM = (LM)^g$ for some $g \in G$. If $p \in \pi$, then HM and LM are Hall π-subgroups of G. Thus $HM = H$, $LM = L$, and $H = L^g$ as desired. If $p \notin \pi$, since H and L^g are complements to M in HM, by the conjugacy part of Schur-Zassenhaus theorem, H and L^g are conjugate in HM. Thus, any two Hall π-subgroups of G are conjugate. □

Theorem 2.13. *Let G be a finite solvable group and H be a π-subgroup of G. Then H is contained in a Hall π-subgroup of G.*

Proof. We proceed by induction on $|G|$. Let N be a minimal normal subgroup of G. Then N is an elementary abelian p-group by Lemma 2.10. First assume that $p \in \pi$. Then G/N has a Hall π-subgroup K/N containing HN/N by induction hypothesis. Note that $|K| = |K : N||N|$ is a π-number and $|G : K| = |G : N|/|K : N|$ is a π'-number. Thus K is a Hall π-subgroup of G containing H.

Next assume that $p \notin \pi$. By the same argument HN/N is contained in a Hall π-subgroup K/N of G/N. By Theorem 2.11, N has a complement in K since $(|N|, |K : N|) = 1$, say M. So we have $HN \leq MN$, where M is a Hall π-subgroup of G. It follows that

$$|M \cap HN| = |M||NH|/|MNH| = |M||N||H|/|MN| = |H|.$$

That is $M \cap HN$ is a complement for N in HN. By the conjugacy part in Theorem 2.11, $H = (M \cap HN)^g$ for some $g \in G$. Therefore H is contained in M^g which is a Hall π-subgroup of G. $\square$

The following famous result due to Burnside will be used in the proof of next theorem studying the structure of a finite group whose all Sylow subgroups are cyclic.

Theorem 2.14. *(Burnside) Let $P \in Syl_p(G)$ where G is a finite group. If $P \leq Z(N_G(P))$, then G has a normal p-complement.*

Proof. See Theorem 5.13 in [2]. $\square$

Theorem 2.15. *If all Sylow subgroups of a finite group G are cyclic, then G is solvable.*

Proof. The case $|G| = 1$ is trivial. We assume that $|G| > 1$ and proceed by induction on $|G|$. Let p be the smallest prime divisor of $|G|$ and let $P \in Syl_p(G)$. By Lemma 2.6, $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $Aut(P)$, and so $|N_G(P)/C_G(P)|$ divides $|Aut(P)| = \varphi(p^n) = p^{n-1}(p-1)$ since P is cyclic with $|P| = p^n$. As $P \leq C_G(P)$, we deduce that p^n divides $|C_G(P)|$, and hence $|N_G(P)/C_G(P)|$ is coprime to p. Therefore $|N_G(P)/C_G(P)|$ divides $p-1$ which forces that $N_G(P) = C_G(P)$ as p is the smallest prime divisor of $|G|$. Note that we have $P \leq Z(C_G(P)) = Z(N_G(P))$. By Theorem 2.14, G has a normal p-complement N. Since all Sylow subgroups of N are also Sylow subgroups of G, they are all cyclic. It follows then by induction hypothesis that N is solvable, and hence G is also solvable by Theorem 2.9. $\square$

In a group G, a series of normal subgroups $1 = N_0 \leq N_1 \leq \ldots \leq N_n = G$ is called a **chief series** if N_{i+1}/N_i is a minimal normal subgroup of G/N_i for each $i = 1, \ldots, n-1$. Here N_{i+1}/N_i is said to be a **chief factor** of G for each i. Note that each chief factor of a finite solvable group is elementary abelian by Lemma 2.10.

Let now $\sigma(n)$ denote the number of prime divisors of n for any natural number n. We call a finite group σ**-reduced** if every prime p divides the order of at most one chief factor of G.

Lemma 2.16. *Every finite solvable group G has a σ-reduced subgroup H such that $\sigma(|G|) = \sigma(|H|)$.*

Proof. We proceed by induction on $|G|$. Let N be a minimal normal subgroup of G. Then N is an elementary abelian p-group for some prime $p \in \pi(G)$ Lemma 2.10. Let K be a Hall p'-subgroup of G. By induction hypothesis, K has a σ-reduced subgroup H such that $\sigma(|K|) = \sigma(|H|) = \sigma(|G|) - 1$. Consider a minimal normal subgroup N_1 of NH which is contained in N. Let $1 < N_2 \leq \ldots \leq N_n = H$ be a chief series of H. Then $1 < N_1 \leq N_1N_2 \leq N_1N_3 \leq \ldots N_1N_n = N_1H$ is a chief series of N_1H as

$$N_1N_k/N_1N_{k-1} \cong (N_1N_k/N_1)/(N_1N_{k-1}/N_1) \cong N_k/N_{k-1}$$

is a minimal normal subgroup of $N_1H/N_1N_{k-1} \cong H/N_{k-1}$ for $k = 2, \ldots, n$. Note that $|N_1|$ is a power of p and all other prime divisors of G appears exactly once in this chief series as H is σ-reduced. So we conclude that N_1H is σ-reduced, and $\sigma(|G|) = \sigma(|H|) + 1 = \sigma(|N_1H|)$, as desired. □

2.2.3 Nilpotent Groups

Given any group G, a chain of normal subgroups

$$1 = N_0 \leq N_1 \leq \ldots \leq N_m = G$$

is called a **central series** of G if $N_{i+1}/N_i \leq Z(G/N_i)$ for $i = 0, 1, \ldots, m-1$. Let $\{Z_i\}_{i\geq 0}$ be defined inductively as $Z_0 = 1$, $Z_1 = Z(G)$ and $Z_i/Z_{i-1} = Z(G/Z_{i-1})$ for $i \geq 2$. The resulting central series

$$1 \leq Z_1 \leq Z_2 \leq \ldots$$

of G is called the **upper central series** of G. We say the group G is **nilpotent** if there exists a nonnegative integer r such that $Z_r = G$.

The **lower central series** $\{G^i\}_{i\geq 1}$ is the chain of subgroups defined by of $G^1 = G$ and $G^i = [G^{i-1}, G]$ for $i \geq 2$.

Theorem 2.17. *The following are equivalent:*

(a) G is nilpotent.

(b) $Z_r = G$ for some r.

(c) $G^{r+1} = 1$ for some r.

(d) All nontrivial homomorphic images of G have nontrivial centers.

Proof. See Lemma 1.20 and 1.21 in [2]. □

(The smallest integer c satisfying $G^{c+1} = 1$ is called the **nilpotency class** of G.)

We shall state and prove some important results related to nilpotent groups which will be useful throughout the thesis.

Theorem 2.18. *Let H be a proper subgroup of a nilpotent group G. Then $N_G(H) > H$.*

Proof. Let $\{Z_i\}_{i\geq 0}$ be the upper central series of G. Then there exists a nonnegative integer k such that $Z_k \leq H$ but $Z_{k+1} \nleq H$. We have $Z_{k+1}/Z_k = Z(G/Z_k)$, and so Z_{k+1}/Z_k centralizes H/Z_k. In particular $Z_{k+1}/Z_k \leq N_{G/Z_k}(H/Z_k)$, which implies that $Z_{k+1} \leq N_G(H)$. Since $Z_{k+1} \nleq H$, the result follows. □

Theorem 2.19. *Every Sylow subgroup of a finite nilpotent group G is a normal subgroup of G. Conversely, if all Sylow subgroups of a finite group G is normal, then G is nilpotent.*

Proof. Let G be a finite nilpotent group and $P \in Syl_p(G)$. Assume that $N_G(P)$ is different from G. Then there exists a maximal subgroup M such that $N_G(P) \leq M$. (A proper subgroup M of G is said to be maximal if no subgroup of G can be inserted properly between M and G.) Since $N_G(M) > M$ by Theorem 2.18, and M is

maximal, we have $N_G(M) = G$. On the other hand, $P \leq N_G(P) \leq M$, and hence P is also a Sylow p-subgroup of M. Then, by the Frattini Argument (see Theorem 2.35), we have $G = MN_G(P) = M$, which is a contradiction. Thus $N_G(P) = G$ as desired.

Notice that if all Sylow subgroups of G are normal, then G is the direct product of its Sylow subgroups, and so any homomorphic image of G is isomorphic to a direct product of subgroups of prime power order. It follows that the center of any nontrivial homomorphic image of G is nontrivial, and the result follows by Theorem 2.17. □

Lemma 2.20. *Let M be a nontrivial normal subgroup of a finite nilpotent group G. Then $M \cap Z(G) > 1$.*

Proof. Let $1 = Z_0 \leq Z_1 \leq \ldots \leq G$ be the upper central series of G. Then there exists a nonnegative integer i such that $M \cap Z_i = 1$ and $M \cap Z_{i+1} > 1$. Since $[G, Z_{i+1}] \leq Z_i$ and $[G, M] \leq M$, we get $[G, M \cap Z_{i+1}] \leq M \cap Z_i = 1$. Thus $M \cap Z_{i+1} \leq Z(G)$, that is, $M \cap Z(G)$ is nontrivial. □

Lemma 2.21. *Let M be a minimal normal subgroup of a finite nilpotent group G. Then M is cyclic of prime order.*

Proof. Assume that $G > 1$. By Lemma 2.20, $M \cap Z(G)$ is a nontrivial normal subgroup of G contained in M. It follows that $M = M \cap Z(G)$, and hence $M \leq Z(G)$. Since any subgroup of $Z(G)$ is normal in G, we see that M is cyclic of prime order. □

Lemma 2.22. *Let G be a finite nilpotent group, and let A be maximal among normal abelian subgroups. Then $A = C_G(A)$.*

Proof. Clearly $A \leq C_G(A)$. Since $A \trianglelefteq G$, we have $C_G(A) \trianglelefteq G$. Assume that $A < C_G(A)$. Let B/A be a minimal normal subgroup of G/A contained in $C_G(A)/A$. By Lemma 2.21, B/A is cyclic of prime order. Since $B \leq C_G(A)$, we get $A \leq Z(B)$. Therefore, B is abelian. On the other hand, $B \trianglelefteq G$ as $B/A \trianglelefteq G/A$, contradicting the fact that A is a maximal normal abelian subgroup of G. This completes the proof. □

2.2.4 Frattini Subgroup, Fitting Subgroup and Fitting Series of a Finite Solvable Group

In a finite group G, the intersection of all maximal subgroups is called the **Frattini subgroup** of G, and is denoted by $\Phi(G)$. Here are two results which will be used later.

Lemma 2.23. *In a finite group G if X is a subset of G such that $\langle X \rangle < G$, then $\langle X \cup \Phi(G) \rangle < G$.*

Proof. Let X be a subset of G such that $\langle X \rangle < G$. Then $\langle X \rangle$ is contained in a maximal subgroup M. Since $\Phi(G) \leq M$, we have $\langle X \cup \Phi(G) \rangle \leq M < G$. □

Lemma 2.24. *For any group G of prime power order, $G/\Phi(G)$ is abelian.*

Proof. Let M be a maximal subgroup of G. By Theorem 2.18, M is a normal subgroup of G. Then the group G/M does not contain a nontrivial proper subgroup, and hence it is of prime order. It follows that G/M is abelian, and so $G' \leq M$. Since M is arbitrary, G' is contained in every maximal subgroup of G, that is, $G' \leq \Phi(G)$. Thus $G/\Phi(G)$ is abelian. □

Let $\pi \subseteq \pi(G)$. We denote by $O_\pi(G)$ the unique largest normal π-subgroup of G, which means that every normal π-subgroup of G is contained in $O_\pi(G)$. In case where $\pi = \{p\}$, we simply write $O_p(G)$. The following lemma will be used frequently in the proofs of the main theorems.

Lemma 2.25. *Let G be a finite group. Then $O_\pi(G/O_\pi(G))$ is trivial for any $\pi \subseteq \pi(G)$.*

Proof. Assume that $O_\pi(G/O_\pi(G)) = H/O_\pi(G)$ for some π-subgroup H of G such that $H > O_\pi(G)$. Then $H/O_\pi(G) \trianglelefteq G/O_\pi(G)$ which means that $H \trianglelefteq G$, that is, $H \leq O_\pi(G)$. This contradiction leads to the fact that $O_\pi(G/O_\pi(G))$ is trivial. □

We define the **Fitting subgroup** of G, which is denoted by $F(G)$, to be the product of the subgroups $O_p(G)$ as p runs through $\pi(G)$. Clearly, $F(G)$ is a characteristic subgroup of G, in particular $F(G) \trianglelefteq G$. Also, by Theorem 2.19, $F(G)$ is nilpotent. Some further important properties of $F(G)$ for a solvable group G can be listed as follows.

Lemma 2.26. *Let G be a finite group. Then $F(G)$ contains every normal nilpotent subgroup of G. In particular, $F(G)$ is nontrivial for any nontrivial finite solvable group G.*

Proof. Let N be a normal nilpotent subgroup of G. If P is a Sylow p-subgroup of N, then P is characteristic in N, and hence $P \trianglelefteq G$. Therefore $P \leq O_p(G) \leq F(G)$, that is, every Sylow subgroup of N lies in $F(G)$. Since N can be written as a product of its Sylow subgroups, we get $N \leq F(G)$.

Next let $G^{(m)}$ be the last nontrivial term in the derived series of G. Then $G^{(m)}$ is abelian, and hence nilpotent. It follows by the above paragraph that $G^{(m)} \leq F(G)$ as $G^{(m)} \trianglelefteq G$. □

Theorem 2.27. *Let G be a finite solvable group. Then $C_G(F(G)) \leq F(G)$.*

Proof. Let $C = C_G(F(G))$. The group $K = F(G) \cap C$ is a normal subgroup of C since every element of K commutes with every element of C. Since G is solvable, so is C/K. Consider the derived series of C/K and assume that C/K is nontrivial. Let A/K be the last nontrivial term in this series. Clearly A/K is abelian and hence $A' \leq K$. On the other hand, as every element of A commutes with K, we have $K \leq Z(A)$. It follows that $[A, A, A] = 1$, and hence A is nilpotent.

Notice that A/K is a characteristic subgroup of C/K as a term of its derived series. As $C/K \trianglelefteq G/K$, we get $A/K \trianglelefteq G/K$, and hence $A \trianglelefteq G$. Thus A is a normal nilpotent subgroup of G, which implies that $A \leq F(G) \cap C = K$. This contradicts the fact that $A \neq K$. Thus we have $C/K = 1$, that is, $C_G(F(G)) \leq F(G)$ as desired. □

We now define an invariant, namely the Fitting length, of a finite solvable group G

which measures how far G is from being nilpotent.

Definition 2.28. *We define the* ***(upper) Fitting series***

$$1 = F_0(G) \leq F_1(G) \leq F_2(G) \leq \dots$$

of a finite group G as follows: Set $F_1(G) = F(G)$, and let $F_i(G)$ be the inverse image of $F(G/F_{i-1}(G))$ for each $i \geq 2$.

(Clearly $F_i(G)$ is a characteristic subgroup of G for each i.)

Remark 2.29. *(i) When G is finite solvable, by Lemma 2.26, $F_1(G)$ is nontrivial. In fact $F_{i+1}(G) > F_i(G)$ for each i such that $F_i(G) < G$. That is, there exists an integer k such that $F_k(G) = G$. The smallest such integer is said to be the* ***Fitting length*** *of G, and is denoted by $\ell_F(G)$.*

(ii) One can show that for a finite solvable group G, $\ell_F(G)$ is the minimum amongst the lengths of all series of normal subgroups whose factors are all nilpotent.

Lemma 2.30. *Let H be a finite solvable group with $\ell_F(H) = n$. Then $\ell_F(H/F_i(H)) = n - i$ and $F_j(H/F_i(H)) = F_{j+i}(H)/F_i(H)$ for all $1 \leq j \leq n - i$, $1 \leq i < n$*

Proof. Let i and j be integers such that $1 \leq i < n$ and $1 \leq j \leq n - i$. Set $\overline{H} = H/F_i(H)$. Then we have $F(\overline{H}) = F(H/F_i(H)) = \overline{F_{i+1}(H)}$, and by induction for any $k \leq j$ if $F_{k-1}(\overline{H}) = \overline{F_{i+k-1}(H)}$, we have

$$F_k(\overline{H})/F_{k-1}(\overline{H}) = F(\overline{H}/F_{k-1}(\overline{H})) = F(H/F_{i+k-1}(H)) =$$

$$F_{i+k}(H)/F_{i+k-1}(H) = \overline{F_{i+k}(H)}/F_{k-1}(\overline{H}).$$

It follows that $F_k(\overline{H}) = \overline{H}(F_{k+i}(H))$. That is, $F_j(H/F_i(H)) = F_{j+i}(H)/F_i(H)$ for $k = j$.

In particular for $j = n - i - 1$ and $j = n - i$, we have

$$F_{n-i-1}(H/F_i(H)) = F_{n-1}(H)/F_i(H) < H/F_i(H), \text{ and}$$

$$F_{n-i}(H/F_i(H)) = F_n(H)/F_i(H) = H/F_i(H).$$

Thus $\ell_F(H/F_i(H)) = n - i$ as desired. $\square$

Lemma 2.31. *Let G be a finite solvable group.*

(i) If H is a subgroup of G, then $\ell_F(H) \leq \ell_F(G)$.

(ii) If $G = H \times K$, then $\ell_F(G) = max\{\ell_F(H), \ell_F(K)\}$.

Proof. (i) Let $1 \leq F(G) \leq F_2(G) \leq \ldots \leq F_n(G) = G$ be the Fitting series of G and $n = \ell_F(G)$. Consider the series $1 \leq H \cap F(G) \leq H \cap F_2(G) \leq \ldots H \cap F_n(G) = H$. Clearly each term in this series is normal in H. As we have

$$(H \cap F_i(G))/(H \cap F_{i-1}(G)) \cong (H \cap F_i(G))F_{i-1}(G)/F_{i-1}(G) \leq F_i(G)/F_{i-1}(G),$$

we see that the quotient $(H \cap F_i(G))/(H \cap F_{i-1}(G))$ is nilpotent for $i = 1, \ldots, n$. It follows that $\ell_F(H) \leq n$.

(ii) Let $\pi_1 : G \to H$ and $\pi_2 : G \to K$ be the projection maps onto H and K, respectively. Then for any prime $p \in \pi(G)$, $\pi_1(O_p(G))$ is a normal p-subgroup of H, and hence $\pi_1(O_p(G)) \leq O_p(H)$. Similarly $\pi_2(O_p(G)) \leq O_p(K)$. Thus

$$O_p(G) \leq \pi_1(O_p(G)) \times \pi_2(O_p(G)) \leq O_p(H) \times O_p(K).$$

On the other hand, we clearly have $O_p(H) \times O_p(K) \leq O_p(G)$ since H and K are both normal in G. Consequently, we have $O_p(G) = O_p(H) \times O_p(K)$ for any prime p. It follows that

$$F(G) = \bigtimes_p O_p(G) = \bigtimes_p (O_p(H) \times O_p(K)) = (\bigtimes_p O_p(H)) \times (\bigtimes_p O_p(K)) = F(H) \times F(K).$$

By an induction argument, we observe that $F_i(G) = F_i(H) \times F_i(K)$ for any positive integer i. Let now $max\{\ell_F(H), \ell_F(K)\} = n$. This means that $F_n(G) = F_n(H) \times F_n(K) = H \times K = G$, and $F_{n-1}(G) = F_{n-1}(H) \times F_{n-1}(K) < H \times K = G$. Thus $\ell_F(G) = n$ as desired. □

The following lemma will be useful in some proofs later.

Lemma 2.32. *Let G be a finite solvable group. Then $G/F_2(G)$ is isomorphic to a subgroup of $Out(F_2(G)/F(G))$.*

Proof. Let $F = F(G)$ and $F_2 = F_2(G)$. First notice that by Theorem 2.27, $C_{G/F}(F_2/F) = Z(F_2/F)$. As $N_{G/F}(F_2/F) = G/F$, we see that $(G/F)/Z(F_2/F)$ is isomorphic to a subgroup of $Aut(F_2/F)$ by Lemma 2.6. Note that

$$(F_2/F)/Z(F_2/F) \cong Inn(F_2/F).$$

By the third isomorphism theorem, we have

$$G/F_2 \cong ((G/F)/Z(F_2/F))/((F_2/F)/Z(F_2/F))$$

which yields that G/F_2 is isomorphic to a subgroup of $Aut(F_2/F)/Inn(F_2/F) = Out(F_2/F)$. □

Remark 2.33. *In Lemma 2.32, one can replace $F(G)$ with any normal subgroup X of G and $F_2(G)$ with any normal subgroup E of G containing X, and satisfying the condition $C_{G/X}(E/X) = Z(E/X)$.*

2.2.5 Group Action

Let A be a group and Ω be a nonempty set. We say A **acts** on Ω if there is a map $\Omega \times A \to \Omega$ which sends (α, a) to the uniquely determined element $\alpha \cdot a$ in Ω for $a \in A$ and $\alpha \in \Omega$ such that the following conditions are satisfied:

(1) $\alpha \cdot 1 = \alpha$ for all $\alpha \in \Omega$,

(2) $(\alpha \cdot a) \cdot b = \alpha \cdot (ab)$ for all $\alpha \in \Omega$ and $a, b \in A$.

Remark 2.34. *Let a group A act on the set Ω. Then the map $\theta : A \to S_\Omega$ defined by $\theta(a) = \theta_a$ where $\theta_a(\alpha) = \alpha \cdot a$, is clearly a group homomorphism called the permutation representation of A on Ω with respect to the given action.*

Conversely, each group homomorphism $\pi : A \to S_\Omega$ for a group A and a set Ω, determines an action of A on Ω, namely the map $\Omega \times A \to \Omega$ which sends (α, a) to the image of α under $\pi(a)$. Consequently, there is a one to one correspondence between permutation representations of A on Ω and actions of A on Ω.

This action is said to be **faithful** if the corresponding permutation representation is injective, that is, its kernel $\{a \in A \mid \alpha \cdot a = \alpha \text{ for every } \alpha \in \Omega\}$ contains only the identity element of A. We say that the action is **transitive** if for any $x, y \in \Omega$, there exists $a \in A$ such that $x \cdot a = y$. For an element $\alpha \in \Omega$, the set $\mathcal{O}_\alpha = \{\alpha \cdot a \mid a \in A\}$ is called the **orbit** of α under the given action. It is easy to see that the set $\{a \in A \mid \alpha \cdot a = \alpha\}$ is a subgroup of A which is called the **stabilizer** of α in A, and is denoted by A_α. It is the result of well-known Orbit-Stabilizer Theorem that $|\mathcal{O}_\alpha| = |A : A_\alpha|$.

Theorem 2.35. *(Frattini Argument) Assume that a group A acts on a set Ω and $B \leq A$. If the action of B on Ω is transitive. Then for any $\alpha \in \Omega$, we have $A = A_\alpha B$.*

Proof. Let $a \in A$ and $\alpha \in \Omega$. Set $\alpha \cdot a = \beta$. Since the action of B on Ω is transitive, there exists $b \in B$ such that $\alpha \cdot b = \beta$. Then $\alpha \cdot a = \alpha \cdot b$, and hence $\alpha \cdot ab^{-1} = \alpha$. Thus $ab^{-1} \in A_\alpha$, that is, $A = A_\alpha B$. □

Corollary 2.36. *Let $N \trianglelefteq G$, and P be a Sylow p-subgroup of N for some prime divisor of $|N|$. Then $G = N_G(P)N$.*

Proof. Since G acts on $Syl_p(N)$ by conjugation as $N \trianglelefteq G$, and N acts transitively on $Syl_p(N)$, we get the result by the above theorem. □

Let A and G be two groups. We say that A acts on G **via automorphisms** if A acts on G as a set, and also the condition $(gh) \cdot a = (g \cdot a)(h \cdot a)$ is satisfied for all $g, h \in G$ and for all $a \in A$.

Remark 2.37. *It can be easily observed that there is a one to one correspondence between the set of actions via automorphisms of A on G and the set of all homomorphisms from A into $Aut(G)$, namely the set of all automorphism representations of A on G.*

Because the exponential notation is more convenient for the theory of group actions on groups, from now on we write g^a for $g \cdot a$ for $g \in G$ and $a \in A$ in case A acts on G.

Let A act on G via automorphisms. Then the set of all ordered pairs (g, a) with $a \in A$ and $g \in G$ forms a group with respect to the multiplication defined by

$$(g_1, a_1)(g_2, a_2) = (g_1 g_2^{a_1^{-1}}, a_1 a_2),$$

which is called the semidirect product of G by A with respect to the given action, and is denoted by $G \rtimes A$.

Lemma 2.38. *Suppose that A and B are two groups where $A = P_1 \times \ldots \times P_n$ is a direct product of groups P_i for $i = 1, \ldots, n$.*

(i) Let $B = Q_1 \times Q_2 \times \ldots \times Q_m$ be a direct product of groups Q_i for $i = 1, \ldots, m$. Suppose that $Aut(P_i)$ is abelian, and Q_j acts on P_i via automorphisms for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. This extends to an action of B on P_i via automorphisms for each $i = 1, \ldots, n$.

(ii) Let $B = N \rtimes M$. Suppose that M acts on P_i via automorphisms, and N acts on P_i trivially for all $i = 1, \ldots, n$. This extends to an action of B on P_i via automorphisms for each $i = 1, \ldots, n$.

(iii) Suppose that B is a group acting on P_i via automorphisms for each $i = 1, \ldots, n$. This extends to an action of B on A via automorphisms.

Proof. (i) Suppose that Q_j acts on P_i via automorphisms for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. For any fixed j in $\{1, \ldots, m\}$, there exists a group homomorphism $\alpha_{i,j} : Q_j \to Aut(P_i)$. Let $(\alpha_{i,j})_y$ denote image of $y \in Q_j$ under $\alpha_{i,j}$, and let x^y denote the image of x under $(\alpha_{i,j})_y$ for any $x \in P_i$. We next consider the map $\alpha_i : B \to Aut(P_i)$ where the image of $q_1 q_2 \ldots q_m \in B$ under α_i is the automorphism sending x to $x^{q_1 \cdots q_m} := (\ldots (x^{q_1})^{q_2} \ldots)^{q_m}$ for each $x \in P_i$. We observe that if $b = q_1 \ldots q_m$ and $c = r_1 \ldots r_m$ are elements in B with $q_j, r_j \in Q_j$ for $j = 1, \ldots, m$, then for any $x \in P_i$, we have $(x^{q_j})^{r_k} = (x^{r_k})^{q_j}$ as $Aut(P_i)$ is abelian, and hence

$$x^{bc} = (\ldots (x^{q_1 r_1}) \ldots)^{q_m r_m} = (((\ldots (x)^{q_1} \ldots)^{q_m})^{r_1} \ldots)^{r_m} = (x^b)^c.$$

It follows that α_i is a group homomorphism which determines the corresponding action of B on P_i.

(ii) Suppose that M acts on P_i via automorphisms, and N acts on P_i trivially for all $i = 1, \ldots, n$. So for any fixed i in $\{1, \ldots, n\}$, there exists a group homomorphism $\beta : M \to Aut(P_i)$. Consider the map $\theta : B \to Aut(P_i)$ defined by $\theta(nm) = \beta(m)$ where $n \in N$, $m \in M$. Let $b = nm$ and $b_1 = n_1 m_1$ be elements of B so that $n, n_1 \in N$ and $m, m_1 \in M$. Then $bb_1 = n(n_1)^{m^{-1}} mm_1$. Thus for any $x \in P_i$, we have

$$(x^b)^{b_1} = (x^m)^{m_1} = x^{mm_1} = x^{n(n_1)^{m^{-1}} mm_1} = x^{bb_1},$$

and the result follows.

(iii) Suppose that B acts on P_i via automorphisms for each $i = 1, \ldots, n$. Then there exists a homomorphism $\alpha_i : B \to Aut(P_i)$ corresponding to the action of B on P_i for each $i = 1, \ldots, n$. Consider the map $\alpha : B \to Aut(A)$ where the image of $b \in B$ under α is the automorphism sending $a = p_1 \ldots p_n \in A$ to $a^b := p_1^b \ldots p_n^b$. (Here p_i^b is defined as in part (i).) One can easily observe that $a^{bc} = (a^b)^c$ for any $a \in A$ and $b, c \in B$, and hence the result follows. □

Let A act via automorphisms on G, where A and G are finite groups. We say A **acts coprimely** on G if $(|A|, |G|) = 1$.

Lemma 2.39. *(Glauberman) Let A act coprimely on G. Assume also that at least one of A or G is solvable. Suppose that A and G each acts on some nonempty set Ω, where the action of G is transitive. Finally assume the compatibility condition: $(\alpha \cdot g) \cdot a = (\alpha \cdot a) \cdot g^a$ for all $\alpha \in \Omega$, $a \in A$ and $g \in G$. Then the following hold.*

(a) *There exists an A-invariant element $\alpha \in \Omega$.*

(b) *If α, $\beta \in \Omega$ are A-invariant, then there exists $c \in C_G(A)$ such that $\alpha \cdot c = \beta$.*

Proof. See Lemma 3.24 in [2]. □

Theorem 2.40. *Let A act coprimely on G, where one of finite groups A or G is solvable. Then for any prime divisor p of $|G|$, the following hold:*

(a) There exists an A-invariant Sylow p-subgroup of G.

(b) If S and T are A-invariant Sylow p-subgroups of G, then $S^c = T$ for some $c \in C_G(A)$.

Proof. By Sylow C-theorem, G acts on $Syl_p(G)$ transitively by conjugation. Since any automorphism of G permutes the Sylow p-subgroups of G, we see that A also acts on $Syl_p(G)$. Let $S \in Syl_p(G)$, then we have $(S^g)^a = (S^a)^{g^a}$ since

$$(S^g)^a = (g^{-1}Sg)^a = (g^a)^{-1}S^a g^a = (S^a)^{g^a}.$$

Hence by Lemma 2.39(a), there exists an A-invariant Sylow p-subgroup of G. On the other hand, by Lemma 2.39(b), there exists $c \in C_G(A)$ such that $S^c = T$ for any A-invariant Sylow p-subgroups S and T. □

Lemma 2.41. *Let A act via automorphisms on G, where A and G are finite groups, and write $C = C_G(A)$. Let $H \leq G$ be an A-invariant subgroup, and suppose that A and H have coprime orders, and A or H is solvable. Then the A-invariant left cosets of H in G and the A-invariant right cosets of H in G are exactly those cosets of H that contain some elements of C.*

Proof. Let $a \in A$ and $c \in C$, then $(cH)^a = c^a H^a = cH$. Similarly $(Hc)^a = Hc$, so the cosets containing elements of C are A-invariant.

Now suppose that X is an A-invariant left coset of H in G. Then H acts on the set X transitively by right multiplication. Clearly, A acts on X, and $(xh)^a = x^a h^a$ for $x \in X$, $h \in H$ and $a \in A$ as A acts via automorphisms on G. Hence by Lemma 2.39, we have an A-invariant element of X, which means that X contains an element of C.

Similarly one can show that any A-invariant right coset of H in G contains an element of C. □

Theorem 2.42. *Let A act via automorphisms on G, where A and G are finite groups. Let N be a normal A-invariant subgroup of G. Assume that the orders of A and N are coprime. Writing $\overline{G} = G/N$, we have*

$$C_{\overline{G}}(A) = \overline{C_G(A)}.$$

Proof. Clearly we have $C_{\overline{G}}(A) \geq \overline{C_G(A)}$ On the other hand, every element of $C_{\overline{G}}(A)$ can be represented by an element of $C_G(A)$ by Lemma 2.41, and the result follows. □

Theorem 2.43. *Let A act coprimely on G, and suppose that one of A or G is solvable. Then $G = C_G(A)[G, A]$.*

Proof. Notice that A act trivially on $G/[G, A] = \overline{G}$. By Theorem 2.42, we have $\overline{G} = C_{\overline{G}}(A) = \overline{C_G(A)}$. Hence we conclude that $G = C_G(A)[G, A]$. □

Theorem 2.44. *(Fitting) Let A act coprimely on an abelian group G. Then $G = C_G(A) \times [G, A]$.*

Proof. Since G is abelian both $C_G(A)$ and $[G, A]$ are normal in G. Clearly G is solvable, and hence we obtain $G = C_G(A)[G, A]$ by Theorem 2.43. To complete the proof, it suffices to show that $C_G(A) \cap [G, A] = 1$.

Since G is abelian, we can define the map $\theta : G \to G$ by $\theta(g) = \prod_{b \in A} g^b$. Clearly θ is a homomorphism as G is abelian.

If $g \in C_G(A)$, then $\theta(g) = g^{|A|} \neq 1$ since $|A|$ and $|G|$ are coprime. Thus we have $C_G(A) \cap ker(\theta) = 1$. So it is enough to show that $[G, A] \leq ker(\theta)$. Let $g \in G$ and $a \in A$. Then

$$\theta([g, a]) = \theta(g^{-1})\theta(g^a) = \theta(g)^{-1}\theta(g) = 1$$

since $\theta(g^a) = \prod_{b \in A} g^{ab} = \theta(g)$. This completes the proof. □

2.2.6 Some Groups of Special Type

The **dihedral group** D_{2n} is the symmetry group of a regular n-gon for $n \geq 3$ which is defined by

$$D_{2n} := \langle a, b \,|\, a^n = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle.$$

The **semidihedral group** SD_{2^n} for $n \geq 4$ is defined by

$$SD_{2^n} := \langle a, b \,|\, a^{2^{n-1}} = b^2 = 1,\ b^{-1}ab = a^{2^{n-2}-1} \rangle.$$

The **generalized quaternion group** Q_{4n} for $n \geq 2$ is defined by

$$Q_{4n} := \langle a, b \,|\, a^n = b^2,\, a^{2n} = 1,\, b^{-1}ab = a^{-1} \rangle.$$

Note that $|D_{2n}| = 2n$, $|SD_{2^n}| = 2^n$ and $|Q_{4n}| = 4n$.

Lemma 2.45. *(i)* $Out(Q_8) \cong S_3$.

(ii) $Out(Q_{2^n})$ *is abelian for* $n \geq 4$.

Proof. (i) Q_8 has the presentation $\langle -1, i, j, k \,|\, i^2 = j^2 = k^2 = ijk = -1 \rangle$. It is easy to check that the conjugacy classes of i, j and k are $\{i, -i\}$, $\{j, -j\}$ and $\{k, -k\}$. So one can easily conclude that an automorphism of Q_8 is inner if and only of it fixes the subgroups $\langle i \rangle$, $\langle j \rangle$ and $\langle k \rangle$. Notice also that any automorphism of Q_8 permutes the subgroups of order 4, namely $\langle i \rangle$, $\langle j \rangle$ and $\langle k \rangle$. This induces a homomorphism $\phi : Aut(Q) \to S_3$. Clearly ϕ is surjective and $\ker(\phi) = Inn(Q_8)$. Thus $Out(Q_8) = Aut(Q_8)/Inn(Q_8) \cong S_3$ as desired.

(ii) By [7], we have

$$Aut(Q_{2^n}) \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,:\, a \in \mathbb{Z}^*_{2^{n-1}},\, b \in \mathbb{Z}_{2^{n-1}} \right\}, \text{ and}$$

$$Inn(Q_{2^n}) \cong D_{2^{n-1}} \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,:\, a \in \mathbb{Z}^*_4,\, b \in \mathbb{Z}_{2^{n-2}} \right\}.$$

Thus

$$Out(Q_{2^n}) \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,:\, a \in \mathbb{Z}^*_{2^{n-2}},\, b \in \mathbb{Z}_2 \right\}$$

which is abelian. □

We end the discussion of dihedral, generalized quaternion and semidihedral groups by proving two theorems which provide some conditions for a p-group to be one of these groups of special type.

Theorem 2.46. *Let P be a nonabelian 2-group having a cyclic subgroup $C = \langle c \rangle$ of index 2. Let z be the unique involution in C and let $a \in P \setminus C$. If $c^a = c^{-1}$, then P is dihedral or generalized quaternion, and if $c^a = zc^{-1}$, then P is semidihedral.*

Proof. Since P is nonabelian and $P = \langle a, c\rangle$, a and c cannot commute. In the case $c^a = c^{-1}$, we have $o(c) \geq 4$, and in the case $c^a = zc^{-1}$, we have $o(c) \geq 8$. First assume that $c^a = c^{-1}$. Then $(c^i)^a = (c^i)^{-1}$ for each i, that is, a inverts every element in C. Since $a^2 \in C$ is centralized by a, we get $a^2 = (a^2)^{-1}$. Then P is dihedral in the case where $o(a) = 2$, and generalized quaternion in the case where $o(a) = 4$. Next assume that $c^a = zc^{-1}$. Then $(c^2)^a = (c^2)^{-1}$, that is, a inverts every element of order 4 in C. By the same argument $o(a) \leq 4$, because otherwise a^2 has a power whose order is 4 and $a^2 \in \langle c\rangle$, which is a contradiction. If $o(a) = 2$, then P is semidihedral. If $o(a) = 4$, then $a^2 = z$. It follows that $c^a = zc$ and $(ca)^2 = caca = czc^a = 1$. We can replace a with ca, and get $o(ca) = 2$ with $c^{(ca)} = c^a = zc^{-1}$, that is, P is semidihedral again. □

The following lemma will be used in proving Theorem 2.48.

Lemma 2.47. *Let P be a p-group such that $|P| \neq 8$ and $|P : Z(P)| = p^2$. Let C be a cyclic subgroup with $Z(P) < C < P$. Then P has a characteristic elementary abelian subgroup of order p^2.*

Proof. Clearly $P/Z(P)$ is abelian as it is of order p^2. Since $|P : C| = p$, we have P/C is cyclic. Applying Lemma 4.6 in [2], we get $P' \cong C/C \cap Z(P) = C/Z(P)$. Thus $|P'| = p$. Notice that $[P', P] \neq P'$ because P is nilpotent, and so $P' > [P', P] = 1$. Then P has nilpotency class 2.

Assume first that p is odd. Now $[y, x]^p = 1$ as $|P'| = p$. One can observe that the map $\theta(x) = x^p$ from T to itself is a homomorphism (see Theorem 4.8 in [2]). Let $K = ker(\theta)$. Clearly $K = \{x \in T \,|\, x^p = 1\}$ is a characteristic subgroup of P, because each automorphism permutes the elements of order p. To complete the proof, it suffices to show that $|K| = p^2$. Note that $P/Z(P)$ is not cyclic since P is not abelian, whih implies that $P/Z(P)$ is an elementary abelian group of order p^2. This yields that $\theta(P) = P^p \leq Z(P)$. It follows from the first isomorphism theorem that

$$|K| = |P|/|\theta(P)| \geq |P|/|Z(P)| = p^2.$$

Notice that by the second isomorphism theorem we have

$$|K : K \cap C| = |KC : C| \leq |P : C| = p.$$

On the other hand, $K \cap C$ is both cyclic and elementary abelian, and so $|K \cap C| \leq p$. As a result $|K| = p^2$ when p is odd.

Finally assume that $p = 2$. Because P is not abelian and $P' \leq Z(P)$, we deduce that P/P' is not cyclic. Now the Fundamental Theorem of Finite Abelian Groups implies that P/P' has a unique and hence characteristic elementary abelian subgroup E/P' of order 4 as P/P' has a cyclic subgroup of index 2, namely C/P'. Clearly E is characteristic in P and $|E| = 8$. Since E is not cyclic, we have $E \neq C$. Also $|P| \neq 8$ implies that $E \neq P$. As $|P : C| = 2$, we get $C \nleq E$. That is, $E \cap C$ is a proper subgroup of C, and so $CE = P$. It follows that $|E \cap C| = 4$. Note that $Z(P)$ is the unique maximal subgroup of C as $|C : Z(P)| = p$. Hence $E \cap C \leq Z(P)$, which implies that E is abelian because $E/(E \cap C)$ is cyclic. Then, applying Fundamental Theorem of Finite Abelian Groups to the group E, we conclude that E has a characteristic elementary abelian subgroup K of order 4. Since E is characteristic in P, we see that K is also characteristic in P, which establishes the claim. □

Theorem 2.48. *Let P be a p-group in which every normal abelian subgroup is cyclic. Then either P is cyclic, or else $p = 2$ and P is dihedral, generalized quaternion or semidihedral.*

Proof. Let C be maximal among normal abelian subgroups of P. Assume that P is not cyclic, then $C < P$. By Lemma 2.22, we get $C = C_P(C)$. Then P/C is isomorphic to a subgroup of $Aut(C)$, and so P/C is abelian by Lemma 2.5 and Lemma 2.6. If $|C| = 4$, then $|Aut(C)| = 2$, which implies that $|P : C| = 2$. It follows that $|P| = 8$, an hence P is isomorphic to Q_8 or D_8. So we can assume that $|C| \neq 4$, and let $|C| = p^e$.

Let T/C be a subgroup of P/C of order p. Since $T/C \trianglelefteq P/C$ we have $T \trianglelefteq P$. Observe that T is not abelian, because otherwise it would be a normal abelian subgroup of P

which properly contais C. This yields that $|T : Z(T)| \geq p$ Also $|T| \neq 8$ since $|C| \neq 4$. Write $C = \langle c \rangle$. Assume that c^p is central in T. Then $|T : Z(T)| \leq |T : \langle c^p \rangle| = p^2$. It follows that $|T : Z(T)| = p^2$ since $|T : Z(T)| > p$. Now by Lemma 2.47, T must have a characteristic elementary abelian subgroup of order p^2, which is a contradiction. Hence we conclude that c^p is not central in T which yields that $(c^p)^a \neq c^p$, for some $a \in T - C$.

Write $c^a = c^i$ for some integer i. Since $a^p \in C$, we have $c = c^{a^p} = c^{i^p}$. Then $i^p \equiv 1 \pmod{p^e}$. On the other hand, $c^p \neq (c^p)^a = (c^p)^i$, and c^p has order p^{e-1}. Thus $i \not\equiv 1 \pmod{p^{e-1}}$.

Let $i - 1 = mp^d$ where m is not divisible by p. By Fermat's Theorem $d \geq 1$. If p is odd,

$$i^p \equiv 1 + pmp^d + \binom{p}{2} m^2 p^{2d} \pmod{p^{d+1}}.$$

Hence p^{d+1} is the greatest power of p dividing i^p. Then $d \geq e - 1$, and so $i \equiv 1 \pmod{p^{e-1}}$, which is a contradiction. We conclude that $p = 2$, and $i^2 \equiv 1 \pmod{2^e}$, and $(i-1)(i+1) \equiv 0 \pmod{2^e}$. Since $i \not\equiv 1 \pmod{2^{e-1}}$, and one of $i - 1$ and $i + 1$ is not divisible by 4, we have $i \equiv -1 \pmod{2^{e-1}}$. Then $2i \equiv -2 \pmod{2^e}$, and $(c^2)^a = (c^2)^{-1}$.

Clearly P acts on C by conjugation and $C = C_P(C)$ is the kernel of this action. Hence P/C acts faithfully on C. Let Ca_1 be an involution in P/C, and set $T_1/C = \langle Ca_1 \rangle$. By the second paragraph, it follows by replacing T with T_1 that $(c^2)^{a_1} = (c^2)^{-1}$. Hence every involution in P/C inverts c^2. Since P/C is abelian, if there were two involutions in P/C, their product would be also an involution centralizing c^2, which is impossible as $o(c^2) \geq 4$. This implies that P/C has only one involution Ca, and hence it is abelian, and P/C is cyclic.

Assume that $|P/C| > 2$. Then $Ca = Cd^2$ for some $d \in P$ with $(c^2)^a = (c^2)^{d^2} = (c^2)^{-1}$. Let $c^d = c^j$, then $(c^2)^{d^2} = c^{2j^2} = c^{-2}$, and so $2j^2 \equiv -2 \pmod{2^e}$. In particular $j^2 \equiv -1 \pmod 4$ since $|C| > 4$, which cannot hold. We conclude that $|P/C| = 2$, and $P = T$. Since $i \equiv -1 \pmod{2^{e-1}}$, we have either $i \equiv -1 \pmod{2^e}$ or $i \equiv 2^{e-1} - 1 \pmod{2^e}$. In the former case, we have $c^a = c^{-1}$ and so P is

either generalized quaternion or dihedral by Theorem 2.46. In the latter case, we have $c^a = c^{2^{e-1}}c^{-1}$ and P is semidihedral by Theorem 2.46. This completes the proof. $\square$

CHAPTER 3

STRUCTURE OF FROBENIUS AND 2-FROBENIUS GROUPS

This chapter is devoted to Frobenius and 2-Frobenius groups which are indispensable in the study of prime graphs. On this occasion we also present a very new, yet not published, character-free proof of famous Frobenius' theorem obtained by He and Xu in [11] recently. We should note that all groups are finite throughout Chapter 3, Chapter 4 and Chapter 5.

3.1 Frobenius Groups

Let G be a group acting on the set Ω with $|\Omega| > 1$. We say G is a **Frobenius group** on Ω if

(i) G acts transitively on Ω,

(ii) $G_\alpha \neq 1$ for each $\alpha \in \Omega$,

(iii) $G_\alpha \cap G_\beta = 1$ whenever $\alpha \neq \beta$, for all $\alpha, \beta \in \Omega$.

Let G be a Frobenius group on $\Omega = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then the set $N = (G \setminus \bigcup_{i=1}^{n} G_{\alpha_i}) \cup \{1\}$ is called the **Frobenius kernel** of G and the subgroup G_{α_i} is called a **Frobenius complement** of G for each i. Clearly, G can be represented as the disjoint union $N \cup G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}$ which is called the **Frobenius decomposition** of G.

In 1901, Frobenius proved that a Frobenius kernel is a normal subgroup of G as a

celebrated application of the character theory of finite groups. This result is known as Frobenius' theorem. During the last century, finding a purely group theoretical proof of Frobenius' theorem has ever been one of the challenging problems of finite group theory. It is very interesting that a complete rather elementary proof is provided in 2020 in [11], which we present below. Firstly some observations are needed.

Lemma 3.1. *Assume that G is a Frobenius group on Ω with kernel N, and let $1 \neq g \in N$. If g has a factorization $g = g_\pi g_{\pi'}$ where $o(g_\pi)$ is a π-number and $o(g_{\pi'})$ is a π'-number, then both g_π and $g_{\pi'}$ lie in N.*

Proof. Let $\Omega = \{\alpha_1, \ldots, \alpha_n\}$. If either one of g_π and $g_{\pi'}$ is trivial, then the result follows. Assume that both of them are nontrivial and that g_π is not in N. Then there exists $\alpha_i \in \Omega$ such that $g_\pi \in G_{\alpha_i}$. Note that $g_{\pi'} \notin G_{\alpha_i}$ because otherwise $g \in G_{\alpha_i} \cap N = 1$. Assume that $\alpha_i \cdot g_{\pi'} = \alpha_j$ where $i \neq j$. Clearly, $g_\pi \notin G_{\alpha_j}$ because $G_{\alpha_i} \cap G_{\alpha_j} = 1$. Let $\alpha_j \cdot g_\pi = \alpha_k$ with $k \neq j$. It follows that

$$\alpha_i \cdot g = \alpha_i \cdot (g_\pi g_{\pi'}) = (\alpha_i \cdot g_\pi) \cdot g_{\pi'} = \alpha_i \cdot g_{\pi'} = \alpha_j.$$

On the other hand,

$$\alpha_i \cdot g = \alpha_i \cdot (g_{\pi'} g_\pi) = (\alpha_i \cdot g_{\pi'}) \cdot g_\pi = \alpha_j \cdot g_\pi = \alpha_k,$$

which is a contradiction. This yields that g_π lies in N. Replacing the role of g_π by that of $g_{\pi'}$, we get $g_{\pi'}$ lies in N, too. □

Lemma 3.2. *Let G be a Frobenius group on Ω. Then G is not nilpotent.*

Proof. Let $\Omega = \{\alpha_1, \ldots, \alpha_n\}$. Assume that G is nilpotent. Take an element $\alpha_1 \in \Omega$. Then $H = G_{\alpha_1}$ is a Frobenius complement for G. Since G is nilpotent, by Theorem 2.18, $H < N_G(H)$. Pick $n \in N_G(H) \setminus H$ and a nonidentity element $h \in H$. Then there exists $h_1 \in H$ such that $nh = h_1 n$, and hence

$$(\alpha_1 \cdot n) \cdot h = \alpha_1 \cdot (h_1 n) = (\alpha \cdot h_1) \cdot n = \alpha_1 \cdot n.$$

This means that $h \in G_{\alpha_1 \cdot n}$. Since $G_{\alpha_1} \cap G_{\alpha_i} = 1$ for any $i \neq 1$, we have $\alpha_1 \cdot n = \alpha_1$, that is, $n \in H$ which is a contradiction. We conclude that G is not nilpotent. □

Lemma 3.3. *Let G be a Frobenius group on $\Omega = \{\alpha_1, \ldots, \alpha_n\}$ with kernel N and complement $H = G_{\alpha_1}$. Then H is a Hall π-subgroup of G for some $\pi \subset \pi(G)$.*

Proof. Assume that H is not a Hall subgroup. Then there exists a Sylow p-subgroup P_1 of H for some prime p dividing $|H|$, such that P_1 is not a Sylow p-subgroup of G. Then G has a Sylow p-subgroup P which contains P_1 properly by the Sylow's Theorem. Let Ω_1 denote the P-orbit containing α_1. Note that for any two distinct elements $\alpha_i, \alpha_j \in \Omega_1$, we have $P_{\alpha_i} \cap P_{\alpha_j} \leq G_{\alpha_i} \cap G_{\alpha_j} = 1$. Also notice that $P_{\alpha_1} = P \cap H = P_1 > 1$. Since any P_{α_i} and P_{α_1} are P-conjugate due to the transitive action of P on Ω_1, we get $P_{\alpha_i} > 1$ for any $\alpha_i \in \Omega_1$. Thus we conclude that P is a Frobenius group on Ω_1, which contradicts Lemma 3.2. Therefore, H is a Hall subgroup as claimed. □

Lemma 3.4. *Let G be a Frobenius group on Ω, with kernel N and complement H. Let H be a Hall π-subgroup. Then $N = \{g \in G \mid o(g)$ is a π'-number$\}$. In particular, N is a characteristic subset of G, that is, N is invariant under any automorphism of G.*

Proof. Let g be a nonidentity element of N. If $o(g)$ is not a π'-number, then there exists a prime $p \in \pi$ such that $g = g_{p'} g_p$ where $g_p \neq 1$, $o(g_p)$ is a p-number, and $o(g_{p'})$ is a p'-number. Note that $g_p \in N$ by Lemma 3.1. On the other hand, since $o(g_p)$ is a p-number, g_p is contained in a Sylow p-subgroup P of G. By the Sylow's theorem P is conjugate to a Sylow p-subgroup of H, and hence there exists $x \in G$ such that $g_p^x \in H$. Let $H = G_\alpha$ for $\alpha \in \Omega$. Then $g_p^x \in G_\alpha$, that is, $g_p \in G_{\alpha \cdot x^{-1}}$. This contradicts the fact that $g_p \in N$. Therefore, the order of each element in N is a π'-number.

Conversely, an element of G whose order is a π'-number, lies outside any conjugate of the Hall π-subgroup H, and is in N by the Frobenius decomposition. Thus N is the full set of all elements in G of order a π'-number. □

Lemma 3.5. *Let G be a Frobenius group on Ω with kernel N. If G has a subgroup M which properly contains N, then M is a Frobenius group.*

Proof. Let $\Omega = \{\alpha_1, \ldots, \alpha_n\}$. Take an element m from $M \setminus N$. Then m lies in some Frobenius complement, say $m \in G_{\alpha_1}$. Let Ω_1 be the M-orbit containing α_1. Note that M is not contained in G_{α_1} as it contains N, and hence $|\Omega_1| > 1$. Then for any two distinct elements α_i and α_j of Ω_1, we have $M_{\alpha_i} \cap M_{\alpha_j} \leq G_{\alpha_i} \cap G_{\alpha_j} = 1$. Since $M_{\alpha_1} > 1$ as it includes m, $M_{\alpha_i} > 1$ for any $\alpha_i \in \Omega_1$. It follows that M is a Frobenius group on Ω_1. □

Lemma 3.6. *Let G be a Frobenius group on Ω with kernel N and complement H. Then $|N| = |\Omega|$.*

Proof. Set $|\Omega| = n$. Since the action is transitive, by the Orbit-Stabilizer theorem, we have $n = |G : H|$. By the Frobenius decomposition, we have

$$|G| = |N| - 1 + n(|H| - 1) + 1,$$

and hence $|N| = n$, as desired. □

Lemma 3.7. *Let G be a Frobenius group on Ω with kernel N and complement H. Then $G = NH$.*

Proof. By the Frobenius decomposition we have the disjoint union $G = N \cup H \cup H^{x_2} \cup \ldots \cup H^{x_n}$ for some $x_1 = 1, x_2, \ldots, x_n \in G$. Assume that there exists $a \in (H \cup H^{x_2} \cup \ldots \cup H^{x_n}) \setminus NH$. Without loss of generality, we may assume that $a = h^{x_n} \in H^{x_n}$. Then

$$a = h^{x_n} h^{-1} h = [x_n, h^{-1}]h = zh.$$

Since $a \notin NH$, we have z is in some complement H^{x_i}. Clearly, $x_i \neq x_n$ because otherwise $h = z^{-1}a \in H^{x_n}$ which is impossible. Also clearly $z \notin H$. Therefore $x_i \notin \{1, x_n\}$. We can assume $x_i = x_{n-1}$ without loss of generality. Then for some $1 \neq h_1 \in H$, we have

$$a = zh = {h_1}^{x_{n-1}} h = [x_{n-1}, {h_1}^{-1}]h_1 h = z_1 h_1 h.$$

Again, since $a \notin NH$, we have $z_1 \notin N$. Therefore $z_1 \in H^{x_j}$ for some j. We have $x_j \neq x_{n-1}$ because otherwise $h_1 = z_1^{-1} z \in H^{x_{n-1}}$, which is not the case.

Also $x_j \neq x_n$ because otherwise $h_1 h = {z_1}^{-1} a \in H^{x_n}$. Also clearly $z_1 \notin H$. Thus $x_j \notin \{1, x_{n-1}, x_n\}$. Applying the same argument repeatedly, we get $a = z_n h_n \dots h_1 h$ where z_n cannot be contained in any Frobenius complement. Recall that $a \notin NH$. This contradiction completes the proof. $\square$

Now we are ready to prove Frobenius' theorem.

Theorem 3.8. *Let G be a Frobenius group on Ω with kernel N. Then N is a subgroup of G.*

Proof. Let $\Omega = \{\alpha_1, \dots, \alpha_n\}$ and $H_1 = G_{\alpha_1}$. Then $G = NH_1 = H_1 N$ by Lemma 3.4 and Lemma 3.7. It also follows by Lemma 3.4 that N is closed under inverses. Let α_i^A denote the set $\{\alpha_i \cdot x \mid x \in A\}$ for any subset A of G and an element $\alpha_i \in \Omega$. Then

$$\alpha_1^N = (\alpha_1^{H_1})^N = \alpha_1^{H_1 N} = \alpha_1^G = \Omega.$$

Replacing H_1 by G_{α_i} in the above argument, one can conclude that $\alpha_i^N = \Omega$ for each $\alpha_i \in \Omega$. Since $|N| = |\Omega|$, for two distinct elements $x, y \in N$ and $\alpha_i \in \Omega$, we get $\alpha_i \cdot x \neq \alpha_i \cdot y$.

Set $H_i = G_{\alpha_i}$. Assume that for some i, there exist $m, n \in N$ such that $H_i m = H_i n$. Then $m = h_i n$ for some $h_i \in H_i$. Thus $\alpha_i \cdot m = \alpha_i \cdot (h_i n) = \alpha_i \cdot n$, and by the above argument $m = n$. This means that N is a right transversal for each H_i in G.

In order to show that N is a group, it is enough to verify that $xy^{-1} \in N$ for any two elements $x, y \in N$. Assume that $xy^{-1} \notin N$. Clearly $x \neq y$, and by the Frobenius decomposition, xy^{-1} lies in a Frobenius complement H_i. It follows that $H_i x = H_i y$, which contradicts the fact that N is a right transversal for H_i in G. Thus N is a subgroup of G as desired. $\square$

It is an immediate consequence of Lemma 3.4 that a Frobenius kernel is a characteristic subgroup.

We investigate now Frobenius groups from another point of view in order to enrich our knowledge on this concept.

Definition 3.9. *Let H and N be nontrivial groups such that H acts on N via automorphisms. The action of H on N is said to be* ***Frobenius*** *if $n^h \neq n$ whenever $n \in N$ and $h \in H$ are nonidentity elements.*

Lemma 3.10. *Let H and N be groups such that H acts on N Frobeniusly. Then $|N| \equiv 1$ (mod $|H|$), and hence $|H|$ and $|N|$ are coprime.*

Proof. The size of the H-orbit of any nonidentity element $n \in N$ is $|H : C_H(n)|$. By the definition of Frobenius action, $C_H(n) = 1$. That is, every nontrivial orbit has length $|H|$, whence $|N| \equiv 1$ (mod $|H|$). □

Lemma 3.11. *Let N be a normal subgroup of a group G and H be a complement for N in G, that is, $G = NH$ with $N \cap H = 1$. Suppose that $H \cap H^g = 1$ for all elements $g \in G \setminus H$. Then $N = (G \setminus \bigcup_{g \in G} H^g) \cup \{1\}$.*

Proof. The case $H = 1$ is trivial. Assume that $H > 1$. For $x \in N_G(H)$, we have $H \cap H^x = H \neq 1$, implying that $x \in H$. Thus $N_G(H) = H$. Then H has $|G : H|$ distinct G-conjugates. Since any two distinct G-conjugates of H intersect trivially, we have

$$|\bigcup_{g \in G} H^g| = |G : H|(|H| - 1) + 1 = |G| - |G : H| + 1.$$

Thus there are $|G : H| - 1$ elements of G that are not conjugate to any nonidentity element of H. Also $N \cap H = 1$, and so $(N \cap H)^g = N^g \cap H^g = N \cap H^g = 1$ for any $g \in G$. Hence no nonidentity element of N is conjugate to any nonidentity element of H. Since $|N| = |G : H|$, the result follows. □

Theorem 3.12. *Let N be a normal subgroup of a group G and H be a complement for N in G. Then the following are equivalent.*

(a) The conjugation action of H on N is Frobenius.

(b) $H \cap H^g = 1$ for all elements $g \in G \setminus H$.

(c) $C_G(h) \leq H$ for all nonidentity elements $h \in H$.

(d) $C_G(n) \leq N$ for all nonidentity elements $n \in N$.

Proof. $(a \implies b)$ Suppose that (a) holds and that $H \cap H^x > 1$ for some $x \in G$. Since $G = HN$, we have $x = hn$ for some $h \in H$ and $n \in N$. Then $H^x = H^{hn} = H^n$, and so $H \cap H^n > 1$. It follows that there exists a nonidentity element k of H with $k^n \in H$. Since $N \trianglelefteq G$, we have $[k, n] = k^{-1}k^n \in N \cap H = 1$. By the Frobenius action we have $n = 1$, that is, $x \in H$.

$(b \implies c)$ Assume that (b) holds and let $1 \neq h \in H$. If $x \in C_G(h)$, then $h \in H \cap H^x$, which implies that $x \in H$.

$((c \implies a) \wedge (d \implies a))$ Suppose that (c) holds and let $1 \neq h \in H$. Then $C_N(h) = N \leq N \cap H = 1$, which shows that action of H on N is Frobenius. Similarly if $1 \neq n \in N$, then $C_H(n) = H \leq H \cap N = 1$, and hence the action of H on N is Frobenius, that is, (d) implies (a).

$(a \implies d)$ Suppose that (a) holds. Let $1 \neq n \in N$ such that $C_G(n) \nleq N$. Then there exists $c \in C_G(n) \setminus N$. Since (a) implies (b), $H \cap H^g = 1$ for all elements $g \in G \setminus H$. Applying now Lemma 3.11, we have $c = h^{m^{-1}}$ for some $h \in H$ and $m \in G$. Then $h = c^m \in C_G(n)^m = C_G(n^m)$, which contradicts the fact that $C_G(n^m) = 1$. $\square$

Remark 3.13. *Suppose that the action of H on N is Frobenius, and let $G = NH$ be the semidirect product of N and H with respect to this action. Let Ω denote the set of all distinct conjugates of H in G. Clearly, G acts on Ω transitively, and all distinct conjugates of H intersect trivially by Theorem 3.12(b). Thus two definitions of Frobenius groups coincide.*

Lemma 3.14. *Every Frobenius group has trivial center..*

Proof. Let G be a Frobenius group with kernel N and complement H. Let $1 \neq z \in Z(G)$. By Theorem 3.12(d) $z \notin N$ because otherwise $G = C_G(z) \leq N$. Thus by Lemma 3.11 $z \in H^g$ for some $g \in G$. Since H^g is also a Frobenius complement for G, we have $G = C_G(z) \leq H^g$ by Theorem 3.12(c), which is a contradiction. Therefore $Z(G) = 1$ as claimed. □

Lemma 3.15. *Let G be a Frobenius group with kernel N and complement H and $M \trianglelefteq G$. Then $M < N$ or $N \leq M$.*

Proof. Suppose that $M \nleq N$. Since $N = (G - \bigcup_{g \in G} H^g) \cup \{1\}$, we have $N \cap H^g \neq 1$ for some $g \in G$. Notice that

$$M \cap H^g = M^g \cap H^g = (M \cap H)^g.$$

Thus $M \cap H \neq 1$. Since $(|N|, |H|) = 1$, by Theorem 2.43, we have

$$N = [N, M \cap H] C_N(M \cap H).$$

Since the action of $M \cap H$ on N is Frobenius, we get $C_N(N \cap H) = 1$. Therefore $N = [N, M \cap H]$. On the other hand, $[N, M] \leq M$ as $N \trianglelefteq G$. It follows that $N \leq M$. □

Lemma 3.16. *Let H act on N via automorphisms and let M be an H-invariant normal subgroup of N. If the action of H on N is Frobenius, then the induced action of H on N/M is also Frobenius.*

Proof. Suppose that the action of H on N is Frobenius, and set $N/M = \overline{N}$. If $C_{\overline{N}}(h)$ is nontrivial for some $1 \neq h \in H$, then there exists a nonidentity $\overline{n} \in C_{\overline{N}}(h)$. We have $C_{\overline{N}}(h) = \overline{C_N(h)}$ by Theorem 2.42. It follows that $C_N(h) \neq 1$, which is a contradiction. □

In 1959, Thompson proved in his doctoral thesis that all Frobenius kernels are nilpotent (see Chapter 6 and 7 in [2]). Assuming the solvability of the kernel shortens the proof greatly. We present a proof of this fact below under this assumption.

Theorem 3.17. *(Thompson) Frobenius kernels are nilpotent.*

Proof. Let $G = NH$ be a Frobenius group with solvable kernel N. Since any subgroup of H also acts on N Frobeniusly, we can assume that H is of prime order p.

Suppose that N is a minimal counter-example to the theorem. Let U be a minimal normal subgroup of G contained in N. Then by Lemma 3.16, H acts on N/U Frobeniusly, and hence N/U is nilpotent by induction hypothesis. Let N^∞ denote the minimal term in the lower central series of N. By nilpotency of N/U, we have $N^\infty \leq U$. Then minimality of U gives $N^\infty = U$, which means that U is the unique minimal normal subgroup of G contained in N.

Since N is solvable, U is an elementary abelian q-group for some prime q by Lemma 2.10. Clearly N is not a q-group since it is not nilpotent. Then there exists a prime r, different from q, dividing $|N|$. By Theorem 2.40, N has an H-invariant Sylow r-subgroup, say R. If R were normal in N, it would be characteristic, and hence normal in G, which forces that $U \leq R$, a contradiction. Therefore R is not normal in N.

Since RU is H-invariant, H induces a Frobenius action on RU, and hence RU is a solvable Frobenius kernel. If $RU < N$, then RU is nilpotent by induction hypothesis, which implies that R is characteristic in RU. Observe that RU/U is a Sylow r-subgroup of the nilpotent group N/U, and so $RU/U \trianglelefteq N/U$. Then we get $RU \trianglelefteq N$, which forces that $R \trianglelefteq N$. This contradiction shows that $RU = N$.

The group RH is also a Frobenius group, and so it has a partition Π consisting of R and all conjugates of H in RH. Then $|\Pi| - 1 = |R|$ is a power of r, and hence $(|U|, |\Pi|-1) = 1$. By Lemma 6.8 in [2], for some member $X \in \Pi$ we have $C_U(X) > 1$. We have $X = R$, because every conjugate of H acts on U Frobeniusly. We deduce that

$$1 < C_U(R) \leq Z(RU) = Z(N).$$

Since U is the unique minimal normal subgroup we have $U \leq Z(N)$, and so $R \trianglelefteq UR = N$. This contradiction completes the proof. $\square$

An immediate consequence of Thompson's theorem is the following.

Corollary 3.18. *Let G be a Frobenius group with kernel N. Then $N = F(G)$.*

Proof. By Theorem 3.17, $N \leq F(G)$. Assume that $N < F(G)$. Pick $h \in F(G) \setminus N$. Then h lies in some Frobenius complement, say $h \in H$. Since $(|N|, |H|) = 1$ and $h \in F(G)$, we conclude that h centralizes N, which contradicts the Frobenius action of H on N. Thus we have $N = F(G)$. □

Lemma 3.19. *No Sylow subgroup of a Frobenius kernel can be generalized quaternion.*

Proof. Let H act on N Frobeniously. By Theorem 3.17, N has a unique Sylow 2-subgroup, say Q. If Q is generalized quaternion, the unique involution in Q must be fixed by every element of H, contradicting the Frobenius action of H on N. □

Lemma 3.20. *Let M be a normal subgroup of a Frobenius group G such that $M < F(G)$. Then G/M is Frobenius.*

Proof. Let H be a complement for $F(G)$ in G. By Lemma 3.16, H acts Frobeniously on $F(G)/M$. Clearly $G/M = F(G)H/M$ is a Frobenius group with kernel $F(G)/M$ and complement HM/M. □

Lemma 3.21. *Let NH be a Frobenius group with kernel N. If $|H|$ is even, then there exists a unique involution $t \in H$ such that $n^t = n^{-1}$ for all $n \in N$, and N is abelian, and $t \in Z(H)$.*

Proof. Let $t \in H$ such that $t^2 = 1$, and define the map $\alpha : x \mapsto x^{-1}x^t$. Assume that $x^{-1}x^t = y^{-1}y^t$ for some $x, y \in N$. Then $yx^{-1} = y^t(x^t)^{-1} = (yx^{-1})^t$, and hence $yx^{-1} = 1$, due to the Frobenius action of H on N. This shows that α is an injection. Since N is finite, α is also surjective. Therefore for any $y \in N$, there exists $x \in N$ such that $y = x^{-1}x^t$, and hence

$$y^t = (x^{-1}x^t)^t = (x^{-1})^t x = (x^{-1}x^t)^{-1} = y^{-1}.$$

Assume next that $s^2 = 1$ for some $s \in H$. Then we see by the same argument that $y^s = y^{-1}$ for any $y \in N$. Thus $y^s = y^t$, and hence $y^{ts^{-1}} = 1$. This forces that $ts^{-1} = 1$, due to the Frobenius action of H on N. Thus t is the unique involution in H.

Finally we observe that for any $x, y \in N$,

$$xy = (y^{-1}x^{-1})^{-1} = (y^{-1}x^{-1})^t = (y^{-1})^t(x^{-1})^t = yx,$$

that is, N is abelian. By the uniqueness of t, the group generated by t is a normal subgroup of H of order 2, and hence it is central in H. □

The following lemmas provide information about the structure of Frobenius complements.

Lemma 3.22. *Let H be a Frobenius complement. Then no subgroup of H is an elementary abelian p-group of order exceeding p, and no solvable subgroup of H is a Frobenius group. Also every subgroup of H having order pq, where p and q are possibly equal primes, is cyclic.*

Proof. Let H act on N Frobeniusly. We assume that H has a subgroup which is an elementary abelian p-group of order exceeding p or a solvable Frobenius group. We can replace H by this subgroup since every subgroup of H acts Frobeniusly on N. If H is elementary abelian, we can assume that $|H| = p^2$ by replacing it with a subgroup of order p^2. Due to coprime action of H on N, the group N has an H-invariant Sylow r-subgroup R for each prime divisor r of $|N|$ by Theorem 2.40. Since $Z(R)\, char\, R$, the subgroup $Z(R)$ is also H-invariant, and hence we can replace N by $Z(R)$, that is, we can assume that N is abelian.

Let $E = C_p \times C_p$ denote the elementary abelian p-group of order p_2, and set $\langle c \rangle = C_p$. Then E has a partition $\bigcup_{i=0}^{p-1} \langle (c, c^i) \rangle \cup \langle (1, c) \rangle$ of cardinality $p + 1$. Since H is either Frobenius or elementary abelian, it has a partition Π of cardinality $n+1$ where n is the order of the kernel of Frobenius group H by Lemma 3.11, or if H is an elementary abelian p-group of order p^2, then $n = p$. In any case n divides $|H|$, and so it is

coprime to $|N|$. Then for any nonidentity element u of N, we have $u^n \neq 1$. It follows by Lemma 6.8 in [2] that $C_N(X) > 1$ for some $X \in \Pi$, which is a contradiction as H acts on N Frobeniusly. We conclude that H cannot contain an elementary abelian p-group of order exceeding p or a solvable Frobenius group.

Finally assume that B is a subgroup of H of order pq for some primes p and q. If $p = q$, we conclude that B is cyclic by the above observation. If $p > q$ then B has a normal Sylow p- subgroup P by Sylow's theorem. If a Sylow q-subgroup Q acts on P nontrivially, no element of P is centralized by any element of Q, since any nonidentity element of P is a generator. Thus PQ is a solvable Frobenius group which is a contradiction. Then Q acts on P trivially, and hence B is cyclic as claimed. □

Lemma 3.23. *Let H be a Frobenius complement. For each prime p dividing $|H|$, a Sylow p-subgroup of H has a unique subgroup of order p.*

Proof. Let P be a Sylow p-subgroup of H, and let Z be a subgroup of $Z(P)$ of order p. If P has another subgroup U of order p, then $Z \cap U = 1$ and hence we have ZU is a subgroup of order p^2. Note that ZU is not cyclic because it contains two distinct subgroups of order p, which is a contradiction by Lemma 3.22. Therefore Z is the unique subgroup of P of order p. □

Lemma 3.24. *Let H be a Frobenius complement. Then each Sylow subgroup of H is either cyclic or generalized quaternion.*

Proof. Firstly by Lemma 3.23, every Sylow p-subgroup of H has a unique subgroup of order p. By fundamental theorem of finite abelian groups, any finite abelian p-group having a unique subgroup of order p, must be cyclic. In particular, every normal abelian subgroup of a Sylow p-subgroup of H is cyclic. Applying Theorem 2.48, we deduce that a Sylow p-subgroup is either cyclic or $p = 2$ and it is generalized quaternion, dihedral or semidihedral. If $p \neq 2$, the only possible case is that the Sylow subgroup is cyclic. If $p = 2$, dihedral and semidihedral 2-groups have at least 2 involutions, one in the cyclic group of index 2 and one outside of it. Hence if a Sylow 2-subgroup is not cyclic, it is generalized quaternion. □

The next two lemmas will be used to show the fact that a Frobenius complement of odd order contains a unique subgroup of order p for each prime p dividing its order.

Lemma 3.25. *Let P be a cyclic Sylow subgroup of a group G. Then p divides exactly one of $|G'|$ or $|G : G'|$.*

Proof. Let $N = N_G(P)$. Since $(|N : P|, |P|) = 1$, there exists a Hall p'-subgroup of N, say K such that $PK = N$. As P is abelian and K acts on P by conjugation, $P = [P, K] \times C_P(K)$ by Theorem 2.44. Firstly assume that $[P, K] = 1$, that is, P is central in N. Theorem 2.14 yields that G has a normal p-complement, say M. The group $G/M = PM/M$ is isomorphic to a Sylow p-subgroup of G, and so it is abelian. Then $G' \leq M$ and p does not divide G'. Assume next that $[P, K] \neq 1$. Then $[P, K]$ contains the unique subgroup of order p in P and hence $C_P(K)$ cannot have a subgroup of order p, that is, $C_P(K) = 1$. It follows that $P = [P, K] \leq G'$, and so p does not divide $|G : G'|$ completing the proof. □

Lemma 3.26. *If all Sylow subgroups of a group G are cyclic, then both G' and G/G' are cyclic, and $(|G'|, |G : G'|) = 1$.*

Proof. We may assume that $|G| > 1$. By Lemma 3.25, no prime divisor of $|G|$ divides both $|G'|$ and $|G : G'|$, which means that $(|G'|, |G : G'|) = 1$. Since all Sylow subgroups of G are cyclic, G is solvable by Theorem 2.15. As G/G' is abelian and its Sylow subgroups are all cyclic, and hence $|G/G'|$ is cyclic. We shall proceed by induction on $|G|$. Note that $G' < G$ as G is solvable. Then we obtain by induction hypothesis that $|G''|$ is cyclic. Clearly G'/G'' is abelian and so cyclic because its all Sylow subgroups are cyclic. Notice that $Aut(G'')$ is abelian by Lemma 2.5 and $G/C_G(G'')$ can be embedded into $Aut(G'')$ by Lemma 2.6. It follows that $G/C_G(G'')$ is also abelian, that is, $G' \leq C_G(G'')$ and $G'' \leq C_G(G')$. So $G'' \leq Z(G')$ and G'/G'' is cyclic. It follows that G' is abelian and hence cyclic, completing the proof. □

Lemma 3.27. *Let H be a Frobenius complement of odd order. Then H has a unique subgroup of order p for each prime p dividing $|H|$.*

Proof. Let r be a prime divisor of $|H'|$. Since all Sylow subgroups of H are cyclic, by Lemma 3.26, we get $(|H : H'|, r) = 1$ and so every subgroup of order r in H lies in H'. Note that H' is cyclic by Lemma 3.26 and hence H' has a unique subgroup of order r, say R, which is also the unique subgroup of H. In particular, $R \trianglelefteq H$.

Let now q be a prime divisor of $|H|$ coprime to $|H'|$. Then the cyclic group H/H' contains a unique subgroup X/H' of order q. Therefore, for any subgroup Q of H of order q we have $H'Q = X$ and Q is a Sylow q-subgroup of X, as q does not divide $|H'|$. Hence, it is enough to show that $Q \trianglelefteq X$ to conclude the uniqueness of Q. Let R be an arbitrary subgroup of H' of prime order r. Then RQ is a subgroup of order rq and so cyclic by Lemma 3.22. That is $[Q, R] = 1$. Thus $C_{H'}(Q)$ contains every subgroup of H' of prime order.

As H' is abelian and $(|H|, q) = 1$, by Fitting's theorem (Theorem 2.44), we get $H' = [H', Q] \times C_{H'}(Q)$ which forces that $[H', Q] = 1$ because $C_{H'}(Q)$ contains every subgroup of H' of prime order. Therefore $Q \trianglelefteq H'Q = X$ as desired. □

Lemma 3.28. *Let H be a Frobenius complement of odd order. Then*

(a) H is metacyclic.

(b) If y is an element of H of prime order p, then $\langle y \rangle \trianglelefteq H$, and $y \in H'$ or $y \in Z(H)$.

Proof. Since $|H|$ is odd, all Sylow subgroups of H are cyclic by Lemma 3.24. Then H' and H/H' cyclic and have coprime orders by Lemma 3.26. Schur-Zassenhaus Theorem shows that $G = \langle a \rangle \langle b \rangle$ where $H' = \langle a \rangle$ and $H/H' \cong \langle b \rangle$ for some $a, b \in G$. Let p be a prime divisor of $|H|$. Then, by Lemma 3.27, the group H has a unique subgroup of order p, say $\langle y \rangle$. Clearly $\langle y \rangle \trianglelefteq H$ due to uniqueness. Suppose that $y \notin H'$. Then we can assume that $y \in \langle b \rangle$. By Lemma 3.22, every subgroup of H of order pq is cyclic. Thus $C_{H'}(\langle y \rangle)$ contains all elements of H' of prime order. By Fitting Theorem $H' = [H', \langle y \rangle] \times C_{H'}(\langle y \rangle)$. Since all subgroups of H' of prime order lie in $C_{H'}(\langle y \rangle)$, we get $[H', \langle y \rangle] = 1$ and hence $y \in Z(H)$. □

The following lemma will be used in the proof of Lucido's result which will be stated

in the next chapter.

Theorem 3.29. *Let H be a Frobenius complement. Then $Z(H) \neq 1$*

Proof. If $|H|$ is even, then by Lemma 3.21, H has a unique involution which lies in $Z(H)$. If $|H|$ is odd, we have $(|H : H'|, |H|) = 1$ and H' is cyclic by Lemma 3.26. As H' is cyclic, we get $H' < H$. Then by Lemma 3.28, for any element y of H of prime order p dividing $|H : H'|$, we have $y \in Z(H)$ and so $Z(H) \neq 1$ as claimed. □

3.2 2-Frobenius Groups

Definition 3.30. *A group G is called **2-Frobenius** if there exist two normal subgroups F and K of G such that K is a Frobenius group with kernel F, and G/F is a Frobenius group with kernel K/F. Here K and G/F are called lower Frobenius and upper Frobenius groups, where F and K/F are called lower kernel and upper kernel, respectively.*

Remark 3.31. *We observe that $F = F(K) = F(G)$ and $K = F_2(G)$:*

$F = F(K)$ and $K/F = F(G/F)$ follow by Corollary 3.18. Clearly F char $K \trianglelefteq G$ and so $F \leq F(G)$, and $F = K \cap F(G)$. Observe that $[F(G)/F, K/F] \leq F(G)/F \cap K/F = 1$ as $F(G)/F$ and K/F are normal in G/F and hence $F(G)/F \leq C_{G/F}(K/F)$. On the other hand $C_{G/F}(K/F) \leq K/F$ by Theorem 2.27, and hence $F = F(G)$ and $K = F_2(G)$.

Lemma 3.32. *A 2-Frobenius group has trivial center.*

Proof. Assume the contrary and let $1 \neq z \in Z(G)$. Since $F_2(G)$ is a Frobenius group, we have $z \notin F_2(G)$ by Lemma 3.14. Then $zF(G)$ is a nonidentity central element of $G/F(G)$, which is impossible by Lemma 3.14 as $G/F(G)$ is a Frobenius group. Therefore $Z(G) = 1$ as desired. □

The following lemma provides a precise description of the structure of solvable 2-Frobenius groups.

Lemma 3.33. *Let G be a solvable 2-Frobenius group. Then*

(i) $G/F_2(G)$ and $F_2(G)/F(G)$ are cyclic groups, and they are isomorphic to some subgroups of G.

(ii) No Sylow subgroup of $F(G)$ is a cyclic group. In particular, $F(G)$ is not cyclic.

(iii) The upper kernel of G is a cyclic group of odd order.

Proof. Recall that $F_2(G)$ and $G/F(G)$ are Frobenius groups with kernels $F(G)$ and $F_2(G)/F(G)$, respectively. Set $F_1 = F(G)$ and $F_2 = F_2(G)$. By Schur–Zassenhaus theorem, we have $F_2 = F_1B$ where B is a subgroup of F_2 with $F_1 \cap B = 1$. Observe that

$$F_1B = F_2 = {F_2}^g = {F_1}^g B^g = F_1B^g$$

for any $g \in G$. Then B and B^g are conjugate in F_2 by the conjugacy part of Schur-Zassenhaus Theorem, that is, there exists h in F_2 such that $B^h = B^g$. This means that F_2 acts transitively on the set of conjugates of B. It follows by the Frattini Argument that $G = F_2G_B = F_1N_G(B)$.

Observe now that $[N_{F_1}(B), B] \leq F_1 \cap B = 1$, and hence $N_{F_1}(B) = C_{F_1}(B)$. Due to the Frobenius action of B on F_1, we get $N_{F_1}(B) = 1$. Thus $N_G(B) \cap F_1 = 1$ whence $G/F_1 \cong N_G(B)$. Since G/F_1 is a Frobenius group with kernel $F_2/F_1 \cong B$, we get that B has a complement C in $N_G(B)$, that is, $N_G(B) = BC$ is a Frobenius group with kernel B and complement C.

We conclude that B is both a Frobenius kernel and a Frobenius complement. If $|B|$ is even, then it has a unique involution by Lemma 3.21 and this unique involution must be fixed by C. This contradiction shows that $|B|$ is odd. By Theorem 3.17 and Lemma 3.24, the group B is nilpotent and its all Sylow subgroups are cyclic, that is, B is cyclic.

There is a homomorphism π from C to $Aut(B)$. Since no nonidentity element of C can fix every element of B, $\ker(\pi) = 1$. Hence C is isomorphic to a subgroup of $Aut(B)$. Notice that the automorphism group of a cyclic group is abelian and so C

cannot contain a generalized quaternion group as a Sylow subgroup. By Lemma 3.24, all Sylow subgroups of C are cyclic which shows that C is cyclic.

Assume that a Sylow subgroup P of F_1 is cyclic. Then $P \trianglelefteq G$ and C normalizes PB. Now PBC is a 2-Frobenius group with lower kernel P, that is, $P = F(PBC)$. There is a homomorphism α from BC to $Aut(P)$. By Theorem 2.27, we know that $C_{BC}(P) = 1$. Hence $\ker(\alpha) = C_{BC}(P) = 1$, that is, BC is isomorphic to a subgroup of $Aut(P)$. Then BC is abelian which contradicts the fact that BC is a Frobenius group. Consequently, no Sylow subgroup of F_1 is cyclic. □

Lemma 3.34. *Let G be a 2-Frobenius group and $N \trianglelefteq G$. Then we have either $N < F_2(G)$ or $F_2(G) \leq N$. In case where $N < F_2(G)$, we see that G/N is 2-Frobenius if $N < F(G)$, and G/N is Frobenius if $F(G) \leq N$.*

Proof. Set $\overline{G} = G/F(G)$. Assume that $N \not< F_2(G)$. By Lemma 3.15, we have either $\overline{N} \leq \overline{F_2(G)}$ or $\overline{F_2(G)} \leq \overline{N}$. By our assumption, we get $\overline{F_2(G)} \leq \overline{N}$, that is, $F_2(G) \leq NF(G)$. Note that $N \cap F_2(G)$ is a normal subgroup of the Frobenius group $F_2(G)$. Hence by Lemma 3.15, we have either $N \cap F_2(G) < F(G)$ or $F(G) \leq N \cap F_2(G)$. Also note that we have $F_2(G) = (N \cap F_2(G))F(G)$ by Dedekind's rule. If $N \cap F_2(G) < F(G)$, then $F_2(G) = (N \cap F_2(G))F(G) < F(G)$, which is a contradiction. Thus we have $F(G) \leq N$, and hence $F_2(G) \leq N$ as claimed.

Assume now that $N < F_2(G)$. Again by Lemma 3.15, we have either $N < F(G)$ or $F(G) \leq N$. Firstly if $F(G) \leq N$, we get $N/F(G)$ is a subgroup of $G/F(G)$ with the condition $N < F_2(G)$, and so by Lemma 3.20, $(G/F(G))/(N/F(G)) \cong G/N$ is a Frobenius group as claimed. Finally suppose that $N < F(G)$. Then $F_2(G)/N$ is a Frobenius group with kernel $F(G)/N$ by Lemma 3.20. On the other hand we have $(G/N)/(F(G)/N) \cong G/F(G)$ which is Frobenius. As a result, G/N is 2-Frobenius. □

We need to distinguish the following types of solvable 2-Frobenius groups for future purposes.

Definition 3.35. *If G is a 2-Frobenius group for which there are primes p and q such that $G/F_2(G)$ and $F(G)$ are p-groups and $F_2(G)/F(G)$ is a q-group, we say that G is a 2-Frobenius group of type (p, q, p).*

Lemma 3.36. *Suppose H is a 2-Frobenius group of type (p, q, p) for primes p and q. If P is a Sylow p-subgroup of H, then P is a semidirect product of two nontrivial subgroups, in particular P is not a Frobenius complement. If Q is a Sylow q-subgroup of H, then Q is cyclic.*

Proof. Clearly, a Sylow q-subgroup Q is isomorphic to the upper kernel of H and hence cyclic by Lemma 3.33. Also by Lemma 3.33, a Sylow p-subgroup of H is isomorphic to semidirect product of $F(H)$ by an upper complement of H. Then P has more than one subgroup of order p and so P cannot be Frobenius by Lemma 3.27. □

Definition 3.37. *Suppose that there exist distinct primes p, q and r so that $G = PQR$, where P, Q, R are Sylow p-, q-, r-subgroups respectively, PQ is a Frobenius group with kernel P, and QR is either a Frobenius group with kernel Q or a 2-Frobenius group of type (r, q, r). Then we say G is a 2-Frobenius group of type (p, q, r).*

Lemma 3.38. *Suppose that p, q, r are distinct primes and $G = PQR$ is a 2-Frobenius group of type (p, q, r), where P, Q and R are Sylow subgroups p-, q-, r-subgroups, respectively. Then P is not cyclic, Q is cyclic and R is not generalized quaternion.*

Proof. By Lemma 3.33, Q is isomorphic to the upper kernel of G and hence cyclic by Lemma 3.33. If r does not divide $|F(G)|$, then R is isomorphic to a Frobenius complement of $G/F(G)$ by Lemma 3.33, and hence cyclic. If r divides $|F(G)|$, then a Hall $\{q, r\}$-subgroup H_{qr} is 2-Frobenius of type (r, q, r). If R were generalized quaternion then it would have a unique subgroup of order r by Lemma 3.36, which is not possible. This completes the proof. □

CHAPTER 4

A CHARACTERIZATION OF PRIME GRAPHS OF SOLVABLE GROUPS

4.1 Prime Graphs and Some General Results

In this section, we present some important results due to Lucido, Gruenberg and Kegel that can be regarded as the main inspirations of research on prime graphs of solvable groups.

Definition 4.1. *We say Γ_G is the **prime graph** of G if $V(\Gamma_G) = \pi(G)$, and $\{p, q\} \in E(\Gamma_G)$ for p, $q \in \pi(G)$ if and only if there exists $g \in G$ such that $o(g) = pq$.*

Firstly, we state the following elegant result due to Lucido which serves as a key in this area of research.

Theorem 4.2. *(Lucido [8], Lemma 8) Let G be a solvable group. If p, q, r are three distinct primes dividing the order of G, then G contains an element of order the product of two of these primes. Equivalently, the complement of Γ_G for any solvable group G is triangle-free.*

Proof. Assume the contrary and let G be a minimal counter example to the theorem. Let N be a minimal normal subgroup of G. Since G is solvable N is elementary abelian by Lemma 2.10. Assume that pqr divides $|G/N|$. By Theorem 2.9, G/N is solvable, and hence G/N satisfies the hypothesis of the theorem. By minimality of G, the group G/N contains an element of order product of two of these primes, say

gN of order pq. It follows that $g^{pq} \in N$. If $o(g^{pq}) = d$ then $o(g^d) = pq$, which is not possible. This shows that one of the primes p, q, r, say p does not divide $|G/N|$. This forces that N is an elementary abelian p-group.

Let H be a Hall-$\{q, r\}$ subgroup of G. Suppose first that $Z(H) \neq 1$ and $|Z(H)|$ is divisible by q. By Cauchy's Theorem $Z(H)$ contains an element a of order q. Cauchy's Theorem also implies that there is an element b of order r in H. Then $o(ab) = qr$ which is impossible. Thus we have $Z(H) = 1$.

Since N is normal in G, H acts on N by conjugation. Assume that there is a non-identity element h in H such that $C_N(h) \neq 1$. Let $1 \neq n \in C_N(h)$. Now $o(n) = p$ and $o(h) = q^\alpha r^\beta$ for some nonnegative integers α and β. Without loss of generality, let $\alpha > 0$. Then $o(nh^{q^{\alpha-1}r^\beta}) = pq$, which is not the case. Hence $C_N(h) = 1$ for each nonidentity $h \in H$, that is, the action of H on N is Frobenius. Now Theorem 3.29 forces that the solvable Frobenius complement H has nontrivial center. This contradiction completes the proof. □

The following lemmas will be used throughout the thesis without any reference.

Lemma 4.3. *Let G be a solvable group with a disconnected prime graph, and π_1 and π_2 be subsets of $\pi(G)$ associated with two disconnected components in Γ_G. Then any π_1-subgroup of G acts Frobeniusly on any normal π_2-subgroup of G.*

Proof. Let N be a normal π_2-subgroup of G. For any $1 \neq x \in G$ whose order is a π_1-number we have $C_N(x) = 1$ by the same argument as in the last paragraph of the proof of Theorem 4.2. □

Lemma 4.4. *Let G be a solvable group with a disconnected prime graph.*

(i) Then Γ_G consists of two complete components Γ_1 and Γ_2.

(ii) Let π_1 and π_2 be the sets of primes associated with Γ_1 and Γ_2, respectively. Then $F(G)$ is either $O_{\pi_1}(G)$ or $O_{\pi_2}(G)$.

(iii) If $F(G) = O_{\pi_1}(G)$, then $F_2(G)$ is a Frobenius group with kernel $F(G)$, (and complement isomorphic to $O_{\pi_2}(G/F(G))$ in case G is not Frobenius).

Proof. Since Γ_G is disconnected, we can consider Γ_G as a disjoint union of two nonempty subgraphs Γ_1 and Γ_2 with $V(\Gamma_1) = \pi_1$ and $V(\Gamma_1) = \pi_2$. Then there is no edge between Γ_1 and Γ_2. This is equivalent to the fact that there exists no pair of elements x and y such that x is a π_1-element, y is a π_2-element, and $[x, y] = 1$. On the other hand,

$$[O_{\pi_1}(G), O_{\pi_2}(G)] \leq O_{\pi_1}(G) \cap O_{\pi_2}(G) = 1,$$

and so at least one of $O_{\pi_1}(G)$ and $O_{\pi_2}(G)$ is trivial. However, at least one of $O_{\pi_1}(G)$ and $O_{\pi_2}(G)$ is nontrivial as G is solvable. Thus we may assume that $O_{\pi_1}(G) > 1$ and $O_{\pi_2}(G) = 1$.

Let $p \in \pi_1$ and $q \in \pi_2$. Clearly, for any $p_1 \in \pi_1$, neither pq nor p_1q is contained in $E(\Gamma_G)$. Hence $p_1p \in E(\Gamma_G)$ by Theorem 4.2. It follows that Γ_1 is complete. Similarly one can show that Γ_2 is complete, and (*i*) follows.

Notice that $C_{O_{\pi_1}(G)}(x) = 1$ for any x of order $q \in \pi_2$. Therefore, the group $O_{\pi_1}(G)\langle x\rangle$ is Frobenius, and hence $O_{\pi_1}(G)$ is nilpotent by Theorem 3.17. In particular, we have $F(G) = O_{\pi_1}(G)$, which establishes *(ii)*.

By Lemma 2.25, we get $O_{\pi_1}(G/F(G)) = 1$. Thus, we have $O_{\pi_2}(G/F(G)) > 1$. If $G/F(G) = O_{\pi_2}(G/F(G))$, then $G/F(G)$ acts Frobeniusly on $F(G)$ by Lemma 4.3, and hence G is Frobenius. In case G is not Frobenius we have $G/F(G) > O_{\pi_2}(G/F(G))$, and hence $O_{\pi_2}(G/F(G))$ is nilpotent by Lemma 4.3. It follows that $F_2(G)/F(G) = O_{\pi_2}(G/F(G))$. By Theorem 2.11, $F(G)$ has a complement in $F_2(G)$, say H. Note that $C_{F(G)}(h) = 1$ for any nonidentity $h \in H$ as no π_2-element can centralize a π_1-element. Thus $F_2(G) = F(G)H$ is a Frobenius group as claimed in *(iii)*. □

Next result to which we frequently appeal in the proofs is a special case of Gruenberg-Kegel Theorem.

Proposition 4.5. *([15], Lemma 4.2) Let G be a solvable group. If Γ_G is disconnected, then G is either Frobenius or 2-Frobenius. Furthermore, if G is 2-Frobenius, Γ_G consists of exactly two components, one of which consists of the primes dividing the order of the upper kernel $F_2(G)/F(G)$.*

Proof. First assume that Γ_G is disconnected. Then, by Lemma 4.4, it consists of two complete components Γ_1 and Γ_2. Let π_1 and π_2 be the sets of primes corresponding to Γ_1 and Γ_2, respectively. Set $F(G) = F$. By Lemma 4.4, we have that $F_2(G)$ is a Frobenius group with kernel $F = O_{\pi_1}(G)$. If $G = F_2(G)$, then G is a Frobenius group.

Assume that G is not Frobenius. It follows that $G > F_2(G)$. Set $G/F = G_1$. Then Γ_{G_1} is a subgraph of Γ_G. Since $O_{\pi_1}(G_1) = 1$, Lemma 4.4 yields that $O_{\pi_2}(G_1) = F_2(G)/F > 1$. Observe that $G_1 > O_{\pi_2}(G_1)$ because otherwise $G = F_2(G)$. Then $O_{\pi_2}(G_1/O_{\pi_2}(G_1)) = 1$, and hence $O_{\pi_1}(G_1/O_{\pi_2}(G_1)) > 1$ by the solvability of G. This forces that the intersections $V(\Gamma_{G_1}) \cap \pi_1$ and $V(\Gamma_{G_1}) \cap \pi_2$ are both nonempty, and hence Γ_{G_1} is disconnected. Applying Lemma 4.4, we conclude that $F_2(G_1)$ is a Frobenius group with kernel $F(G_1)$. Therefore $F_3(G)$ is 2-Frobenius since $F_2(F_3(G)) = F_2(G)$ and $F_3(G)/F = F_2(G_1)$ are Frobenius groups. Notice that the upper kernel of $F_3(G)$ is $F(G_1)$, which is cylic by Lemma 3.33. If $G_1 = F_2(G_1)$, then G is a 2-Frobenius group.

Lastly, assume that G is not 2-Frobenius. It follows that $G_1 > F_2(G_1)$. Set $G_2 = G_1/F(G_1)$. Then Γ_{G_2} is a subgraph of Γ_G. Since $O_{\pi_2}(G_2) = 1$ by Lemma 2.25, the above lemma yields that $O_{\pi_1}(G_2) = F_2(G_1)/F(G_1) > 1$. Observe that $G_2 > O_{\pi_1}(G_2)$ because otherwise $G_1 = F_2(G_1)$. Then $O_{\pi_1}(G_2/O_{\pi_1}(G_2)) = 1$, and hence $O_{\pi_2}(G_2/O_{\pi_1}(G_2)) > 1$ by the solvability of G_2. This forces that the intersections $V(\Gamma_{G_2}) \cap \pi_1$ and $V(\Gamma_{G_2}) \cap \pi_2$ are both nonempty, and hence Γ_{G_2} is disconnected. Therefore, by Lemma 4.4, $F_2(G_2) = F_3(G_1)/F(G_1)$ is a Frobenius group with kernel $F(G_2) = F_2(G_1)/F(G_1)$, that is, $F_3(G_1)$ is a 2-Frobenius group since both $F_2(F_3(G_1)) = F_2(G_1)$ and $F_3(G_1)/F(G_1)$ are Frobenius groups. Then, by Lemma 3.33, the lower kernel $F(G_1)$ of $F_3(G_1)$ is not cyclic. This contradiction shows that G_1 cannot properly contain $F_2(G_1)$. Thus G is either Frobenius or 2-Frobenius, completing the proof of the first statement of the proposition.

Assume now that G is a solvable 2-Frobenius group, and let $F = F(G)$. Then there exist subgroups B and C of G such that $F_2(G)/F \cong B$ and $G/F_2(G) \cong C$ as in

the proof of Lemma 3.33. Let Ω_1 and Ω_2 be the induced subgraphs of Γ_G associated with $\pi(F) \cup \pi(C)$ and $\pi(B)$, respectively. Assume that there is an edge between Ω_1 and Ω_2. This means that there is an element $1 \neq b \in B$ such that $C_G(b) \nleq B$. Pick $g \in C_G(b) \setminus B$. Then g can be uniquely written in the form fb_1c, where $f \in F$, $b_1 \in B$ and $c \in C$. Also since B normalizes F and C normalizes B, we can write $bf = f_1b$, and $cb = b_2c$ for some $f_1 \in F$ and $b_2 \in B$. Then

$$bg = bfb_1c = f_1bb_1c \quad \text{and} \quad gb = fb_1cb = fb_1b_2c.$$

It follows that $f_1 = f$ is centralized by b. Then $f = 1$ since FB is a Frobenius group. This forces that $b_1c \in BC$ centralizes b, and hence $c = 1$ since BC is Frobenius group. Now we have $g \in B$, which is a contradiction. Consequently, Γ_G is the disjoint union of two subgraphs Ω_1 and Ω_2 each of which is complete by Lemma 4.4. $\square$

We shall present another contribution of Lucido to the study of prime graphs which is closely related to the main theorem of Chapter 5.

Theorem 4.6. *([4], Proposition 3) Let G be a solvable group with $diam(\Gamma_G) = 3$. Then either $\ell_F(G) \leq 3$ or $\ell_F(G) = 4$, and G has a normal section isomorphic to $2O$.*

Proof. Since $diam(\Gamma_G) = 3$, we have a 3-path $p_1p_2p_3p_4$ in Γ_G such that $d(p_1, p_4) = 3$. By Theorem 4.2, each vertex in $\pi(G)$ is connected to one of p_1 or p_4. Note that no vertex in $\pi(G)$ is connected to both p_1 and p_4 because otherwise $d(p_1, p_4) = 2$, which is not the case. Now we consider $\pi(G)$ as the disjoint union of subsets

$$\pi = \{p \in \pi(G) \mid d(p_1, p) = 1\} \cup \{p_1\} \quad \text{and} \quad \pi' = \{p \in \pi(G) \mid d(p_4, p) = 1\} \cup \{p_4\}.$$

Then accordingly we write $\pi = \pi_1 \cup \pi_2$ and $\pi' = \pi_3 \cup \pi_4$ where

$$\pi_1 = \{p \in \pi \mid d(p, p_4) = 3\},$$

$$\pi_2 = \{p \in \pi \mid d(p, p_4) = 2\},$$

$$\pi_3 = \{p \in \pi' \mid d(p, p_1) = 2\},$$

$$\pi_4 = \{p \in \pi' \mid d(p, p_1) = 3\}.$$

Set $\sigma = \pi_1 \cup \pi_4$ and $\sigma' = \pi_2 \cup \pi_3$. Observe that $p_i \in \pi_i$ for $i = 1, 2, 3, 4$. Notice that there is no edge between π_1 and π' or there is no edge between π_4 and π, because otherwise there would be a vertex in π_1 whose distance from p_4 is 2 or a vertex in π_4 whose distance from p_1 is 2.

Since G is solvable, we see that either $O_\sigma(G)$ or $O_{\sigma'}(G)$ is nontrivial. Assume first that $O_\sigma(G) > 1$ and that $O_{\pi_1}(G) > 1$ without loss of generality. Then a p_4-subgroup of G acts Frobeniusly on $O_\pi(G)$, and hence $O_\pi(G)$ is nilpotent by Theorem 3.17. Now $O_{\pi_1}(G) \leq O_\pi(G) \leq F(G)$. So there is no π'-element in $F(G)$ which implies that $F(G) = O_\pi(G)$. Then $F_2(G)/F(G)$ is a π'-group by Lemma 2.25. Notice that any Sylow q-subgroup Q of G, for $q \in \pi'$, acts Frobeniusly on $O_{\pi_1}(G) > 1$, and hence is cyclic or generalized quaternion by Lemma 3.24. Also by Lemma 2.32, we have that $G/F_2(G)$ is isomorphic to a subgroup of $Out(F_2(G)/F(G))$. If all Sylow subgroups of $F_2(G)/F(G)$ are cyclic or generalized quaternion aside from the quaternion group, then $G/F_2(G)$ is abelian by Lemma 2.31, and hence $\ell_F(G) \leq 3$. Assume that the Sylow 2-subgroup $Q/F(G)$ of $F_2(G)/F(G)$ is isomorphic to the quaternion group Q_8. Then $Out(F_2(G)/F(G)) \cong S_3 \times T$ where T is an abelian group by Lemma 2.7, Lemma 2.45 and Lemma 2.5.

In case where $G/F_2(G)$ is nonabelian, it has a normal subgroup isomorphic to S_3, say $X/F_2(G)$. This gives that $X \trianglelefteq G$. Now $G/F_2(G) \cong S_3 \times T_0$ for an abelian subgroup T_0 of T, and hence $\ell_F(G/F_2(G)) = \ell_F(S_3) = 2$ by Lemma 2.31. It follows that $\ell_F(G) = 4$. For simplicity, we set $G/F(G) = \overline{G}$ and $\overline{S} = O_{2'}(\overline{F_2(G)})$. Let $\overline{Q}$ be the Sylow 2-subgroup of $\overline{F_2(G)}$. Clearly, $Q_8 \cong \overline{Q} \leq \overline{F_2(G)} \leq \overline{X}$. Observe that

$$Q_8 \cong \overline{Q} \cong \overline{QS}/\overline{S} = \overline{F_2(G)}/\overline{S} \trianglelefteq \overline{X}/\overline{S},$$

and so

$$(\overline{X}/\overline{S})/(\overline{QS}/\overline{S}) \cong \overline{X}/\overline{F_2(G)} \cong X/F_2(G) \cong S_3.$$

Since Sylow 2-subgroup of $\overline{X}/\overline{S} \cong X/S$ is generalized quaternion, it has a unique subgroup of order 2, and hence X/S is a non-split extension of S_3 by Q_8. On the other hand, if $\overline{X}/\overline{S}$ has a normal Sylow 3-subgroup $\overline{Y}/\overline{S}$, then $\overline{Y} \trianglelefteq \overline{G}$, and hence $\overline{Y} \leq \overline{F_2(G)}$. This gives a contradiction since 3 divides $|F_3(G)/F_2(G)|$. It follows

that X/S cannot be isomorphic to $C_3 \rtimes Q_{16}$. So we are left with the case $X/S \cong 2O$ by [30]. This means that X/S is a normal section of G (that is, X and S are both normal subgroups of G) which is isomorphic to $2O$.

Next we can assume that $O_\sigma(G) = 1$. Let P be a Sylow p-subgroup of G for $p \in \pi_1$. It follows that $O_p(G) = 1$. Firstly, if $O_{\pi'}(G) > 1$, then P acts Frobeniusly on $O_{\pi'}(G)$, and hence P is a Frobenius complement. On the other hand, if $O_{\pi'}(G) = 1$, then $O_\pi(G) > 1$, and hence $O_{\pi'}(G/O_\pi(G)) > 1$. Clearly P acts Frobeniusly on $O_{\pi'}(G/O_\pi(G))$. Thus P is a Frobenius complement in any case, and so it is either cyclic or generalized quaternion by Lemma 3.24. In the same way, one can see that any Sylow q-subgroup of G for $q \in \pi_4$ is also either cyclic or generalized quaternion.

Now, without loss of generality, suppose that the Fitting subgroup of a Hall $\{p_1, p_4\}$-subgroup of G is a p_1-group. Since $O_\sigma(G) = 1$, we have $N = O_{\sigma'}(G) > 1$. Let N_2 and N_3 be some Hall π_2- and π_3-subgroups of N, respectively. Then we can write $N = N_2N_3$. Notice that any G-conjugate of N_2 is also a Hall π_2-subgroup of N as N is normal in G. By the Frattini argument (Theorem 2.35), we have $G = NN_G(N_2)$. As p_1 and p_4 are both coprime to $|N|$, a Hall $\{p_1, p_4\}$-subgroup U of $N_G(N_2)$ is also a Hall $\{p_1, p_4\}$-subgroup of G. Notice that any subgroup of order p_4 of U acts Frobeniusly on N_2, and hence N_2 is nilpotent. Since a Sylow p_1-subgroup of U is either cyclic or generalized quaternion, and $F(U)$ is a p_1-group, we see that $F(U)$ has a unique subgroup L of order p_1.

Let Q be any Sylow subgroup of N_2. If Q is abelian, by Theorem 2.44, we have $Q = C_Q(L) \times [Q, L]$. Notice that U normalizes $[Q, L]$. First suppose that $[Q, L] > 1$. If U acts Frobeniusly on $[Q, L]$, then U has nontrivial center by Theorem 3.29, yielding the existence of an element of order p_1p_4, which is impossible. Notice also that since p_4 and π_2 are disconnected in Γ_G, we have $C_{[Q,L]}(x) = 1$ for any element x of U whose order is divisible by p_4. This shows that $C_{[Q,L]}(u) > 1$ for some nonidentity p_1-element u of U. Clearly $L = \langle u^t \rangle$ for some positive integer t. Then $C_{[Q,L]}(L) > 1$ which is a contradiction as $C_{[Q,L]}(L) \leq C_Q(L) \cap [Q, L] = 1$. Therefore $[Q, L] = 1$.

If Q is not abelian, we replace Q by $Q/\Phi(Q)$ in the above paragraph and conclude

that $[Q/\Phi(Q), L] = 1$. This means that $[Q, L] \leq \Phi(Q)$, and so $Q = C_Q(L)\Phi(Q)$ by Theorem 2.43. It follows by Lemma 2.23 that $Q = C_Q(L)$. Consequently, L centralizes all Sylow subgroups of N_2 as Q is arbitrary, that is, $[N_2, L] = 1$.

Notice that L permutes the Hall π_3-subgroups of N by conjugation, We have $(|L|, |N|) = 1$ as $|L| = p_1$ and $|N|$ is a σ'-number. Then by Lemma 2.39, there exists a Hall π_3-subgroup of N which is normalized by L. Therefore we may assume that N_3 is L-invariant. Now $[N, L] = [N_2N_3, L] = [N_3, L] \leq N_3$. Notice that L acts Frobeniusly on N_3 as L is a p_1-subgroup and N_3 is a π_3-subgroup of G, and hence N_3 is nilpotent. Then for any Sylow subgroup R of N_3, we have $C_R(L) = 1$, and hence, by Theorem 2.43, $R = [R, L]$. This yields that $[N_3, L] \geq R$ for any Sylow subgroup R of N_3, that is, $[N, L] = [N_3, L] = N_3$. It follows by Lemma 2.8 that N_3 is a normal subgroup of N. In particular N_3 is characteristic in N as N_3 is a Hall subgroup of N. Let $F/N = F(G/N)$. Then F/N is a nilpotent σ-group. Recall that for any prime $p \in \sigma$, a Sylow p-subgroup of G is either cyclic or generalized quaternion. Hence every Sylow subgroup of F/N is either cyclic or generalized quaternion. If 2 divides $|F : N|$, then a subgroup of order 2 of F/N is characteristic in the Sylow 2-subgroup of F/N which is characteristic in G/N. In particular, it is normal, and hence central in G/N. Thus each vertex in σ is connected to 2 in $\Gamma_{G/N}$, and $diam(G/N) \leq 2$. However, both p_1 and p_4 divide G/N, and $d(p_1, p_4) = 3$, which is a contradiction. Therefore F/N is a cyclic group of odd order. It follows that $C_{G/N}(F/N) = F/N$ by Theorem 2.27. Since $G/F \cong (G/N)/(F/N)$ is isomorphic to a subgroup of $Aut(F/N)$ by Lemma 2.6, and $Aut(F/N)$ is abelian by Lemma 2.5, we see that the group G/F is abelian.

Since F/N is nilpotent, it is either a π_1- or a π_4-group. Suppose that F/N is a π_4-group. Consider a Hall σ-subgroup V of G containing U. Since F/N is contained in a Hall σ-subgroup of G, and all Hall σ-subgroups of G are conjugate, we see that $F/N \leq VN/N$. In fact $F/N \leq F(VN/N) \cong F(V)$, that is, $F(V)$ is a π_4-group. Since Γ_V consists of disconnected components associated to π_1 and π_4, we get V is either Frobenius or 2-Frobenius by Proposition 4.5. Notice that for any prime p dividing $|F : N|$, the Sylow p-subgroup of $F(V)$ is cyclic. This forces that V is

not 2-Frobenius by Lemma 3.33, and hence V is Frobenius. There exists a Hall π_1-subgroup K of V which contains $F(U)$. Clearly K acts Frobeniusly on $F(V)$, and hence the group $F(V)K = G$ is Frobenius with complement K. It follows that for any element g of order p_4 in U, we have $K \cap K^g \geq F(U)$, which contradicts Theorem 3.12 as $g \notin K$. Therefore, we conclude that F/N is a π_1-subgroup of G/N. Then $|F : N_3| = |F : N||N : N_3|$ is a π-number. Recall that N_3 is characteristic in N, and so it is normal in F. Hence any p_4-subgroup of G acts Frobeniusly on F/N_3, that is, F/N_3 is nilpotent. This shows that $\ell_F(G) \leq 3$ as claimed. □

4.2 Frobenius Digraphs

Proposition 4.5 guarantees the existence of a missing edge in the prime graph of a given solvable group G by which one can take into account fixed point free actions of the Sylow subgroups of the Hall $\{p, q\}$-subgroups for primes p, q dividing the order of G. It should be noted that this enables us to define a 3-path free orientation on the complementary graph.

Definition 4.7. *Define an orientation of $\overrightarrow{\Gamma_G}$ for a solvable group G as follows: For each edge $pq \in \overline{\Gamma_G}$, a Hall $\{p, q\}$-subgroup H_{pq} is either a Frobenius or 2-Frobenius by Proposition 4.5. If H_{pq} is a Frobenius group with complement a Sylow p-subgroup and kernel a Sylow q-subgroup, we direct the edge pq in $\overrightarrow{\Gamma_G}$ by $p \to q$. If H_{pq} is a 2-Frobenius group of type (p, q, p), we direct the edge pq in $\overrightarrow{\Gamma_G}$ by $p \to q$. We call this orientation the **Frobenius digraph** of G, denoted by $\overrightarrow{\Gamma_G}$.*

The following lemma gives information about Hall subgroups associated to 2-paths in the Frobenius digraph.

Lemma 4.8. *If $r \to q \to p$ is a 2-path in $\overrightarrow{\Gamma_G}$ for a solvable group G, then a Hall $\{p, q, r\}$-subgroup H_{pqr} is 2-Frobenius of type (p, q, r).*

Proof. Let $H = H_{pqr}$. The prime graph of H is disconnected since q is an isolated vertex and so the group H is either Frobenius or 2-Frobenius by Proposition 4.5.

Suppose first that H is a Frobenius group with kernel K and complement C. Each of p, q, r divides either $|K|$ or $|C|$. If q and some other prime, say p, divide $|K|$, then K has an element of order pq as a nilpotent group, which is a contradiction. Similarly if q and some other prime p divide $|C|$, then C has an element of order pq as $Z(C) \neq 1$ by Theorem 3.29. Therefore K or C is a q-group.

Let C be a q-group. Any Sylow subgroup of K is C-invariant, and hence C acts on any Sylow subgroup of K Frobeniusly. Then for any Sylow r-subgroup R of K, RC is a Frobenius group with kernel R. This implies $q \to r$ in the digraph, which is not the case. If K is a q-group then for any Sylow p-subgroup P of C, we get KP is a Frobenius group with kernel K. This forces that $p \to q$, which is a contradiction. Thus we can assume that H is a 2-Frobenius group.

It remains to show that H is 2-Frobenius of type (p, q, r). Set $F_1 = F(H)$ and $F_2/F_1 = F(H/F_1)$. By Proposition 4.5, either F_2/F_1 is a q-group or both F_1 and H/F_2 are q-groups. Firstly suppose that F_1 and H/F_2 are q-groups. Let Q be a complement to F_2 in H and R be a Sylow r-subgroup of H. Since H/F_1 is a Frobenius group, F_1R is normal in H. Thus the Hall $\{q, r\}$-subgroup F_1RQ of H is a 2-Frobenius group of type (q, r, q), whence $q \to r$, which is a contradiction. We conclude that F_2/F_1 is a Sylow q-subgroup of H/F_1 which is isomorhic to a Sylow q-subgroup of H, and cyclic by Lemma 3.33. By the definition of Frobenius digraph, we know that a Hall $\{q, r\}$-subgroup of H is either Frobenius with a q-group kernel or 2-Frobenius of type (r, q, r). If a Hall $\{p, q\}$-subgroup of H is a Frobenius group, the proof is complete. Assume now that a Hall $\{p, q\}$-subgroup of H is a 2-Frobenius group of type (q, p, q). Then its lower kernel is not cyclic and hence a Sylow q-subgroup of Hall $\{p, q\}$-subgroup cannot be cyclic. Notice that it is also a Sylow q-subgroup of H which is not the case as observed above. This completes the proof.

□

Lemma 4.9. *Let G be a solvable group. If $p_1 \to p_2 \to p_3$ is a 2-path in $\overrightarrow{\Gamma_G}$, then for every prime $q \in \overrightarrow{F}[N^1_{\uparrow}(p_3)]$, a Hall$\{q, r\}$-subgroup H_{qr} is a Frobenius group for*

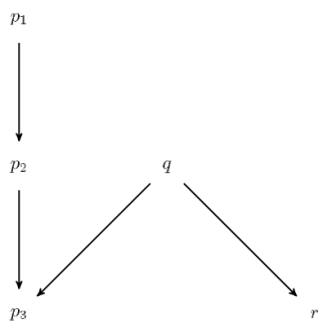

Figure 4.1: Setup of Lemma 4.9

every prime $r \in N^1_{\downarrow}(q)$ (See Figure 4.1).

Proof. Consider a Hall $\{p_1, p_2, p_3\}$-subgroup $H_{p_1p_2p_3}$ of G. By Lemma 4.8, $H_{p_1p_2p_3}$ is 2-Frobenius of type (p_3, p_2, p_1). We know that a Hall $\{p_2, p_3\}$-subgroup $H_{p_2p_3}$ of G contained in $H_{p_1p_2p_3}$ is Frobenius by Lemma 4.8 and hence a Sylow p_3-subgroup P_3 of G is not cyclic. On the other hand, prime graph of a Hall $\{q, r\}$-subgroup H_{qr} is disconnected, and hence H_{qr} is either Frobenius or 2-Frobenius of type (q, r, q). We claim that it is Frobenius with complement Q: Assume that H_{qr} is 2- Frobenius for some $r \in N^1_{\downarrow}(q)$. If $r = p_3$, then P_3 is cyclic by Lemma 3.33, which is a contradiction. Therefore H_{qr} is Frobenius with complement Q. Let now $r \neq p_3$. If H_{qr} is 2-Frobenius of type (q, r, q), then Q cannot be a Frobenius complement by Lemma 3.36. Consequently, H_{qr} is Frobenius group for every prime $r \in N^1_{\downarrow}(q)$. $\square$

We are now ready to prove an immediate but very important consequence of Lemma 4.8.

Corollary 4.10. *The Frobenius digraph of a solvable group G cannot contain a directed 3-path.*

Proof. Suppose that $p_1 \to p_2 \to p_3 \to p_4$ is a 3-path in the Frobenius digraph of G. A Hall $\{p_1, p_2, p_3\}$-subgroup $H_{p_1p_2p_3}$ is 2-Frobenius of type (p_3, p_2, p_1) by Lemma 4.8. Then a Sylow p_3-subgroup P_3 is not cyclic by Lemma 3.33. Also a Hall $\{p_2, p_3, p_4\}$-subgroup $H_{p_2p_3p_4} = H$ is 2-Frobenius of type (p_4, p_3, p_2). Then the upper kernel

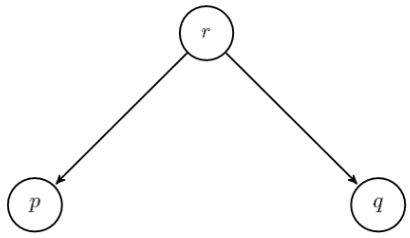

Figure 4.2: Lemma 4.11

of H is isomorphic to a Sylow p_3-subgroup and also cyclic by Lemma 3.33. This contradiction completes the proof. □

Lemma 4.11. *Let G be a solvable group and p, q and r be distinct primes dividing $|G|$. Suppose that a Hall $\{p, r\}$-subgroup of G is a Frobenius group whose kernel is a Sylow p-subgroup of G. Suppose additionally $r \to q$ in $\overline{\Gamma_G}$ (See Figure 4.2). Then a Hall $\{p, q, r\}$-subgroup of G is a Frobenius group whose kernel is a Hall $\{p, q\}$-subgroup of G. In particular, some Sylow p-subgroup and some Sylow q-subgroup of G centralize each other.*

Proof. Let P, Q and R be members of a Sylow system of G as p-, q-, r-subgroups respectively. We know that PR is a Frobenius group with kernel P and complement R. Also QR is either Frobenius or 2-Frobenius of type (r, q, r) since $r \to q$. If QR is 2-Frobenius by Lemma 3.36, then R cannot be a Frobenius complement, which is not the case. Thus QR is Frobenius with kernel Q. Then R normalizes PQ, and so PQR is a subgroup. Clearly, PQR is either Frobenius or 2-Frobenius by Proposition 4.5. Assume that it is 2-Frobenius. So, by Proposition 4.5 again, either the upper kernel of PQR is isomorphic to R or the lower kernel is an r-subgroup. It follows that an r-subgroup is normalized by some p- or q-subgroup, which is impossible. Thus PQR is a Frobenius group and R cannot lie in the kernel by the same argument. Also neither P nor Q is contained in a complement by Theorem 3.29. This forces that PQR is a Frobenius group with kernel PQ, and hence P and Q centralize each other. □

Lemma 4.12. *Let G be a solvable group and $\pi(G) = \{p, q, r\}$ with $r \to p$ and $r \to q$ (See Figure 4.2). Suppose that a Hall $\{p, r\}$-subgroup of G is 2-Frobenius. Then G is 2-Frobenius with upper kernel isomorphic to a Hall $\{p, q\}$-subgroup of G.*

Proof. As $r \to p$ and a Hall $\{p, r\}$-subgroup H_{pr} of G is 2-Frobenius, we see that H_{pr} is of type (r, p, r). First suppose that G is Frobenius. Then r divides the order of a Frobenius complement as $r \to p$, and p divides the order of its kernel as in the proof of Lemma 4.8. In this case a Sylow r-subgroup acts Frobeniusly on a Sylow p-subgroup, contradicting the fact that H_{pr} is 2-Frobenius. Thus by Proposition 4.5, G is 2-Frobenius and either pq or r divides the order of the upper kernel. In the latter case, by Lemma 3.33, some p- or q-subgroup of G acts Frobeniusly on a Sylow r-subgroup, which forces that $p \to r$ or $q \to r$. This contradiction shows that the former holds, that is, the upper kernel is a $\{p, q\}$-group and hence isomorphic to a Hall $\{p, q\}$-subgroup of G.

□

Lemma 4.13. *Let G be a solvable group and $\pi(G) = \{p, q, r\}$ with $p \to r$ and $q \to r$. Suppose that a Hall $\{r, p\}$-subgroup of G is 2-Frobenius. Then G is 2-Frobenius with upper kernel isomorphic to a Sylow r-subgroup of G.*

Proof. As $p \to r$ and a Hall $\{r, p\}$-subgroup H_{pr} of G is 2-Frobenius, we see that H_{pr} is of type (p, r, p). First suppose that G is Frobenius. Then r divides the order of the Frobenius kernel as $p \to r$, and hence a Sylow p-subgroup acts Frobeniusly on a Sylow q-subgroup, contradicting the fact that H_{pr} is 2-Frobenius. Thus by Proposition 4.5, G is 2-Frobenius, and either pq or r divides the order of the upper kernel. First assume that pq divides the order of the upper kernel. Then r cannot divide the order of an upper complement as we have $p \to r$. It follows that the upper kernel of G is an r-group as claimed. □

4.3 Characterization

This section is devoted to the proof of an interesting result which characterizes the prime graphs of solvable groups. We first state a well-known number theoretical result due to Dirichlet which will be used to construct a group admitting a given graph as its prime graph.

Theorem 4.14. *(Dirichlet) Let m and n be positive integers such that $(m, n) = 1$. Then there exist infinitely many prime numbers in the form $mk + n$ where k is an integer.*

Proof. See Chapter 7 in [5]. □

Next lemma will serve the construction of 2-Frobenius groups.

Lemma 4.15. *Let G be a solvable group of square-free order. Assume that $|F(G)|$ divides $p - 1$ for a prime number p. Then there exists a faithful irreducible $\mathbb{F}_pG$-module V, and $F(G)$ acts Frobeniusly on V.*

Proof. See Lemma 1.8 in [6]. □

We are now ready to construct a solvable group admitting a given 3-colorable and triangle free graph as the complement of its prime graph.

Proposition 4.16. *([1], Theorem 2.8) For any 3-colorable, triangle-free (unlabeled) graph F, there exists a solvable group G for which F is isomorphic to the complement of the prime graph Γ_G of G. Furthermore, there exists an acyclic orientation of F that does not contain a directed 3-path, and given any orientation $\overrightarrow{F}$ of F, which does not contain a 3-path, there exists a solvable group G for which $\overrightarrow{F}$ is isomorphic to the Frobenius digraph of G.*

Proof. Let F be a 3-colorable graph and assign colors 1, 2 or 3 to each of its vertices so that none of two adjacent vertices are assigned to the same color. Directing all edges in ascending order, we get an acyclic, 3-path free orientation $\overrightarrow{F}$ of F.

We next define the sets $\mathcal{O}$, $\mathcal{D}$ and $\mathcal{I}$ as follows:

$$\mathcal{O} = \{p \in V(\overrightarrow{F}) \mid N^1_{\uparrow}(p) = \emptyset, N^1_{\downarrow}(p) \neq \emptyset\},$$

$$\mathcal{D} = \{p \in V(\overrightarrow{F}) \mid N^1_{\uparrow}(p) \neq \emptyset, N^1_{\downarrow}(p) \neq \emptyset\},$$

$$\mathcal{I} = \{p \in V(\overrightarrow{F}) \mid N^1_{\downarrow}(p) = \emptyset\}.$$

Namely, $\mathcal{O}$ consists of all vertices with only and at least one outgoing edges; $\mathcal{I}$ consists of all vertices with no outgoing edges; and $\mathcal{D}$ consists of all other vertices. Denote the number of vertices in each of these sets by n_o, n_d, n_i respectively.

Let now $\mathcal{P} = \{p_j \in \mathbb{P} : j = 1, ..., n_o\}$ be a set of n_o distinct primes and put $p = p_1 p_2 ... p_{n_o}$. By Dirichlet's Theorem (Theorem 4.14), we can form a set $\mathcal{Q} = \{q_k : k = 1, ..., n_d\}$ for which $q_k \equiv 1 \ (mod\ p)$, $k = 1, \ldots, n_d$. We fix a bijection from $\mathcal{O} \cup \mathcal{D}$ onto $\mathcal{P} \cup \mathcal{Q}$ mapping vertices in $\mathcal{O}$ to primes in $\mathcal{P}$ and vertices in $\mathcal{D}$ to primes in $\mathcal{Q}$. This clearly extends to a graph isomorphism $\Phi : \overrightarrow{F}[\mathcal{O} \cup \mathcal{D}] \to \overrightarrow{A}$ for a directed graph $\overrightarrow{A}$ with vertex set $\mathcal{P} \cup \mathcal{Q}$ and the edge set defined by the image of Φ.

Set the groups $T = \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_{n_o}}$ and $U = \mathbb{Z}_{q_1} \times ... \times \mathbb{Z}_{q_{n_d}}$, and the invariants

$$m_{j,k} = \begin{cases} q_k - 1 & \text{if } p_j q_k \notin \overrightarrow{A} \\ (q_k - 1)/p_j & \text{if } p_j q_k \in \overrightarrow{A} \end{cases}$$

for each pair of primes $p_j \in \mathcal{P}$ and $q_k \in \mathcal{Q}$. Define an action of $\mathbb{Z}_{p_j}$ on $\mathbb{Z}_{q_k}$ by $y^x := c^{x m_{j,k}} y \ (mod\ q_k)$ for $x \in \mathbb{Z}_{p_j}$, $y \in \mathbb{Z}_{q_k}$, where c is a primitive root modulo q_k. Note that this action is well-defined since $c^{(x+p_j)m_{j,k}} \equiv c^{x m_{j,k}} \ (mod\ q_k)$ as $q_k - 1 \mid p_j m_{j,k}$.

Assume that we have $y^x = y$ for some nonidentity elements $x \in \mathbb{Z}_{p_j}$ and $y \in \mathbb{Z}_{q_k}$. Then $y \equiv c^{x m_{j,k}} y \ (mod\ q_k)$, and so $c^{x m_{j,k}} \equiv 1 \ (mod\ q_k)$, that is, $q_k - 1$ divides $x m_{j,k}$. In case where $p_j q_k \in \overrightarrow{A}$, we have $m_{j,k} = (q_k - 1)/p_j$, and hence p_j divides x, which is a contradiction. It follows that this action is Frobenius if $p_j q_k \in \overrightarrow{A}$. In case where $p_j q_k \notin \overrightarrow{A}$, we have $m_{j,k} = q_k - 1$, and hence $c^{x m_{j,k}} \equiv 1 \ (mod\ q_k)$. It follows that this action is trivial if $p_j q_k \notin \overrightarrow{A}$. Appealing to Lemma 2.38(i) and (iii), we obtain an action of T on U, and define the semidirect product $K = U \rtimes T$ with respect to this

action. Then $\overrightarrow{A}$ is the Frobenius digraph of K. Notice that if a Sylow subgroup P of K contained in T acts trivially on a Sylow subgroup Q of U, then any conjugate P^k of P for $k \in K$ acts trivially on $Q^k = Q$. Hence if $p_j q_k \notin \overrightarrow{A}$, then any Sylow p_j-subgroup of K acts trivially on the Sylow q_k-subgroup of K.

For each vertex $v \in \mathcal{I}$, let $\Phi_1(v)$ and $\Phi_2(v)$ denote the set of primes appearing as vertices in the images of $\overrightarrow{F}[N^1_\uparrow(v)]$ and $\overrightarrow{F}[N^2_\uparrow(v)]$ under Φ, respectively. Set $\Phi(v) = \Phi_1(v) \cup \Phi_2(v)$. In the case $\Phi(v) \neq \emptyset$, let H_v be a Hall $\Phi(v)$-subgroup of the solvable group K containing the Hall $\Phi(v)$-subgroup of T. Consider a Hall $\Phi_1(v)$-subgroup V of H_v containing the Hall $\Phi_1(v)$-subgroup of T. Notice that Sylow subgroups of V corresponding to the primes in $\mathcal{Q}$ are also Sylow subgroups of U, and that T contains a Sylow subgroup of V corresponding to each prime in $\mathcal{P}$. Since F is triangle-free, there is no edge between the vertices in $N^1_\uparrow(v)$. Therefore, all Sylow subgroups of V corresponding to the primes in $\mathcal{Q}$ and Sylow subgroups of V contained in the Hall $\Phi_1(v)$-subgroup of T commute. Thus V is nilpotent.

Let S be a Sylow s-subgroup of V. If $s \in \mathcal{Q}$, then S is characteristic in U, and hence normal in H_v. If $s \in \mathcal{P}$, we can assume that S lies in the Hall $\Phi_1(v)$-subgroup of T, and hence S is centralized by T. Notice that T contains a Sylow subgroup of H_v corresponding to each prime in $\Phi_2(v)$. So there exists a Sylow subgroup of H_v for every prime in $\Phi_2(v)$ centralizing S, that is, S is central in V. This means that V is normal in H_v, and hence $V \leq F(H_v)$.

Notice that $F(H_v)$ is cyclic, and also that every prime in $\Phi_2(v)$ is connected to at least one prime in $\Phi_1(v)$ by the definition of $N^2_\uparrow(v)$. It follows that there is no element $y \in F(H_v) \setminus V$ of prime order, because otherwise $[V, y] = 1$. Therefore, $F(H_v) = V$ which is a cyclic Hall $\Phi_1(v)$-subgroup.

By Dirichlet's theorem, we can choose a set $\mathcal{R} = \{r_j \in \mathbb{P} : j = 1, ..., n_i\}$ of distinct primes so that $\mathcal{R} \cap (\mathcal{P} \cup \mathcal{Q}) = \emptyset$ and each vertex $v_j \in \mathcal{I}$ is associated with a unique prime r_j satisfying the condition $r_j \equiv 1 (mod\ |F(H_{v_j})|)$. For the case $\Phi(v_j) = \emptyset$, define $R_{v_j} = \mathbb{Z}_{r_j}$. For $v_j \in \mathcal{I}$ if $\Phi(v_j) \neq \emptyset$, by applying [7, Lem. 1.8], we obtain a faithful irreducible $\mathbb{F}_{r_j} H_{v_j}$-module R_{v_j} such that $F(H_{v_j})$ acts Frobeniusly on R_{v_j}.

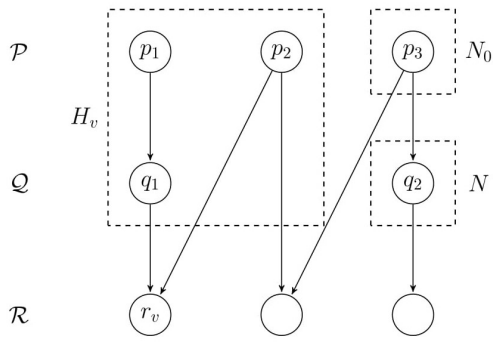

Figure 4.3: An illustration of Hall subgroups H_v, N_0 and N

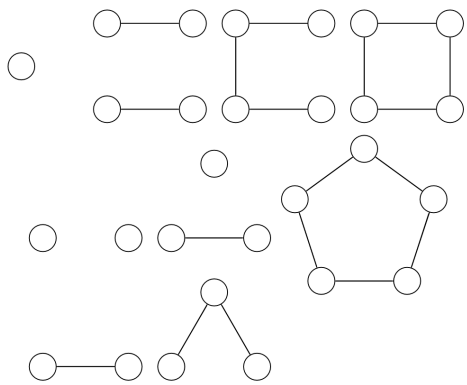

Figure 4.4: Exceptions to Corollary 4.17

Set $J = R_{v_1} \times ... \times R_{v_{n_i}}$. Let any subgroup $\mathbb{Z}_s$ of K for $s \in \mathcal{P} \cup \mathcal{Q}$ act on R_{v_j} by the corresponding module action if $s \in \Phi(v)$, and trivially otherwise.

We show that this extends to an action of K on J. Notice that there is no edge between $\mathcal{P} \setminus \Phi(v)$ and $\Phi(v)$. Thus the Hall $\mathcal{P} \setminus \Phi(v)$-subgroup N_0 of T and H_v centralize each other. So by Lemma 2.38(ii), $M = N_0 \times H_v$ acts on R_v. Next consider the Hall $\mathcal{Q} \setminus \Phi(v)$-subgroup N of U. N is characteristic in U, and hence normal in K. Also notice that M is a complement for N in K. So $K = N \rtimes M$ acts on R_v by Lemma 2.38(ii). Since v is arbitrary, K acts on each R_{v_j} where $v_j \in \mathcal{I}$. By Lemma 2.38(iii), this extends to an action of K on J. Let $G = J \rtimes K$ with respect to this action. It follows that $\overrightarrow{F}$ is isomorphic to the Frobenius digraph of G. $\square$

The next corollary gives a noteworthy information about the group structure although its proof is purely graph theoretical.

Corollary 4.17. *The prime graph of any solvable group has girth* 3 *aside from the following exceptions: the* 4*-cycle, the* 5*-cycle and the seven unique forests that do not contain an independent set of size* 3 *(See Figure 4.4).*

Proof. It is easy to verify that each of the exceptions listed above has a triangle-free and 3-colorable complement. So by Proposition 4.16, they are prime graphs of some solvable groups. One can check that any other graph with at most 4 vertices either contains a triangle or not a prime graph of a solvable group by Lucido's result. Now assume that Γ is a graph with $|V(\Gamma)| = n > 4$, and $\overline{\Gamma}$ is triangle-free and 3-colorable. Suppose first that Γ is a cycle or a path, except for Γ is the 5-cycle. Let $v_1v_2 \dots v_n$ be an $(n-1)$-path in Γ. Then v_1, v_3 and v_5 are three mutually nonadjacent vertices, which is a contradiction as $\overline{\Gamma}$ is triangle-free. Next assume that Γ is not connected. Then it consists of two components, because otherwise it would contain an independent set of size 3. Notice also that any two vertices in the same component must be connected, because otherwise these two vertices and any vertex from the other component would form an independent set of size 3. It follows that each component is a clique and one of them has order at least 3, that is, Γ contains a triangle. Thus we may assume that Γ is a connected graph which is neither a cycle nor a path. Then Γ has a vertex v of degree at least 3 by Lemma 2.2. If Γ is triangle-free, then three neighbors of v form an independent set, which is a contradiction. Hence Γ contains a triangle, that is, Γ has girth 3 aside from the exceptions. □

Combining Corollary 4.10 and Proposition 4.16 we obtain the following.

Theorem 4.18. *An unlabelled graph* F *is isomorphic to the prime graph of some solvable group if and only if its complement* $\overline{F}$ *is* 3*-colorable and triangle-free.*

Proof. If F is isomorphic to the prime graph Γ_G of a solvable group G, then the corresponding Frobenius digraph $\overrightarrow{\Gamma_G}$ does not contain a 3-path, and hence $\overline{F}$ is 3-colorable by Theorem 2.4. Also $\overline{\Gamma}_G$ is triangle-free by Lucido's theorem.

Conversely suppose that $\overline{F}$ is 3-colorable and triangle-free. Then F is isomorphic to the prime graph of a solvable group G by Proposition 4.16. $\square$

CHAPTER 5

SOLVABLE GROUPS WITH MINIMAL PRIME GRAPHS

As observed in Theorem 4.18, a graph with many edges is more likely to be the prime graph of a solvable group because conditions of being triangle-free and 3-colorable on $\overline{\Gamma_G}$ can be violated by removing some edges from the graph. Therefore it is natural to study tight prime graphs, namely the graphs in which the removal of any edge disrupts at least one of the properties of being triangle-free or 3-colorable. In this direction the notion of a minimal prime graph is introduced as follows.

5.1 Minimal Prime Graphs

Definition 5.1. *Suppose that G is a solvable group and Γ_G satisfies the following:*

(a) $|V(\Gamma_G)| > 1$,

(b) Γ_G *is connected,*

(c) $\Gamma_G \backslash \{pq\}$ *is not the prime graph of any solvable group for any* $p, q \in \Gamma_G$.

Then Γ_G is said to be a ***minimal prime graph****.*

Lemma 5.2. *Let G be a solvable group. If Γ_G is a minimal graph, then $\overline{\Gamma}_G$ contains no isolated vertex.*

Proof. Any isolated vertex in $\overline{\Gamma}_G$ can be connected to any other vertex by adding a new edge without making a triangle or increasing the chromatic number of $\overline{\Gamma}_G$, contradicting the minimality of Γ_G. This completes the proof. □

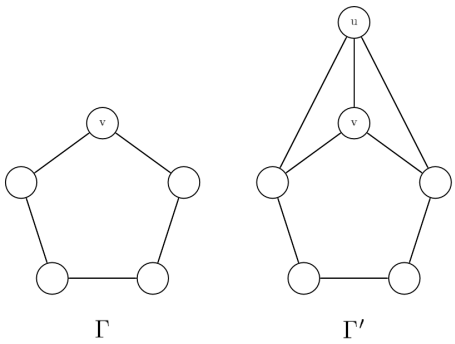

Figure 5.1: Proposition 3.2

Proposition 5.3. *Any minimal prime graph is an induced subgraph of some minimal prime graph of greater order.*

Proof. Suppose that Γ is a minimal prime graph and $v \in \Gamma$. Let Γ' be the graph obtained from Γ by adding

- a new vertex u,
- a new edge uv,
- the edges ux such that $x \in V(\Gamma)$ and $vx \in E(\Gamma)$.

We claim that Γ' is a minimal prime graph. Γ' is connected since Γ is connected and u is connected to v. Note that $\overline{\Gamma}$ is 3-colorable and triangle-free by Theorem 4.18. Then $\overline{\Gamma'}$ is 3-colorable by the same coloring of $\overline{\Gamma}$ and coloring u with the same color as v. Consider the graph $\Omega = \Gamma' \setminus \{v\}$ and the bijective map $\alpha : V(\Gamma) \to V(\Omega)$, defined by $\alpha(v) = u$ and $\alpha(x) = x$ if $x \neq v$. Notice that $xy \in E(\Gamma)$ if and only if $\alpha(x)\alpha(y) \in E(\Omega)$. Therefore Γ and Ω are isomorphic.

Assume now that $\overline{\Gamma'}$ contains a triangle T. If all vertices of T lie in $\overline{\Omega}$, then $\overline{\Gamma}$ also contains a triangle, which is not the case. It follows that u and v lie on T, which contradicts the fact that uv is not an edge in $\overline{\Gamma'}$. Therefore $\overline{\Gamma'}$ is triangle-free, and hence is a prime graph of some solvable group by Theorem 4.18.

In order to show the minimality of Γ', it only remains to check if the addition of a new edge e to $\overline{\Gamma'}$ makes a triangle or increases the chromatic number $\chi(\overline{\Gamma'})$. There are three cases:

- the vertices of e lie in $\overline{\Gamma}$,
- the vertices of e lie in $\overline{\Omega}$,
- the vertices of e are u and v.

In the first case by minimality of Γ we would have a triangle in $\overline{\Gamma} \cup \{e\}$ (also in $\overline{\Gamma'} \cup \{e\}$) or $\chi(\overline{\Gamma} \cup \{e\}) > \chi(\overline{\Gamma})$ (and $\chi(\overline{\Gamma'} \cup \{e\}) > \chi(\overline{\Gamma'})$). In the second case the same result follows since $\Omega \cong \Gamma$. In the last case the new edge is uv. Then by Lemma 5.2, there exists a vertex $x \in V(\Gamma)$ such that $ux \notin E(\Gamma)$, and hence $vx \notin E(\Gamma)$. The addition of uv to $\overline{\Gamma'}$ gives a triangle with vertices u, v and x, and the claim follows. $\square$

Repeated applications of the process described in the proof of the above proposition to the 5-cycle can be used to produce infinitely many minimal prime graphs. However the following observation shows that one cannot obtain all minimal prime graphs in this way: Recall that the Grötzsch graph is the smallest triangle-free graph with chromatic number 4. We can build the complement of a minimal prime graph Λ from the Grötzsch graph by applying a removal and an addition as shown in Figure 5.2. Notice that Λ does not contain any pair u, v of connected vertices such that $N^1(u) \setminus \{v\} = N^1(v) \setminus \{u\}$. Thus Λ cannot be obtained by the process described in Proposition 5.3, for which there exists a pair u, v of connected vertices in the resulting graph with the property that $N^1(u) \setminus \{v\} = N^1(v) \setminus \{u\}$.

Grötzsh graph is not the smallest minimal prime graph which is not derived from the 5-cycle, but it is the smallest example proving that the 3-colorability condition in Theorem 4.18 is not redundant.

Lemma 5.4. *Let G be a solvable group having a minimal prime graph and $K \leq G$ and $N \trianglelefteq G$. Then*

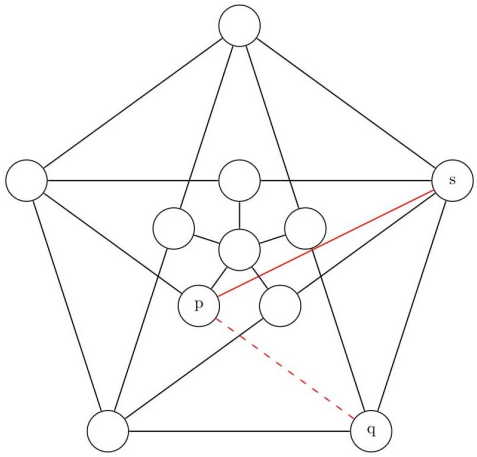

Figure 5.2: Remove pq from the the Grötzsch graph and then add ps

(i) $\Gamma_K = \Gamma_G$ *if and only if* $\pi(K) = \pi(G)$.

(ii) $\Gamma_{G/N} = \Gamma_G$ *if and only if* $\pi(G/N) = \pi(G)$.

Proof. (i) Clearly Γ_K is a subgraph of Γ_G. Since Γ_G is minimal, removing one or more edges from Γ_G does not give a prime graph of some solvable group. Hence if $\pi(K) = \pi(G)$, then we have $\Gamma_K = \Gamma_G$ by the solvability of K.

(ii) Notice that $o(gN) \mid o(g)$ for any $g \in G$. Hence $\Gamma_{G/N}$ is a subgraph of Γ_G. If $\pi(G/N) = \pi(G)$, then $\Gamma_{G/N} = \Gamma_G$ by the solvability of G/N as in the above argument. □

Lemma 5.5. *Let G be a solvable group such that Γ_G is minimal. Then $\overline{\Gamma_G}$ is not 2-colorable.*

Proof. Suppose that $\overline{\Gamma_G}$ is bipartite. Since Γ_G is connected, there exists a non-edge between the color classes in $\overline{\Gamma_G}$. Then removing this edge from Γ_G still makes $\overline{\Gamma_G}$ bipartite, that is, the new graph is also a prime graph contradicting the fact that Γ_G is minimal. □

The following lemma is another pure graph theoretical consequence of Theorem 4.18, and will be used in bounding the Fitting length of a solvable group with minimal prime graph.

Lemma 5.6. *Every minimal prime graph contains an induced 5-cycle.*

Proof. Let Γ be a minimal prime graph. We see that $\overline{\Gamma}$ is 3-colorable and triangle-free by Theorem 4.18. Furthermore, $\overline{\Gamma}$ is not 2-colorable by Lemma 5.5, and hence it contains an odd cycle by Lemma 2.3. Let $v_1v_2\dots v_{2n+1}$ be an odd cycle in $\overline{\Gamma}$ with the shortest length. If $n = 2$, there is nothing to prove. Assume that $n > 2$. Then v_1v_4 and v_1v_5 are edges in $\overline{\Gamma}$ because otherwise there would have been a smaller odd cycle in $\overline{\Gamma}$. Also notice that the addition of the edges v_1v_4 or v_1v_5 to $\overline{\Gamma}$ does not make a triangle, because otherwise if v is the third corner of the triangle, either $vv_1v_2v_3v_4$ or $vv_5v_6\dots v_{2n+1}v_1$ would have been a smaller odd cycle. Let Γ have a 3-coloring. In this coloring at least one of the colors of v_4 and v_5 must be different from v_1's color, say v_i's color $i \in \{4, 5\}$. This shows that the addition of v_1v_i to $\overline{\Gamma}$, neither makes a triangle nor increases the chromatic number of $\overline{\Gamma}$, contradicting the fact that Γ is minimal. $\square$

Definition 5.7. *For a solvable group G, the sets $\mathcal{O}$, $\mathcal{D}$, $\mathcal{I}$ and Π are defined as follows:*

$\mathcal{O} = \{p \in \overrightarrow{\overline{\Gamma_G}} \mid N^1_{\uparrow}(p) = \emptyset\}$,

$\mathcal{D} = \{p \in \overrightarrow{\overline{\Gamma_G}} \mid N^1_{\uparrow}(p) \neq \emptyset, N^1_{\downarrow}(p) \neq \emptyset\}$,

$\mathcal{I} = \{p \in \overrightarrow{\overline{\Gamma_G}} \mid N^1_{\downarrow}(p) = \emptyset\}$,

$\Pi = \{p \in \overrightarrow{\overline{\Gamma_G}} \mid N^2_{\uparrow}(p) \neq \emptyset\}$.

If Γ_G is minimal, there is no isolated vertex in $\overline{\Gamma_G}$, and hence $\{\mathcal{O}, \mathcal{D}, \mathcal{I}\}$ gives a partition of $V(\overrightarrow{\overline{\Gamma_G}})$. Also notice that $\Pi \subseteq \mathcal{I}$ since there is no 3-path in $\overrightarrow{\overline{\Gamma_G}}$.

Next result shows that having a minimal prime graph provides much information about the group structure.

Proposition 5.8. *Suppose that G is a solvable group. If Γ_G is minimal, then a Hall Π-subgroup H_Π lies in $F(G)$.*

Proof. It suffices to show that a Sylow p-subgroup P of G is normal in G for every prime $p \in \Pi$. We shall achieve this by observing that $N_G(P)$ contains a Sylow s-subgroup of G for every prime $s \in \pi(G)$. Since any set of representatives of Sylow subgroups generates G, this will yield that $N_G(P) = G$. Note that as G is solvable, we can choose a Hall π-subgroup H_π of G containing P for any $p \in \pi$.

Suppose first that $s \to p$ in $\overrightarrow{\Gamma_G}$. Then H_{sp} is Frobenius by Lemma 4.11, and hence a Sylow s-subgroup of H_{sp} normalizes P.

Suppose next that $sp \in \Gamma_G$. By minimality of Γ_G, the removal of sp should increase the chromatic number of $\overline{\Gamma_G}$ or make a triangle in $\overline{\Gamma_G}$.

First assume that the removal of sp makes a triangle with vertices s, p, t. If $s \in \mathcal{O}$, then $s \to t \to p$ in $\overrightarrow{\Gamma_G}$, which means that H_{stp} is a 2-Frobenius group of type (p, t, s). So a Sylow s-subgroup normalizes P. If $s \in \mathcal{D}$, then $t \to p$ and $t \to s$ are in $\overrightarrow{\Gamma_G}$, that is, H_{tp} is Frobenius by Lemma 4.9. Then a Sylow s-subgroup centralizes P by Lemma 4.9.

Suppose now that the removal of sp increases the chromatic number of $\overline{\Gamma_G}$. So the only possibility is that $s \in \mathcal{I}$. If there exists a vertex $q \in N^1_\uparrow(p) \cap N^1_\uparrow(s)$ in $\overline{\Gamma_G}$, then H_{pq} is Frobenius by Lemma 4.9, and so a Sylow s-subgroup centralizes P by Lemma 4.11. Therefore, we may assume that $N^1_\uparrow(p) \cap N^1_\uparrow(s)$ is empty. Since $p \in \Pi$, there exists a path $r \to q \to p$ in $\overrightarrow{\Gamma_G}$ where $q \in \mathcal{D}$ and $r \in \mathcal{O}$. Since $N^1_\uparrow(p) \cap N^1_\uparrow(s)$ is empty, we have $qs \in \Gamma_G$. By minimality of Γ_G, the removal of qs gives a triangle, and hence there exists $t \in \mathcal{O}$ such that $t \to s$ and $t \to q$ in $\overrightarrow{\Gamma_G}$. Let $H = H_{psqt}$. Then Γ_H has diameter 3 as Γ_H is a 3-path $tpsq$. Theorem 4.6 implies that we have either $\ell_F(H) \leq 3$ or $\ell_F(H) = 4$ and the binary octahedral group $2O$ is isomorphic to a normal section of H.

Firstly we consider the case $\ell_F(G) = 4$ for which there exist normal subgroups M and N of H such that $M/N \cong 2O$. Note that $|Z(M/N)| = 2$, and hence $Z(M/N) \leq Z(H/N)$. Now $2 \in \{p, q, s, t\}$. By Lemma 3.33, $q \neq 2$.

Assume that $s = 2$. Let $L = H_{2t}$ be a Hall $\{2, t\}$-subgroup of $H = H_{psqt}$. As we

have $t \to 2$, the group L is either Frobenius or 2-Frobenius. If L is 2-Frobenius, the upper kernel is a 2-group, which is not the case by Lemma 3.33. Suppose that L is Frobenius. Then by Lemma 3.15, we have either $N \cap L \leq F(L)$ or $F(L) \leq N \cap L$. If $N \cap L \leq F(L)$, then LN/N is Frobenius with kernel $F(L)N/N$. If $F(L) \leq N \cap L$, then LN/N is a t-group. In either case LN/N cannot have a central subgroup of order 2. This contradiction shows that $s \neq 2$.

Set $H_{tqp} = K$. Assume that $F_2(K) \leq N$. Note that K is 2-Frobenius of type (p, q, t), and hence $K/F_2(K)$ is a cyclic t-group by Lemma 3.33. It follows that $(K/F_2(K))/(N/F_2(K)) \cong K/N$ is a cyclic Sylow t-subgroup of H/N. Since $2O$ is isomorphic to a subgroup of H/N, we get $\{2, 3\} \subseteq \pi(H/N) = \{t, s\}$, which enforces that $t = 2$. This contradicts the fact that a Sylow 2-subgroup of $2O$ is not cyclic. Hence by Lemma 3.34, we have $N < F_2(K)$. Since K is 2-Frobenius and $N < F_2(K)$, we have either KN/N is Frobenius or 2-Frobenius by Lemma 3.34. In either case the center of KN/N is trivial by Lemma 3.14 and Lemma 3.32, which is not the case since $2 \mid |K|$ and H/N has a central subgroup of order 2. This contradiction shows that $\ell_F(G) \neq 4$. Then $\ell_F(G) \leq 3$.

If P is not normalized by a Sylow s-subgroup, then $P \nleq F(H)$. Note that K is a 2-Frobenius group with upper kernel a Sylow q-subgroup of K. Thus $1 = O_q(K) \geq O_q(H)$ and $T > O_t(K) \geq O_t(H)$ for a Sylow t-subgroup T of K. Then ptq divides $|H/F(H)|$. Recall that $t \to q \to p$ in $\overrightarrow{\Gamma_H}$. Hence $H/F(H)$ has a 2-Frobenius subgroup which has Fitting length 3. This forces that $\ell_F(H) > 3$, which is a contradiction.

Consequently, a Sylow s-subgroup of G normalizes P for every $s \in \pi(G)$, and hence $P \leq F(G)$ as desired. □

We shall close this section by a partial answer to a conjecture in group theory.

Definition 5.9. *For a group G, define $\sigma(G) = max\{\sigma(o(g)) \mid g \in G\}$.*

Conjecture 5.10. *Let G be a solvable group. Then $\sigma(|G|) \leq 3\sigma(G)$.*

We can prove this conjecture for solvable groups with minimal prime graphs.

Theorem 5.11. *([1], Theorem 3.6) For any solvable group G for which Γ_G is minimal, $\sigma(|G|) \leq 3\sigma(G)$.*

Proof. By Lemma 2.16, G contains a σ-reduced subgroup H so that $\sigma(|H|) = \sigma(|G|)$. Since Γ_G is minimal, we have $\Gamma_G \cong \Gamma_H$ by Lemma 5.4. Clearly $\sigma(H) \leq \sigma(G)$, and so we can assume without loss of generality that G is σ-reduced.

Notice that every prime divides the order of at most one chief factor of G and every chief factor is elementary abelian by Lemma 2.10 because G is solvable. Also note that every Sylow subgroup of G is elementary abelian. If an edge $p \to q$ in $\overrightarrow{\Gamma_G}$ is associated to a Hall $\{p, q\}$-subgroup H_{pq} which is 2-Frobenius of type (p, q, p), then the lower kernel $F(H_{pq})$ is a proper normal subgroup of a Sylow p-subgoup P of G. Since $C_{H_{pq}}(F(H_{pq})) \leq F(H_{pq})$ by Theorem 2.27, we see that $F(H_{pq})$ is not centralized by P. This contradiction shows that all edges in $\overrightarrow{\Gamma_G}$ are associated to some Hall subgroups which are Frobenius. Therefore every vertex in $\mathcal{O} \cup \mathcal{D}$ serves as a Frobenius complement in some Hall subgroup of G. Since a Frobenius complement has at most one subgroup of order p for any prime p, all Sylow subgroups corresponding to primes in $\mathcal{O} \cup \mathcal{D}$ are cyclic of prime order.

We claim that $H_{\mathcal{O}}$, $H_{\mathcal{D}}$ and $H_{\mathcal{I}}$ are all nilpotent: Any pair p, q with $p, q \in \mathcal{O}$ or $p, q \in \mathcal{D}$ defines an edge in Γ_G, and hence an element of order pq generates a cyclic Hall $\{p, q\}$-subgroup of G. Fix a Sylow p-subgroup P of $H_{\mathcal{O}}$. For any prime $q \in \mathcal{O}$ such that $q \neq p$, a Hall $\{p, q\}$-subgroup of $H_{\mathcal{O}}$ which contains P is cyclic. So P is centralized by some Sylow q-subgroup of $H_{\mathcal{O}}$. Since p is arbitrary, $H_{\mathcal{O}}$ is nilpotent. Similarly one can see that $H_{\mathcal{D}}$ is also nilpotent. To prove $H_{\mathcal{I}}$ is nilpotent, it is enough to show that H_{ps} is nilpotent for any $p \in I$ and $s \in \mathcal{I} \setminus \Pi$ since H_{Π} is in $F(G)$ by Proposition 5.8. Recall that $\mathcal{O}$, $\mathcal{D}$ and $\mathcal{I}$ are three independent sets in $\overline{\Gamma_G}$. We also know that $N^1_{\uparrow}(s) \cap D = \emptyset$, and hence $\mathcal{O}$, $\mathcal{D} \cup \{s\}$ and $\mathcal{I} \setminus \{s\}$ are also three independent sets in $\overline{\Gamma_G}$, which define a 3-coloring for which p and s are in different color classes. Thus the addition of edge ps to $\overline{\Gamma_G}$ does not disturb 3-colorability. By minimality, the addition of ps to $\overline{\Gamma_G}$ must give a triangle in $\overline{\Gamma_G}$, that is, there exists a vertex $q \in N^1_{\uparrow}(p) \cup N^1_{\uparrow}(s)$. Recall that H_{qp} is Frobenius from the previous paragraph.

Therefore by Lemma 4.11, we get that H_{qps} is Frobenius with kernel H_{ps}, and hence H_{ps} is nilpotent.

So there exist elements whose orders are divisible by $|\mathcal{O}|$ primes in $H_{\mathcal{O}}$, by $|\mathcal{D}|$ primes in $H_{\mathcal{D}}$ and by $|\mathcal{I}|$ primes in $H_{\mathcal{I}}$. We conclude that $n := max\{|\mathcal{O}|, |\mathcal{D}|, |\mathcal{I}|\} \leq \sigma(G)$, and so $\sigma(|G|) = |\mathcal{O}| + |\mathcal{D}| + |\mathcal{I}| \leq 3n \leq 3\sigma(G)$. $\square$

5.2 Fitting Lengths and Minimal Prime Graphs

In this section we deal with bounding the Fitting length of a solvable group with minimal prime graph.

Lemma 5.12. *Let G be a solvable group. Assume that $H = H_{srq}$ is a Hall $\{q, r, s\}$-subgroup of G and that H_{sr} and H_{sq} are Hall $\{s, r\}$- and $\{s, q\}$-subgroups of H, respectively.*

(a) If $s \to r \to q$ in $\overrightarrow{\Gamma_G}$, then $O_s(H) = O_s(H_{sr})$.

(b) If $s \to r$ and $s \to q$ in $\overrightarrow{\Gamma_G}$, then $O_s(H) = O_s(H_{sr})$=$O_s(H_{sq})$.

(c) If $s \to r$ and $q \to r$ in $\overrightarrow{\Gamma_G}$, then $O_s(H) = O_s(H_{sr})$.

Proof. Notice that in the first and third cases r is an isolated vertex, and in the other case s is an isolated vertex in Γ_H. Hence in each case Γ_H is disconnected, which yields that H is either Frobenius or 2-Frobenius.

(a) Since $O_s(H)$ is a normal s-subgroup of H, and all Hall $\{s, r\}$-subgroups are H conjugate, we have $O_s(H) \leq O_s(H_{sr})$. By Lemma 4.8, H is 2-Frobenius of type (q, r, s) in this case, and $F(H) = Q \times O_s(H)$ for a Sylow q-subgroup Q of H. Since $H_{sr}Q = H$, we have $H_{sr} \cong H/Q$, and hence $O_s(H_{sr}) \cong O_s(H/Q)$. By Lemma 3.34 we have the following: If $O_s(H) = 1$, then H/Q is Frobenius with kernel isomorphic to a Sylow r-subgroup of H, and so $O_s(H_{sr}) = 1$ too. If $O_s(H) > 1$, then H/Q is 2-Frobenius with lower kernel isomorphic to $O_s(H)$, and so $O_s(H_{sr}) \cong O_s(H)$, which yields the equality as $O_s(H) \leq O_s(H_{sr})$.

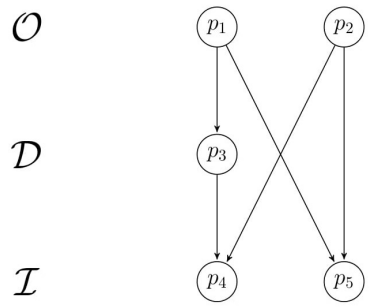

Figure 5.3: Frobenius digraph of G

(b) First assume that H is Frobenius. By Lemma 4.11, $F(H) = O_r(H) \times O_q(H)$. Hence $O_s(H) = 1$. It also follows by Lemma 4.11 that H_{sr} and H_{sq} are Frobenius groups with kernels $O_r(H_{sr})$ and $O_q(H_{sq})$, respectively. Thus $O_s(H_{sr}) = O_s(H_{sq}) = 1 = O_s(H)$. So we can assume that H is 2-Frobenius. It follows by Lemma 4.11 and Lemma 4.12 that H_{sr} and H_{sq} are 2-Frobenius, and $F(H) = O_s(H) > 1$. So $F(H) \leq O_s(H_{sr})$ and $F(H) \leq O_s(H_{sq})$. Then $H_{sr}/F(H)$ and $H_{sq}/F(H)$ are Frobenius groups with kernels $O_r(H_{sr}/F(H))$ and $O_q(H_{sq}/F(H))$, respectively. Hence $O_s(H_{sr}/F(H)) = 1 = O_s(H_{sq}/F(H))$, which implies that $F(H) = O_s(H_{sr}) = O_s(H_{sq})$ as desired.

(c) Suppose first that $O_s(H) = 1$. Then s divides exactly one of $F_2(H)/F(H)$ and $H/F_2(H)$, and hence a Sylow s-subgroup of H is a Frobenius complement. By Lemma 3.36, H_{sr} cannot be 2-Frobenius. Thus H_{sr} is Frobenius with kernel $O_r(H_{sr})$, that is, $O_s(H_{sr}) = 1$. So we may assume that $O_s(H) > 1$. Since $O_s(H_{sr}) \geq O_s(H)$ and $s \to r$, we see that H_{sr} is 2-Frobenius. This implies by Lemma 4.13 that H is a 2-Frobenius group with lower kernel $F(H) = O_s(H) \times O_q(H)$, and so $H_{sr} \cap F(H) = O_s(H)$. Then $H_{sr}/H_{sr} \cap F(H) \cong H_{sr}F(H)/F(H)$ is a Frobenius group with kernel isomorphic to a Sylow r-subgroup of H. So $O_s(H_{sr}/H_{sr} \cap F(H)) = 1$, which implies that

$$O_s(H_{sr}) \leq H_{sr} \cap F(H) = O_s(H) \leq O_s(H_{sr}).$$

□

Proposition 5.13. *([1], Theorem 4.4) Let G be a solvable group such that Γ_G is isomorphic to the 5-cycle. Then $\ell_F(G) = 3$.*

Proof. Let $p_1, \ldots, p_5$ be the prime divisors of $|G|$. Because $\overline{\Gamma_G}$ is also isomorphic to the 5-cycle, there exists only one possible Frobenius digraph of G up to isomorphism as in Figure 5.3. Since a Hall $\{p_1, p_3, p_4\}$-subgroup of G is 2-Frobenius by Lemma 4.8, we see that $\ell_F(G) \geq 3$.

According to the definitions of $\mathcal{O}$, $\mathcal{D}$, $\mathcal{I}$ and Π in Definition 5.7 we have $\Pi = \{p_4\}$, $\mathcal{I} = \{p_4, p_5\}$, $\mathcal{D} = \{p_3\}$, $\mathcal{O} = \{p_1, p_2\}$. By Proposition 5.8, there is a unique Sylow p_4-subgroup P_4, that is, $P_4 \leq F(G)$. Note that the primes dividing $|F(G)|$ must be adjacent to each other in Γ_G.

Firstly suppose that $H_{p_1p_3}$ is 2-Frobenius where $H_{p_1p_3}$ is a Hall $\{p_1, p_3\}$-subgroup of G. Since $p_1 \to p_3$, we see that $J := O_{p_1}(H_{p_1p_3}) = F(H_{p_1p_3})$ is nontrivial . We claim that J is normal in G. To see this we shall show that $N_G(J)$ contains a Sylow subgroup of G for every prime divisor of $|G|$. As G is solvable, there exists a Hall $\{p_1, p_3, p_4\}$-subgroup $H_{p_1p_3p_4}$ containing $H_{p_1p_3}$. Notice that $p_1 \to p_3 \to p_4$. Applying Lemma 5.12(a), we get $J = O_{p_1}(H_{p_1p_3}) = O_{p_1}(H_{p_1p_3p_4})$. Hence J is normalized by a Sylow p_3- and a Sylow p_4-subgroup of G. Similarly by Lemma 5.12(b), the existence of $p_1 \to p_5$ shows that $J = O_{p_1}(H_{p_1p_3p_5}) = O_{p_1}(H_{p_1p_5})$ for a Hall $\{p_1, p_3, p_5\}$-subgroup of G. This means that a Sylow p_5-subgroup of G normalizes J. Finally, as $p_2 \to p_5$, Lemma 5.12(c) implies that $J = O_{p_1}(H_{p_1p_5}) = O_{p_1}(H_{p_1p_2p_5})$ for a Hall $\{p_1, p_2, p_5\}$-subgroup of G, and hence a Sylow p_2-subgroup of G normalizes J. As a result J is normal in G, and hence $J \leq O_{p_1}(G) \leq O_{p_1}(H_{p_1p_3}) = J$. Thus we have $F(G) = J \times P_4$.

If we prove that all Sylow subgroups of $G/F(G)$ are cyclic, by Lemma 3.26, we get $(G/F(G))'$ and $(G/F(G))/(G/F(G))'$ are cyclic, and hence $G/F(G)$ has Fitting length at most 2. That is, G has Fitting length 3. Observe that $F(G) = F(H_{p_1p_3p_4})$ and since $F(H_{p_1p_3p_4})$ is 2-Frobenius, we know by Lemma 3.33 that Frobenius kernel and complement of $H_{p_1p_3p_4}/F(G)$ are both cyclic. Hence Sylow p_1- and p_3-subgroups of $G/F(G)$ are cyclic. On the other hand, as $p_1 \to p_5$ and $J = O_{p_1}(H_{p_1p_5}) > 1$, we

see that $H_{p_1p_5}$ is also 2-Frobenius with cyclic upper kernel isomorphic to a Sylow p_5-subgroup of G. As $p_2 \to p_4$ where $p_4 \in \Pi$, and $p_2 \to p_5$, we see by Lemma 4.9 that $H_{p_2p_5}$ is Frobenius, and a Sylow p_2-subgroup of $H_{p_2p_5}$ is isomorphic to a subgroup of $Aut(P_5)$ which is abelian. Then P_2 cannot be a generalized quaternion group, and so by Lemma 3.24, P_2 is cyclic. Therefore all Sylow subgroups of $G/F(G)$ are cyclic implying that $\ell_F(G) = 3$ when $H_{p_1p_3}$ is 2-Frobenius.

Finally we suppose that $H_{p_1p_3}$ is a Frobenius group and claim that a Sylow p_5-subgroup P_5 is normal in G. To see this we shall proceed as in the third paragraph in which the normality of J is shown. Recall that $H_{p_2p_5}$ is a Frobenius group by Lemma 4.9, and hence a Sylow p_2-subgroup normalizes P_5. Also $H_{p_2p_4p_5}$ is Frobenius whose kernel is a Hall $\{p_4, p_5\}$-subgroup. It follows that the Sylow p_4-subgroup P_4 centralizes P_5. As $H_{p_1p_3}$ is Frobenius, $p_1 \to p_3$ and $p_1 \to p_5$, Lemma 4.11 implies that $H_{p_1p_5}$ and $H_{p_1p_3p_5}$ are Frobenius groups and so some Sylow p_1- and p_3-subgroups normalize P_5. Thus we have $P_5 \leq F(G)$. It follows that $F = P_4 \times P_5$ since all primes dividing $|F(G)|$ must be adjacent in Γ_G.

If all Sylow subgroups of $G/F(G)$ are cyclic, we are done. Sylow p_1- and p_3-subgroups of G are cyclic by Lemma 3.33. Thus we can assume that a Sylow p_2-subgroup of $G/F(G)$ is not cyclic. Then it is generalized quaternion by Lemma 3.24 and so $p_2 = 2$. Recall that $H_{p_1p_3}$ is a Frobenius group whose kernel is a Sylow p_3-subgroup P_3, and hence a Sylow p_1-subgroup is isomorphic to a subgroup of $Aut(P_3)$. If $p_3 = 3$ we have $|Aut(P_3)| = 2 \cdot 3^n$ for some nonnegative integer n by Lemma 2.5. Thus $p_1 = 2$, which contradicts the fact that $p_2 = 2$. So we conclude that $p_3 \neq 3$.

Let $F_2(G)/F(G) = F(G/F(G))$. By Lemma 2.32, $G/F_2(G)$ is isomorphic to a subgroup of $Out(F_2/F(G))$. If $F_2(G)/F(G)$ were cyclic, then $Out(F_2(G)/F(G))$ would be abelian by Lemma 2.5 and G would have Fitting length 3. Thus we can assume that $F_2(G)/F(G)$ is not cyclic.

Let $Q/F(G)$ be the Sylow 2-subgroup of $F_2(G)/F(G)$ and let $D/F(G)$ be a Hall $2'$-subgroup of $F_2(G)/F(G)$. Then $Q/F(G)$ is isomorphic to a subgroup of a generalized quaternion group, and hence it is either generalized quaternion or cyclic by

Lemma 3.24. Note that $D/F(G)$ is cyclic since all Sylow subgroups of $G/F(G)$ other than Sylow 2-subgroups are cyclic. Also we have $F_2(G)/F(G) = Q/F(G) \times D/F(G)$. Thus $G/F_2(G)$ is isomorphic to a subgroup of $Out(Q/F(G)) \times Out(D/F(G))$ by Lemma 2.7. Notice that $Out(D/F(G))$ is abelian, and $Out(Q/F(G))$ is abelian unless $Q/F(G)$ is isomorphic to Q_8 by Lemma 2.45. Therefore $G/F_2(G)$ is abelian unless $Q/F(G)$ is isomorphic to Q_8. Since $\ell_F(G) = 3$ in case $G/F_2(G)$ is abelian, we may assume that $Q/F(G)$ is isomorphic to Q_8, in particular $G/F_2(G)$ is nonabelian and contains a subgroup $H/F_2(G)$ which is isomorphic to S_3. Since $H'F_2(G)/F_2(G)$ has order 3, G' has a nontrivial Sylow 3-subgroup. As $p_3 \neq 3$ and the only primes dividing $|G : F(G)|$ are p_1, 2 and p_3, we get $p_1 = 3$.

Let $C/F(G) = C_{G/F(G)}(Q/F(G))$. Then G/C is isomorphic to a subgroup of $Aut(Q_8)$, and hence $|G : C|$ divides 24 by Lemma 2.45. Notice that p_3 does not divide $|G : C|$, and so C contains a Sylow p_3-subgroup P_3 of G. As $H_{p_1p_3} \cap F(G) = 1$, we see that $H_{p_1p_3}F(G)/F(G) \cong H_{p_1p_3}$ is a Frobenius group whose kernel is a p_3-group. Observe that $O_{p_1}(G/F(G)) \leq O_{p_1}(H_{p_1p_3}F(G)/F(G)) = 1$, which means that $p_1 = 3$ does not divide $|F_2(G) : F(G)|$. Thus $D/F(G)$ is a p_3-group, and so $D/F(G) \leq P_3F(G)/F(G)$. Since $P_3 \cong P_3F(G)/F(G)$ is cyclic, $P_3F(G)/F(G)$ and $D/F(G)$ commute. Notice also that $P_3F(G)/F(G)$ centralizes $Q/F(G)$ since $P_3 \leq C$. Thus

$$P_3F(G)/F(G) \leq C_{G/F(G)}(F_2(G)/F(G)) \leq F_2(G)/F(G)$$

by Theorem 2.27. It follows that $P_3F(G)/F(G) = D/F(G)$. Let $B/F(G) = C_{G/F(G)}(D/F(G))$. Then G/B is isomorphic to a subgroup of $Aut(D/F(G))$ which is abelian since $D/F(G)$ is cyclic. Therefore $G' \leq B$. On the other hand, $H_{p_1p_3}F(G)/F(G)$ is a Frobenius group with kernel $D/F(G)$, and hence 3 does not divide $|B : F(G)|$. This contradicts the fact that G' has a nontrivial 3-subgroup. □

Definition 5.14. *Let G be a solvable group and $\mathcal{O}$, $\mathcal{D}$, $\mathcal{I}$ and Π be defined as in Definition 5.7. For any $p \in \mathcal{I}$, let $\mathcal{O}_1(p) = N^1_\uparrow(p) \cap \mathcal{O}$. Then we define the following sets*

$$\mathcal{O}_1 = \bigcup_{p\in\Pi} \mathcal{O}_1(p) \quad \mathcal{O}_1^* = \bigcap_{p\in\Pi} \mathcal{O}_1(p) \quad \mathcal{O}_2 = \bigcap_{p\in\Pi} N^2_\uparrow(p) \quad \mathcal{O}_2^* = \bigcup_{p\in\Pi} N^2_\uparrow(p)$$

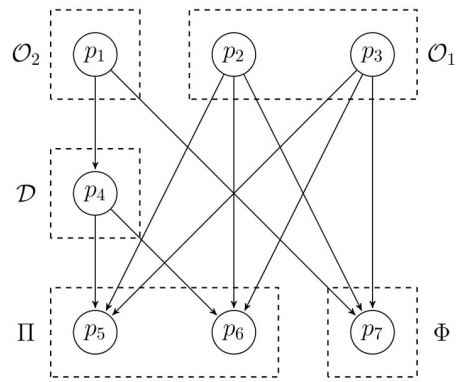

Figure 5.4: A demonstration of $\mathcal{O}_1$, $\mathcal{O}_2$, Π and Φ

and define $\Phi = \{q \in \mathcal{I} \mid N^2_\uparrow(q) = \emptyset\}$ *(See Figure 5.4).*

Remark 5.15. *Clearly we have* $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 = \mathcal{O}^*_1 \cup \mathcal{O}^*_2$ *and* $\mathcal{I} = \Pi \cup \Phi$ *are disjoint unions.*

Lemma 5.16. *Let* G *be a solvable group with a minimal prime graph.*

(a) $\mathcal{O} \subseteq N^1_\uparrow(p) \cup N^2_\uparrow(p)$ *for any prime* $p \in \mathcal{I}$.

(b) $\mathcal{O} \subseteq N^1_\uparrow(p)$ *for any prime* $p \in \Phi$.

(c) If $s \in \mathcal{O}_2$*, then* $\mathcal{D} \subseteq N^1_\downarrow(s)$.

Proof. (a) Assume there exists a vertex $q \in \mathcal{O} \setminus N^1_\uparrow(p)$. Since q and p are in different color classes, addition of the edge pq in $\overline{\Gamma_G}$ does not increase the chromatic number of $\overline{\Gamma_G}$. Therefore it should give a triangle which means there exists $t \in \mathcal{D}$ such that $q \in N^1_\uparrow(t)$ and $t \in N^1_\uparrow(p)$. Thus we have $q \in N^2_\uparrow(p)$, and (a) follows.

(b) By definition of Φ, $N^2_\uparrow(p)$ is empty for any $p \in \Phi$. Hence by part (a), $\mathcal{O} \subseteq N^1_\uparrow(p)$ as claimed.

(c) Let $s \in \mathcal{O}_2$. Assume that $q \in \mathcal{D} \setminus N^1_\downarrow(s)$. Then addition of the edge qs does not increase the chromatic number of $\overline{\Gamma_G}$, and so it should give a triangle with vertices s, q and t in $\overline{\Gamma_G}$. Since $s \in N^1_\uparrow(t)$, we have $t \notin \mathcal{O}$. Also there is an edge between q and

t, and so $t \notin \mathcal{D}$ and $t \notin \Phi$. Hence we are left with the case $t \in \Pi$ and $s \in N^1_{\uparrow}(t)$, which contradicts the fact that $s \in \mathcal{O}_2$. $\square$

Lemma 5.17. *Let G be a solvable group such that Γ_G is a minimal prime graph.*

(a) If $q \in \mathcal{D}$ and $p \in N^1_{\downarrow}(q)$, then $q \neq 2$, a Sylow q-subgroup of G is cyclic, and a Hall $\{p, q\}$-subgroup of G is a Frobenius group.

(b) If $s \in \mathcal{O}_1$ and $t \in N^1_{\downarrow}(s)$, then a Hall $\{t, s\}$-subgroup of G is a Frobenius group. In particular, a Sylow s-subgroup of G is either cyclic or generalized quaternion.

(c) If $s \in \mathcal{O}^*_2$, then a Sylow s-subgroup of S is not generalized quaternion.

Proof. (a) Since $q \in \mathcal{D}$, there is a vertex $r \in \mathcal{I}$ such that $q \to r$. Hence we get $p \to q \to r$. By Lemma 4.8, a Hall $\{p, q\}$-subgroup of G is a Frobenius group.

(b) Since $s \in \mathcal{O}_1$, there exists $u \in \Pi$ such that $s \to u$. Then by Lemma 4.9, a Hall $\{s, u\}$-subgroup of G is a Frobenius group. It follows by Lemma 4.11 that a Hall $\{t, s\}$-subgroup of G is also a Frobenius group with Frobenius complement a Sylow s-subgroup of G. Then a Sylow s-subgroup of G is either cyclic or generalized quaternion by Lemma 3.24.

(c) Since $s \in \mathcal{O}^*_2$, there exist $u \in \mathcal{D}$ and $v \in \Pi$ such that $s \to u \to v$. Then a Hall $\{s, u, v\}$-subgroup of G is 2-Frobenius of type (v, u, s). Lemma 3.38 implies that a Sylow s-subgroup of G is not generalized quaternion. $\square$

Lemma 5.18. *Suppose that G is a solvable group such that Γ_G is minimal. If a Sylow 2-subgroup of G is generalized quaternion, then $2 \in \mathcal{O}^*_1$.*

Proof. Let $2 \in \pi(G)$. Assume first that $2 \in \mathcal{I} \cup \mathcal{D}$. Then there exist a vertex q such that $q \to 2$. Hence Hall $\{2, q\}$-subgroup is either Frobenius, or 2-Frobenius of type $(q, 2, q)$. Then a Sylow 2-subgroup of G is a Frobenius kernel, and hence it cannot be generalized quaternion by Lemma 3.19. Finally if $2 \in \mathcal{O}^*_2$, a Sylow 2-subgroup is not generalized quaternion by Lemma 5.17(c), and hence if a Sylow 2-subgroup is generalized quaternion, and hence $2 \in \mathcal{O}^*_1$ as claimed. $\square$

Using Theorem 4.6, it is not too difficult to see that 5 is an upper bound for the Fitting length of a solvable group with minimal prime graph. We shall see that a more detailed analysis of the situation leads to the best possible upper bound, namely 4.

Theorem 5.19. *([1], Theorem 5.4) Let G be a solvable group such that Γ_G is a minimal prime graph. Then $3 \leq \ell_F(G) \leq 4$. Furthermore, if $\ell_F(G) = 4$, then $2O$ is isomorphic to a normal section of G.*

Proof. As Γ_G contains an induced 5-cycle by Lemma 5.6, there exists a 2-path in $\overrightarrow{\Gamma_G}$. Since a Hall subgroup associated to a 2-path in the Frobenius digraph is 2-Frobenius by Lemma 4.8, we see that $\ell_F(G) \geq 3$.

Firstly we observe that it is sufficient to find a normal nilpotent subgroup X of G such that all Sylow subgroups of G/X are either cyclic or generalized quaternion: If such a subgroup X exists, then all Sylow subgroups of $E/X = F(G/X)$ are also either cyclic or generalized quaternion by Lemma 3.24. Then by Remark 2.33, G/E is isomorphic to a subgroup of $Out(E/X)$. Since $Out(E/X)$ is isomorphic to the direct product of outer automorphism groups of its Sylow subgroups, $Out(E/X)$ has Fitting length at most 2 since all terms in this direct product have Fitting length at most 2 by Lemma 2.45. Therefore G/E also has Fitting length at most 2 by Lemma 2.31. Since E/X and X are both nilpotent, we get $\ell_F(G) \leq 4$.

Firstly, if $\ell_F(G) = 4$, then $\ell_F(G/X) = 3$, and as in the proof of Theorem 4.6, we get that $2O$ is isomorphic to a normal section of G.

Now we partition $\mathcal{O}_2$ into two subsets, namely $\mathcal{C}$ and $\mathcal{N}$, where

$$\mathcal{C} = \{s \in \mathcal{O}_2 \,|\, G \text{ has a cyclic Sylow } s\text{-subgroup.}\}, \text{ and } \mathcal{N} = \mathcal{O}_2 \backslash \mathcal{C}.$$

We divide the proof into two cases.

Case 1. $\mathcal{N} = \emptyset$.

In this case, every prime divisor of $|G : H_{\mathcal{I}}|$ lies in $\mathcal{D} \cup \mathcal{C} \cup \mathcal{O}_1$. By Lemma 5.17, for $p \in \mathcal{D} \cup \mathcal{O}_1$, a Sylow p-subgroup is cyclic or generalized quaternion, and for

$p \in \mathcal{C}$, a Sylow p-subgroup is cyclic by definition. Then it is enough to show that $H_{\mathcal{I}}$ is nilpotent and normal in G because this would imply by the second paragraph that $\ell_F(G) \leq 4$.

Recall that $\mathcal{I} = \Pi \cup \Phi$. Note that H_Π is nilpotent and normal by Proposition 5.8. Therefore we may assume that $\Phi \neq \emptyset$. It is enough to show that for each $p \in \Phi$, a Sylow p-subgroup is normal in G. This will be done by observing that $N_G(P)$ contains a Sylow s-subgroup for every $s \in \pi(G)$ and for any Sylow p-subgroup P of G.

Let $s \in \mathcal{O}$. Then $s \to p$ by Lemma 5.16(b). We know that a Sylow s-subgroup is either cyclic or generalized quaternion, particularly a Sylow s-subgroup cannot be written as a semidirect product of its nontrivial subgroups. Then by Lemma 3.36, a Hall $\{p, s\}$-subgroup H_{ps} which contains P cannot be 2-Frobenius, and hence H_{ps} is Frobenius. It follows that a Sylow s-subgroup of G normalizes P.

Let $s \in \mathcal{D}$. Then there exists a prime $r \in \mathcal{O}$ such that $r \to s$. If $r \in \mathcal{O}_2 = \mathcal{C}$, then any Sylow r-subgroup R of G is cyclic. If $r \notin \mathcal{O}_2$, then there exists a prime $u \in \Pi$ such that $r \to u$. Then by Lemma 4.9, a Hall $\{r, s\}$-subgroup containing R is a Frobenius group with complement R. Also $r \to p$ since $p \in \Phi$. Then by Lemma 4.11, a Hall $\{r, s, p\}$-subgroup containing P is a Frobenius group with kernel H_{ps} which implies that a Sylow s-subroup of G centralizes P.

Let $s \in \mathcal{I} \setminus \Pi'$ where $\Pi' = \{q \in \mathcal{I} \mid \mathcal{O}_1(q) = \emptyset\}$. Then there exists a prime $r \in \mathcal{O}_1(s)$. By Lemma 5.16(b), we have $r \to p$. Then a Hall $\{s, p, r\}$-subgroup containing P is either a Frobenius group whose kernel is a Hall $\{s, p\}$-subgroup by Lemma 4.11 or a 2-Frobenius group with upper kernel isomorphic to a Hall $\{s, p\}$-subgroup by Lemma 4.12. In either case a Sylow s-subgroup centralizes P.

Assume that there exists $s \in \Pi'$. Clearly $\mathcal{D} \cap N^1_\uparrow(p) = \emptyset$, and $\mathcal{D} \geq N^1_\uparrow(s)$. Hence $\mathcal{O}$, $\mathcal{D} \cup \{p\}$ and $\mathcal{I} \setminus \{p\}$ are three independent sets, and the addition of ps to $\overrightarrow{\Gamma_G}$ neither gives a triangle nor increases the chromatic number, contradicting that Γ_G is minimal. It follows that Π' is empty, and this completes the case 1.

Case 2. $\mathcal{N} \neq \emptyset$.

Let $L = H_\Pi \times \prod_{s\in\mathcal{N}} O_s(G)$. Clearly each factor in this product is normal and nilpotent in G, whence $L \leq F(G)$. We show next that every Sylow subgroup of G/L is either cyclic or generalized quaternion, and hence we will have $\ell_F(G) \leq 4$ by the first paragraph with $X = L$. Observe that the primes dividing $|G : L|$ lie in $\mathcal{N} \cup \mathcal{C} \cup \mathcal{O}_1 \cup \mathcal{D} \cup \Phi$. By Lemma 5.17, for the primes in $\mathcal{C} \cup \mathcal{O}_1 \cup \mathcal{D}$, Sylow subgroups of G are cyclic or generalized quaternion. Hence it remains only to consider primes in $\mathcal{N} \cup \Phi$.

Firstly, let $q \in \Phi$. For a fixed $s \in \mathcal{N}$ we have $s \to q$ by Lemma 5.16(b). By definition of $\mathcal{N}$, a Sylow s-subgroup S of G is not cyclic. Also S is not generalized quaternion because otherwise $s \in \mathcal{O}_1^* \leq \mathcal{O}_1$ by Lemma 5.18, which is a contradiction as $\mathcal{N} \cap \mathcal{O}_1 = \emptyset$. Therefore a Hall $\{s, q\}$-subgroup H_{sq} cannot be Frobenius, and hence H_{sq} is 2-Frobenius of type (s, q, s). It follows by Lemma 3.38 that a Sylow q-subgroup is cyclic. As q is arbitrary, all Sylow subgroups for primes in Φ are cyclic. Since $s \in \mathcal{O}_2$, there exists a prime $r \in \mathcal{D}$ such that $s \to r$. Let H_{sr} be a Hall $\{s, r\}$-subgroup containing S, and set $J = O_s(H_{sr})$. Clearly H_{sr} is not a Frobenius group since S is neither cyclic nor generalized quaternion. This forces that H_{sr} is 2-Frobenius of type (s, r, s) with upper complement isomorphic to S/J. Note that S/J is cyclic by Lemma 3.33. If $J = O_s(G)$, then $S \cap L = J$, and so $S/J \cong SL/L$ is cyclic. This shows that once it has been proven that $J = O_s(G)$, we can choose $X = L$ and conclude that all Sylow subgroups of G/X are cyclic or generalized quaternion, which completes the proof.

Since $O_s(G) \trianglelefteq H_{sr}$, we have $O_s(G) \leq J$. It remains to prove that $J \trianglelefteq G$ in order to have $J = O_s(G)$. This will be done by showing that $N_G(J)$ contains a Sylow p-subgroup of G for every prime $p \in \pi(G)$.

For $p \in \mathcal{D} \cup \Phi$ by Lemma 5.16, we have $s \to p$. Then by Lemma 5.12(b), $J = O_s(H_{sp})$ where H_{sp} is a Hall $\{s, p\}$-subgroup containing S. In particular, a Sylow p-subgroup normalizes J.

For $p \in \Pi$, since $s \in \mathcal{O}_2$, we have $q \in \mathcal{D}$ such that $s \to q \to p$. Let H_{sqp} be a Hall $\{s, q, p\}$-subgroup, and H_{sq} be a Hall $\{s, q\}$-subgroup contained in H_{sqp}. By previous paragraph $J = O_s(H_{sq})$, and by Lemma 5.12(a) $J = O_s(H_{sqp})$. Hence J is centralized by the Sylow p-subgroup of G.

Let $p \in \mathcal{O}$. Suppose that $N^1_{\downarrow}(s) \cap N^1_{\downarrow}(p) \neq \emptyset$. Then $\mathcal{D} \cap N^1_{\downarrow}(p) = \emptyset$ by Lemma 5.16(c), and $\mathcal{O} \setminus \{p\}$, $\mathcal{D} \cup \{p\}$ and $\mathcal{I}$ are three independent sets. Thus the addition of sp to $\overrightarrow{\Gamma_G}$ neither gives a triangle nor increases the chromatic number, contradicting the minimality of Γ_G. Therefore, there exists $u \in N^1_{\downarrow}(s) \cap N^1_{\downarrow}(p)$, and by the previous paragraphs, Lemma 5.12(a) and (c), we have $J = O_s(H_{su}) = O_s(H_{psu})$ for a Hall $\{s, u\}$-subgroup H_{su}, and a Hall $\{p, s, u\}$-subgroup H_{psu} of G. This implies that a Sylow p-subgroup normalizes J completing the proof.

$\square$

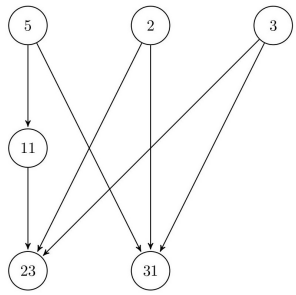

Figure 5.5: Frobenius digraph of G

5.3 Examples

1. Every solvable group whose minimal prime graph is a 5-cycle has Fitting length 3 by Proposition 5.13. As indicated in the proof of Theorem 4.18, one can construct such a solvable group.

2. We construct below a group G with minimal prime graph such that $\ell_F(G) = 4$: Let $K = C_{11} \rtimes C_5$, be a Frobenius group and $L = K \times 2O$. Let V_1 be an irreducible $\mathbb{F}_{23}[L]$-module so that $C_{11} \times 2O$ acts fixed point freely on V_1. The smallest such V_1 has dimension 10. Let V_2 be an absolutely irreducible V_2 be an absolutely irreducible $\mathbb{F}_{31}[L]$-module so that $C_5 \times 2O$ acts fixed point freely on V_2. The smallest such V_2 has dimension 2. Then the group $G = (V_1 \times V_2) \rtimes L$ has Fitting length 4, and it has a minimal prime graph.

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