GRAVITATIONAL WAVES AND 2.5 PN GRAVITATIONAL WAVE MEMORY

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ABSTRACT

GRAVITATIONAL WAVES AND 2.5 PN GRAVITATIONAL WAVE MEMORY

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Since the discovery of general relativity, the Einstein field equations have explained many phenomenon (e.g. precession of orbits) and predicted others (e.g. gravitational lensing, gravitational time dilation and black holes). Among its many predictions, the wave-like nature of the linearized theory has garnered a lot of attention due to the possiblity of gravitational wave propagation; their effects, their observability and the complicated nature of gravitational radiation. Many scientists (that can be found in references) have worked on ways to approach the non-linear limit of this theory through methods such as post-Newtonian (PN) expansion and post-Monkowskian expansion. From these methods, one can find that for weakly self-gravitating object, we can approximate the production of waves up to corrections of $\mathcal{O}(v/c)^n$. These corrections also predicts observable gravitational corrections in the far zone called "memory effect". This effect corresponds to a permanent change in the metric after the wave has long passed the observer. In this thesis, I will go through the process of deriving these intriguing predictions which are soon to be put to the test by free-fall interferometers (e.g. LISA and DECIGO) or matter-wave interferometers (e.g. MIGA).

Keywords: General Relativity, Gravitational Waves, Post-Newtonian expansion, Gravitational Wave Memory via Blanchet-Damour method

ÖZ

YERÇEKİMSEL DALGALAR VE 2.5 PN YERÇEKİMSEL DALGA HAFIZASI

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Einstein alan denklemleri genel göreceliğin keşfinden beri bazı doğa olaylarını açıklamış (örn. yörünge ötelenmeleri), bazılarını ise öngürmüştür (örn. yer çekimsel lensleme, yerçekimsel zaman genişlemesi ve karadelikler). Bu alan denklemlerini doğrusallaştırarak elde ettiğimiz dalga denklemleri olası yerçekimsel dalgalara işaret ettiği için ilgi çekiyordu; böyle dalgaların etkileri, gözlemlenebilirliği ve yerçekimsel radyasyonun karışık doğası öncelikli olmak üzere. Bu yerçekim teorisinin hakkında birçok bilim insanı yöntemler geliştirmişlerdir (referanslarda da görülebileceği üzere), bunların arasında post-Newtonian (PN) açılımı ve post-Minkowskian açılımı ise bu dalgaların doğrusal olmayan düzeltmelerinin incelenmesini sağlar. Bu metotlar sayesinde, yerçekimsel kuvvetlerin egemen olmadığı kaynakların yarattığı dalgaların üzerinde $\mathcal{O}(v/c)^n$ mertebelerinde düzeltmeler yapılabilir. Bu düzeltmelerin arasında da gözlemlenebilir "hafıza etkisi" bulunmaktadır. Bu etki yerçekimsel dalgaların geçişinden sonra uzay-zaman üzerinde kalan kalıcı değişimdir. Bu tezde amacım, yakında serbest-düşüş interferometreler (LISA ve DECIGO gibi) ve madde-dalga interferometreleri (MIGA gibi) tarafından gerçekliği teyit edilecek bu dalga öngörlerinin nasıl çıkartıldığının üzerinden gitmek.

Anahtar Kelimeler: Genel Görelilik, Yerçelimsel Dalgalar, Post-Newton açılımı, Blanchet-Damour yöntemi ile yerçekimsel hafıza

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STF	Symmetric Trace Free
PN	Post-Newtonian
TT	Transverse Traceless

CHAPTER 1

INTRODUCTION

In 1916, 3 years after the publication of his papers introducing General Relativity, Einstein had predicted the existence of gravitational waves [1] for the first time. This was achieved through linearization of the Einstein field equations, a method which is now refered to as the leading term expansion in the Post-Minkowskian approximation. 2 years later in 1918, Einstein had calculated the energy loss rate of a source with such gravitational emissions [2] . Even though this approach was questionned by some authors [3] (e.g. Infeld and Plebanski argued that the analysis they made did not include exact wave-like solutions), the existence of mechanics resulting in the decay of orbit in star binaries were in fact eventually observed [4] (one such observation is the Hulse-Taylor binary, which is in accordance with the predicted waves).
The next breakthrough came in 1959 by Fock. He argued in his book [5] that the asymptotic behaviour of the spacetime can be linked to the motion inside the source through the matching between the gravitational field in a region outside the source and another gravitational field including the inside of the source. A method called Bondi-Sachs-Penrose was revised between 1959-1984 by multiple authors (Trautman, Newman, Geroch, Schmidt, Ashtekar, etc. being among ones that are not cited into the name of the method), but was later discredited due to its inconsistency on a global scale along with some physically inappropriate definitions [6].
Among the analytical approaches devised to remedy this problem were post-Newtonian expansion (which had its own set of problems in the form of divergent integrals; see Kerlick (1980)), which was later improved (Persides 1971, Winicour 1983, Futumase and Schutz 1983, Schäfer 1985). The post-Minkowskian methods were also vital to the advancement of the field (Kovacs and Thorne 1977, Damour 1983), but fell short as they were limited in the types of sources they could be worked on.

This was later fixed by revision of this method (Bonnor 1959, Bonnor and Rotenberg 1966, Couch et. al. 1968 Thorne 1977,1980,1983), which now expanded the post-Minkowskian expansion into its multipoles (which is sometimes refered to as Multipolar-post-Minkowskian expansion). Later on, works by Blanchet and Damour (1986, 1987, 1988, 1997) reconsiled these two PN and MPM approaches in order to expand the scope of both approaches, and to ultimately use PN expansion in the radiative field of a source.

In this paper, I am to introduce the reader to some general understanding of the conventions used by introducing to some basic concepts of differential geometry and to tie this knowledge to the General relativistic picture by deriving the Einstein Field equations. The introduction chapter is thus dedicated to such a task.

The second chapter will be aimed at linearizing the Einstein field equations and exploring the gauge freedom's choice that will ultimately lead us to some wave equation. The waves described by these wave equations will then be converted to some very special gauge, in which they take a simple form that allows our intuitions to understand more of what is actually happening -in terms of space-time- when one of those waves passes through. We will then make general calculations for some source containing low-velocity matter at a linearized level, after which an example will be solved to understand what the implications of these calculations are.

The third chapter is a preparation chapter for the next one. We introduce briefly the solutions of wave equations in terms of multipolar expansions for different types of entities (such as scalars, vectors and two-tensors).

The fourth chapter will focus on the methodologies explained in this introduction; such as the post-Newtonian approach (velocity expansion of a source), and how it relates to the Einstein Field Equations. We will then recast the Einstein field equations by defining a new variable, one that will be more suitable for the introduction of the post-Minkowskian expansion (G parameter expansion of space-time). I will then use the post-Newtonian expansion to this new variable, whose solutions will be put into multipolar expansion. After a brief analysis of the structures of the Multipolar-post-Minkowskian solutions, I will show how to match the solutions found in the PN-multipolar expansion to the MPM expansion, which will give us the radiative gravitational fields coming from any source. One of the terms in this radiative fields will be the "non-linear memory effect". The fifth and last chapter will about the conclusions and remarks on our findings inside the previous chapter.

1.1 A brief lesson in Differential Geometry

1.1.1 Vectors and Dual Vectors

Vector spaces in differential geometry are defined at each point of the manifold separatly, as opposed to the cartesian vector spaces we so often see in classical physics. These entities live inside the *tangent space* at p of some manifold M, denoted as T_p (also written T_pM to specify which manifold this point belongs to).
These abstract vectors' existence do not depend on whether or not we have chosen our coordinates, although one usually has to pick a basis in order to make calculations. Given a vector V and a coordinate basis with basis vectors $\hat{e}_\mu = \partial_\mu$, the abstract vector can be expressed as

$$V = V^\mu \hat{e}_\mu. \tag{1.1}$$

Where the summation of repeated indices are implied. We call V^μ the vector components of our abstract vector in the basis $\hat{e}_\mu$.
A generic example of a vector component can be expressed with the help of a parametrized curve $x^\mu(\lambda)$, where the tangent vector's component $V^\mu(\lambda)$ can be expressed as

$$V^\mu = \frac{dx^\mu}{d\lambda}. \tag{1.2}$$

The abstract vector V itself can be expressed with the use of (1.1).
Given a different basis $\hat{e}_{\nu'}$, one can transform this vector components in one basis into the vector components in the other. Most notably in the case of two frames connected via Lorentz transformations, one can express this transformation via the Jacobian matrix $\frac{\partial x^{\nu'}}{\partial x^\mu}$

$$V^\mu(x) \to V^\mu(x'(x)) = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x). \tag{1.3}$$

With the help of this vector space, we define another entity called *dual vector space*. This new space is in fact a new vector space, and is defined with the knowledge of the tangent space at some point T_p. It is denoted as T_p^* and is called *cotangent space* at p. This dual space is the set of all linear maps from the original vector space to the real numbers $\mathbb{R}$.

$$\begin{aligned} \omega :& T_p \to \mathbb{R} \\ & \omega(V) = a \qquad \text{with } V \in T_p \,, \omega \in T_p^* \text{ and } a \in \mathbb{R}. \end{aligned} \tag{1.4}$$

We can define basis vectors $\hat{\theta}^\mu = \mathbf{d}x^\mu$ in some coordinate basis for this new construction, and demand that the following holds

$$\hat{\theta}^\mu(\hat{e}_\nu) = \delta^\mu_{\ \nu}. \tag{1.5}$$

This in turn allows one to write the abstract dual vector (occasionally called one-form) as

$$\omega = \omega_\mu \theta^\mu. \tag{1.6}$$

Upon which the action of a one-form on a vector can be expressed as

$$\begin{aligned} \omega(V) &= \omega_\mu \hat{\theta}^\mu(V^\nu \hat{e}_\nu) \\ &= \omega_\mu V^\nu \hat{\theta}^\mu(\hat{e}_\nu) \\ &= \omega_\mu V^\mu. \end{aligned} \tag{1.7}$$

The last line also demonstrates how the resulting sum is $\in \mathbb{R}$.
The components of this dual vector can be transformed similarly to our regular vector but with the inverse tranformation matrix;

$$\omega_\mu(x) \to \omega_\mu(x'(x)) = \frac{\partial x^\nu}{\partial x'^\mu} \omega_\nu(x). \tag{1.8}$$

1.1.2 Tensors

The generalization of vectors and dual vectors can be summed up as *tensors*. A tensor of rank (k, l) are multilinear maps of the following kind

$$T : \underbrace{T_p^* \times \cdots \times T_p^*}_{k \text{ times}} \times \underbrace{T_p \times \cdots \times T_p}_{l \text{ times}} \to \mathbb{R}. \tag{1.9}$$

This abstract tensor can be expressed in any coordinate with vector basis $\hat{e}_\mu$ and one-form basis $\hat{\theta}^\mu$ as

$$T = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \, \hat{e}_{\mu_1} \otimes \cdots \otimes \hat{e}_{\mu_k} \otimes \hat{\theta}^{\nu_1} \cdots \otimes \hat{\theta}^{\nu_l}. \tag{1.10}$$

The coordinate transformation on the components of this tensor can be implied through the transformation of the individual bases, and is

$$T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}(x) \to T'^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}(x'(x)) = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_k}}{\partial x^{\alpha_k}} \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_l}}{\partial x'^{\nu_l}} T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_l}(x). \tag{1.11}$$

Considering an infinitesimal transformation $x^\mu \to x'^\mu = x^\mu - \xi^\mu(x)$, the Jacobian matrix $\frac{\partial x'^\mu}{\partial x^\alpha}$ and its inverse $\frac{\partial x^\alpha}{\partial x'^\mu}$ corresponding to such a transformation can be found as

$$\begin{aligned} \frac{\partial x'^\mu}{\partial x^\alpha} &= \delta^\mu{}_\alpha - \frac{\partial \xi^\mu}{\partial x^\alpha}, \\ \frac{\partial x^\alpha}{\partial x'^\mu} &= \delta^\alpha{}_\mu + \frac{\partial \xi^\alpha}{\partial^\mu}. \end{aligned} \tag{1.12}$$

When we plug these into our tensor component transformation relation (1.11), we get

$$\begin{aligned} \delta_\xi T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} = \xi^\sigma \partial_\sigma T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} & \\ - T^{\sigma \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \partial_\sigma \xi^{\mu_1} - \cdots - T^{\mu_1 \ldots \sigma}{}_{\nu_1 \ldots \nu_l} \partial_\sigma \xi^{\mu_k} & \\ + T^{\mu_1 \ldots \mu_k}{}_{\sigma \ldots \nu_l} \partial_{\nu_1} \xi^\sigma + \cdots + T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \partial_{\nu_l} x^\sigma & \end{aligned} \tag{1.13}$$

Where we've made use of $\delta_\xi T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} = T'^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}(x) - T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}(x)$. The right hand side of the equation is then defined as the *lie derivative* denoted as

$$\mathcal{L}_\xi T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \tag{1.14}$$

1.1.3 Metric Tensor

The Metric tensor is a two-form describing the distances inside a manifold. The coordinate dependent components of the metric tensor are denoted as $g_{\mu\nu}$ and can be used to calculate the inner product of two vectors

$$< V, W >= g_{\mu\nu} V^\mu W^\nu = V^\mu W_\mu. \tag{1.15}$$

Where upon lowering one of the indices, one can notice the similarity with (1.7). This is also why the inner product can be defined as the operation taking one vector and one dual vector to give a real number.

Considering an infinitesimal vector ds, we can also calculate its length using this inner product

$$< ds, ds >= g_{\mu\nu} dx^\mu dx^\nu. \tag{1.16}$$

1.1.4 Connection and Covariant Derivative

A covariant derivative ∇ is a multilinear mapping defined as

$$\nabla : T(M) \times T(M) \to T(M), \tag{1.17}$$

Which satisfies the Leibniz's rule. The specific definition of this derivative is dependent on a chosen *connection*, whose choice is arbitrary.
The motivation for this new derivative comes from the fact that basis vectors are not shared between points on the manifold. Generally speaking, we can investigate the change of an abstract vector V as

$$\begin{aligned} dV &= d(V^\mu \hat{e}_\mu) \\ &= dV^\mu \hat{e}_\mu + V^\mu d\hat{e}_\mu \\ &= \frac{\partial V^\mu}{\partial x^\alpha} dx^\alpha \hat{e}_\mu + V^\mu \frac{\partial \hat{e}_\mu}{\partial x^\alpha} dx^\alpha. \end{aligned} \tag{1.18}$$

We note that $\frac{\partial \hat{e}_\mu}{\partial x^\alpha}$ can be represented as a linear combination of basis vectors, thus $\frac{\partial \hat{e}_\mu}{\partial x^\alpha} = \Gamma^\sigma{}_{\alpha\mu} \hat{e}_\sigma$, meaning that after relabeling some of the dummy indices, (1.18) becomes

$$\begin{aligned} dV &= \big(\frac{\partial V^\mu}{\partial x^\alpha} + \Gamma^\mu{}_{\alpha\beta} V^\beta\big) dx^\alpha \hat{e}_\mu \\ &= \nabla_\alpha V^\mu dx^\alpha \hat{e}_\mu \end{aligned} \tag{1.19}$$

One can check that the transformations of the connection's coefficients are non-tensorial, meaning that they transform as

$$\Gamma^\mu{}_{\nu\sigma}(x) \to \Gamma^\mu{}_{\nu\sigma}(x'(x)) = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\sigma} \Gamma^\alpha{}_{\beta\gamma}(x) - \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\sigma} \frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta}. \tag{1.20}$$

The last term in this transformation is what makes the connection coefficients non-tensorial. An important remark is that since this last term is symmetric in the $\nu \leftrightarrow \sigma$ indices, the new quantity defined as

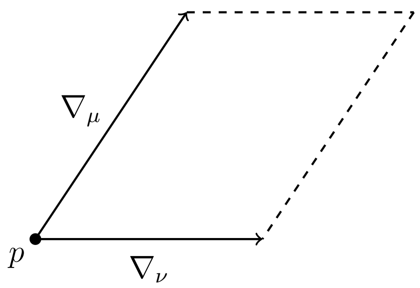
$$T^\mu_{\ \nu\sigma} = \Gamma^\mu{}_{\nu\sigma} - \Gamma^\mu{}_{\sigma\nu}. \tag{1.21}$$

is in fact a tensorial, and is called *torsion tensor*.

1.1.5 Curvature

The curvature tensor (or Riemann tensor) is denoted as $R^{\mu}{}_{\nu\sigma\lambda}$. All of its entries vanish when the metric is flat. This tensor manifests itself in the calculation of a multitude of quantities, such as when we consider the parallel transport of a vector V living in T_p along some loop defined as in the figure below

Figure 1.1



Where the commutator of the covariant derivatives acting on this vector V gauges how much the vector differs when we parallel transport it one way along the loop, compared to when we transport it along the other way.

$$[\nabla_\mu, \nabla_\nu]V^\sigma = R^{\sigma}{}_{\lambda\mu\nu}V^\lambda - T^{\lambda}_{\mu\nu}\nabla_\lambda V^\sigma, \tag{1.22}$$

Where one can calculate the Riemann tensor to be

$$R^{\sigma}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\sigma}{}_{\nu\lambda} - \partial_\nu \Gamma^{\sigma}{}_{\mu\lambda} + \Gamma^{\sigma}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\lambda} - \Gamma^{\sigma}{}_{\nu\alpha}\Gamma^{\alpha}{}_{\mu\lambda}. \tag{1.23}$$

Where not all of its symmetries are apparent in this mixed form, but once written in its fully covariant form, one can Another calculation where the Riemann tensor arises is when we calculate geodesic deviations of test masses in curved spaces which will be studied in detail in chapter 2.4.

1.2 General Relativity

In this chapter, we shall derive the General relativistic framework starting from a Lagrangian. We shall make use of the Levi-Civita connection (No torsion and metric compatible connection). The specific value of this connection can be derived by a method called the Palattini approach in Appendix C).

We want to first derive the linearized theory in order to see for ourselves how the existence of gravitational waves is predicted by this theory and what happens at a physical level when one of them passes through a spacetime with test charges on it.

To this end, let us start with the action to our gravitational theory

$$S = \int d^n x \, \mathcal{L}(g_{\alpha\beta}, \nabla_\mu g_{\alpha\beta}, x, \partial x, \tau). \tag{1.24}$$

Where $\mathcal{L}$ can be decomposed as

$$\mathcal{L} = c_H \mathcal{L}_H(g_{\alpha\beta}, \nabla_\mu g_{\alpha\beta}) + c_M \mathcal{L}_M. \tag{1.25}$$

Where c_H and c_M are some proportionality constants to our geometric ($\mathcal{L}_H$) and matter ($\mathcal{L}_M$) contribution to the lagrangian. We take $\mathcal{L}_H = \sqrt{-g}R$, where $g = \det(g_{\mu\nu})$ and R is the Ricci Scalar.

To find the dynamical equations, let us vary the action S with respect to the dynamical field $g_{\mu\nu}$.

$$\begin{aligned} g_{\mu\nu} \to g'_{\mu\nu} &= g_{\mu\nu} + \delta g_{\mu\nu} \\ S \to S' &= S + \delta S. \end{aligned} \tag{1.26}$$

Where

$$\begin{aligned} \delta S &= \int d^n x \, \delta\mathcal{L}(g_{\alpha\beta}, \nabla_\mu g_{\alpha\beta}) \\ &= c_H \int d^n x \, \delta\mathcal{L}_H + c_M \int d^n x \, \delta\mathcal{L}_M \\ &= c_H \int d^n x \, \delta(\sqrt{-g})R + c_H \int d^n x \, \sqrt{-g} R^{\mu\nu} \delta g_{\mu\nu} \\ &\qquad + c_H \int d^n x \, \sqrt{-g} g_{\mu\nu} \delta R^{\mu\nu} + c_M \int d^n x \, \delta\mathcal{L}_M. \end{aligned} \tag{1.27}$$

We need to develop both the varied quantities, to this end, starting with the identity valid for a matrix $A = \log B$

$$\begin{aligned} e^{\mathrm{Tr}[A]} &= \det[e^A], \\ e^{\mathrm{Tr}[\log B]} &= \det[B], \\ \mathrm{Tr}[\log B] &= \log\big(\det[B]\big), \\ \mathrm{Tr}[B^{-1}\delta B] &= \big(\det[B]\big)^{-1} \delta \det[B], \\ \frac{\delta g}{g} &= g_{\mu\nu} \delta g^{\mu\nu}, \end{aligned} \tag{1.28}$$

Where we have replaced $B = g_{\mu\nu}$ and $B^{-1} = g^{\mu\nu}$ in the last line. This allows the calculation of the relation

$$
\begin{aligned}
\delta(\sqrt{-g}) &= -\frac{\delta g}{2\sqrt{-g}} \\
&= \frac{g g^{\mu\nu}\delta g_{\mu\nu}}{2\sqrt{-g}} \\
&= -\frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}.
\end{aligned} \tag{1.29}
$$

The second varied quantity in our action variation is the $\delta R^{\mu\nu}$ term. We start with the connection

$$
\begin{aligned}
{\Gamma^{\alpha}}_{\beta\gamma} &= \frac{1}{2}g^{\alpha\lambda}(\partial_\beta g_{\lambda\gamma} + \partial_\gamma g_{\lambda\beta} - \partial_\lambda g_{\beta\gamma}) \\
\delta{\Gamma^{\alpha}}_{\beta\gamma} &= \frac{1}{2}\delta g^{\alpha\lambda}(\partial_\beta g_{\lambda\gamma} + \partial_\gamma g_{\lambda\beta} - \partial_\lambda g_{\beta\gamma}), \\
&\quad + \frac{1}{2}g^{\alpha\lambda}(\partial_\beta \delta g_{\lambda\gamma} + \partial_\gamma \delta g_{\lambda\beta} - \partial_\lambda \delta g_{\beta\gamma}), \\
\delta{\Gamma^{\alpha}}_{\beta\gamma} &= -\frac{1}{2}\Big(g_{\lambda\beta}\nabla_\gamma(\delta g^{\lambda\alpha}) + g_{\lambda\gamma}\nabla_\beta(\delta g^{\lambda\alpha}) - g_{\beta\lambda}g_{\gamma\rho}\nabla^\alpha(\delta g^{\lambda\rho})\Big).
\end{aligned} \tag{1.30}
$$

Which can be used, in turn, to find

$$
\begin{aligned}
{R^{\mu}}_{\nu\rho\sigma} &= \partial_\rho {\Gamma^{\mu}}_{\nu\sigma} - \partial_\sigma {\Gamma^{\mu}}_{\nu\rho} + {\Gamma^{\mu}}_{\rho\lambda}{\Gamma^{\lambda}}_{\nu\sigma} - {\Gamma^{\mu}}_{\sigma\lambda}{\Gamma^{\lambda}}_{\nu\rho}, \\
\delta{R^{\mu}}_{\nu\rho\sigma} &= \partial_\rho \delta{\Gamma^{\mu}}_{\nu\sigma} - \partial_\sigma \delta{\Gamma^{\mu}}_{\nu\rho} + \delta{\Gamma^{\mu}}_{\rho\lambda}{\Gamma^{\lambda}}_{\nu\sigma} + {\Gamma^{\mu}}_{\rho\lambda}\delta{\Gamma^{\lambda}}_{\nu\sigma} - \delta{\Gamma^{\mu}}_{\sigma\lambda}{\Gamma^{\lambda}}_{\nu\rho} - {\Gamma^{\mu}}_{\sigma\lambda}\delta{\Gamma^{\lambda}}_{\nu\rho}, \\
\delta{R^{\mu}}_{\nu\rho\sigma} &= \nabla_\rho \delta{\Gamma^{\mu}}_{\nu\sigma} + \cancelto{i}{{\Gamma^{\mu}}_{\lambda\rho}\delta{\Gamma^{\lambda}}_{\nu\sigma}} - \cancelto{ii}{{\Gamma^{\lambda}}_{\rho\nu}\delta{\Gamma^{\mu}}_{\lambda\sigma}} - \cancelto{iii}{{\Gamma^{\lambda}}_{\rho\sigma}\delta{\Gamma^{\mu}}_{\nu\lambda}} \\
&\qquad - \nabla_\sigma \delta{\Gamma^{\mu}}_{\nu\rho} - \cancelto{ii}{{\Gamma^{\mu}}_{\lambda\sigma}\delta{\Gamma^{\lambda}}_{\nu\rho}} + \cancelto{i}{{\Gamma^{\lambda}}_{\sigma\nu}\delta{\Gamma^{\mu}}_{\lambda\rho}} + \cancelto{iii}{{\Gamma^{\lambda}}_{\sigma\rho}\delta{\Gamma^{\mu}}_{\nu\lambda}} \\
&\qquad + \cancelto{iv}{\delta{\Gamma^{\mu}}_{\rho\lambda}{\Gamma^{\lambda}}_{\nu\sigma}} - \cancelto{iv}{{\Gamma^{\mu}}_{\rho\lambda}\delta{\Gamma^{\lambda}}_{\nu\sigma}} - \cancelto{v}{\delta{\Gamma^{\mu}}_{\sigma\lambda}{\Gamma^{\lambda}}_{\nu\rho}} + \cancelto{v}{{\Gamma^{\mu}}_{\sigma\lambda}\delta{\Gamma^{\lambda}}_{\nu\rho}} \\
&= \nabla_\rho \delta{\Gamma^{\mu}}_{\nu\sigma} - \nabla_\sigma \delta{\Gamma^{\mu}}_{\nu\rho}, \\
g^{\nu\sigma}\delta R_{\nu\sigma} &= g^{\nu\sigma}\delta{R^{\mu}}_{\nu\mu\sigma} \\
&= \nabla_\lambda\Big(g_{\mu\nu}\nabla^\lambda(\delta g^{\mu\nu}) - \nabla_\sigma(\delta g^{\lambda\sigma})\Big).
\end{aligned} \tag{1.31}
$$

What we should notice is that this is a full derivative term, which corresponds to a boundary term when integrated. The action variation (1.27) thus becomes Thus, from Hamilton's principle $\delta S = 0$

$$
\delta S = c_H \int d^n x \sqrt{-g}\Big(-\frac{1}{2}g^{\mu\nu}R + R^{\mu\nu}\Big)\delta g_{\mu\nu} + c_M \int d^n x \left(\frac{\partial \mathcal{L}_M}{\partial g_{\mu\nu}} - \nabla_\sigma \frac{\partial \mathcal{L}_M}{\partial(\partial_\sigma g_{\mu\nu})}\right)\delta g_{\mu\nu}. \tag{1.32}
$$

From which, if we define the matter Stress-Energy tensor

$$\frac{\partial \mathcal{L}_M}{\partial g_{\mu\nu}} - \nabla_\sigma \frac{\partial \mathcal{L}_M}{\partial(\partial_\sigma g_{\mu\nu})} \propto \sqrt{-g}T^{\mu\nu}. \tag{1.33}$$

We end up with

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \kappa T^{\mu\nu}, \tag{1.34}$$

Or contracting the indices

$$R = -\kappa T, \tag{1.35}$$

Which implies another form of the equation

$$R^{\mu\nu} = \kappa\big(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\big). \tag{1.36}$$

Now in order to find the proportionality κ, we need to check its validity for the newtonian limit.
Knowning that in the newtonian limit, the $g_{\mu\nu}$ field is weak and is asymtotically flat. We might thus expand the field $g_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{1.37}$$

Using the property $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu$, up to first order in $h_{\mu\nu}$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2). \tag{1.38}$$

We also assume the gravitating body is made of space dust, which is just uninteracting particles but particles in the local bunches moving at the same speed. This description fits into the stress energy tensor as follows

$$T_{\mu\nu} = \rho U_\mu U_\nu. \tag{1.39}$$

Knowning that the four-velocities are normalized as

$$g_{\mu\nu}U^\mu U^\nu = -1, \tag{1.40}$$

Which gives us the contraction for $T_{\mu\nu}$

$$\begin{aligned} g^{\mu\nu}T_{\mu\nu} &= \rho\, \overbrace{g^{\mu\nu}U^\mu U^\nu}^{-1} \\ &= -\rho. \end{aligned} \tag{1.41}$$

And the four velocity of the small local bunches

$$
\begin{aligned}
\text{(At rest frame)} \rightarrow \; U^\mu &= (U^0, 0, 0, 0), \\
(-1 + h_{00})\big(U^0\big)^2 &= -1, \\
U^0 &= (1 - h_{00})^{-1/2}, \\
U^0 &= 1 + \frac{1}{2} h_{00} + \mathcal{O}(h^2).
\end{aligned} \tag{1.42}
$$

Our next trick comes from the fact that we are talking about weak fields. This implies ρ is relatively small compared to other quantities, say of the order of $h_{\mu\nu}$. While looking for ways to express T_{00} in terms of ρ, we can disregard the $\mathcal{O}(h)$ terms.

$$
\begin{aligned}
T_{00} &= \rho\big(g_{00}\big)^2\big(U^0\big)^2 \\
&= \rho.
\end{aligned} \tag{1.43}
$$

Now, let us find the Ricci tensor's R_{00}[th] component

$$
\begin{aligned}
R_{00} &= R^\lambda_{\ 0\lambda 0} \\
&= \partial_\lambda \Gamma^\lambda_{00} - \cancel{\partial_0 \Gamma^\lambda_{0\lambda}} + \mathcal{O}(h^2) \\
&= \frac{1}{2} \partial_\lambda g^{\lambda\sigma} (\cancelto{0}{\partial_0 g_{\sigma 0}} + \cancelto{0}{\partial_0 g_{\sigma 0}} - \partial_\sigma g_{00}).
\end{aligned} \tag{1.44}
$$

Noting that in the λ contraction, the $\lambda = 0$ case gives us the $R^0_{\ 000}$ term, which is known to be 0 in the static case. So we might skip this null term, and reduce the summation to a spatial one ($\lambda \rightarrow i$)

$$
\begin{aligned}
R_{00} &= -\frac{1}{2} \partial_i (g^{i\sigma} \partial_\sigma h_{00}) \\
&= -\frac{1}{2} \partial_i (\delta^{ij} \partial_j h_{00}) \\
&= -\frac{1}{2} \nabla^2 h_{00},
\end{aligned} \tag{1.45}
$$

Which finally, coupled with $T = -\rho$ and $T_{00} = \rho$, gives us a familiar equation.

$$
\begin{aligned}
R_{00} &= \kappa\big(T_{00} - \frac{1}{2} g_{00} T\big) \\
\nabla^2 h_{00} &= -\kappa \rho.
\end{aligned} \tag{1.46}
$$

We now need to express h_{00} in terms of newtonian potential ϕ.

For this, consider the geodesic equation

$$
\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\ \nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \tag{1.47}
$$

In the newtonian limit

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}, \tag{1.48}$$

So the 4 dynamical equations become

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\Big(\frac{dt}{d\tau}\Big)^2 = 0. \tag{1.49}$$

In the static case that we're looking at $\partial_0 g_{\mu\nu} = 0$, where the Christoffel symbols Γ^μ_{00} can simply be written as

$$\begin{aligned} \Gamma^\mu_{00} &= \frac{1}{2}g^{\mu\lambda}(\cancelto{0}{\partial_0 g_{\lambda 0}} + \cancelto{0}{\partial_0 g_{\lambda 0}} - \partial_\lambda g_{00}) \\ &= -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00}, \end{aligned} \tag{1.50}$$

With which, evaluating Γ^μ_{00}

$$\begin{aligned} \Gamma^\mu_{00} &= -\frac{1}{2}(\eta^{\mu\lambda} - h^{\mu\lambda})\partial_\lambda(g_{00} + h_{00}) \\ &= -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00} + \mathcal{O}(h^2). \end{aligned} \tag{1.51}$$

Noting the $\mu = 0$ case will simply give us

$$\begin{aligned} \frac{d^2x^0}{d\tau^2} &= \frac{1}{2}\eta^{0\lambda}\partial_\lambda h_{00}\Big(\frac{dt}{d\tau}\Big)^2 \\ &= \frac{1}{2}\eta^{00}\cancel{\partial_0 h_{00}}\Big(\frac{dt}{d\tau}\Big)^2. \end{aligned} \tag{1.52}$$

Where we have used the fact that the metric is static ($\partial_0 h_{\mu\nu} = 0$).

The dynamical equations thus becomes

$$\begin{aligned} \frac{d^2x^i}{d\tau^2} &= \frac{1}{2}\eta^{i\lambda}\partial_\lambda h_{00}\Big(\frac{dt}{d\tau}\Big)^2 \\ \frac{d^2x^i}{dt^2} &= \frac{1}{2}\partial_i h_{00}. \end{aligned} \tag{1.53}$$

We recognize that this would be equivalent to the Newtonian mechanics dynamical equations, since they obey those equations in the *Newtonian* domain, which should read

$$\frac{d^2x^i}{dt^2} = -\partial_i\phi, \tag{1.54}$$

Where ϕ is the gravitational potential, as understood in the classical limit.

The above lets us reconsile this classically defined term, and see its place in the new, relativistic domain. We have discovered that in our static, weak field case

$$h_{00} = -2\phi. \tag{1.55}$$

Or as the entire metric

$$g_{00} = -(1 + 2\phi). \tag{1.56}$$

We can now find the proportionality constant

$$\begin{aligned} -2\nabla^2\phi &= -\kappa\rho \\ \nabla^2\phi &= \frac{\kappa}{2}\rho. \end{aligned} \tag{1.57}$$

This here, is the kinematic description of the field in the weak limit. Recall that in Newtonian mechanics, we have a field equation that goes ($c = 1$)

$$\nabla^2\phi = 4\pi G\rho. \tag{1.58}$$

Which tells us to set $\kappa = 8\pi G$, such that we get our normalized field equations.

$$\boxed{R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G T^{\mu\nu}}, \tag{1.59}$$

$$\boxed{R^{\mu\nu} = 8\pi G\big(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\big)}. \tag{1.60}$$

Or we can rewrite them in covariant form by simply lowering indices

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}}, \tag{1.61}$$

$$\boxed{R_{\mu\nu} = 8\pi G\big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\big)}. \tag{1.62}$$

CHAPTER 2

GRAVITATIONAL WAVE PREDICTIONS AND LINEARIZED THEORY ANALYSIS

2.1 Linearization of Einstein Equations on flat background

The linearization of the Einstein equations upon a flat background metric can be expressed by decomposing a "weak" metric into its Minkowski background $\eta_{\mu\nu}$ and a small perturbation around it $h_{\mu\nu}$. The weakness of the field simply implies the pertubation to be small.

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \\ g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2). \end{aligned} \tag{2.1}$$

We investigate what our field equations would look like up to first order in this limit. For this part, we adopt the notation that a quantity written as $[A_{\mu\nu}]^{(n)}$ simply denotes the part of the tensor $A_{\mu\nu}$ which includes the terms of order $O(h^n)$. We thus begin to decompose the Einstein equations up to their linear

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= 8\pi G T_{\mu\nu}, \\ [R_{\mu\nu}]^{(0)} + [R_{\mu\nu}]^{(1)} - \frac{1}{2}[g_{\mu\nu}R]^{(0)} - \frac{1}{2}[g_{\mu\nu}R]^{(1)} + \mathcal{O}(h^2) &= 8\pi G T_{\mu\nu}, \\ \cancelto{0}{[R_{\mu\nu}]^{(0)}} + [R_{\mu\nu}]^{(1)} - \frac{1}{2}[g_{\mu\nu}]^{(0)}\cancelto{0}{[R]^{(0)}} - \frac{1}{2}[g_{\mu\nu}]^{(1)}\cancelto{0}{[R]^{(0)}} & \\ - \frac{1}{2}[g_{\mu\nu}]^{(0)}[R]^{(1)} + \mathcal{O}(h^2) &= 8\pi G T_{\mu\nu}, \\ [R_{\mu\nu}]^{(1)} - \frac{1}{2}[g_{\mu\nu}]^{(0)}[R]^{(1)} + \mathcal{O}(h^2) &= 8\pi G T_{\mu\nu}, \\ [R_{\mu\nu}]^{(1)} - \frac{1}{2}\eta_{\mu\nu}[R]^{(1)} + \mathcal{O}(h^2) &= 8\pi G T_{\mu\nu}. \end{aligned} \tag{2.2}$$

Where the flat space-time Riemann tensor of the background is simplified. Now, to attain $[R_{\mu\nu}]^{(1)}$ and $[R]^{(1)}$, one must first find $[R^{\alpha}{}_{\beta\mu\nu}]^{(1)}$, then contract indices accordingly.

To achieve this, we first expand our connections up to first order as

$$\begin{aligned}\Gamma^{\alpha}_{\ \mu\nu} &= \frac{1}{2}g^{\alpha\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \\ &= \frac{1}{2}(\eta^{\alpha\lambda} - h^{\alpha\lambda})(\cancelto{0}{\partial_\mu \eta_{\lambda\nu}} + \cancelto{0}{\partial_\nu \eta_{\lambda\mu}} - \cancelto{0}{\partial_\lambda \eta_{\mu\nu}}) \\ &\qquad + \frac{1}{2}\eta^{\alpha\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) + \mathcal{O}(h^2) \\ &= \frac{1}{2}\eta^{\alpha\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) + \mathcal{O}(h^2).\end{aligned} \tag{2.3}$$

Which is simply composed of $\mathcal{O}(h)$ and higher order terms, meaning $[\Gamma^{\alpha}_{\ \beta\gamma}]^0 = 0$. Thus our Riemann Tensor's $\mathcal{O}(h)$ terms becomes

$$\begin{aligned}[R^{\alpha}_{\ \beta\mu\nu}]^{(1)} &= [\partial_\mu \Gamma^{\alpha}_{\ \beta\nu}]^{(1)} - [\partial_\nu \Gamma^{\alpha}_{\ \beta\mu}]^{(1)} + [\Gamma^{\alpha}_{\ \lambda\mu}\Gamma^{\lambda}_{\ \beta\nu}]^{(1)} - [\Gamma^{\alpha}_{\ \lambda\nu}\Gamma^{\lambda}_{\ \beta\mu}]^{(1)} \\ &= \partial_\mu[\Gamma^{\alpha}_{\ \beta\nu}]^{(1)} - \partial_\nu[\Gamma^{\alpha}_{\ \beta\mu}]^{(1)} + \cancelto{0}{[\Gamma^{\alpha}_{\ \lambda\mu}]^{(0)}}[\Gamma^{\lambda}_{\ \beta\nu}]^{(1)} + [\Gamma^{\alpha}_{\ \lambda\mu}]^{(1)}\cancelto{0}{[\Gamma^{\lambda}_{\ \beta\nu}]^{(0)}} \\ &\qquad - \cancelto{0}{[\Gamma^{\alpha}_{\ \lambda\nu}]^{(0)}}[\Gamma^{\lambda}_{\ \beta\mu}]^{(1)} - [\Gamma^{\alpha}_{\ \lambda\nu}]^{(1)}\cancelto{0}{[\Gamma^{\lambda}_{\ \beta\mu}]^{(0)}} \\ &= \frac{1}{2}\eta^{\alpha\lambda}\Big(\partial_\mu(\partial_\beta h_{\lambda\nu} + \partial_\nu h_{\lambda\beta} - \partial_\lambda h_{\beta\nu}) - \partial_\nu(\partial_\beta h_{\lambda\mu} + \partial_\mu h_{\lambda\beta} - \partial_\lambda h_{\beta\mu})\Big) \\ &= \frac{1}{2}\Big(\partial_\mu\partial_\beta h^{\alpha}_{\ \nu} - \partial_\mu\partial^\alpha h_{\beta\nu} - \partial_\nu\partial_\beta h^{\alpha}_{\ \mu} + \partial_\nu\partial^\alpha h_{\beta\mu}\Big).\end{aligned} \tag{2.4}$$

Where we have used the fact that $\partial_\alpha \eta_{\mu\nu} = 0$ and lowering/raising indices with the flat background metric introduces corrections to higher order terms.

So finally, contracting to get $[R_{\mu\nu}]^{(1)}$

$$\begin{aligned}[R_{\mu\nu}]^{(1)} &= [R^{\alpha}_{\ \mu\alpha\nu}]^{(1)} \\ &= \frac{1}{2}\Big(\partial^\alpha\partial_\mu h_{\alpha\nu} + \partial^\alpha\partial_\nu h_{\mu\alpha} - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h\Big),\end{aligned} \tag{2.5}$$

While $[R]^{(1)}$ becomes

$$\begin{aligned}[R]^{(1)} &= [R_{\alpha\beta}g^{\alpha\beta}]^{(1)} \\ &= [R_{\alpha\beta}]^{(1)}[g^{\alpha\beta}]^{(0)} + \cancelto{0}{[R_{\alpha\beta}]^{(0)}}[g^{\alpha\beta}]^{(1)} \\ &= \frac{1}{2}\Big(\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^\alpha\partial_\alpha h^{\beta}_{\ \beta} - \partial^\beta\partial_\beta h^{\alpha}_{\ \alpha} + \partial^\beta\partial^\alpha h_{\alpha\beta}\Big) \\ &= \partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^2 h,\end{aligned} \tag{2.6}$$

And thus, plugging these back into (2.2) to get

$$
\begin{aligned}
[R_{\mu\nu}]^{(1)} - \frac{1}{2}\eta_{\mu\nu}[R]^{(1)} + \mathcal{O}(h^2) &= 8\pi G T_{\mu\nu} \\
\eta_{\mu\nu}\partial^2 h + \partial^\alpha\partial_\mu h_{\alpha\nu} + \partial^\alpha\partial_\nu h_{\mu\alpha} - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} &= 16\pi G T_{\mu\nu}.
\end{aligned}
\tag{2.7}
$$

And once we substitute $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \Rightarrow h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overline{h}$ and $h = -\overline{h}$ for convenience, we get

$$
\begin{aligned}
-\eta_{\mu\nu}\partial^2\overline{h} + \partial^\alpha\partial_\mu(\overline{h}_{\alpha\nu} - \frac{1}{2}\eta_{\alpha\nu}\overline{h}) \\
+\partial^\alpha\partial_\nu(\overline{h}_{\mu\alpha} - \frac{1}{2}\eta_{\mu\alpha}\overline{h}) - \partial^2(\overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overline{h}) \\
+\partial_\mu\partial_\nu\overline{h} - \eta_{\mu\nu}\partial^\alpha\partial^\beta(\overline{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\overline{h}) = 16\pi G T_{\mu\nu}, \\
\cancel{-\eta_{\mu\nu}\partial^2\overline{h}}^{\,i} + \partial^\alpha\partial_\mu\overline{h}_{\alpha\nu} - \cancel{\frac{1}{2}\partial_\mu\partial_\nu\overline{h}}^{\,ii} + \partial^\alpha\partial_\nu\overline{h}_{\mu\alpha} \\
-\cancel{\frac{1}{2}\partial_\mu\partial_\nu\overline{h}}^{\,ii} - \partial^2\overline{h}_{\mu\nu} + \cancel{\frac{1}{2}\eta_{\mu\nu}\partial^2\overline{h}}^{\,i} + \cancel{\partial_\mu\partial_\nu\overline{h}}^{\,ii} \\
-\eta_{\mu\nu}\partial^\alpha\partial^\beta\overline{h}_{\alpha\beta} + \cancel{\frac{1}{2}\eta_{\mu\nu}\partial^2\overline{h}}^{\,(a)} = 16\pi G T_{\mu\nu}.
\end{aligned}
\tag{2.8}
$$

Which leads to

$$
\boxed{\partial^\alpha\partial_\mu\overline{h}_{\alpha\nu} + \partial^\alpha\partial_\nu\overline{h}_{\mu\alpha} - \partial^2\overline{h}_{\mu\nu} - \eta_{\mu\nu}\partial^\alpha\partial^\beta\overline{h}_{\alpha\beta} = 16\pi G T_{\mu\nu}}
\tag{2.9}
$$

Which is the linearized Einstein equations in arbitrary gauge.

2.2 Gauge freedom

Let us consider a slowly varying coordinate transformation such that

$$
x^\mu \to x'^\mu = x^\mu + \xi^\mu(x). \tag{2.10}
$$

Where $|\partial_\alpha\xi^\beta| \leq \mathcal{O}(h)$.

Knowning the metric two form's components transforms as follows

$$
g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\lambda\sigma}(x). \tag{2.11}
$$

We inquire as to how the perturbation on the flat background metric $h_{\mu\nu}(x)$ varies in this case.

$$
\begin{aligned}
g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \eta_{\mu\nu} + h'_{\mu\nu}(x') &= \frac{\partial x^{\lambda}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\lambda\sigma}(x) \\
\eta_{\mu\nu} + h'_{\mu\nu}(x') &= (\delta^{\lambda}_{\ \mu} - \partial_{\mu}\xi^{\lambda})(\delta^{\sigma}_{\ \nu} - \partial_{\nu}\xi^{\sigma})(\eta_{\lambda\sigma} + h_{\lambda\sigma}(x)) \\
\eta_{\mu\nu} + h'_{\mu\nu}(x') &= \eta_{\mu\nu} + h_{\mu\nu}(x) - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu},
\end{aligned}
\tag{2.12}
$$

Which yields the variation for $h_{\mu\nu}$

$$
h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}. \tag{2.13}
$$

Now let us remark something about the variation of the linearized Riemann Tensor under the above diffeomorphism.

$$
\begin{aligned}
R_{\alpha\beta\mu\nu}(x) \to R'_{\alpha\beta\mu\nu}(x') &= \frac{1}{2}\Big(\partial_{\mu}\partial_{\beta}h'_{\alpha\nu}(x') - \partial_{\mu}\partial_{\alpha}h'_{\beta\nu}(x') \\
&\qquad - \partial_{\nu}\partial_{\beta}h'_{\alpha\mu}(x') + \partial_{\nu}\partial_{\alpha}h'_{\beta\mu}(x')\Big) \\
R'_{\alpha\beta\mu\nu}(x') &= R_{\alpha\beta\mu\nu}(x) + \frac{1}{2}\Big(\cancelto{i}{-\partial_{\mu}\partial_{\beta}\partial_{\alpha}\xi_{\nu}}\cancelto{ii}{-\partial_{\mu}\partial_{\beta}\partial_{\nu}\xi_{\alpha}}\cancelto{i}{+\partial_{\mu}\partial_{\alpha}\partial_{\beta}\xi_{\nu}} \\
&\quad \cancelto{iii}{+\partial_{\mu}\partial_{\alpha}\partial_{\nu}\xi_{\beta}}\cancelto{iv}{+\partial_{\nu}\partial_{\beta}\partial_{\alpha}\xi_{\mu}}\cancelto{ii}{+\partial_{\nu}\partial_{\beta}\partial_{\mu}\xi_{\alpha}}\cancelto{iv}{-\partial_{\nu}\partial_{\alpha}\partial_{\beta}\xi_{\mu}}\cancelto{iii}{-\partial_{\nu}\partial_{\alpha}\partial_{\mu}\xi_{\beta}}\Big) \\
R'_{\alpha\beta\mu\nu}(x') &= R_{\alpha\beta\mu\nu}(x).
\end{aligned}
\tag{2.14}
$$

The Riemann Tensor's contractions are essentially the dynamical part of our Field equations (C.16), and since it is invariant under these transformations, we have a freedom to choose whichever is more convenient among these gauges.

Now to figure out how we can manipulate the linearized equation (2.9), we first look into how $\overline{h}_{\mu\nu}$ varies

$$
\begin{aligned}
\overline{h}_{\mu\nu}(x) \to \overline{h}'_{\mu\nu}(x') &= h'_{\mu\nu}(x') - \frac{1}{2}\eta_{\mu\nu}\, h'(x') \\
&= h_{\mu\nu}(x) - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} - \frac{1}{2}\eta_{\mu\nu}\big(h(x) - 2\partial_{\alpha}\xi^{\alpha}\big) \\
&= \overline{h}_{\mu\nu}(x) - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} + \eta_{\mu\nu}\partial_{\alpha}\xi^{\alpha} \\
&= \overline{h}_{\mu\nu}(x) - \xi_{\mu\nu}.
\end{aligned}
\tag{2.15}
$$

Or the term that appears more often in the Field equations :

$$
\begin{aligned}
\partial^{\mu}\overline{h}_{\mu\nu}(x) \to \partial^{\mu'}\overline{h}'_{\mu\nu}(x') &= \partial^{\mu'}x_{\beta}\, \partial^{\beta}\big(\overline{h}_{\mu\nu}(x) - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} + \eta_{\mu\nu}\partial_{\alpha}\xi^{\alpha}\big) \\
&= \partial^{\mu}\overline{h}_{\mu\nu} - \Box\xi_{\nu} + \mathcal{O}(h^{2}).
\end{aligned}
\tag{2.16}
$$

This implies that no matter what $\partial^{\mu}\overline{h}_{\mu\nu} = f_{\mu}(x)$ is in our gauge, we can swap to a Lorentz gauge where $\partial^{\mu'}\overline{h}'_{\mu\nu} = 0$ by simply setting $\Box\xi_{\nu} = f_{\mu}(x)$. This simplifies the field equation to

$$\Box\overline{h}^{\mathrm{Tr}}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{2.17}$$

Picking this gauge, we have set 4 restrictions onto our initial 10 degrees of freedom, reducing them to 6.

One should note that generally, we cannot set a specific component of $h_{\mu\nu}$ to 0 over the manifold in a certain coordinate system. The proof is simple, suppose we want to fix a specific component of $\overline{h}_{\mu\nu}$, say $\overline{h}_{ab}$ where a and b are specific values among $\{0, 1, 2, 3\}$ assume we can find a ξ_{μ} for which the function ξ_{ab} is exactly equal to that function h_{ab} everywhere on the manifold. Then this function should satisfy $\Box\xi_{ab} = -16\pi G T_{ab}$. But this would break our Lorentz gauge, which we do not want to break if we want the linearized field equations to have this simple *sourced wave equation* form. Thus, unless $T_{\mu\nu} = 0$, we cannot fix any further component of $\overline{h}_{\mu\nu}$.

2.3 Vacuum solutions to the Linearized Einstein Equations in the Transverse Traceless gauge

For the special case of vacuum ($T_{\mu\nu} = 0$), the gravitational wave's propagation can be regarded as easy. We could pick a further gauge transformations, without breaking the Lorentz gauge, such that $\overline{h}$ (the trace of our perturbation field) and $\overline{h}^{0i}$ vanishes. Seeing how the trace is varied,

$$\overline{h}(x) \to \overline{h}'(x') = \overline{h}(x) + 2\partial_{\mu}\xi^{\mu}. \tag{2.18}$$

we could pick ξ^{μ} such that $\partial_{\mu}\xi^{\mu} = -\frac{1}{2}\overline{h}$ and $\partial^{0}\xi^{i} + \partial^{i}\xi^{0} = \overline{h}^{0i}$ are satisfied. A quick look at equations (2.15) and (2.18) tells us that these conditions will make sure those components vanish.

The Lorentz condition in this gauge reads

$$\begin{aligned} \partial^{0}\overline{h}_{00} + \partial^{i}\cancelto{0}{\overline{h}_{i0}} &= 0 \\ \partial^{0}\overline{h}_{00} &= 0. \end{aligned} \tag{2.19}$$

We can interpret the independence of the $\overline{h}_{00}$ from time as this being the gravitational effect's static part. The gravitational waves are expected to be time dependent phenomena, so as long as we work in flat spacetime - far from any $T_{\mu\nu}$ components -, this component can be set to 0.
This introduces 4 new restrictions on the remaining 6 independent components of $\overline{h}_{\mu\nu}$.
The gauge studied above, *the Transverse Traceless gauge (TT gauge)*, is the choice of $h_{\mu\nu}$ that sets

$$\partial^j \overline{h}_{ij} = 0, \qquad \overline{h}^{0\mu} = 0, \qquad \overline{h} = 0. \tag{2.20}$$

The reader might also note that in this gauge, since $\overline{h} = 0$, we automatically have $\overline{h}_{\mu\nu} = h_{\mu\nu}$, so our gauge conditions along with the linearized field equation becomes

$$\partial^j h_{ij} = 0, \qquad h^{0\mu} = 0, \qquad h = 0. \tag{2.21}$$

We can also resume our analysis in this gauge by disregarding the $h^{0\mu} = h^{\mu 0} = 0$ components of our metric perturbation, denoting the spatial indices with latin letters. Far away from the source, we are able to put this solution into the *TT gauge* by the usage of a projection tensor $\Lambda_{ij,kl}(\hat{\mathbf{n}})$, dependent on the propagation direction $\hat{\mathbf{n}} = \frac{\mathbf{k}}{|\mathbf{k}|}$ (as a side note, $n^\alpha n_\alpha = -1$).
We conclude that in order for this $\overline{h}_{ij}^{\text{TT}}$ to be a solution to the field equation $\Box \overline{h}_{ij}^{\text{TT}} = 0$, we also require $\Box \overline{h}_{ij} = 0$ to be true, thus the initial metric perturbation should at least be in *Lorentz gauge*. The vacuum equation $\Box \overline{h}_{ij}^{\text{TT}} = 0$ admits the following solutions (which can be chequed through Fourier analysis)

$$h_{ij}^{\text{TT}}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} (\mathcal{A}_{ij}(\vec{k}) e^{ikx} + \mathcal{A}_{ij}^*(\vec{k}) e^{-ikx}). \tag{2.22}$$

Upon subtituting the components of the 4-vector $k^\mu = (\omega, \vec{k})$ where $|\vec{k}| = \omega$ and $\frac{\vec{k}}{|\vec{k}|} = \hat{n}$ (From which we can obtain the volume element to be $d^3\vec{k} = |\vec{k}|^2 d|\vec{k}| d^2\hat{n} = \omega^2 d\omega d^2\hat{n}$) :

$$h_{ij}^{\text{TT}}(t, \vec{x}) = \frac{1}{(2\pi)^3} \int_0^\infty d\omega\, \omega^2 \int d^2\hat{n}\, \left(\mathcal{A}_{ij}(\vec{k}) e^{-i\omega(t - \hat{n}\cdot\vec{x})} + \mathcal{A}_{ij}^*(\vec{k}) e^{i\omega(t - \hat{n}\cdot\vec{x})} \right). \tag{2.23}$$

When we consider the waves propagated from a singular astrophysical source, the direction of propagation of the wave, say $\hat{n}_0$, is a specific direction in the sky, thus we can rewrite

$$\mathcal{A}_{ij}(\vec{k}) = A_{ij}(\omega) \delta^{(2)}(\hat{n} - \hat{n}_0). \tag{2.24}$$

Upon which we can rewrite

$$h_{ij}^{\mathrm{TT}}(t,\vec{x}) = \frac{1}{(2\pi)^3}\int_0^\infty d\omega\,\left(\tilde{h}_{ij}(\omega,\vec{x})e^{-i\omega t} + \tilde{h}_{ij}^*(\omega,\vec{x})e^{i\omega t}\right). \tag{2.25}$$

Where the $\tilde{h}_{ij}(\omega,\vec{x})$ are

$$\begin{aligned}\tilde{h}_{ij}(\omega,\vec{x}) &= \omega^2\int d^2\hat{n}\mathcal{A}_{ij}(\omega,\hat{n})e^{i\omega\hat{n}\cdot\vec{x}}\\ &= \omega^2 A_{ij}(\omega)e^{i\omega\hat{n}_0\cdot\vec{x}}.\end{aligned} \tag{2.26}$$

Another thing one that is facilitated in this TT gauge is the general assumptions one can make about the wave by taking into account the transversality equation $\partial^j h_{ij} = 0$; we can reformulate it with the help of (2.23) as $n^j h_{ij} = 0$. If we take the propagation direction of the wave towards the z direction $\left(n^j = \delta^j{}_3\right)$, we can further deduce the vanishing components $h_{i3} = h_{3i} = 0$. The symmetric property of the metric perturbation, along with its tracelessness in this gauge, we obtain that $h_{11} = -h_{22}$ and $h_{12} = h_{21}$. So finally, it is possible to write the metric perturbation in the TT gauge as

$$h_{ij}^{\mathrm{TT}}(t,\vec{x}) = \begin{pmatrix} h_+(t,\vec{x}) & h_\times(t,\vec{x}) & 0\\ h_\times(t,\vec{x}) & -h_+(t,\vec{x}) & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{2.27}$$

It is also convenient to define new polarization vectors by considering two unit vectors $\hat{u}$ and $\hat{v}$ which are perpendicular to the propagation direction $\hat{n}$ and to eachother. The new polarization tensors e_{ij}^A (Where $A \in +, \times$ labels the two different polarizations) is defined as

$$\begin{aligned} e_{ij}^+(\hat{n}) &= \hat{u}_i\hat{u}_j - \hat{v}_i\hat{v}_j,\\ e_{ij}^\times(\hat{n}) &= \hat{u}_i\hat{v}_j + \hat{v}_i\hat{u}_j,\end{aligned} \tag{2.28}$$

Where one can notice that upon picking $\hat{u} = \hat{x}$ and $\hat{v} = \hat{y}$, with the propagation vector being in the z direction again, one gets

$$\begin{aligned} e_{ij}^+(\hat{z}) &= \begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}.\\ e_{ij}^\times(\hat{z}) &= \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.\end{aligned} \tag{2.29}$$

Which facilitates the expression (2.30) into a more compact form

$$h_{ij}^{\mathrm{TT}}(t,\vec{x}) = h_A(t,\vec{x})e_{ij}^A. \tag{2.30}$$

Where a sum over the counting indice A is implied.

2.4 Gravitational waves passing by test masses

The equation of motion for free particles inside a spacetime endowed with metric $g_{\mu\nu}$ can be found as our geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}(x)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0. \tag{2.31}$$

We consider a neighboring geodesic line $x^\mu(\tau)+\xi^\mu(\tau)$ near our initial geodesic $x^\mu(\tau)$, such that we are mereley considering $0 \neq \xi^\mu(\tau) \ll 1$ -the lines are only differing by a non-zero, small perturbation-. The newly introduced geodesic line obeys its own geodesic equation

$$\begin{aligned} \frac{d^2(x^\mu+\xi^\mu)}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}(x+\xi)\frac{d(x^\alpha+\xi^\alpha)}{d\tau}\frac{d(x^\beta+\xi^\beta)}{d\tau} &= 0, \\ \frac{d^2x^\mu}{d\tau^2} + \frac{d^2\xi^\mu}{d\tau^2} + \left(\Gamma^\mu{}_{\alpha\beta}(x) + \xi^\sigma\partial_\sigma\Gamma^\mu{}_{\alpha\beta}(x)\right)\left(\frac{dx^\alpha}{d\tau}+\frac{d\xi^\alpha}{d\tau}\right)\left(\frac{dx^\beta}{d\tau}+\frac{d\xi^\beta}{d\tau}\right) + \mathcal{O}(\xi^2) &= 0, \\ \underbrace{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}(x)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}_{=0} + \frac{d^2\xi^\mu}{d\tau^2} + \xi^\sigma\partial_\sigma\Gamma^\mu{}_{\alpha\beta}(x)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + 2\Gamma^\mu{}_{\alpha\beta}(x)\frac{dx^\alpha}{d\tau}\frac{d\xi^\beta}{d\tau} &= 0. \end{aligned} \tag{2.32}$$

Where we have also assumed that the derivative $\frac{d\xi^\mu}{d\tau} = \mathcal{O}(\xi)$. This last equation is called *the deviation equation*. we denote the derivatives $u^\mu(\tau) \equiv \frac{dx^\mu}{d\tau}$

$$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu{}_{\alpha\beta}u^\alpha\frac{d\xi^\beta}{d\tau} + \partial_\sigma\Gamma^\mu{}_{\alpha\beta}\xi^\sigma u^\alpha u^\beta = 0. \tag{2.33}$$

Which can be simplified through the introduction of the covariant derivative along the curve $x^\mu(\tau)$

$$\frac{D\xi^\mu}{d\tau} = \nabla_u \xi^\mu = u^\alpha \nabla_\alpha \xi^\mu$$
$$= \frac{d\xi^\mu}{d\tau} + \Gamma^\mu_{\ \alpha\beta} u^\alpha \xi^\beta,$$
$$\frac{D^2\xi^\mu}{d\tau^2} = \frac{d}{d\tau}\big(\frac{d\xi^\mu}{d\tau} + \Gamma^\mu_{\ \alpha\beta} u^\alpha \xi^\beta\big) + \Gamma^\mu_{\ \alpha\beta} u^\alpha \big(\frac{d\xi^\beta}{d\tau} + \Gamma^\beta_{\ \sigma\lambda} u^\sigma \xi^\lambda\big) \tag{2.34}$$
$$= \frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\ \alpha\beta} u^\alpha \frac{d\xi^\beta}{d\tau} + \partial_\sigma \Gamma^\mu_{\ \alpha\beta} \xi^\beta u^\sigma u^\alpha$$
$$+ \Gamma^\mu_{\ \alpha\beta} \frac{du^\alpha}{d\tau} \xi^\beta + \Gamma^\mu_{\ \alpha\beta} \Gamma^\beta_{\ \sigma\lambda} \xi^\lambda u^\alpha u^\sigma.$$

Upon using the geodesic equation $\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\ \sigma\lambda} u^\sigma u^\lambda$ and changing some of the contracted indices, we use this last line to express the deviation equation (2.33) covariantly

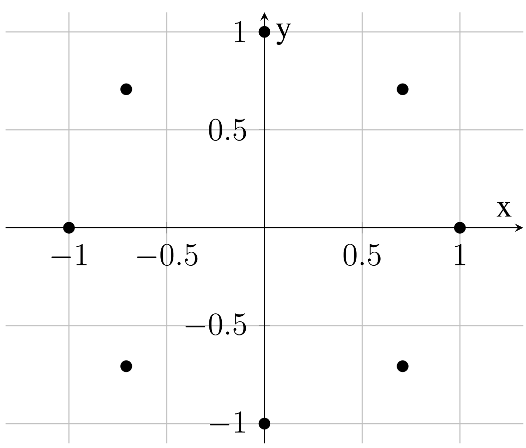
$$\frac{D^2\xi^\mu}{d\tau^2} = \Big(\partial_\alpha \Gamma^\mu_{\ \beta\nu} - \partial_\beta \Gamma^\mu_{\ \alpha\nu} + \Gamma^\mu_{\ \alpha\lambda} \Gamma^\lambda_{\ \beta\nu} - \Gamma^\mu_{\ \beta\lambda} \Gamma^\lambda_{\ \alpha\nu}\Big) \xi^\beta u^\nu u^\alpha. \tag{2.35}$$

The term in the paranthesis can be recognized as the coordinate components of the Riemann tensor, that is $R^\mu_{\ \nu\alpha\beta}$

$$\boxed{\frac{D^2\xi^\mu}{d\tau^2} = R^\mu_{\ \nu\alpha\beta} \xi^\beta u^\nu u^\alpha}. \tag{2.36}$$

Let's say we want to investigate what the effects of gravitational wave propagating in the z direction has on a collection of static masses spread around the space in the following way :

Figure 2.1



We shall first analyse the effect the gravitational wave on the above system in the TT gauge. For this analysis, I will use equation (2.33) with the premise that the masses are initially at rest. We are, for now, only concerned about the physical seperation of the masses, so the proper time evolution of the ith component of the separation vector ξ^i is

$$\Big[\frac{d^2\xi^i}{d\tau^2}\Big]_{\tau=0} = -2\big[\Gamma^i{}_{\alpha\beta}u^\alpha\frac{d\xi^\beta}{d\tau}\big]_{\tau=0} - \big[\partial_\sigma\Gamma^\mu{}_{\alpha\beta}\xi^\sigma u^\alpha u^\beta\big]_{\tau=0}. \tag{2.37}$$

Where in the TT gauge, given our initial conditions, we have

$$\begin{aligned}
[u^0]_{\tau=0} &= 1,\\
[u^i]_{\tau=0} &= 0,\\
\Gamma^\mu{}_{00} &= \frac{1}{2}g^{\mu\lambda}\big(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}\big) = 0 \qquad \text{in the TT gauge,}\\
\Gamma^i{}_{0\alpha} &= \frac{1}{2}g^{i\lambda}\big(\partial_0 g_{\lambda\alpha} + \partial_\alpha g_{0\lambda} - \partial_\lambda g_{0\alpha}\big)\\
&= \frac{1}{2}(\eta^{i\lambda} - h^{i\lambda} + \mathcal{O}(h^2))\partial_0 h_{\lambda\alpha}\\
&= -\frac{1}{2}\partial_0 h_{i0} + \frac{1}{2}\partial_0 h_{ij} + \mathcal{O}(h^2)\\
&= \frac{1}{2}\partial_0 h_{ij}.
\end{aligned} \tag{2.38}$$

Our deviation equation in this gauge then becomes

$$\Big[\frac{d^2\xi^i}{d\tau^2}\Big]_{\tau=0} = -\big[\partial_0 h_{ij}\frac{d\xi^j}{d\tau}\big]_{\tau=0}. \tag{2.39}$$

This equation tells us that for test masses initially at rest with respect to eachother in the TT gauge frame (i.e. $\big[\frac{d\xi^j}{d\tau}\big]_{\tau=0} = 0$), they will always remain at rest in this frame (i.e. $\frac{d^2\xi^i}{d\tau^2} = 0$ for all τ). But what about their physical separation ? Assume the events $E_1 = (t, x_1, y_1, z_1)$ and $E_2 = (t, x_2, y_2, z_2)$, which represent the measurement events of two of the masses sitting on the x axis simultaneously with respect to the TT frame. If these masses are at rest initially, we know that the coordinate distance $(x_2 - x_1, y_2 - y_1, z_2 - z_1) = \vec{L}$ will not change. We express the proper distance between these two events as

$$ds^2 = (\eta_{\mu\nu} + h^{\text{TT}}_{\mu\nu})dx^\mu dx^\nu. \tag{2.40}$$

For the events E_1 and E_2, they are simultaneous ($dt = 0$), meaning the proper distance can be expressed as

$$
\begin{aligned}
ds^2|_{E_1E_2} &= (\eta_{ij} + h_{ij}^{\text{TT}})dx^i dx^j, \\
ds|_{E_1E_2} &= (\eta_{ij} + h_{ij}^{\text{TT}})^{1/2} dx^i dx^j \\
&\approx (\eta_{ij} + \frac{1}{2}h_{ij}^{\text{TT}})dx^i dx^j + \mathcal{O}(h^2), \\
s|_{E_1E_2} &\approx L + \frac{1}{2L}h_{ij}L_iL_j,
\end{aligned}
\tag{2.41}
$$

Where upon defining $R_i = \frac{L_i}{L}$ and $s = s_iR_i$, we can quantify the proper distance's rate of change

$$
\ddot{s}_i|_{E_1E_2} = \frac{1}{2}\ddot{h}_{ij}L_j. \tag{2.42}
$$

2.5 Quadrupole Gravitational waves generated by weak-field sources with arbitrary velocity

We have seen that in the lorentz gauge, the equations gouverning a pertubation field $h_{\mu\nu}$ around a minkowskian background metric has the form

$$
\Box \overline{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{2.43}
$$

Let us generally inspect gravitational waves propagating from a central gravitational phenomenon, where $T_{\mu\nu}(x) = 0$ in the distant regions outside the source. Using the Lorentz gauge, but noting that we cannot force the TT gauge on a general space with energy-matter, our field equation can be solved by making use of radiation type Green's function; *retarded Green's function.*

$$
\begin{aligned}
&\Box_x G(x - x') = \delta^{(4)}(x - x'), \\
&\quad G(x - x') = -\frac{1}{4\pi|\vec{x} - \vec{x'}|}\delta(x^0_{\text{ret}} - x'^0) \qquad \text{where } x^0_{\text{ret}} = x^0 - |\vec{x} - \vec{x'}|.
\end{aligned}
\tag{2.44}
$$

With the use of which we can infer the non-homogeneous solutions as

$$
\begin{aligned}
\overline{h}_{\mu\nu}(x^0, \vec{x}) &= -16\pi G \int d^4x' G(x - x') T_{\mu\nu}(x') \\
&= 4G \int d^3\vec{x'} \frac{T_{\mu\nu}(x^0_{\text{ret}}, \vec{x'})}{|\vec{x} - \vec{x'}|}.
\end{aligned}
\tag{2.45}
$$

For an observer sufficiently far away from the localized source (where $T_{\mu\nu} \approx 0$), we had also mentionned that it is possible to further tweak our perturbation field using the gauge freedoms mentioned, where a very simple form can be attained by passing to the transverse traceless gauge.

$$\begin{aligned} h_{\mu\nu}^{\mathrm{TT}}(x^0,\vec{x}) &= -16\pi G\, \Lambda_{\mu\nu}{}^{\alpha\beta}(\hat{n}) \int d^4x' G(x-x') T_{\alpha\beta}(x') \\ &= 4G\, \Lambda_{\mu\nu}{}^{\alpha\beta}(\hat{n}) \int d^3\vec{x'} \frac{T_{\alpha\beta}(x^0_{\text{ret}},\vec{x}')}{|\vec{x}-\vec{x'}|}. \end{aligned} \tag{2.46}$$

Expanding

$$\begin{aligned} |\vec{x}-\vec{x'}| &= \big((\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')\big)^{\frac{1}{2}} \\ &= \big(|\vec{x}|^2 - 2\vec{x}\cdot\vec{x}' + |\vec{x}'|^2\big)^{\frac{1}{2}} \\ &= |\vec{x}|\big(1 - 2\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2} + \frac{|\vec{x}'|^2}{|\vec{x}|^2}\big)^{\frac{1}{2}} \\ &= |\vec{x}|\big(1 + \frac{1}{2}(-2\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2} + \frac{|\vec{x}'|^2}{|\vec{x}|^2}) + \frac{\frac{1}{2}(-\frac{1}{2})}{2}(-2\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2} + \frac{|\vec{x}'|^2}{|\vec{x}|^2})^2 + \dots\big) \\ &= |\vec{x}| - \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|} + \mathcal{O}(\frac{|\vec{x}'|^2}{|\vec{x}|}). \end{aligned} \tag{2.47}$$

We can thus express the retarded time inside the stress energy's arguments up to leading term $x^0_{\text{ret}} = x^0 - |\vec{x}| + \hat{x}\cdot\vec{x}'$, while a similar treatment for the term

$$\begin{aligned} \frac{1}{|\vec{x}-\vec{x'}|} &= \big((\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')\big)^{-\frac{1}{2}} \\ &= \big(|\vec{x}|^2 - 2\vec{x}\cdot\vec{x}' + |\vec{x}'|^2\big)^{-\frac{1}{2}} \\ &= \frac{1}{|\vec{x}|}\big(1 - 2\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2} + \frac{|\vec{x}'|^2}{|\vec{x}|^2}\big)^{-\frac{1}{2}} \\ &= \frac{1}{|\vec{x}|}\big(1 + (-\frac{1}{2})(-2\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2} + \frac{|\vec{x}'|^2}{|\vec{x}|^2}) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(-2\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2} + \frac{|\vec{x}'|^2}{|\vec{x}|^2})^2 + \dots\big) \\ &= \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3} + \mathcal{O}(\frac{|\vec{x}'|^2}{|\vec{x}|^2}). \end{aligned} \tag{2.48}$$

Since we are interested in the metric perturbation far away from the source, we generally speak of distances where $|\vec{x}'| \ll |\vec{x}|$ (For any $|\vec{x}'|$ comparable to $|\vec{x}|$, $T_{\alpha\beta}(x^0_{\text{ret}},\vec{x}')$ itself vanishes by assumption.)

We can also express the stress energy tensor itself in the momentum space with its fourier transform

$$T_{\alpha\beta}(t,\vec{x}) = \int \frac{d\omega}{2\pi}\frac{d^3\vec{k}}{(2\pi)^3}\tilde{T}_{\alpha\beta}(\omega,\vec{k})e^{-i\omega t + i\vec{k}\cdot\vec{x}}. \tag{2.49}$$

Such that, far away from a localized source, we can use the the above approximations to get

$$\begin{aligned}\int d^3\vec{x}' T_{\alpha\beta}(x^0 - |\vec{x}| + \hat{x}\cdot\vec{x}', \vec{x'}) &= \int d^3\vec{x}' \int \frac{d\omega}{2\pi}\frac{d^3\vec{k}}{(2\pi)^3}\tilde{T}_{\alpha\beta}(\omega,\vec{k})e^{-i\omega(x^0-|\vec{x}|+\hat{x}\cdot\vec{x}')+i\vec{k}\cdot\vec{x}'} \\ &= \int d^3\vec{x}' \int \frac{d\omega}{2\pi}\frac{d^3\vec{k}}{(2\pi)^3}\tilde{T}_{\alpha\beta}(\omega,\vec{k})e^{-i\omega(x^0-|\vec{x}|)+i(\vec{k}-\omega\hat{x})\cdot\vec{x}'} \\ &= \int \frac{d\omega}{2\pi} d^3\vec{k}\,\tilde{T}_{\alpha\beta}(\omega,\vec{k})e^{-i\omega(x^0-|\vec{x}|)}\delta^{(3)}(\vec{k}-\omega\hat{x}) \\ &= \int \frac{d\omega}{2\pi}\tilde{T}_{\alpha\beta}(\omega,\omega\hat{x})e^{-i\omega(x^0-|\vec{x}|)}.\end{aligned} \tag{2.50}$$

Where $\hat{n}$ is the direction in which the gravitational wave radiation will be propagating. The metric perturbation $h^{\mathrm{TT}}_{\mu\nu}$, to leading order, is then

$$h^{\mathrm{TT}}_{\mu\nu}(x^0,\vec{x}) = \frac{4G}{|\vec{x}|}\Lambda_{\mu\nu}{}^{\alpha\beta}(\hat{n})\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{T}_{\alpha\beta}(\omega,\omega\hat{n})e^{-i\omega(x^0-|\vec{x}|)}. \tag{2.51}$$

We can calculate the stress energy tensor of the gravitational wave with [7]

$$t_{\mu\nu} = \frac{1}{32\pi G} < \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} > . \tag{2.52}$$

We have also seen its invariance under gauge transformation ξ^μ. This means we can simply calculate it using any gauge. Picking the TT gauge, we see that only the physical α and β indices makes contributions to the overall quantity, and that $\partial^\mu h_{\mu\nu} = 0$, we are only left with the time derivatives, yielding

$$t_{\mu\nu} = t_{00} = \frac{1}{32\pi G} < \dot{h}^{\mathrm{TT}}_{ij}\dot{h}^{\mathrm{TT}}_{ij} > . \tag{2.53}$$

2.6 Low velocity expansion

Let us make the assumption that the velocities of matter making up the stress energy tensor are sufficiently small. For a periodically moving piece of matter with angular frequency ω_s inside the source, the relevant velocities are $v \approx \omega_s d \ll 1$, where d is the approximate size of the mentioned source. The angular frequency of the radiation emited from such a source is of the same order as the source's angular frequency, $\omega_s \approx \omega$. If we take back the relation (2.49) found previously for a more general case, applying the current low velocity regime gives us

$$T_{\alpha\beta}(x^0 - |\vec{x}| + \hat{n}\cdot\vec{x}', \vec{x}') = \int \frac{d\omega}{2\pi}\frac{d^3\vec{k}}{(2\pi)^3}\tilde{T}_{\alpha\beta}(\omega,\vec{k})e^{-i\omega(x^0-|\vec{x}|)+i(\vec{k}-\omega\hat{n})\cdot\vec{x}'}. \tag{2.54}$$

We know that the source is localized within $|\vec{x}'| \le d$. Using all of the assumptions above ($\omega_s d \ll 1$, $\omega_s \approx \omega$ and $|\vec{x}'| \le d$), we can relate the exponential ω term to another one; $|\omega \hat{n} \cdot \vec{x}'| \le \omega_s d \ll 1$. This makes sense; the leading contribution to the stress energy will come from areas where $|\vec{x}'| \le d$, because that is the condition which reduces the oscillatory term inside the ω integral the most. We can assume from now on that the main part contributing to the above Fourier reverse transform is the part where $|\omega \hat{n} \cdot \vec{x}'| \ll 1$, so we can expand (defining $u' = x^0 - |\vec{x}| + \hat{n} \cdot \vec{x}'$)

$$
\begin{aligned}
T_{\alpha\beta}(u', \vec{x}') &= \int \frac{d\omega}{2\pi} \frac{d^3\vec{k}}{(2\pi)^3} \tilde{T}_{\alpha\beta}(\omega, \vec{k}) e^{-i\omega(x^0 - |\vec{x}|) + i\vec{k}\cdot\vec{x}'} e^{-i\omega\hat{n}\cdot\vec{x}'} \\
&= \int \frac{d\omega}{2\pi} \frac{d^3\vec{k}}{(2\pi)^3} \tilde{T}_{\alpha\beta}(\omega, \vec{k}) e^{-i\omega(x^0 - |\vec{x}|) + i\vec{k}\cdot\vec{x}'} \\
&\qquad \left(1 - i\omega\hat{n}\cdot\vec{x}' - \frac{1}{2}\omega^2(\hat{n}\cdot\vec{x}')^2 + \dots\right) \\
&= \int \frac{d\omega}{2\pi} \frac{d^3\vec{k}}{(2\pi)^3} \tilde{T}_{\alpha\beta}(\omega, \vec{k}) e^{-i\omega(x^0 - |\vec{x}|) + i\vec{k}\cdot\vec{x}'} \\
&\qquad - i\hat{n}\cdot\vec{x}' \int \frac{d\omega}{2\pi} \frac{d^3\vec{k}}{(2\pi)^3} \tilde{T}_{\alpha\beta}(\omega, \vec{k}) e^{-i\omega(x^0 - |\vec{x}|) + i\vec{k}\cdot\vec{x}'} \omega \\
&\qquad - \frac{1}{2}(\hat{n}\cdot\vec{x}')^2 \int \frac{d\omega}{2\pi} \frac{d^3\vec{k}}{(2\pi)^3} \tilde{T}_{\alpha\beta}(\omega, \vec{k}) e^{-i\omega(x^0 - |\vec{x}|) + i\vec{k}\cdot\vec{x}'} \omega^2 + \dots \Big) \\
&= \int \frac{d\omega}{2\pi} \frac{d^3\vec{k}}{(2\pi)^3} \tilde{T}_{\alpha\beta}(\omega, \vec{k}) e^{-i\omega(x^0 - |\vec{x}|) + i\vec{k}\cdot\vec{x}'} \\
&\qquad + \hat{n}\cdot\vec{x}' \partial_0 \int \frac{d\omega}{2\pi} \frac{d^3\vec{k}}{(2\pi)^3} \tilde{T}_{\alpha\beta}(\omega, \vec{k}) e^{-i\omega(x^0 - |\vec{x}|) + i\vec{k}\cdot\vec{x}'} \\
&\qquad + \frac{1}{2}(\hat{n}\cdot\vec{x}')^2 \partial_0^2 \int \frac{d\omega}{2\pi} \frac{d^3\vec{k}}{(2\pi)^3} \tilde{T}_{\alpha\beta}(\omega, \vec{k}) e^{-i\omega(x^0 - |\vec{x}|) + i\vec{k}\cdot\vec{x}'} + \dots \Big) \\
&= T_{\alpha\beta}(x^0 - |\vec{x}|, \vec{x}') + \hat{n}\cdot\vec{x}' \partial_0 T_{\alpha\beta}(x^0 - |\vec{x}|, \vec{x}') \\
&\qquad\qquad + \frac{1}{2}(\hat{n}\cdot\vec{x}')^2 \partial_0^2 T_{\alpha\beta}(x^0 - |\vec{x}|, \vec{x}') + \dots,
\end{aligned}
\tag{2.55}
$$

Which amounts to expanding the initial time argument $x^0 - |\vec{x}| + \hat{n}\cdot\vec{x}'$ around $x^0 - |\vec{x}|$, taking $\hat{n}\cdot\vec{x}'$ to be small. I have chosen to derive it from here to emphasize the need for $\omega_s d \ll 1$ in our expansion.

If we define new momenta for the stress energy tensor as follows

$$\begin{aligned} S^{\alpha\beta}(x^0) &= \int d^3\vec{x}\, T^{\alpha\beta}(x^0,\vec{x}), \\ S^{\alpha\beta,i}(x^0) &= \int d^3\vec{x}\, T^{\alpha\beta}(x^0,\vec{x})x^i, \\ S^{\alpha\beta,ij}(x^0) &= \int d^3\vec{x}\, T^{\alpha\beta}(x^0,\vec{x})x^i x^j. \\ &\dots \end{aligned} \tag{2.56}$$

Going back to the metric perturbation far outside the source, in TT gauge and replacing the stress energy integrals with the newly defined momenta:

$$\begin{aligned} h_{ij}^{\mathrm{TT}}(t,\vec{x}) &= \frac{1}{|\vec{x}|}4G\Lambda_{ij,kl}(\hat{n})\int d^3\vec{x}'\; T^{kl}(x^0-|\vec{x}|+\hat{n}\cdot\vec{x}',\vec{x}'), \\ &= \frac{1}{|\vec{x}|}4G\Lambda_{ij,kl}(\hat{n})\int d^3\vec{x}'\Big(T^{kl}(x^0-|\vec{x}|,\vec{x}')+n_m x^m\partial_0 T^{kl}(x^0-|\vec{x}|,\vec{x}') \\ &\qquad + \frac{1}{2}n_m n_p x^m x^p \partial_0^2 T^{kl}(x^0-|\vec{x}|,\vec{x}')+\dots\Big), \\ &= \frac{1}{|\vec{x}|}4G\Lambda_{ij,kl}(\hat{n})\Big(S^{kl}(x^0-|\vec{x}|)+n_m\dot{S}^{kl,m}(x^0-|\vec{x}|)+ \\ &\qquad \frac{1}{2}n_m n_p \ddot{S}^{kl,mp}(x^0-|\vec{x}|)+\dots\Big). \end{aligned} \tag{2.57}$$

Just to see if this expansion makes sense in our current domain of validity ($\omega_s d \approx v \ll 1$), one can check that compared to S^{kl}, $\dot{S}^{kl,m}$ has an extra factor of $\mathcal{O}(d)$ coming from the x^m term and $\mathcal{O}(\omega) \approx \mathcal{O}(\omega_s)$, giving us a total factor of $\mathcal{O}(\omega_s d) \approx \mathcal{O}(v)$. Similarly, compared to S^{kl}, $\ddot{S}^{kl,mp}$ has an extra factor of $\mathcal{O}(v^2)$, so this is indeed a low velocity expansion.

To simplify things further, let us define the following momenta

$$\begin{aligned} M(x^0) &= \int d^3\vec{x}\, T^{00}(x^0,\vec{x}), \\ M^i(x^0) &= \int d^3\vec{x}\, T^{00}(x^0,\vec{x})x^i, \\ M^{ij}(x^0) &= \int d^3\vec{x}\, T^{00}(x^0,\vec{x})x^i x^j. \end{aligned} \tag{2.58}$$

And

$$\begin{aligned} P^i(x^0) &= \int d^3\vec{x}\, T^{0i}(x^0,\vec{x}), \\ P^{i,j}(x^0) &= \int d^3\vec{x}\, T^{0i}(x^0,\vec{x})x^j, \\ P^{i,jk}(x^0) &= \int d^3\vec{x}\, T^{00}(x^0,\vec{x})x^j x^k. \end{aligned} \tag{2.59}$$

Using the conservation of the stress energy tensor $\partial_\mu T^{0\mu} = 0$

$$\begin{aligned} \partial_0 T^{00} &= -\partial_i T^{0i}, \\ \partial_0 \int_V d^3\vec{x}\, T^{00} &= -\int_V d^3\vec{x}\, \partial_i T^{0i}, \\ \dot{M} &= -\oint_{\partial V} dS_i T^{0i} \\ &= 0. \end{aligned} \tag{2.60}$$

Where we have taken the space V such that $T^{\alpha\beta}$ vanishes at its boundaries ∂V. Starting off from the conservation relation yet again, we can relate the the time derivatives of the momenta $M^{\cdots}$ as follows

$$\begin{aligned} \partial_0 \int_V d^3\vec{x}\, T^{00} x^i &= -\int_V d^3\vec{x}\, \partial_j T^{0j} x^i, \\ \dot{M}^i &= -[T^{0j} x^i]_{j\ \partial V} + \int_V d^3\vec{x}\, T^{0j} \delta_j^{\ i} \\ &= \int_V d^3\vec{x}\, T^{0i} \\ &= P^i. \end{aligned} \tag{2.61}$$

$$\begin{aligned} \partial_0 \int_V d^3\vec{x}\, T^{00} x^i x^j &= -\int_V d^3\vec{x}\, \partial_k T^{0k} x^i x^j, \\ \dot{M}^{ij} &= -[T^{0k} x^i x^j]_{k\ \partial V} + \int_V d^3\vec{x}\, T^{0k} \delta_k^{\ i} x^j + \int_V d^3\vec{x}\, T^{0k} x^i \delta_k^{\ j} \\ &= \int_V d^3\vec{x}\, T^{0i} x^j + \int_V d^3\vec{x}\, T^{0j} x^i \\ &= P^{i,j} + P^{j,i}. \end{aligned} \tag{2.62}$$

$$\begin{aligned} \partial_0 \int_V d^3\vec{x}\, T^{00} x^i x^j x^k &= -\int_V d^3\vec{x}\, \partial_l T^{0l} x^i x^j x^k, \\ \dot{M}^{ijk} &= -[T^{0l} x^i x^j x^k]_{l\ \partial V} + \int_V d^3\vec{x}\, T^{0l} \delta_l^{\ i} x^j x^k \\ &\quad + \int_V d^3\vec{x}\, T^{0l} x^i \delta_l^{\ j} x^k + \int_V d^3\vec{x}\, T^{0l} x^i x^j \delta_l^{\ k} \\ &= \int_V d^3\vec{x}\, T^{0i} x^j x^k + \int_V d^3\vec{x}\, T^{0j} x^i x^k + \int_V d^3\vec{x}\, T^{0k} x^i x^j \\ &= P^{i,jk} + P^{j,ik} + P^{k,ij}. \end{aligned} \tag{2.63}$$

While the time derivatives of $P^{i,\cdots}$ can be found through

$$\begin{aligned}\partial_0 \int_V d^3\vec{x}\, T^{0i} &= \int_V d^3\vec{x}\partial_0 T^{0i},\\ \dot{P}^i &= -\int_V d^3\vec{x}\partial_j T^{ji}\\ &= -\oint_{\partial V} dS_j T^{ji}\\ &= 0.\end{aligned} \tag{2.64}$$

With the same assumptions over the boundaries of V as previously mentionned. Repeating the same procedure, we find

$$\begin{aligned}\partial_0 \int_V d^3\vec{x}\, T^{0i}x^j &= \int_V d^3\vec{x}\partial_0 T^{0i}x^j,\\ \dot{P}^{i,j} &= -\int_V d^3\vec{x}\partial_k T^{ki}x^j\\ &= -[T^{ki}x^j]_{k\ \partial V} + \int_V d^3\vec{x}\, T^{ki}\delta^j_k\\ &= \int_V d^3\vec{x}\, T^{ji}\\ &= S^{ji} = S^{ij}.\end{aligned} \tag{2.65}$$

$$\begin{aligned}\partial_0 \int_V d^3\vec{x}\, T^{0i}x^jx^k &= \int_V d^3\vec{x}\partial_0 T^{0i}x^jx^k,\\ \dot{P}^{i,jk} &= -\int_V d^3\vec{x}\partial_l T^{li}x^jx^k\\ &= -[T^{li}x^jx^k]_{l\ \partial V} + \int_V d^3\vec{x}\, T^{li}\delta_l{}^j x^k + \int_V d^3\vec{x}\, T^{li}x^j\delta_l{}^k\\ &= \int_V d^3\vec{x}\, T^{ji}x^k + \int_V d^3\vec{x}\, T^{ki}x^j\\ &= S^{ij,k} + S^{ik,j}.\end{aligned} \tag{2.66}$$

Using what we've derived here to substitute all $S^{ij,\cdots}$ in (2.57).
Using (2.62) and (2.65)

$$\begin{aligned}S^{ij} &= \frac{1}{2}\big(\dot{P}^{i,j} + \dot{P}^{j,i}\big)\\ &= \frac{1}{2}\ddot{M}^{ij}.\end{aligned} \tag{2.67}$$

Using (2.63), then applying the results of (2.66)

$$
\begin{aligned}
\dot{M}^{ijk} &= P^{i,jk} + P^{j,ik} + P^{k,ij}, \\
\ddot{M}^{ijk} &= \dot{P}^{i,jk} + \dot{P}^{j,ik} + \dot{P}^{k,ij} \\
&= S^{ij,k} + S^{ik,j} + S^{ji,k} + S^{jk,i} + S^{ki,j} + S^{kj,i} \\
&= 2\big(S^{ij,k} + S^{jk,i} + S^{ki,j}\big), \\
\dddot{M}^{ijk} &= 2\big(\dot{S}^{ij,k} + \dot{S}^{jk,i} + \dot{S}^{ki,j}\big), \\
\frac{1}{6}\dddot{M}^{ijk} &= \frac{1}{3}\big(\dot{S}^{ij,k} + \dot{S}^{jk,i} + \dot{S}^{ki,j}\big) \\
&= \dot{S}^{ij,k} + \frac{1}{3}\big(\dot{S}^{jk,i} + \dot{S}^{ki,j} - 2\dot{S}^{ij,k}\big) \\
&= \dot{S}^{ij,k} + \frac{1}{3}\big(\dot{S}^{kj,i} + \dot{S}^{ki,j} + \dot{S}^{ki,j} + \dot{S}^{kj,i} - \dot{S}^{ki,j} - \dot{S}^{kj,i} - 2\dot{S}^{ij,k}\big) \\
&= \dot{S}^{ij,k} + \frac{1}{3}\big(2\ddot{P}^{k,ij} - \dot{S}^{ik,j} - \dot{S}^{ij,k} - \dot{S}^{jk,i} - \dot{S}^{ji,k}\big) \\
&= \dot{S}^{ij,k} + \frac{1}{3}\big(2\ddot{P}^{k,ij} - \ddot{P}^{i,jk} - \ddot{P}^{j,ik}\big),
\end{aligned}
\tag{2.68}
$$

Which requires that the following be true;

$$
\dot{S}^{ij,k} = \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}\big(\ddot{P}^{i,jk} + \ddot{P}^{j,ik} - 2\ddot{P}^{k,ij}\big). \tag{2.69}
$$

We can now express our metric perturbation in the TT gauge h_{ij}^{TT} up to the second highest contribution term

$$
\begin{aligned}
h_{ij}^{\mathrm{TT}}(x^0, \vec{x}) &= \frac{1}{|\vec{x}|} 4G\Lambda_{ij,kl}(\hat{n})\Big(S^{kl} + n_m \dot{S}^{kl,m} + \dots\Big) \\
&= \frac{1}{|\vec{x}|} 4G\Lambda_{ij,kl}(\hat{n})\Big(\frac{1}{2}\ddot{M}^{kl} \\
&\qquad + \frac{1}{3} n_m \big(\frac{1}{2}\dddot{M}^{klm} + \ddot{P}^{k,lm} + \ddot{P}^{l,km} - 2\ddot{P}^{m,kl}\big) + \dots\Big).
\end{aligned}
\tag{2.70}
$$

Some knowledge of Group Theory would hint at the fact that what we did here was mere decomposition of S^{kl} and $S^{kl,m}$ into their irreducible representations of SO(3). Since S^{kl} is symmetric in k and l, it can only be decomposed into a spin-2 representation ($\ddot{M}^{kl}$) called "Mass Quadrupole". $S^{kl,m}$'s treatment must include the fact that k an l indices are symmetric, while m doesn't have any property under interchange of indices, meaning the initial $S^{kl,m}$ can be decomposed into a spin-3 representation ($\dddot{M}^{klm}$) called "Mass Octupole" and a spin-2 representation ($-\big(2\ddot{P}^{m,kl} - \ddot{P}^{k,lm} - \ddot{P}^{l,km}\big)$) called "Current Quadrupole". This analysis could also be made through the use of Young Diagrams, which facilitates the decomposition process.

2.7 Gravitational Radiation -Simplifications-

We now dwelve into how to treat (2.70) in order to simplify them as much as possible. We will treat the leading and up-to-leading terms separately

Leading term - Mass Quadrupole

The leading term in our expansion is

$$[h_{ij}^{\mathrm{TT}}(x^0,\vec{x})]_{\text{Leading}} = \frac{1}{|\vec{x}|}2G\Lambda_{ij,kl}(\hat{n})\ddot{M}^{kl}. \tag{2.71}$$

Since we can write out our symmetric tensor M^{kl} as a traceless spin-2 tensor field and a scalar trace part,

$$\begin{aligned} M^{kl} &= \left(M^{kl} - \frac{1}{3}\delta^{kl}M\right) + \frac{1}{3}\delta^{kl}M \\ &= Q^{kl} + \frac{1}{3}\delta^{kl}M. \end{aligned} \tag{2.72}$$

Where we have defined the symmetric trace free tensor (STF) $Q^{kl} \equiv M^{kl} - \frac{1}{3}\delta^{kl}M$. When contracted with δ^{kl}, our TT projection operator $\Lambda_{ij,kl}$ vanishes by construction. This allows us to neglect the trace part of the above expansion, so our leading term reads

$$[h_{ij}^{\mathrm{TT}}(x^0,\vec{x})]_{\text{Leading}} = \frac{1}{|\vec{x}|}2G\Lambda_{ij,kl}(\hat{n})\ddot{Q}^{kl}, \tag{2.73}$$

Up to leading term - Mass Octupole

The up-to-leading mass octupole term in our expansion is

$$[h_{ij}^{\mathrm{TT}}(x^0,\vec{x})]_{\text{Octupole}} = \frac{1}{|\vec{x}|}\frac{2G}{3}\Lambda_{ij,kl}(\hat{n})n_m\dddot{M}^{klm}. \tag{2.74}$$

Similarly to the Mass Quadrupole treatment, one can write out this tensor as

$$\begin{aligned} M^{klm} &= \Big(M^{klm} - \frac{1}{5}\big(\delta^{kl}M^{nnm} + \delta^{km}M^{nln} + \delta^{lm}M^{knn}\big)\Big) \\ &\qquad + \frac{1}{5}\big(\delta^{kl}M^{nnm} + \delta^{km}M^{nln} + \delta^{lm}M^{knn}\big) \\ &= \mathcal{O}^{klm} + \frac{1}{5}\big(\delta^{kl}M^{nnm} + \delta^{km}M^{nln} + \delta^{lm}M^{knn}\big). \end{aligned} \tag{2.75}$$

Where we have defined another STF tensor $\mathcal{O}^{klm} \equiv M^{klm} - \frac{1}{5}\big(\delta^{kl}M^{nnm} + \delta^{km}M^{nln} + \delta^{lm}M^{knn}\big)$.

When our $\Lambda_{ij,kl}n_m$ hits on δ^{kl}, it vanished as argued in the previous section concerning the leading term. One can also check that upon hitting either δ^{km} or δ^{lm}, the indice of n_m is raised to either k or l, which vanishes when hitting $\Lambda_{ij,kl}$ due to the transversal projection nature of the operator. Both of the above cancelations lead to

$$[h_{ij}^{\text{TT}}(x^0,\vec{x})]_{\text{Octupole}} = \frac{1}{|\vec{x}|}\frac{2G}{3}\Lambda_{ij,kl}(\hat{n})n_m\dddot{\mathcal{O}}^{klm}. \tag{2.76}$$

Up to leading term - Current Quadrupole

The up-to-leading current quadrupole term in our expansion is

$$[h_{ij}^{\text{TT}}(x^0,\vec{x})]_{\text{C.Quadrupole}} = \frac{1}{|\vec{x}|}\frac{4G}{3}\Lambda_{ij,kl}(\hat{n})n_m\big(\ddot{P}^{k,lm} + \ddot{P}^{l,km} - 2\ddot{P}^{m,kl}\big). \tag{2.77}$$

Let's first try to understand what the quantity $P^{k,lm} + P^{l,km} - 2P^{m,kl}$ corresponds to physically.

$$\begin{aligned} P^{k,lm} + P^{l,km} - 2P^{m,kl} &= \int d^3\vec{x}\big(T^{0k}x^lx^m + T^{0l}x^kx^m - 2T^{0m}x^kx^l\big) \\ &= \int d^3\vec{x}\Big(x^l\big(x^mT^{0k} - x^kT^{0m}\big) + x^k\big(x^mT^{0l} - x^lT^{0m}\big)\Big). \end{aligned} \tag{2.78}$$

One might notice the quantities of the form $x^mT^{0k} - x^kT^{0m}$ to be angular momentum densities associated with the (m,k) plane. Denoting them as $j^{mk} \equiv x^mT^{0k} - x^kT^{0m}$,

$$P^{k,lm} + P^{l,km} - 2P^{m,kl} = \int d^3\vec{x}\big(x^lj^{mk} + x^kj^{ml}\big). \tag{2.79}$$

Reminding ourselves that the angular momentum density vector $j^l = \epsilon^{ijl}x^ip^j = \epsilon^{ijl}x^iT^{0j}$ can be contracted as follows

$$\begin{aligned} \epsilon^{mkl}j^l &= \epsilon^{mkl}\epsilon^{ijl}x^iT^{0j} \\ &= (\delta^{mi}\delta^{kj} - \delta^{mj}\delta^{ki})x^iT^{0j} \\ &= x^mT^{0k} - x^kT^{0m} \\ &= j^{mk}. \end{aligned} \tag{2.80}$$

So finally, with the definition of the angular momentum moment $J^{i,j} \equiv \int d^3\vec{x}j^ix^j$,

$$\begin{aligned} P^{k,lm} + P^{l,km} - 2P^{m,kl} &= \int d^3\vec{x}\big(x^l\epsilon^{mkp}j^p + x^k\epsilon^{mlp}j^p\big) \\ &= \epsilon^{mkp}J^{p,l} + \epsilon^{mlp}J^{p,k}. \end{aligned} \tag{2.81}$$

This means we can express the Current Quadrupole term in our metric perturbation as moments of the angular momentum symmetrized over k and l;

$$[h_{ij}^{\rm TT}(x^0,\vec{x})]_{\rm C.Quadrupole} = \frac{1}{|\vec{x}|}\frac{4G}{3}\Lambda_{ij,kl}(\hat{n})n_m\big(\epsilon^{mkp}J^{p,l}+\epsilon^{mlp}J^{p,k}\big). \tag{2.82}$$

2.8 Gravitational Radiation -Radiated Energy and Angular Momentum-

We shall now calculate what each of the above terms contribute to the loss of energy and angular momentum of a system. For this, we have the relation for the power radiated by gravitational wave [8]

$$\frac{dP}{d\Omega} = \frac{|\vec{x}|^2}{32\pi G} < \dot{h}_{ij}^{\rm TT}\dot{h}_{ij}^{\rm TT} > . \tag{2.83}$$

The time derivative of the linear momentum P^k was also described to be

$$\frac{dP^k}{dt} = -\frac{1}{32\pi G}\int d\Omega |\vec{x}|^2 < \dot{h}_{ij}^{\rm TT}\partial^k \dot{h}_{ij}^{\rm TT} >, \tag{2.84}$$

While the time derivative of the total angular momentum J^i of such a system was

$$\frac{dJ^i}{dt} = \underbrace{-\frac{1}{32\pi G}\epsilon^{ijk}\int d\Omega |\vec{x}|^2\big\langle \dot{h}_{lm}^{\rm TT}x^j\partial^k h_{lm}^{\rm TT}\big\rangle}_{=\frac{dL^i}{dt}} + \underbrace{\frac{1}{32\pi G}\epsilon^{ijk}\int d\Omega |\vec{x}|^2\big\langle 2\dot{h}_{lj}^{\rm TT}h_{lk}^{\rm TT}\big\rangle}_{=\frac{dS^i}{dt}} . \tag{2.85}$$

Expressing these losses as an expansion of velocity $\mathcal{O}(v^n)$, we study the contribution of the leading mass quadrupole contribution.

Leading term - Mass Quadrupole

The power relation for the leading term reads

$$\begin{aligned}\left[\frac{dP}{d\Omega}\right]_{\rm Leading} &= \frac{|\vec{x}|^2}{32\pi G} < [\dot{h_{ij}^{\rm TT}}]_{\rm Lead}[\dot{h_{ij}^{\rm TT}}]_{\rm Lead} > \\ &= \frac{G}{8\pi}\Lambda_{ij,kl} < \dddot{Q}^{ij}\dddot{Q}^{kl} > .\end{aligned} \tag{2.86}$$

Using the property of the TT projector $\Lambda_{ij,kl}$ (Appendix A)

$$\int d\Omega \Lambda_{ij,kl} = \frac{2\pi}{15}\big(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}\big). \tag{2.87}$$

one can integrate the Power radiated per unit solid angle to find the total power radiated due to the leading term.

$$
\begin{aligned}
P_{\text{Leading}} &= \frac{G}{8\pi}\frac{2\pi}{15}\big(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}\big) < \dddot{Q}^{ij}\dddot{Q}^{kl} > \\
&= \frac{G}{5} < \dddot{Q}^{ij}\dddot{Q}^{ij} > .
\end{aligned}
\tag{2.88}
$$

For the damped linear momentum for the leading term, we take the above relation

$$
\begin{aligned}
\frac{dP^m}{dt} &= -\frac{1}{32\pi G}\int d\Omega|\vec{x}|^2 < [\dot{h}_{ij}^{\text{TT}}]_{\text{Lead}}\partial^k[\dot{h}_{ij}^{\text{TT}}]_{\text{Lead}} > \\
&= -\frac{1}{32\pi G}\int d\Omega\ \Lambda_{ij,kl} < \dddot{Q}^{ij}\partial^m\dddot{Q}^{kl} > .
\end{aligned}
\tag{2.89}
$$

Noting that under the parity transformation $\vec{x} \to -\vec{x}$, the STF Quadrupole tensor is even $Q^{ij} \to Q^{ij}$ while the partial derivative is odd $\partial^k \to -\partial^k$. This implies that the leading linear momentum loss is equal to minus itself, thus the leading term contribution vanishes identically.
In turn, the change in total angular momentum can be expressed as the above relation,

$$
\frac{dJ^i}{dt} = \frac{1}{32\pi G}\epsilon^{ijk}\int d\Omega|\vec{x}|^2\big\langle -[\dot{h}_{lm}^{\text{TT}}]_{\text{Lead}}x^j\partial^k[h_{lm}^{\text{TT}}]_{\text{Lead}} + 2[\dot{h}_{lj}^{\text{TT}}]_{\text{Lead}}[h_{lk}^{\text{TT}}]_{\text{Lead}}\big\rangle. \tag{2.90}
$$

We first treat some of the terms this includes;

$$
\begin{aligned}
\partial^k[h_{lm}^{\text{TT}}]_{\text{Lead}} &= 2G\partial^k\big(\frac{1}{|\vec{x}|}\Lambda_{lm,pq}(\hat{n})\ddot{Q}^{pq}\big) \\
&= 2G\big(\frac{x^k}{|\vec{x}|^3}\Lambda_{lm,pq}(\hat{n})\ddot{Q}^{kl} + \frac{1}{|\vec{x}|}\partial^k\Lambda_{lm,pq}(\hat{n})\ddot{Q}^{pq} \\
&\qquad + \frac{1}{|\vec{x}|}\Lambda_{lm,pq}(\hat{n})\frac{\partial|\vec{x}|}{\partial x_k}\frac{d}{d|\vec{x}|}\ddot{Q}^{pq}\big).
\end{aligned}
\tag{2.91}
$$

Where upon using the fact that Q^{pq} is a function of the retarded time $x_0 - |\vec{x}|$, we can equate $\frac{d}{d|\vec{x}|}\ddot{Q}^{pq} = -\dddot{Q}^{pq}$. We also can relate $\frac{\partial|\vec{x}|}{\partial x_k} = \frac{x^k}{|\vec{x}|}$, upon which it becomes obvious that among the three terms presented above, those with $\epsilon^{ijk}x^jx^k$ vanishes. We are thus left with

$$
\partial^k[h_{lm}^{\text{TT}}]_{\text{Lead}} = \frac{2G}{|\vec{x}|}\partial^k\Lambda_{lm,pq}(\hat{n})\ddot{Q}^{pq}. \tag{2.92}
$$

This can in turn be simplified even further with the knowledge that $n^l = \frac{x^l}{|\vec{x}|} \to \frac{\partial n^s}{\partial x_k} =$

$\frac{\delta^{sk}}{|\vec{x}|} - \frac{x^s x^k}{|\vec{x}|^3} = \frac{P^{sk}}{|\vec{x}|}$, which implies

$$\begin{aligned}
\partial^k \Lambda_{lm,pq} &= \frac{\partial n^s}{\partial x_k}\frac{\partial}{\partial n^s}\Lambda_{lm,pq} \\
&= -\frac{P^{sk}}{|\vec{x}|}\Big(\big(\delta_{ls}n_p + \delta_{ps}n_l\big)P_{mq} + P_{lp}\big(\delta_{ms}n_q + \delta_{qs}n_m\big) \\
&\qquad - \frac{1}{2}\big(\delta_{ls}n_m + \delta_{ms}n_l\big)P_{pq} - \frac{1}{2}P_{lm}\big(\delta_{ps}n_q + \delta_{qs}n_p\big)\Big) \\
&= -\frac{1}{|\vec{x}|}\Big(\big(P_l{}^k n_p + P_p{}^k n_l\big)P_{mq} + P_{lp}\big(P_m{}^k n_q + P_q{}^k n_m\big) \\
&\qquad - \frac{1}{2}\big(P_l{}^k n_m + P_m{}^k n_l\big)P_{pq} - \frac{1}{2}P_{lm}\big(P_p{}^k n_q + P_q{}^k n_p\big)\Big) \\
&= -\frac{1}{|\vec{x}|}\Big(n_l\big(P_p{}^k P_{mq} - \frac{1}{2}P_m{}^k P_{pq}\big) + n_m\big(P_q{}^k P_{lp} - \frac{1}{2}P_l{}^k P_{pq}\big) \\
&\qquad + n_p\big(P_l{}^k P_{mq} - \frac{1}{2}P_q{}^k P_{lm}\big) + n_q\big(P_m{}^k P_{lp} - \frac{1}{2}P_p{}^k P_{lm}\big)\Big) \\
&= -\frac{1}{|\vec{x}|}\Big(n_l\Lambda^k{}_{m,pq} + n_m\Lambda^k{}_{l,pq} + n_p\Lambda^k{}_{q,lm} + n_q\Lambda^k{}_{p,lm}\Big).
\end{aligned} \tag{2.93}$$

Since the TT Projector is transverse to $\hat{n}$ on all indices, the terms proportional to n^l and n^m will cancel out with the projector inside $[\dot{h}^{\rm TT}_{lm}]_{\rm Lead}$ in the orbital momentum part of $\frac{dJ^i}{dt}$.

$$\begin{aligned}
\frac{dL^i}{dt} &= \frac{G}{2}\epsilon^{ijk} < \dddot{Q}_{rs}\ddot{Q}_{pq} > \int \frac{d\Omega}{4\pi}\Lambda_{lm,rs}n_j\big(n_p\Lambda_{lm,kq} + n_q\Lambda_{lm,kp}\big) \\
&= \frac{G}{2}\epsilon^{ijk} < \dddot{Q}_{rs}\ddot{Q}_{pq} > \int \frac{d\Omega}{4\pi}n_j\big(n_p\Lambda_{rs,kq} + n_q\Lambda_{rs,kp}\big).
\end{aligned} \tag{2.94}$$

Expanding $\Lambda_{rs,kq}$'s in terms of n's, while noting that any term containting n_k will eventually cancel out due to $\epsilon^{ijk}n_j n_k = 0$, coupled with the fact that under the averaging out operation, we can use integration by part to equate $< \dddot{Q}_{rs}\ddot{Q}_{pq} >=< \ddot{Q}_{rs}\dddot{Q}_{pq} >$ where the full derivative part will vanish due to averaging. We can then express the above as

$$\begin{aligned}
\frac{dL^i}{dt} &= \frac{G}{2}\epsilon^{ijk}\big(\frac{8}{15} < \dddot{Q}_{kr}\ddot{Q}_{jr} > -\frac{1}{5} < \dddot{Q}_{kj}\ddot{Q}_{rr} > \big) \\
&= \frac{4G}{15}\epsilon^{ijk} < \dddot{Q}_{jr}\ddot{Q}_{kr} > .
\end{aligned} \tag{2.95}$$

Where we have used the symmetry of Q_{kj} in the k and j indices to cancel out with ϵ^{ijk} .

The spin contribution is straightforward

$$\begin{aligned}
\frac{dS^i}{dt} &= \frac{1}{32\pi G}\epsilon^{ijk}\int d\Omega|\vec{x}|^2\big\langle 2[\dot{h}^{\rm TT}_{lj}]_{\rm Lead}[h^{\rm TT}_{lk}]_{\rm Lead}\big\rangle \\
&= G\epsilon^{ijk} < \dddot{Q}_{rs}\ddot{Q}_{pq} > \int \frac{d\Omega}{4\pi}\Lambda_{lj,rs}\Lambda_{lk,pq}.
\end{aligned} \tag{2.96}$$

Upon expanding the first Λ traceless projector in terms of P's, we make use of the identity $P_{lr}\Lambda_{lk,pq} = \Lambda_{rk,pq}$. Reminding ourselves that due to the symmetric nature of the Λ traceless projector in the first two and last two indicese, we have $\epsilon^{ijk}\Lambda_{jk,pq} = 0$. Making use of these, we can express our spin contribution as

$$\begin{aligned} \frac{dS^i}{dt} &= G\epsilon^{ijk} < \dddot{Q}_{rs}\ddot{Q}_{pq} > \int \frac{d\Omega}{4\pi}\big(P_{lr}P_{js} - \frac{1}{2}P_{lj}P_{rs}\big)\Lambda_{lk,pq} \\ &= G\epsilon^{ijk} < \dddot{Q}_{rs}\ddot{Q}_{pq} > \int \frac{d\Omega}{4\pi}\big(P_{js}\Lambda_{rk,pq} - \cancel{\frac{1}{2}P_{rs}\Lambda_{jk,pq}}\big). \end{aligned} \tag{2.97}$$

After expanding both P_{js} and $\Lambda_{rk,pq}$ in terms of n's, we carry out the operations similarly to the orbital angular momentum contribution, we end up with

$$\frac{dS^i}{dt} = \frac{2G}{15}\epsilon^{ijk} < \dddot{Q}_{jr}\ddot{Q}_{kr} > . \tag{2.98}$$

Yielding a total angular momentum total derivative of

$$\frac{dJ^i}{dt} = \frac{dL^i}{dt} + \frac{dS^i}{dt} = \frac{2G}{5}\epsilon^{ijk} < \dddot{Q}_{jr}\ddot{Q}_{kr} > . \tag{2.99}$$

2.9 Examplification of an oscillatory newtonian system

We will now compute the gravitational radiation of a single mass in circular orbit using what we have learned so far. As we have not yet introduced the reader to what effects on the background a self-gravitating object has, we ignore the non-linearities of the theory for now, and simply focus on the quadrupole radiation emitted on a flat background. Let's start by defining our problem; a mass μ is rotating around the origin, such that its coordinates are given as

$$\begin{aligned} x_1^p(t) &= R\cos(\omega_s t), \\ x_2^p(t) &= R\sin(\omega_s t), \\ x_3^p(t) &= 0. \end{aligned} \tag{2.100}$$

This might be understood as a two body binary with reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ obeying Newtonian gravity, or simply as a particle of mass μ rotating inside a star, forced to move at some radius R. The latter case is instructive, as it will give us some insight as to what sort of a radiation one might expect in the case of a star (even though the theory is non-linear, and the amplitudes of the different modes cannot simply be added up to the whole star's contribution, it gives us good insight as to what frequency modes are at play.)

We first find the Mass Quadrupole moment of the system

$$\begin{aligned} M_{ij} &= \int d^3\vec{x}\, T^{00}(t,\vec{x}) x_i x_j \\ &= \mu x_i^p x_j^p. \end{aligned} \tag{2.101}$$

Where we have used the stress energy tensor derived in Appendix B, using its non-relativistic ($p^0 \approx \mu$) and flat spacetime ($\sqrt{-g} = 1$) limits.

This eventually leads to the following Mass Quadrupole

$$\begin{aligned} M_{ij} &= \mu R^2 \begin{bmatrix} \cos^2(\omega_s t) & \sin(\omega_s t)\cos(\omega_s t) & 0 \\ \sin(\omega_s t)\cos(\omega_s t) & \sin^2(\omega_s t) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{ij} \\ &= \frac{1}{2}\mu R^2 \begin{bmatrix} 1+\cos(2\omega_s t) & \sin(2\omega_s t) & 0 \\ \sin(2\omega_s t) & 1-\cos(2\omega_s t) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{ij} . \end{aligned} \tag{2.102}$$

As we have calculated previously, the quantity that is of importance to us is the second time derivative of this figure, which is

$$\ddot{M}_{ij} = -2\mu\omega_s^2 R^2 \begin{bmatrix} \cos(2\omega_s t) & \sin(2\omega_s t) & 0 \\ \sin(2\omega_s t) & -\cos(2\omega_s t) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{ij} , \tag{2.103}$$

One can check that since this is already traceless, $\ddot{Q}_{ij} = \ddot{M}_{ij}$.

The Mass Quadrupole term inside the metric perturbation, $[h_{ij}^{\mathrm{TT}}(x^0,\vec{x})]_{\text{Leading}}$, in this case is then

$$[h_{ij}^{\mathrm{TT}}(x^0,\vec{x})]_{\text{Leading}} = -\frac{4G}{r}\mu\omega_s^2 R^2 \begin{bmatrix} \cos(2\omega_s t) & \sin(2\omega_s t) & 0 \\ \sin(2\omega_s t) & -\cos(2\omega_s t) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{ij} . \tag{2.104}$$

For an observer far away on the x_3 axis. We have bypassed the projector Λ since our matrix was already Symmetric and Trace-free. We thus end up with the two polarization amplitudes h_+ and $h_\times$

$$\begin{aligned} h_+ &= -\frac{4G}{r}\mu\omega_s^2 R^2 \cos(2\omega_s t), \\ h_\times &= -\frac{4G}{r}\mu\omega_s^2 R^2 \sin(2\omega_s t). \end{aligned} \tag{2.105}$$

In order to find the polarization amplitudes for an observer arbitrarely placed far away from the source (i.e. in order to find the angular dependency of the radiation), one can first rotate the Mass Quadrupole by $R^{x_2}_{ik} R^{x_3}_{jl} Q_{ij} = Q'_{kl}$ before equation (2.104). Even though an angular dependence will appear, the frequency of the radiation remains the same; the gravitational radiation's frequency of repetition is two times the frequency of the source. This knowledge will be useful later on when making our assumptions when dealing with Post-Newtonian corrections to the source.

CHAPTER 3

GENERALIZATION TO FIND THE MULTIPOLE EXPANSION UP TO ARBITRARY ORDER

It is useful to introduce a generalization that allows one to make a multipole expansion to the solution of a sourced wave equation (such as our linearized Einstein field equation) up to arbitrary order. [8]
Some new notation has to be introduced in this section, such as the multi index notation (introduced by Blanchet & Damour [9]) which facilitates the index notation of a tensor with l separate indices into a more compact form

$$F_L \equiv F_{i_1 i_2 \ldots i_l}. \tag{3.1}$$

The Symmetric Trace Free (STF) projection of a tensor can also be denoted by brakets around the projected indices.

$K_{<L>}$ or $\hat{K}_L$ Describes the K tensor's STF components through all of its l indices.

$\epsilon_{ij<k}K_{L-1>l}$ Describes the STF components along the indices inside the brackets.

3.1 Scalar Fields

Let us first consider a scalar field ϕ satisfying the relativistic wave equation

$$\Box\phi(t, \vec{x}) = -4\pi\rho(t, \vec{x}), \tag{3.2}$$

Where the generally time dependent source $\rho(t, \vec{x})$ is considered to be localized in some region ($\rho(t, \vec{x}) = 0$ when $|\vec{x}| > d$ for some finite d).
Then one could start with the ansatz that the general solution can be found through

some arbitrary function $F_L(x^0 - |\vec{x}|)$

$$\phi(t,\vec{x}) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \Big(\frac{F_L(t-|\vec{x}|)}{|\vec{x}|} \Big). \tag{3.3}$$

Where a summation between the indices of the partial derivatives ∂_L with the indices of the arbitrary function $F_L(t - |\vec{x}|)$ is implied.

One way we could check that this does indeed satisfy the wave equation is to check

$$\begin{aligned} \Box\Big(\frac{F_L(t-|\vec{x}|)}{|\vec{x}|}\Big) &= -\frac{1}{|\vec{x}|}\frac{\partial^2}{\partial t^2}F_L(t-|\vec{x}|) + \nabla^2\frac{F_L(t-|\vec{x}|)}{|\vec{x}|} \\ &= -\frac{1}{|\vec{x}|}\frac{\partial^2}{\partial r^2}F_L(t-|\vec{x}|) + \frac{1}{|\vec{x}|^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial}{\partial r}\frac{F_L(t-|\vec{x}|)}{|\vec{x}|}\big) \\ &\qquad - \frac{L^2}{|\vec{x}|^2}\frac{F_L(t-|\vec{x}|)}{|\vec{x}|} \\ &= 0. \end{aligned} \tag{3.4}$$

Another way we can rewrite the general solution is by using the Green's function method, which in the case of a radiation problem, corresponds to the retarded Green's function

$$\phi(t,\vec{x}) = \int d^3\vec{y} \frac{1}{|\vec{x}-\vec{y}|}\rho(t - |\vec{x}-\vec{y}|, \vec{y}) \tag{3.5}$$

The explicit calculation is done in *Blanchet & Damour 1986*, Appendix B, and the function $F_L(t - |\vec{x}|)$ satisfying the equality of both these solution is

$$F_L(u) = \int d^3\vec{y}\, y_{<L>} \int_{-1}^{1} dz \delta_l(z) \rho(u + z|\vec{y}|, \vec{y}) \qquad \text{Where } u \equiv t - |\vec{x}| \text{ from now onwards,} \tag{3.6}$$

Where the function $\delta_l(z)$ defined as

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1} l!}(1-z^2)^l, \tag{3.7}$$

also satisfies the following properties

$$\begin{aligned} &\int_{-1}^{1} dz \delta_l(z) = 1, \\ &\lim_{l->\infty} \delta_l(z) = \delta(z). \end{aligned} \tag{3.8}$$

3.2 Vector Fields

Before making our way to rank two tensors satisfying the wave equation, let's first investigate an Electromagnetic vector field A^μ in the Lorentz gauge ($\partial_\mu A^\mu = 0$). This field is known to satisfy

$$\Box A^{\mu}(t,\vec{x}) = -4\pi J^{\mu}(t,\vec{x}). \tag{3.9}$$

Where the vector source is defined as $J^{\mu}(t,\vec{x}) = (\rho(t,\vec{x}), \vec{J}(t,\vec{x}))$, which is taken to be time dependent but localized (in the same manner the source of our scalar field was). Then each component of A^{μ} can be written out as scalar quantities obeying the scalar wave equation

$$\begin{aligned} \Box A^{0}(t,\vec{x}) &= -4\pi\rho(t,\vec{x}), \\ \Box A^{i}(t,\vec{x}) &= -4\pi J^{i}(t,\vec{x}). \end{aligned} \tag{3.10}$$

Which, as we demonstrated for the scalar case, have as general solution

$$\begin{aligned} A^{0}(t,\vec{x}) &= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\partial_L\Big(\frac{F_L(u)}{|\vec{x}|}\Big), \\ A^{i}(t,\vec{x}) &= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\partial_L\Big(\frac{G_{iL}(u)}{|\vec{x}|}\Big). \end{aligned} \tag{3.11}$$

Where the functions $F_L(u)$ and $G_{iL}(u)$ can be found through the expression of the solution in terms of Retarded Green's function, as was done previously.

$$\begin{aligned} F_L(u) &= \int d^3\vec{y}\, y_{<L>} \int_{-1}^{1} dz\ \delta_l(z)\rho(u + z|\vec{y}|, \vec{y}), \\ G_{iL}(u) &= \int d^3\vec{y}\, y_{<L>} \int_{-1}^{1} dz\ \delta_l(z)J_i(u + z|\vec{y}|, \vec{y}). \end{aligned} \tag{3.12}$$

Where $\delta_l(z)$ has the same definition as in the scalar case.
We also note here that G_{iL} is STF in the L indices, but it isn't symmetric nor trace-free between i and any of the L indices, and is thus further reducible into its irreducible represensentation of the $SO(3)$ rotation group using the newly defined irreducible tensors

$$\begin{aligned} U_{L+1} &= G_{<L+1>}, \\ C_L &= G_{ab<L-1}\epsilon_{i_l>ab}, \\ D_{L-1} &= G_{aaL-1}. \end{aligned} \tag{3.13}$$

When decomposed, G_{iL} becomes

$$G_{iL} = U_{iL} + \frac{l}{l+1}\epsilon_{ia<i_l}C_{L-1>a} + \frac{2l-1}{2l+1}\delta_{i<i_l}D_{L-1>}, \tag{3.14}$$

Then A^i becomes the sum of all of these STF components, among which the term containing the tensor D can be gauged away ($A^\mu \to A^\mu - \partial^\mu\theta$ all the while keeping the Lorentz gauge through $\Box\theta = 0$).

The components of our vector field solution A^μ thus become

$$\begin{aligned} A^0(t,\vec{x}) &= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\partial_L\Big(\frac{Q_L(u)}{|\vec{x}|}\Big), \\ A^i(t,\vec{x}) &= -\sum_{l=1}^{\infty} \frac{(-1)^l}{l!}\partial_{L-1}\Big(\frac{Q^{(1)}_{iL-1}(u)}{|\vec{x}|} + \frac{l}{l+1}\epsilon_{iab}\partial_{aL-1}\big(\frac{M_{bL-1(u)}}{|\vec{x}|}\big)\Big). \end{aligned} \tag{3.15}$$

Where $Q^{(1)}_{iL-1}(u) \equiv \frac{d}{du}Q_{iL-1}(u)$, and the new tensors' expressions in terms of he source vector is

$$\begin{aligned} Q_L(u) &= \int d^3\vec{y}\int_{-1}^{1} dz\Big[\delta_l(z)\rho(u+z|\vec{y}|,\vec{y}) \\ &\qquad - \frac{2l+1}{(l+1)(2l+3)}\delta_{l+1}(z)y_{<iL>}J_i^{(1)}(u+z|\vec{y}|,\vec{y})\Big] \\ M_L(u) &= \int d^3\vec{y}\int_{-1}^{1} dz\delta_l(z)\hat{y}_{<L-1}m_{i_l>}(u+z|\vec{y}|,\vec{y}) \end{aligned} \tag{3.16}$$

Where $m_i = \epsilon_{ijk}y_jJ_k$ is the *magnetization density* of the source. We have thus shown that the Electrimagnetic Vector potential can be expressed in terms of a sum of STF, time dependent multipole moments, notably the Electric moments $Q_L(u)$ and the Magnetic moments $M_L(u)$

3.3 Tensor Fields of rank 2

Consider the linearized gravitational field $\overline{h}_{\mu\nu}$ in the Lorentz gauge $\partial_\mu\overline{h}^{\mu\nu} = 0$. As we've previously seen in detail, this weak field limit of our gravitational theory obeys a wave equation of the form

$$\Box\overline{h}^{\mu\nu}(t,\vec{x}) = -16\pi GT^{\mu\nu}(t,\vec{x}). \tag{3.17}$$

As we did for the vector potential case, we can write out three separate equation which behave similarly to the scalar case [10]

$$\begin{aligned} \Box\overline{h}^{00}(t,\vec{x}) &= -16\pi GT^{00}(t,\vec{x}), \\ \Box\overline{h}^{0i}(t,\vec{x}) &= -16\pi GT^{0i}(t,\vec{x}), \\ \Box\overline{h}^{ij}(t,\vec{x}) &= -16\pi GT^{ij}(t,\vec{x}). \end{aligned} \tag{3.18}$$

The computation can be followed through *Damour & Iyer (1991) [9]*, where the solutions can be expressed as

$$\begin{aligned}
\overline{h}^{00} &= 4G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\Big(\frac{M_L(u)}{r}\Big),\\
\overline{h}^{0i} &= -4G\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\partial_{L-1}\Big(\frac{M^{(1)}_{iL-1}(u)}{r}+\frac{l}{l+1}\epsilon_{iab}\partial_a\big(\frac{S_{bL-1}(u)}{r}\big)\Big),\\
\overline{h}^{ij} &= 4G\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\partial_{L-2}\Big(\frac{M^{(2)}_{ijL-2}(u)}{r}+\frac{2l}{l+1}\partial_a\big(\frac{\epsilon_{ab(i}S^{(1)}_{j)bL-2}(u)}{r}\big)\Big).
\end{aligned} \tag{3.19}$$

Where we have defined two new STF tensors, M_L and S_L, as follows

$$\begin{aligned}
M_L(u) = &\int d^3\vec{y}\int_{-1}^{1}dz\Big(\delta_l(z)\hat{x}_L\sigma-\frac{4(2l+1)\delta_{l+1}(z)}{(l+1)(2l+3)}\hat{x}_{iL}\sigma^{(1)}_i,\\
&+\frac{2(2l+1)\delta_{l+2}(z)}{(l+1)(l+2)(2l+5)}\hat{x}_{ijL}\sigma^{(2)}_{ij}\Big)(u+z|\vec{y}|,\vec{y}),\\
S_L(u) = &\int d^3\vec{y}\int_{-1}^{1}dz\epsilon_{ab<i_l}\Big(\delta_l(z)\hat{x}_{L-1>a}\sigma_b\\
&-\frac{(2l+1)\delta_{l+1}(z)}{(l+2)(2l+3)}\hat{x}_{L-1>ac}\sigma^{(1)}_{bc}\Big)(u+z|\vec{y}|,\vec{y}),
\end{aligned} \tag{3.20}$$

Where σ, σ_i and σ_{ij} are expressed in terms of stress-energy tensor as

$$\begin{aligned}
\sigma &= T^{00}+T^{ii},\\
\sigma_i &= T^{0i},\\
\sigma_{ij} &= T^{ij}.
\end{aligned} \tag{3.21}$$

CHAPTER 4

MULTIPOLAR POST-MINKOWSKIAN SOLUTION OUTSIDE A POST-NEWTONIAN SOURCE

4.1 PN Expansion of the Einstein field equations

For this section of the chapter, we shall follow the methods employed in Weinberg (Chapter 9) [11] to argue the validity of the post-Newtonian approach.
We first begin by arguing that the velocities inside the source are such that $\epsilon = \frac{v}{c} \ll 1$. If we neglect the radiation emission, we can safely assume that a system subject to conservative forces will be invariant under time reversal. Since the velocity itself is odd under such a transformation, while g_{00} and g_{ij} are even and g_{0i} is odd. This implies that there is such a coordinate for which, under time reversal, the metric components of the system can be expanded into the velocities of the system involved as

$$\begin{aligned} g_{00} &= -1 +^{(2)} g_{00} +^{(4)} g_{00} + \dots, \\ g_{0i} &=^{(3)} g_{0i} +^{(5)} g_{0i} + \dots, \\ g_{ij} &= \delta_{ij} +^{(2)} g_{ij} +^{(4)} g_{ij} + \dots, \end{aligned} \tag{4.1}$$

Where ${}^{(n)}g_{\mu\nu}$ denotes terms of order ϵ^n inside $g_{\mu\nu}$. The inverse metric reads similarly

$$\begin{aligned} g^{00} &= -1 +^{(2)} g^{00} +^{(4)} g^{00} + \dots, \\ g^{0i} &=^{(3)} g^{0i} +^{(5)} g^{0i} + \dots, \\ g^{ij} &= \delta_{ij} +^{(2)} g^{ij} +^{(4)} g^{ij} + \dots. \end{aligned} \tag{4.2}$$

One thus finds, through the contraction of two metric tensors $g_{\mu\alpha}g^{\alpha\nu} = \delta_{\mu}^{\ \nu}$, that the leading corrections to the metric and the inverse metric are connected through the following relations

$$
\begin{aligned}
{}^{(2)}g_{00} &= -{}^{(2)}g^{00}, \\
{}^{(3)}g_{0i} &={}^{(3)} g_{0i}, \\
{}^{(2)}g_{ij} &= -{}^{(2)}g_{ij}.
\end{aligned} \tag{4.3}
$$

The scales of distances and of times inside our system are of order a and $\frac{a}{v}$, respectively. We can thus give an order of magnitude estimate to the following derivatives when they act on objects of our system

$$
\begin{aligned}
\frac{\partial}{\partial x^i} &= \mathcal{O}(\frac{1}{a}), \\
\frac{\partial}{\partial t} &= \mathcal{O}(\frac{v}{a}).
\end{aligned} \tag{4.4}
$$

Using this, coupled with (4.1) , (4.2) and (4.3), we find that the connections' expansions goes as

$$
\begin{aligned}
\Gamma^{\mu}_{\ \alpha\beta} &={}^{(2)} \Gamma^{\mu}_{\ \alpha\beta} +{}^{(4)} \Gamma^{\mu}_{\ \alpha\beta} + \ldots \qquad \text{For } \Gamma^{i}_{\ 00}, \Gamma^{i}_{\ jk} \text{ and } \Gamma^{0}_{\ i0}, \\
\Gamma^{\mu}_{\ \alpha\beta} &={}^{(3)} \Gamma^{\mu}_{\ \alpha\beta} +{}^{(5)} \Gamma^{\mu}_{\ \alpha\beta} + \ldots \qquad \text{For } \Gamma^{i}_{\ j0}, \Gamma^{0}_{\ 00} \text{ and } \Gamma^{0}_{\ ij},
\end{aligned} \tag{4.5}
$$

While the lowest order corrections to the Riemann tensor can be found to be

$$
\begin{aligned}
R_{00} &= \frac{{}^{(2)}\Gamma^{i}_{\ 00}}{\partial x^i} + \frac{{}^{(3)}\Gamma^{i}_{\ i0}}{\partial t} - \frac{{}^{(4)}\Gamma^{i}_{\ 00}}{\partial x^i} +{}^{(2)} \Gamma^{0}_{\ i0}\, {}^{(2)}\Gamma^{i}_{\ 00} -{}^{(2)} \Gamma^{i}_{\ 00}\, {}^{(2)}\Gamma^{j}_{\ ij} + \ldots, \\
R_{i0} &= \frac{{}^{(2)}\Gamma^{j}_{\ ij}}{\partial t} + \frac{{}^{(3)}\Gamma^{j}_{\ i0}}{\partial x^j} + \ldots, \\
R_{ij} &= \frac{{}^{(2)}\Gamma^{0}_{\ i0}}{\partial x^j} + \frac{{}^{(2)}\Gamma^{k}_{\ ik}}{\partial x^j} - \frac{{}^{(2)}\Gamma^{k}_{\ ij}}{\partial x^k} + \ldots,
\end{aligned} \tag{4.6}
$$

Which, when simplified to their metric components and expressed in harmonic coordinates $g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu} = 0$, yields the following expressions

$$
\begin{aligned}
R_{00} &= \frac{1}{2}\nabla^2\, {}^{(2)}g_{00} + \frac{1}{2}\nabla^2\, {}^{(4)}g_{00} - \frac{1}{2}\frac{\partial^2\, {}^{(2)}g_{00}}{\partial t^2} \\
&\qquad - \frac{1}{2}\, {}^{(2)}g_{ij}\frac{\partial^2\, {}^{(2)}g_{00}}{\partial x^i \partial x^j} + \frac{1}{2}\big(\nabla^2\, {}^{(2)}g_{00}\big)^2 + \ldots, \\
R_{i0} &= \frac{1}{2}\nabla^2\, {}^{(3)}g_{i0} + \ldots, \\
R_{ij} &= \frac{1}{2}\nabla^2\, {}^{(2)}g_{ij} + \ldots,
\end{aligned} \tag{4.7}
$$

Where the ∇^2 are Laplacian operators.

The matter stress energy tensor can be expanded in a similar fashion

$$
\begin{aligned}
T^{00} &={}^{(0)} T^{00} +{}^{(2)} T^{00} + \ldots, \\
T^{0i} &={}^{(1)} T^{0i} +{}^{(3)} T^{0i} + \ldots, \\
T^{ij} &={}^{(2)} T^{ij} +{}^{(4)} T^{ij} + \ldots.
\end{aligned} \tag{4.8}
$$

In order to facilitate the task ahead, let us simplify our task by using the Einstein field equations cast into the following form

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\big). \tag{4.9}$$

Which, after plugging in the expansions we've mentionned above, gives us an equation for each order in the expansion. The first few orders equations are given as

$$\begin{aligned}
\nabla^2\,{}^{(2)}g_{00} &= \frac{8\pi G}{c^4}{}^{(0)}T^{00},\\
\nabla^2\,{}^{(2)}g_{ij} &= \frac{8\pi G}{c^4}\delta_{ij}\,{}^{(0)}T^{00},\\
\nabla^2\,{}^{(3)}g_{0i} &= -\frac{16\pi G}{c^4}{}^{(0)}T^{0i},\\
\nabla^2\,{}^{(4)}g_{00} &= \Big(\frac{\partial\,{}^{(2)}g_{00}}{\partial x^i}\Big)\Big(\frac{\partial\,{}^{(2)}g_{00}}{\partial x^i}\Big) - \frac{1}{c^2}\frac{\partial^2\,{}^{(2)}g_{00}}{\partial t^2} -{}^{(2)}g_{ij}\frac{\partial^2\,{}^{(2)}g_{00}}{\partial x^i\partial x^j},\\
&\quad + \frac{8\pi G}{c^4}\Big({}^{(2)}T^{00} + {}^{(2)}T^{00} - 2\,{}^{(2)}g_{00}\,{}^{(0)}T^{00}\Big).
\end{aligned} \tag{4.10}$$

The first equation is at Newtonian order (0PN), and admits as a solution

$${}^{(2)}g_{00} = -2\phi(t,\vec{x}) \qquad \text{With } \phi(t,\vec{x}) = \frac{G}{c^4}\int d^3\vec{x}'\frac{{}^{(0)}T^{00}(t,\vec{x}')}{|\vec{x}-\vec{x}'|}. \tag{4.11}$$

From which one can notice the similarity with the weak field metric we've postulated in Chapter 1.

The next three equations in (4.10) are said to be of (1PN) order. Solving for ${}^{(2)}g_{ij}$ and ${}^{(3)}g_{0i}$ first

$$\begin{aligned}
{}^{(2)}g_{ij} &= -2\delta_{ij}\phi(t,\vec{x}) \qquad \text{With } \phi(t,\vec{x}) = \frac{G}{c^4}\int d^3\vec{x}'\frac{{}^{(0)}T^{00}(t,\vec{x}')}{|\vec{x}-\vec{x}'|},\\
{}^{(3)}g_{ij} &= \zeta(t,\vec{x}) \qquad \text{With } \zeta(t,\vec{x}) = \frac{4G}{c^4}\int d^3\vec{x}'\frac{{}^{(1)}T^{0i}(t,\vec{x}')}{|\vec{x}-\vec{x}'|}.
\end{aligned} \tag{4.12}$$

For the solution of ${}^{(4)}g_{00}$, we make use of the vector calculus identity $\partial_i\phi\partial_i\phi = \frac{1}{2}\nabla^2\phi^2 - \phi\nabla^2\phi$, and take the solution ansatz

$${}^{(4)}g_{00} = -2\big(\phi^2(t,\vec{x}) + \varphi(t,\vec{x})\big). \tag{4.13}$$

We then find that the postulated $\varphi(t,\vec{x})$ function satisfies the following equation

$$\nabla^2\varphi = \partial_0^2\phi + \frac{4\pi G}{c^4}\Big[{}^{(2)}T^{00} +{}^{(2)}T^{ii}\Big]. \tag{4.14}$$

Requiring the field vanishes at infinities, we have the following solution

$$\varphi(t,\vec{x}) = -\int d^3\vec{x}'\frac{\partial_0^2\phi + \frac{4\pi G}{c^4}\Big[{}^{(2)}T^{00} +{}^{(2)}T^{ii}\Big]}{|\vec{x}-\vec{x}'|}. \tag{4.15}$$

It is more convenient to work with the quantity $V = c^2(\phi + \varphi)$, since under this field and up to this order, we might write the g_{00} part of the metric as follows [8]

$$g_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \mathcal{O}(\frac{1}{c^6}), \tag{4.16}$$

Where upon the inspection of our metric equations (4.10), one also notes that this new field $V(t, \vec{x})$ satisfies the following equation (Up to 1 PN order)

$$\Box V = -4\pi G\sigma. \tag{4.17}$$

Where we have defined $\sigma = \frac{1}{c^2}\mathrm{Tr}\big[T^{\mu\nu}\big]$.

Since the retardation effects were negligible up to 1PN order, we can express the solution in terms of the retarded integral

$$V(t, \vec{x}) = G\int d^3\vec{x}' \frac{\sigma(t - |\vec{x} - \vec{x}'|/c, \vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{4.18}$$

The same treatment can be applied to the ζ_i field, where find a new field V_i, and upon defining $\sigma_i = \frac{1}{c}T^{0i}$, one finds that

$$\Box V_i = -4\pi G\sigma_i. \tag{4.19}$$

Which then implies

$$V_i(t, \vec{x}) = G\int d^3\vec{x}' \frac{\sigma_i(t - |\vec{x} - \vec{x}'|/c, \vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{4.20}$$

Which is the same as our ζ_i field up to 1PN order, so we can merely replace it with this new vector field.

So finally, our metric in terms of these scalar V and vector V_i potentials becomes

$$\begin{aligned} g_{00} &= -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \mathcal{O}(\frac{1}{c^6}), \\ g_{0i} &= -\frac{4}{c^3}V_i + \mathcal{O}(\frac{1}{c^5}), \\ g_{ij} &= \delta_{ij}\big(1 + \frac{1}{c^2}V\big) + \mathcal{O}(\frac{1}{c^4}). \end{aligned} \tag{4.21}$$

4.2 Relaxed Einstein Equations

We have studied the general behaviour of the linearized theory on a flat background in Chapter 2 of this thesis. We will now generalize the method by first deriving the relaxed Einstein field equations by following the methodology of *Landau Lifshitz 1994* [12].

We first notice how the Einstein equations inherits a property from the second bianchi identity,

$$
\begin{aligned}
R_{\mu\nu\alpha\beta;\sigma} + R_{\mu\nu\sigma\alpha;\beta} + R_{\mu\nu\beta\sigma;\alpha} &= 0, \\
R^{\alpha\beta}{}_{\alpha\beta;\sigma} + R^{\alpha\beta}{}_{\sigma\alpha;\beta} + R^{\alpha\beta}{}_{\beta\sigma;\alpha} &= 0, \qquad \leftarrow \times g^{\mu\alpha} g^{\nu\beta} \\
R_{;\sigma} - R^{\beta}_{\sigma;\beta} - R^{\alpha}_{\sigma;\alpha} &= 0, \\
R^{\alpha}_{\sigma;\alpha} - \frac{1}{2} R_{;\sigma} &= 0, \\
R^{\alpha}_{\sigma;\alpha} - \frac{1}{2} g^{\alpha}_{\sigma} R_{;\alpha} &= 0, \\
\left(R^{\alpha}_{\sigma} - \frac{1}{2} g^{\alpha}_{\sigma} R\right)_{;\alpha} &= 0, \\
G^{\alpha}_{\sigma;\alpha} &= 0,
\end{aligned}
\tag{4.22}
$$

Which encodes the information that the stress energy tensor $T^{\mu\nu}$ is covariantly conserved ($T^{\mu\nu}{}_{;\mu}$). Playing around with the expression, we have

$$
\begin{aligned}
T^{\mu}_{\nu;\mu} &= 0, \\
\partial_\mu T^{\mu}_{\nu} + \Gamma^{\mu}_{\mu\alpha} T^{\alpha}_{\nu} - \Gamma^{\alpha}_{\mu\nu} T^{\mu}_{\alpha} &= 0, \\
\partial_\mu T^{\mu}_{\nu} + \frac{1}{2} g^{\mu\beta} \left(g_{\beta\mu,\alpha} + g_{\beta\alpha,\mu} - g_{\mu\alpha,\beta}\right) T^{\alpha}_{\nu} & \\
- \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right) T^{\mu}_{\alpha} &= 0, \\
\partial_\mu T^{\mu}_{\nu} + \frac{1}{2} g^{\mu\beta} \left(g_{\beta\mu,\alpha} + \cancelto{i}{g_{\mu\alpha,\beta}} - \cancelto{i}{g_{\mu\alpha,\beta}}\right) T^{\alpha}_{\nu} & \\
- \frac{1}{2} \left(g_{\beta\mu,\nu} + \cancelto{ii}{g_{\mu\nu,\beta}} - \cancelto{ii}{g_{\mu\nu,\beta}}\right) T^{\mu\beta} &= 0,
\end{aligned}
\tag{4.23}
$$

Where on the last line, the symmetry of $g^{\mu\beta}$ and $T^{\mu\beta}$ in their indices were used to change the dummy indices inside the parenthesis. We've derived in Appendix C that $g^{\mu\nu} g_{\mu\nu,\sigma} = 2\frac{\partial_\sigma \sqrt{-g}}{\sqrt{-g}}$, using this, we can simplify the above even further

$$
\begin{aligned}
T^{\mu}_{\nu;\mu} = \partial_\mu T^{\mu}_{\nu} + \frac{\partial_\mu \sqrt{-g}}{\sqrt{-g}} T^{\mu}_{\nu} - \frac{1}{2} g_{\alpha\mu,\nu} T^{\alpha\mu} &= 0, \\
\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} T^{\mu}_{\nu}\right) - \frac{1}{2} g_{\alpha\mu,\nu} T^{\alpha\mu} &= 0.
\end{aligned}
\tag{4.24}
$$

Let's consider a special coordinate frame ($\partial_\alpha g_{\mu\nu} = 0$ is satisfied but $g_{\mu\nu} = 0$ isn't necessarily true), the above relation then becomes

$$
T^{\mu}_{\nu,\mu} = 0. \tag{4.25}
$$

Where we have also used the fact that in these coordinate systems, we have $\partial_\mu \sqrt{-g} = 0$.

Or in the totally contravariant form

$$T^{\mu\nu}_{\ \ \ ,\mu} = 0. \tag{4.26}$$

We can then define $T^{\mu\nu}$ through a new quantity $\eta^{\mu\nu\alpha}$

$$T^{\mu\nu} = \eta^{\mu\nu\alpha}_{\ \ \ \ \ ,\alpha}. \tag{4.27}$$

We can assess an anti-symmetry to this new tensor

$$T^{\mu\nu}_{\ \ \ ,\mu} = \eta^{\mu\nu\alpha}_{\ \ \ \ \ ,\alpha\mu} = 0 \rightarrow \eta^{\mu\nu\alpha} = -\eta^{\alpha\nu\mu} \text{ or since } T^{\mu\nu} \text{ is symmetric } \eta^{\nu\mu\alpha} = -\eta^{\nu\alpha\mu}. \tag{4.28}$$

We can put the stress energy tensor in this form by using the Einstein equation, but first, I want to prove an identity relating the fully covariant Riemann tensor to the metric.

For a general coordinate frame, we have the following indentity (we introduce the Christoffel's symbol of the first kind $\Gamma_{\mu\nu\alpha} = g_{\mu\sigma}\Gamma^{\sigma}_{\ \nu\alpha}$)

$$\begin{aligned}
R_{\mu\nu\alpha\beta} &= g_{\mu\lambda}R^{\lambda}_{\ \nu\alpha\beta} = g_{\mu\lambda}\big(\Gamma^{\lambda}_{\ \beta\nu,\alpha} - \partial_\beta\Gamma^{\lambda}_{\ \alpha\nu} + \Gamma^{\lambda}_{\ \alpha\sigma}\Gamma^{\sigma}_{\ \beta\nu} - \Gamma^{\lambda}_{\ \beta\sigma}\Gamma^{\sigma}_{\ \alpha\nu}\big) \\
&= g_{\mu\lambda}\big((g^{\lambda\sigma}\Gamma_{\sigma\beta\nu})_{,\alpha} - (g^{\lambda\sigma}\Gamma_{\sigma\alpha\nu})_{,\beta}\big) + g^{\sigma\lambda}\big(\Gamma_{\mu\alpha\sigma}\Gamma_{\lambda\beta\nu} - \Gamma_{\mu\beta\sigma}\Gamma_{\lambda\alpha\nu}\big) \\
&= g_{\mu\lambda}g^{\lambda\sigma}_{\ \ \ ,\alpha}\Gamma_{\sigma\beta\nu} + \delta_{\mu}^{\ \sigma}\Gamma_{\sigma\beta\nu,\alpha} - g_{\mu\lambda}g^{\lambda\sigma}_{\ \ \ ,\beta}\Gamma_{\sigma\alpha\nu} \\
&\qquad\qquad - \delta_{\mu}^{\ \sigma}\Gamma_{\sigma\alpha\nu,\beta} + g^{\sigma\lambda}\big(\Gamma_{\mu\alpha\sigma}\Gamma_{\lambda\beta\nu} - \Gamma_{\mu\beta\sigma}\Gamma_{\lambda\alpha\nu}\big) \\
&= \Gamma_{\mu\beta\nu,\alpha} - \Gamma_{\mu\alpha\nu,\beta} + \cancelto{0}{(\delta_{\mu}^{\ \sigma})_{,\alpha}}\Gamma_{\sigma\beta\nu} - g_{\mu\lambda,\alpha}g^{\lambda\sigma}\Gamma_{\sigma\beta\nu} \\
&\qquad\qquad - \cancelto{0}{(\delta_{\mu}^{\ \sigma})_{\beta}}\Gamma_{\sigma\alpha\nu} + g_{\mu\lambda,\beta}g^{\lambda\sigma}\Gamma_{\sigma\alpha\nu} + g^{\sigma\lambda}\big(\Gamma_{\mu\alpha\sigma}\Gamma_{\lambda\beta\nu} - \Gamma_{\mu\beta\sigma}\Gamma_{\lambda\alpha\nu}\big) \\
&= \Gamma_{\mu\beta\nu,\alpha} - \Gamma_{\mu\alpha\nu,\beta} + g^{\sigma\lambda}\big(\Gamma_{\lambda\beta\nu}\underbrace{(\Gamma_{\mu\alpha\sigma} - g_{\mu\sigma,\alpha})}_{=-\Gamma_{\sigma\alpha\mu}} - \Gamma_{\lambda\alpha\nu}\underbrace{(\Gamma_{\mu\beta\sigma} - g_{\mu\sigma,\beta})}_{=-\Gamma_{\sigma\beta\mu}}\big) \\
&= \Gamma_{\mu\beta\nu,\alpha} - \Gamma_{\mu\alpha\nu,\beta} + g^{\sigma\lambda}\big(\Gamma_{\lambda\alpha\nu}\Gamma_{\sigma\beta\mu} - \Gamma_{\lambda\beta\nu}\Gamma_{\sigma\alpha\mu}\big) \\
&= \Gamma_{\mu\beta\nu,\alpha} - \Gamma_{\mu\alpha\nu,\beta} + g_{\sigma\lambda}\big(\Gamma^{\lambda}_{\ \alpha\nu}\Gamma^{\sigma}_{\ \beta\mu} - \Gamma^{\lambda}_{\ \beta\nu}\Gamma^{\sigma}_{\ \alpha\mu}\big) \\
&= \frac{1}{2}\big(g_{\beta\mu,\nu\alpha} + \cancelto{i}{g_{\nu\mu,\beta\alpha}} - g_{\beta\nu,\mu\alpha} - g_{\alpha\mu,\nu\beta} - \cancelto{i}{g_{\nu\mu,\alpha\beta}} + g_{\alpha\nu,\mu\beta}\big) \\
&\qquad\qquad\qquad\qquad + g_{\sigma\lambda}\big(\Gamma^{\lambda}_{\ \alpha\nu}\Gamma^{\sigma}_{\ \beta\mu} - \Gamma^{\lambda}_{\ \beta\nu}\Gamma^{\sigma}_{\ \alpha\mu}\big) \\
&= \frac{1}{2}\big(g_{\beta\mu,\nu\alpha} + g_{\alpha\nu,\mu\beta} - g_{\beta\nu,\mu\alpha} - g_{\alpha\mu,\nu\beta}\big) + g_{\sigma\lambda}\big(\Gamma^{\lambda}_{\ \alpha\nu}\Gamma^{\sigma}_{\ \beta\mu} - \Gamma^{\lambda}_{\ \beta\nu}\Gamma^{\sigma}_{\ \alpha\mu}\big).
\end{aligned} \tag{4.29}$$

Returning to our problem in the specially picked coordinate system, we know that we can write the Einstein equation as

$$T^{\mu\nu} = \frac{1}{8\pi G}\big(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\big). \tag{4.30}$$

into which one can plug in the values of $R^{\mu\nu}$ and R from the equality we've just found in (4.29). We have to note that both $\Gamma^{\mu}{}_{\nu\alpha}$ and $\Gamma_{\mu\nu\alpha}$ are vanishing in this frame

$$T^{\mu\nu} = \frac{1}{16\pi G}\partial_\alpha\Big(\frac{1}{(-g)}\partial_\beta\big((-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})\big)\Big). \tag{4.31}$$

In this frame we can simply take the $\frac{1}{(-g)}$ out of the derivative to get

$$(-g)T^{\mu\nu} = \frac{1}{16\pi G}\chi^{\mu\nu\alpha\beta}{}_{,\alpha\beta}. \tag{4.32}$$

Where $\chi^{\mu\nu\alpha\beta} = \big((-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})\big)$.
Let's say that in a general frame, this equation is balanced out by some quantity $t^{\mu\nu}$ defined such that, in such a general frame

$$\tau^{\mu\nu} = gT^{\mu\nu} + \frac{1}{16\pi G}\chi^{\mu\nu\alpha\beta}{}_{,\alpha\beta}. \tag{4.33}$$

From which the symmetry of $\tau^{\mu\nu}$ is apparent due to the symmetries of the quantities $T^{\mu\nu}$ and $\chi^{\mu\nu\alpha\beta}{}_{,\alpha\beta}$ in their μ and ν indices. Please remark that we have made no assumption indicating this might be a tensor.
Using the Einstein field equations, and defining a new variable $\mathfrak{h}^{\mu\nu} = \mathfrak{g}^{\mu\nu} - \eta^{\mu\nu}$, where $\mathfrak{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$ and $\eta^{\mu\nu}$ is the inverse flat minkowski metric, we can express the new quantity which we rename for simplicity $\chi^{\mu\nu\alpha\beta}{}_{,\alpha\beta} = \Lambda^{\mu\nu}$ in harmonic coordinates ($\partial_\alpha \mathfrak{h}^{\alpha\beta} = 0$) as [13]

$$\begin{aligned}\Lambda^{\mu\nu} = &-\mathfrak{h}^{\alpha\beta}\partial_\alpha\partial_\beta\mathfrak{h}^{\mu\nu} + \partial_\alpha\mathfrak{h}^{\mu\beta}\partial_\beta\mathfrak{h}^{\nu\alpha} + \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\partial_\sigma\mathfrak{h}^{\alpha\lambda}\partial_\lambda\mathfrak{h}^{\beta\sigma} \\ &- g^{\mu\alpha}g_{\beta\sigma}\partial_\lambda\mathfrak{h}^{\nu\sigma}\partial_\alpha\mathfrak{h}^{\beta\lambda} - g^{\nu\alpha}g_{\beta\sigma}\partial_\lambda\mathfrak{h}^{\mu\sigma}\partial_\alpha\mathfrak{h}^{\beta\lambda} + g_{\alpha\beta}g^{\sigma\lambda}\partial_\sigma\mathfrak{h}^{\mu\alpha}\partial_\sigma\mathfrak{h}^{\nu\beta} \\ &+ \frac{1}{8}\big(2g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}\big)\big(2g_{\sigma\kappa}g_{\delta\gamma} - g_{\delta\kappa}g_{\sigma\gamma}\big)\partial_\alpha\mathfrak{h}^{\sigma\gamma}\partial_\beta\mathfrak{h}^{\kappa\delta}.\end{aligned} \tag{4.34}$$

Having defined this new quantity, our Einstein field equation can now me recast into the following form (again, using harmonic or de Donder gauge)

$$\Box\mathfrak{h}^{\mu\nu} = 16\pi G\tau^{\mu\nu}(\mathfrak{h}). \tag{4.35}$$

We know that the metric tensor is highly non-linear in the newly defined dynamical variable $\mathfrak{h}^{\mu\nu}$, this means that $\Lambda^{\mu\nu}$ also inherits such a non-linearity. Expressing $\Lambda^{\mu\nu}$

in powers of $\mathfrak{h}^{\mu\nu}$, while noting that we do not have terms of the first order due to how we've defined the quantity, we express

$$\Lambda^{\mu\nu} = N^{\mu\nu}[\mathfrak{h},\mathfrak{h}] + M^{\mu\nu}[\mathfrak{h},\mathfrak{h},\mathfrak{h}] + \mathcal{O}(\mathfrak{h}^4). \tag{4.36}$$

Where $N^{\mu\nu}[\mathfrak{h},\mathfrak{h}]$ and $M^{\mu\nu}[\mathfrak{h},\mathfrak{h},\mathfrak{h}]$ represents all of the terms that are quadratic and cubic in $\mathfrak{h}$, respectively. One can find [14] their full expressions as

$$\begin{aligned}
N^{\mu\nu} = &-\mathfrak{h}^{\alpha\beta}\partial_\alpha\partial_\beta\mathfrak{h}^{\mu\nu} + \frac{1}{2}\partial^\mu\mathfrak{h}_{\alpha\beta}\partial^\nu\mathfrak{h}_{\alpha\beta} - \frac{1}{4}\partial^\mu\mathfrak{h}\partial^\nu\mathfrak{h} \\
&+ \partial_\alpha\mathfrak{h}^{\mu\beta}(\partial^\alpha\mathfrak{h}^\nu{}_\beta + \partial_\beta\mathfrak{h}^{\nu\alpha}) - 2\partial^{(\mu}\mathfrak{h}_{\alpha\beta}\partial^\nu\mathfrak{h}^{\nu)\beta} \\
&+ \eta^{\mu\nu}\Big[-\frac{1}{4}\partial_\alpha\mathfrak{h}_{\beta\sigma}\partial^\alpha\mathfrak{h}^{\beta\sigma} + \frac{1}{8}\partial_\alpha\mathfrak{h}\partial^\alpha\mathfrak{h} + \frac{1}{2}\partial_\alpha\mathfrak{h}_{\beta\sigma}\partial^\beta\mathfrak{h}^{\alpha\sigma}\Big], \\
M^{\mu\nu} = &-\mathfrak{h}^{\alpha\beta}\big(\partial^\mu\mathfrak{h}_{\alpha\sigma}\partial^\nu\mathfrak{h}^\sigma{}_\beta + \partial_\sigma\mathfrak{h}^\mu{}_\alpha\partial^\sigma\mathfrak{h}^\nu{}_\beta - \partial_\alpha\mathfrak{h}^\mu{}_\sigma\partial_\beta\mathfrak{h}^{\nu\sigma}\big) \\
&+ \mathfrak{h}^{\mu\nu}\Big[-\frac{1}{4}\partial_\alpha\mathfrak{h}_{\beta\sigma}\partial^\alpha\mathfrak{h}^{\beta\sigma} + \frac{1}{8}\partial_\alpha\mathfrak{h}\partial^\alpha\mathfrak{h} + \frac{1}{2}\partial_\alpha\mathfrak{h}_{\beta\sigma}\partial^\beta\mathfrak{h}^{\alpha\sigma}\Big] + \frac{1}{2}\mathfrak{h}^{\alpha\beta}\partial^{(\mu}\mathfrak{h}_{\alpha\beta}\partial^{\nu)}\mathfrak{h} \\
&+ 2\mathfrak{h}^{\alpha\beta}\partial_\sigma\mathfrak{h}_\alpha{}^{(\mu}\partial^{\nu)}\mathfrak{h}^\sigma{}_\beta + \mathfrak{h}^{\alpha(\mu}\Big[\partial^{\nu)}\mathfrak{h}_{\beta\sigma}\partial_\alpha\mathfrak{h}^{\beta\sigma} - 2\partial_\beta\mathfrak{h}^{\nu)}_\sigma\partial_\alpha\mathfrak{h}^{\beta\sigma} - \frac{1}{2}\partial^{\nu)}\mathfrak{h}\partial_\alpha\mathfrak{h}\Big] \\
&\eta^{\mu\nu}\Big[\frac{1}{8}\mathfrak{h}^{\alpha\beta}\partial_\alpha\mathfrak{h}\partial_\beta\mathfrak{h} - \frac{1}{4}\mathfrak{h}^{\alpha\beta}\partial_\sigma\mathfrak{h}_{\alpha\beta}\partial^\sigma\mathfrak{h} \\
&\quad - \frac{1}{4}\mathfrak{h}^{\alpha\beta}\partial_\alpha\mathfrak{h}_{\sigma\lambda}\partial_\beta\mathfrak{h}^{\sigma\lambda} - \frac{1}{2}\mathfrak{h}^{\alpha\beta}\partial_\sigma\mathfrak{h}_{\alpha\lambda}\partial^\lambda\mathfrak{h}^\sigma{}_\beta + \frac{1}{2}\mathfrak{h}^{\alpha\beta}\partial_\sigma\mathfrak{h}^\lambda{}_\alpha\partial^\sigma\mathfrak{h}_{\beta\lambda}\Big].
\end{aligned} \tag{4.37}$$

4.3 Iterative multipolar post-Minkowskian solution

In this section, we shall iteratively and approximately solve the field equation (4.35) with the gauge condition $\partial_\alpha\mathfrak{h}^{\alpha\beta} = 0$. Before doing so, I'd like to specify the assumptions under which we undertake such an action:

1. The matter Stress Energy tensor $T^{\mu\nu}$ is localized at some point in space (it is also said that $T^{\mu\nu}$ has spatially compact support). This means that there exists some a for which all points satisfying $a < |\vec{x}|$ also satisfy $T^{\mu\nu}(t,\vec{x}) = 0$.
2. The gravitational source term $\Lambda^{\mu\nu}$ is divergence free outside the source. This means that $\partial_\mu\Lambda^{\mu\nu} = 0$ when $a < |\vec{x}|$
3. The matter distribution inside the source is spatially smooth : $T^{\mu\nu} \in C^\infty(\mathbb{R}^3)$. This also implies this approach does not include black holes, but merely non-singular objects.

4. The matter source can be expanded in post-Newtonian formalism, meaning that a small parameter $\epsilon = \max\left\{ \left|\frac{T^{0i}}{T^{00}}\right|, \left|\frac{T^{ij}}{T^{00}}\right| \right\} \ll 1$ must exist.

5. The gravitational field is required to be time independent before some finite distant time $-\mathcal{T}$. This condition naturally brings the no-incoming radiation condition.

The metric outside the source, in the external domain where $d < r < \infty$ can be expanded into its post-Minkowskian form. For our weakly self-gravitating object where $R_s/d \ll 1$, we can expand the variable $\mathfrak{h}^{\mu\nu}$ as an expansion in R_s/r. Since $R_s = 2Gm$ is proportional to G, we can express this epansion as powers of this constant

$$\begin{aligned} \sqrt{-g}g^{\mu\nu} &= \eta^{\mu\nu} + G\mathfrak{h}_1^{\mu\nu} + G^2\mathfrak{h}_2^{\mu\nu} + \dots, \\ \mathfrak{h}^{\mu\nu} &= \sum_{n=1}^{\infty} G^n \mathfrak{h}_n^{\mu\nu}. \end{aligned} \tag{4.38}$$

We plug this into our relaxed Einstein equation, while noting that outside the source, $T^{\mu\nu} = 0$. Equating terms in the same order of G yields

$$\begin{aligned} \Box\mathfrak{h}^{\mu\nu} &= \Lambda^{\mu\nu}, \\ \Box\mathfrak{h}_n^{\mu\nu} &= \Lambda_n^{\mu\nu}, \end{aligned} \tag{4.39}$$

Where $\Lambda_n^{\mu\nu}$ denotes all the terms inside $\Lambda^{\mu\nu}$ having G^n as its coefficient, writing them out explicitly

$$\begin{aligned} &\Box\mathfrak{h}_1^{\mu\nu} = 0, \\ &\Box\mathfrak{h}_2^{\mu\nu} = N^{\mu\nu}[\mathfrak{h}_1, \mathfrak{h}_1], \\ &\Box\mathfrak{h}_3^{\mu\nu} = M^{\mu\nu}[\mathfrak{h}_1, \mathfrak{h}_1, \mathfrak{h}_1] + N^{\mu\nu}[\mathfrak{h}_1, \mathfrak{h}_2] + N^{\mu\nu}[\mathfrak{h}_2, \mathfrak{h}_1]. \\ &\quad \dots \end{aligned} \tag{4.40}$$

While the gauge condition reads

$$\partial_\mu \mathfrak{h}_n^{\mu\nu} = 0, \tag{4.41}$$

Where one can solve these equations iteratively, starting from the first equation, solving for $\mathfrak{h}_1$, then plugging that answer into the next set of equations and solving it for higher orders.

To this end, one can find the general solution of the first equation in terms of retarded multipolar waves,

$$\mathfrak{h}_1^{\mu\nu} = \sum_{l=0}^{\infty} \partial_L \Big[\frac{1}{r} K_L^{\mu\nu}(t-r)\Big], \tag{4.42}$$

Which does, in fact, solve $\Box \mathfrak{h}_1^{\mu\nu} = 0$. But if we also want to use the $\partial_\mu \mathfrak{h}^{\mu\nu} = 0$ gauge condition, then this solution isn't fit to be used. Instead, we can use the gauge freedom to also satisfy the gauge condition [10, 15]

$$\mathfrak{h}_1^{\mu\nu} = k_1^{\mu\nu} + \partial^\mu \varphi_1^\nu + \partial^\nu \varphi_1^\mu - \eta^{\mu\nu}\partial_\alpha \varphi_1^\alpha. \tag{4.43}$$

Where $k_1^{\mu\nu}$ is defined as

$$\begin{aligned}
k_1^{00} &= -4\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \Big[\frac{1}{r} I_L(t-r)\Big],\\
k_1^{0i} &= 4\sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \Big[\partial_{L-1}\big(\frac{1}{r} I^{(1)}_{iL-1}(t-r)\big) + \frac{l}{l+1}\epsilon_{iab}\partial_{aL-1}\big(\frac{1}{r} J_{bL-1}(t-r)\big)\Big],\\
k_1^{ij} &= -4\sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \Big[\partial_{L-2}\big(\frac{1}{r} I^{(2)}_{ijL-2}(t-r)\big) + \frac{2l}{l+1}\partial_{aL-2}\big(\frac{1}{r} \epsilon_{ab(i}J^{(1)}_{j)bL-2}(t-r)\big)\Big].
\end{aligned} \tag{4.44}$$

Where the functions $I_L(t-r)$ and $J_L(t-r)$ are time dependent functions of the retarded time, except for I, I_i and J_i which are proportional to the mass, the center-of-mass position and the angular momentum of the system, respectively. These functions encode the physical properties of the source.

The gauge vector φ_1^μ is itself defined as

$$\begin{aligned}
\varphi_1^0 &= 4\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \Big[\frac{1}{r} W_L(t-r)\Big],\\
\varphi_1^i &= -4\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_{iL} \Big[\frac{1}{r} X_L(t-r)\Big]\\
&\quad - 4\sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \Big[\partial_{L-1}\big(\frac{1}{r} Y_{iL-1}(t-r)\big) + \frac{l}{l+1}\epsilon_{iab}\partial_{aL-1}\big(\frac{1}{r} Z_{bL-1}(t-r)\big)\Big].
\end{aligned} \tag{4.45}$$

One thing to specify here is that the set of moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$ describing a certain source are not equivalent to the one described by the set $(I_L, J_L, 0, 0, 0, 0)$. What is meant by this is that we cannot simply equate the gauge moments (W_L, X_L, Y_L, Z_L) to zero and hope it describes the same system. But rather, another set of moments, say

$(M_L, P_L, 0, 0, 0, 0)$, that differ from the initial moments I_L and J_L by some non-linear corrections is in fact isometric to our first choice of moments describing the system. The reason why we do not simply give the new M_L and P_L (called the cannonical moments) is that they do not immediatly have a closed form [13] such as the ones we have presented in (4.44) and (4.45).

The iteration can be made from the above first order solution $\mathfrak{h}_1^{\mu\nu}$ to the desired order of the post-Minkowskian expansion. The method, a priori, seems straightforward;

$$\begin{aligned} \Box \mathfrak{h}_n^{\mu\nu} &= \Lambda_n^{\mu\nu}, \\ \mathfrak{h}_n^{\mu\nu} &= \Box_{\text{ret}}^{-1}\left[\Lambda_n^{\mu\nu}\right]. \end{aligned} \tag{4.46}$$

But one notices that in the right hand side of the last equation above, the retarded integral has an integrand which is the product of a multitude of multipoles, meaning its poles are no longer simple. This means that the integral itself will become singular thus breaking this approach. One way to get over this is to regularize the source term $\Lambda^{\mu\nu}$ by multiplying it with some factor r^B, where $B \in \mathbb{C}$. We thus define a new function with argument B as

$$I^{\mu\nu}(B) \equiv \Box_{\text{ret}}^{-1}\left[\tilde{r}^B \Lambda_n^{\mu\nu}\right]. \tag{4.47}$$

We have also defined $\tilde{r} \equiv \frac{r}{r_0}$, where r_0 is some constant. Normally, this integral is only defined when the real part of B is larger than some minimal value. This is due to the fact that $\Re(B)$ is what regularizes the divergent terms inside the source term $\Lambda^{\mu\nu}$. So although the domain of B is limited by this requisite, it also admits a unique analytical continuation into the entirety of the complex plane [6, 14] (although some integers should still be exempt). Knowing this, we write the Laurent expansion of $I^{\mu\nu}(B)$ around $B = 0$ as

$$I^{\mu\nu}(B) = \sum_{p=p_0}^{\infty} \mathfrak{I}_p^{\mu\nu} B^p. \tag{4.48}$$

The order of the poles, p_0, depend on which source term we are refering to (or in other words, it depends on n). Applying the d'Alembertian operator to both sides

$$\begin{aligned} \Box I^{\mu\nu}(B) &= \sum_{p=p_0}^{\infty} B^p \Box \mathfrak{I}_p^{\mu\nu}, \\ e^{\ln r B} \Lambda_n^{\mu\nu} &= \sum_{p=p_0}^{\infty} B^p \Box \mathfrak{I}_p^{\mu\nu}, \\ \sum_{p=0}^{\infty} B^p \frac{\ln r^p}{p!} \Lambda_n^{\mu\nu} &= \sum_{p=p_0}^{\infty} B^p \Box \mathfrak{I}_p^{\mu\nu}. \end{aligned} \tag{4.49}$$

After equating the powers of B among themselves, we get

$$\begin{aligned}\Box\mathfrak{I}^{\mu\nu}_p &= 0 \qquad \text{For } p_0 \leq p < -1,\\ \Box\mathfrak{I}^{\mu\nu}_p &= \frac{\ln r^p}{p!}\Lambda^{\mu\nu}_n \qquad \text{For } p \geq 0.\end{aligned} \tag{4.50}$$

This demonstrates that the finite part coefficient $\mathfrak{I}^{\mu\nu}_0$ is in fact a solution to the desired equation $\Box\mathfrak{I}^{\mu\nu}_0 = \Lambda^{\mu\nu}_n$. Let us rename this solution to $u^{\mu\nu}_n \equiv \mathfrak{I}^{\mu\nu}_0$, in other words

$$u^{\mu\nu}_n = \mathcal{FP}_{B=0}\Box^{-1}_{\text{ret}}\big[\tilde{r}^B\Lambda^{\mu\nu}_n\big]. \tag{4.51}$$

Where $\mathcal{FP}_{B=0}$ indicates the process of finding the finite part coefficient as we did above. This solution does however not satisfy the harmonic gauge condition $\partial_\mu\mathfrak{h}^{\mu\nu}_n = 0$. Instead, the solution's divergence reads

$$w^\nu_n = \partial_\mu u^{\mu\nu}_n = \mathcal{FP}_{B=0}\Box^{-1}_{\text{ret}}\big[B\tilde{r}^B\frac{n_i}{r}\Lambda^{\nu i}_n\big]. \tag{4.52}$$

Where we have used the fact that the source term is divergence free $\partial_\mu\Lambda^{\mu\nu}_n = 0$.
The above expression for w^ν_n is only zero for the special case in which the retarded integral has no simple pole at $B \to 0$. Due to how we've defined this function, it must satisfy the homogeneous solution of the source-free d'Alembertian equation. We can thus express w^ν_n's solutions in terms of four STF-tensorial functions of the retarded time $u = t - r/c$ (call them $N_L(u)$, $P_L(u)$, $Q_L(u)$ and $R_L(u)$)

$$\begin{aligned}w^0_n &= \sum_{l=0}^{\infty}\partial_L\big[\frac{N_L(u)}{r}\big],\\ w^i_n &= \sum_{l=0}^{\infty}\partial_{iL}\big[\frac{P_L(u)}{r}\big] + \sum_{l=1}^{\infty}\Big(\partial_{L-1}\big[\frac{Q_{iL-1}(u)}{r}\big] + \epsilon_{iab}\partial_{aL-1}\big[\frac{R_{bL-1}(u)}{r}\big]\Big).\end{aligned} \tag{4.53}$$

The trick now is to find another function $v^{\mu\nu}_n$ which inherits the property of w^ν_n -which is that it is also a solution to the homogeneous d'Alembertian-, and whose divergence is exactly the negative of w^ν_n, meaning $\partial_\mu v^{\mu\nu} = -w^\nu_n$. Such a function is not unique [16], so we chose a relatively simple form which satisfies these prerequisites

$$\begin{aligned}v^{00}_n &= -\frac{N^{(-1)}}{r} + \partial_a\big[\frac{1}{r}(-N^{(-1)}_a + Q^{(-2)} - 3P_a)\big],\\ v^{0i}_n &= \frac{1}{r}(-Q^{(-1)}_i + 3P^{(1)}_i) - \epsilon_{iab}\partial_a\big[\frac{R^{(-1)}_b}{r}\big] - \sum_{l=2}^{\infty}\partial_{L-1}\big[\frac{N_{iL-1}}{r}\big],\\ v^{ij}_n &= -\delta_{ij}\frac{P}{r} + \sum_{l=2}^{\infty}\Big\{2\delta_{ij}\partial_{L-1}\big[\frac{P_{L-1}}{r}\big] - 6\partial_{L-2(i}\big[\frac{P_{j)L-2}}{r}\big]\\ &\qquad + \partial_{L-2}\big[\frac{1}{r}(N^{(1)}_{ijL-2} + 3P^{(2)}_{ijL-2} - Q_{ijL-2})\big] - 2\partial_{aL-2}\big[\frac{1}{r}\epsilon_{ab(i}R_{j)bL-2}\big]\Big\}.\end{aligned} \tag{4.54}$$

We have used the notation that the superscripts indicate derivatives when they are positive $N^{(2)} = \frac{d^2}{du^2}N(u)$ and an antiderivative when they are negative $N^{(-1)} = \int_{\infty}^{u} dv N(v)$.

We now have a full algorithm in order to find our post-Minkowskian solution up to an arbitrary order n in the harmonic gauge. The solution for each order goes as

$$\mathfrak{h}_n^{\mu\nu} = u_n^{\mu\nu} + v_n^{\mu\nu}. \tag{4.55}$$

Which can indeed be checked that it satisfies both the sourced d'Alembertian (4.40) and the gauge condition (4.41).

4.4 PN Expansion in the near region using the $\mathfrak{h}^{\mu\nu}$ field

As we did in the first section of this chapter, we want to expand the $\mathfrak{h}^{\mu\nu}$ field up to its 1PN order. Following a similar treatment to what we previously did, one obtains the following set of equations [8, 17]

$$\begin{aligned}
\Box\mathfrak{h}^{00} &= \frac{16\pi G}{c^4}\big(1+\frac{4V}{c^2}\big)T^{00} - \frac{14}{c^4}\partial_k V\partial_k V + \mathcal{O}(\frac{1}{c^6}),\\
\Box\mathfrak{h}^{0i} &= \frac{16\pi G}{c^4}T^{0i} + \mathcal{O}(\frac{1}{c^5}),\\
\Box\mathfrak{h}^{ij} &= \frac{16\pi G}{c^4}T^{ij} + \frac{4}{c^4}\Big(\partial_i V\partial_j V - \frac{1}{2}\delta_{ij}\partial_k V\partial_k V\Big) + \mathcal{O}(\frac{1}{c^4}),
\end{aligned} \tag{4.56}$$

whose solutions can be expressed as

$$\begin{aligned}
\mathfrak{h}^{00} &= -\frac{4}{c^2}V + \frac{4}{c^4}\big(W_{kk} - 2V^2\big) + \mathcal{O}(\frac{1}{c^6}),\\
\mathfrak{h}^{0i} &= -\frac{4}{c^3}V_i + \mathcal{O}(\frac{1}{c^5}),\\
\mathfrak{h}^{ij} &= -\frac{4}{c^4}W_{ij} + \mathcal{O}(\frac{1}{c^6}),
\end{aligned} \tag{4.57}$$

Where the V and V_i fields are defined as before, and the newly introduced W_{ij} field is defined as

$$W_{ij}(t,\vec{x}) = G\int d^3\vec{x}'\frac{1}{|\vec{x}-\vec{x}'|}\Big[\sigma_{ij} + \frac{1}{4\pi G}\big(\partial_i V\partial_j V - \frac{1}{2}\delta_{ij}\partial_k V\partial_k V\big)\Big](t-|\vec{x}-\vec{x}'|/c,\vec{x}'). \tag{4.58}$$

One can use the equations that both V and V_i satisfy -(4.17) and (4.20)- in order to expand them in terms of their multipoles, as we saw in Chapter 3. Their general solution can be expressed as

$$V(t,\vec{x}) = G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\Big[\frac{F_L(u)}{r}\Big],$$
$$V(t,\vec{x}) = G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\Big[\frac{G_{iL}(u)}{r}\Big]. \tag{4.59}$$

Where the multipoles $F_L(u)$ and $G_{iL}(u)$ satisfy the equations when

$$F_L(u) = \int d^3\vec{y}\hat{y}_L\int_{-1}^{1}dz\delta_l(z)\sigma(u+z|\vec{y}|/c,\vec{y}),$$
$$G_{iL}(u) = \int d^3\vec{y}\hat{y}_L\int_{-1}^{1}dz\delta_l(z)\sigma_i(u+z|\vec{y}|/c,\vec{y}). \tag{4.60}$$

Up to 1 PN order, these two potentials are enough for our purposes.

4.5 Concerning near and far zone structures of the post-Minkowskian expansion

The general structure of the post-Minkowskian exterior metric in the near zone is of the following type [6]

$$\mathfrak{h}_n^{\mu\nu}(t,\vec{x}) = \sum_{m=m_0}^{N}\sum_{p\leq n-1}\hat{n}_L r^m(\ln r)^p[F_L^{\mu\nu}]_{m,p,n}(t) + o(r^N). \tag{4.61}$$

Where $o(r^N)$ is the symbol defined in Landau means that there is a remainder of the order of r^N. $[F_L^{\mu\nu}]_{m,p,n}(t)$ is a collection of multilinear functional depending on the source multipole moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$. The logarithmic term is troublesome; if we want to reconsile the post-Newtonian expansion -which doesn't have such terms- with our post-Minkowskian one, we need to get rid of these inconvinences if we can.

So let us see about the far zone structure of such a solution. The far zone structure at Minkowskian future null infinity has also been studied [5, 13], and has been found to be

$$\mathfrak{h}_n^{\mu\nu}(t,\vec{x}) = \sum_{k=1}^{N}\sum_{p\leq n-1}\hat{n}_L\frac{(\ln r)^p}{r^k}[G_L^{\mu\nu}]_{m,p,n}(t) + o(\frac{1}{r^N}). \tag{4.62}$$

The logarithmic terms are still present, but thanks to the work of Bondi, Sachs and Penrose [18, 19, 20, 21], these logarithms are known to be an artifact of our chosen coordinates (i.e. of our harmonic gauge). As proven in [22], the most general post-Minkowskian solution compatible with the no-incoming radiation admits some radiative coordinates $(T, \vec{X})$ for which the expansion at future null infinity takes the form

$$H_n^{\mu\nu}(T, \vec{X}) = \sum \frac{\hat{N}_L}{R^k}[K_L^{\mu\nu}]_{k,n}(U) + o(\frac{1}{R^N}). \tag{4.63}$$

Where $[K_L^{\mu\nu}]_{k,n}(U)$ are multipoles that are functionals of the source multipole moments, and which takes as argument the retarded time in this coordinate system $U = T - R/c$. This new radiative metric $H_n^{\mu\nu}(T, \vec{X})$ is a coordinate transformed version of the harmonic gauge metric $\mathfrak{h}_n^{\mu\nu}$, whose first linear term is related via (4.43)

$$H_1^{\mu\nu} = h_1^{\mu\nu} + \partial^\mu \xi_1^\nu + \partial^\nu \xi_1^\mu - \eta^{\mu\nu}\partial_\alpha \xi_1^\alpha. \tag{4.64}$$

Where the gauge vector ξ_1^μ is defined as

$$\xi_1^\mu = \frac{2M}{c^2}\eta^{0\mu}\ln\tilde{r}. \tag{4.65}$$

This gauge transformation makes it so that our new radiative metric's first term $H_1^{\mu\nu}(T, \vec{X})$ no longer satisfies the harmonic gauge.

This new radiative metric corrects the logarithmic deviation of the retarded time in harmonic coordinates at a linearized level $U = u - \frac{2GM}{c^3 \ln\tilde{r}} + \mathcal{O}(G^2)$. This condition need to hold for a linearized metric in order to constitute the linearized approximation to a full post-Minkowskian radiative field [23].

Getting rid of similar effects for higher orders of post-Minkowskian corrections requires one to define the following gauge tensor [22]

$$\xi_n^\mu = \mathcal{FP}\Box_{\text{ret}}^{(-1)}\Big[\frac{k^\mu}{2r^2}\int_{-\infty}^{u} dv\sigma_n(v, \hat{n})\Big]. \tag{4.66}$$

After which the n^{th} post-Minkowskian radiative metric is defined by

$$H_n^{\mu\nu} = U_n^{\mu\nu} + V_n^{\mu\nu} + \partial^\mu \xi_n^\nu + \partial^\nu \xi_n^\mu - \eta^{\mu\nu}\partial_\alpha \xi_n^\alpha. \tag{4.67}$$

Where $U_n^{\mu\nu}$ and $V_n^{\mu\nu}$ are defined analogously to $u_n^{\mu\nu}$ and $v_n^{\mu\nu}$ of the harmonic case ; their sum is divergence free. This, again, breaks the harmonic gauge.

We now have everything to express our gravitational radiation with. These $U^{\mu\nu}$ and $V^{\mu\nu}$ radiative multipoles can be expressed as functionals of M_L and P_L plus the PN corrections which we will soon find.

4.6 Matching of the post-Newtonian regime with the post-Minkowskian one

We have studied the post-Minkowskian expansion outside the source and found their multipole moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$, or equivalently, (M_L, P_L) and have linked them iteratively to any order of the expansion.

On the other hand, we have expressed the post-Newtonian expansion of the source in the near region in terms of their multipole expansions dependent on the source matter. We now note that by comparing these two solutions in the overlapping region (the near zone), we can determine the multipole moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$ of the post-Minkowskian approach in terms of the multipoles of the PN expansion, which is to say, in terms of the source matter.

To achieve this, we start by noting that the harmonic metric up to some order n; $\mathfrak{h}_n^{\mu\nu}$, has terms of order [8, 13]

$$\begin{aligned}\mathfrak{h}_n^{00} &= \mathcal{O}(\frac{1}{c^{2n}}),\\ \mathfrak{h}_n^{0i} &= \mathcal{O}(\frac{1}{c^{2n+1}}),\\ \mathfrak{h}_n^{ij} &= \mathcal{O}(\frac{1}{c^{2n}}),\end{aligned} \tag{4.68}$$

Which implies that working in a certain order in our PN regime also lets us “cut off” the post-Minkowskian expansion up to a certain point.

Comparing the multipolar PN expansion with the multipolar post-Minkowskian one, we can fix the I_L, J_L (also noting that the gauge vector multipoles only come into play at above the 1PN approximation) as [8, 15]

$$\begin{aligned}I_L(u) &= \mathcal{FP}\int d^3\vec{x}\int_{-1}^{1} dz\Big\{\delta_l(z)\hat{x}_L\Sigma - \frac{4(2l+1)\delta_{l+1}(z)}{c^2(l+1)(2l+3)}\hat{x}_{iL}\Sigma_i^{(1)}\\ &\qquad \frac{2(2l+1)\delta_{l+2}(z)}{c^4(l+1)(l+2)(2l+5)}\hat{x}_{ijL}\Sigma_{ij}^{(2)}\Big\}(u+z|\vec{x}|/c,\vec{x}),\\ J_L(u) &= \mathcal{FP}\int d^3\vec{x}\int_{-1}^{1} dz\epsilon_{ab<i_l}\Big\{\delta_l(z)\hat{x}_{L-1>a}\Sigma_b\\ &\qquad -\frac{(2l+1)\delta_{l+1}(z)}{c^2(l+1)(2l+3)}\hat{x}_{L-1>ac}\Sigma_{bc}^{(1)}\Big\}(u+z|\vec{x}|/c,\vec{x}),\end{aligned} \tag{4.69}$$

Where we have defined $\Sigma = \frac{1}{c^2}(\overline{\tau}^{00} + \overline{\tau}^{ii})$, $\Sigma_i = \frac{1}{c}\overline{\tau}^{0i}$ and $\Sigma_{ij} = \overline{\tau}^{ij}$. One can note that despite the non-linearities that we are dealing with, this is the same result as the one found in Chapter 3, where we have essentially solved the linearized theory with no regards of non-linearities -with the only difference being the $\mathcal{FP}$ treatment-. The bar above the τ denotes its PN expansion up to the desired order.

To express our radiative multipoles U_L and V_L in terms of the newly fleshed out I_L and J_L, we first start with our metric (4.43), where we can neglect the gauge vectors (for the same reasoning that they will have consequences only at higher PN orders). We then write, retaining only the two lowest order mass multipole contributions ($I = M$ and $I_{ij} \approx M_{ij}$ until 2.5 PN)

$$\mathfrak{h}_1^{\mu\nu} = \mathfrak{h}_{(M)}^{\mu\nu} + \mathfrak{h}_{(M_{ij})}^{\mu\nu}. \tag{4.70}$$

Where $\mathfrak{h}_{(M)}^{\mu\nu}$ denotes the contribution to the metric from the monopole and $\mathfrak{h}_{(M_{ij})}^{\mu\nu}$, where using (4.44) with these yield

$$\begin{aligned}
\mathfrak{h}_{(M)}^{00} &= -\frac{4M}{c^2 r}, \\
\mathfrak{h}_{(M)}^{0i} &= 0, \\
\mathfrak{h}_{(M)}^{ij} &= 0, \\
\mathfrak{h}_{(M_{ab})}^{00} &= -\frac{2}{c^2}\partial_k\partial_l\big[\frac{M_{kl}(u)}{r}\big], \\
\mathfrak{h}_{(M_{ab})}^{0i} &= \frac{2}{c^3}\partial_k\big[\frac{M_{ki}^{(1)}(u)}{r}\big], \\
\mathfrak{h}_{(M_{ab})}^{ij} &= -\frac{2}{c^4}\frac{1}{r}M_{ij}^{(2)}(u),
\end{aligned} \tag{4.71}$$

Which now allows us to calculate the next iteration $\mathfrak{h}_2^{\mu\nu}$. Since $\Lambda_2^{\mu\nu}$ is quadratic in $\mathfrak{h}_1^{\mu\nu}$, the second order metric will be composed of the squares of the mass monopole and the square of the mass quadrupole, but it will also have a mixed monopole-quadrupole term

$$\mathfrak{h}_2^{\mu\nu} = \mathfrak{h}_{(M^2)}^{\mu\nu} + \mathfrak{h}_{(M\times M_{ij})}^{\mu\nu} + \mathfrak{h}_{(M_{ij}\times M_{kl})}^{\mu\nu}. \tag{4.72}$$

The source term $\Lambda_2^{\mu\nu} = N^{\mu\nu}[\mathfrak{h}_1, \mathfrak{h}_1]$ can then be written out as a sum of each of the $\mathfrak{h}_1^{\mu\nu}$

$$N^{\mu\nu}[\mathfrak{h}_1, \mathfrak{h}_1] = N^{\mu\nu}[\mathfrak{h}_1, \mathfrak{h}_1]_{(M^2)} + N^{\mu\nu}[\mathfrak{h}_1, \mathfrak{h}_1]_{(M\times M_{ij})} + N^{\mu\nu}[\mathfrak{h}_1, \mathfrak{h}_1]_{(M_{ij}\times M_{kl})}. \tag{4.73}$$

Where due to the linearity of our d'Alembertian in $\Box\mathfrak{h}_2^{\mu\nu} = \Lambda_2^{\mu\nu}$, we can solve each contribution to $\mathfrak{h}_2^{\mu\nu}$. Denoting the (M^2) monopole-monopole contribution of the wave equation to $\mathfrak{h}_2^{\mu\nu}$ as $\mathfrak{h}_{(M^2)}^{\mu\nu}$ (and similarly defining them for the other $(M \times M_{ij})$ and $(M_{ij} \times M_{kl})$), we calculate the (M^2) contribution, as an example [15]

$$N^{00}[\mathfrak{h}_1, \mathfrak{h}_1]_{(M^2)} = -\frac{14M^2}{c^4 r^4}. \tag{4.74}$$

Solving for $\mathfrak{h}^{00}_{(M^2)}$ -which doesn't require the $\mathcal{FP}$ approach since the integral converges-yields

$$\mathfrak{h}^{00}_{(M^2)} = -\frac{7M^2}{c^4 r^2}. \tag{4.75}$$

A similar approach for the other components gives [8]

$$\begin{aligned} \mathfrak{h}^{0i}_{(M^2)} &= 0, \\ \mathfrak{h}^{ij}_{(M^2)} &= -n_i n_j \frac{M^2}{c^4 r^2}. \end{aligned} \tag{4.76}$$

But we note that since these fields are all proportional to $\frac{1}{r^2}$, they do not contribute to the radiative field.

For our next step, while we try to find the $\mathfrak{h}^{\mu\nu}_{(M\times M_{ij})}$ contributions, we note that $N^{\mu\nu}[\mathfrak{h}_1, \mathfrak{h}_1]_{(M\times M_{ij})}$ has the following form [15]

$$N^{\mu\nu}[\mathfrak{h}_1, \mathfrak{h}_1]_{(M\times M_{ij})} = \sum_{k=2}^{\infty} n_L \frac{H_L^{(k)}(u)}{r^k}. \tag{4.77}$$

To calculate its contribution to our metric, we need to take the retarded integral of this term, which is divergent. Alas, when one tries to integrate these terms, the integral varies for each value of $k = 2$ and for $k \geq 3$. Since in this case, we are interested in the radiative zone effect, we can disregard terms that diverge faster than $\frac{1}{r^3}$, and focus on the $k = 2$ case [**?**, 24]

$$\Box^{-1}_{\text{ret}}\Big[n_L \frac{H(u)}{r^2}\Big] = -\frac{n_L}{r}\int_r^{\infty} dz Q_l\big(\frac{z}{r}\big) H(t - z/c). \tag{4.78}$$

Where the $Q_l(x)$'s are Legendre functions of the second kind. The asymtotic behaviour of such functions are [8]

$$Q_l(x) = -\frac{1}{2}\ln(\frac{x-1}{2}) - a_l + \mathcal{O}((x-1)\ln(x-1)). \tag{4.79}$$

Where $a_l = \sum_{k=1}^{l} k^{-1}$. Knowing this, after changing the integration variable to $y = (z - r)/c$, we finally get

$$\Box^{-1}_{\text{ret}}\Big[n_L \frac{H(u)}{r^2}\Big] = -\frac{c n_L}{2r}\int_0^{\infty} dy H(t - r/c - y)\Big[\ln\big(\frac{cy}{2r} + 2a_l\big)\Big] + \mathcal{O}(\frac{\ln r}{r^2}). \tag{4.80}$$

Using the asymptotic expansion, we can now compute the contribution of $\mathfrak{h}^{\mu\nu}_2$ to the radiation field at infinity.[15] (The $N^{\mu\nu}[\mathfrak{h}_1, \mathfrak{h}_1]_{(M_{ij}\times M_{kl})}$ contribution can be found in [25])

$$U_L(u) = M_L^{(l)}(u) + \frac{2GM}{c^3}\int_0^{\infty} d\tau M_L^{(l+2)}(u-\tau)\Big[\ln\big(\frac{c\tau}{2r} + \kappa_l\big)\Big] + \mathcal{O}(\frac{1}{c^5}), \tag{4.81}$$

Where $\kappa = \frac{2l^2+5l+4}{l(l+1)(l+2)} + \sum_{k=1}^{l-2} \frac{1}{k}$

To get rid of the logarithmic term, recall our radiative coordinates via the transformation (4.65). Our radiative moments in radiative coordinates are thus given by

$$U_L(U) = M_L^{(l)}(u) + \frac{2GM}{c^3}\int_0^\infty d\tau M_L^{(l+2)}(U-\tau)\Big[\ln\big(\frac{c\tau}{2r_0} + \kappa_l\big)\Big] + \mathcal{O}(\frac{1}{c^5}). \quad (4.82)$$

While the current type multipoles $V_L(U)$ can be calculated similarly [13]

$$V_L(U) = S_L^{(l)}(u) + \frac{2GM}{c^3}\int_0^\infty d\tau S_L^{(l+2)}(U-\tau)\Big[\ln\big(\frac{c\tau}{2r_0} + \pi_l\big)\Big] + \mathcal{O}(\frac{1}{c^5}). \quad (4.83)$$

Where this time, $\pi = \frac{l-1}{l(l+1)} + \sum_{k=1}^{l-1} \frac{1}{k}$ Now that we've seen the methodology behind how to find PN corrections to the radiative fields, let us mention the mass octopole moment U_{ij} up to order 2.5 PN, which illustrates the memory effect of the sources in the radiation zone. [13]

$$\begin{aligned} U_{ij}(U) = M_{ij}^{(2)}(U) &+ \frac{2GM}{c^3}\int_0^\infty d\tau M_{ij}^{(4)}(U-\tau)\Big[\ln\big(\frac{c\tau}{2r_0} + \frac{11}{12}\big)\Big] \\ &- \frac{2G}{7c^5}\int_0^\infty d\tau M_{a<i}^{(3)}(U-\tau)M_{j>a}^{(3)}(U-\tau) \\ &- \frac{G}{c^5}\Big[\frac{2}{7}M_{a<i}^{(3)}M_{j>a}^{(2)} + \frac{5}{7}M_{a<i}^{(4)}M_{j>a}^{(1)} - \frac{1}{3}\epsilon_{ab<i}M_{j>a}^{(4)}S_b\Big]. \end{aligned} \quad (4.84)$$

CHAPTER 5

CONCLUSIONS AND LAST REMARKS

The last result obtained for the mass octopole moment U_{ij} has some classically unexpected results up to its 2.5 PN order. In Electromagnetism, we've known that a wave traveling at the speed of light, detected at some time t_0 is dependent on the instantaneous state of the source at some retarded time $u_0 = t_0 - r/c$. This is reflect in how we calculate the four-vector potential A^μ at some point p through a series of retarded integrals, which gives us the multipole moments all evaluated at the retarded time u_0. The same cannot be said for the gravitational waves. As we see from the last expressions, there are contributions on the radiation which depend on the entire past of the multipoles, rather than just some retarded time U.

The first integral term, for instance is called the hereditary tail integral, it is thought to be the contribution coming from gravitons scattering off the background curvature generated by the mass of the source [8]. It comes from the $\mathfrak{h}^{\mu\nu}_{(M\times M_{ij})}$ contribution in $\mathfrak{h}^{\mu\nu}_2$, and due to the logarithmic term inside, the contribution from distant times are less "weighted".

The second integral in U_{ij} in the last equation is what is called the non-linear memory term. It comes from the integral part of $\mathfrak{h}^{\mu\nu}_{(M_{ij}\times M_{kl})}$ in $\mathfrak{h}^{\mu\nu}_2$. Christodoulou [26] first introduced (although, not in this form) the effect by analyzing the asymptotic behavior of the graviational field at null infinity, which was later attributed by [27] to the graviton-graviton scattering contribution. This term is especially intruiging, since unlike the tail hereditary term, which has diminished contributions from the very distant past of the source (due to its logarithmic integrant piece), this one has “equally weighed” contribution from from its remote past. This is the memory effect we wanted to show the derivation methods of during this thesis.

The last (non integral) term is the only correction up to order 2.5 PN which is instantaneously dependent on some retarded time u. It comes from the instantaneous part of $\mathfrak{h}^{\mu\nu}_{(M_{ij}\times M_{kl})}$ in $\mathfrak{h}^{\mu\nu}_{2}$.

REFERENCES

[1] A. Einstein, "Näherungsweise Integration der Feldgleichungen der Gravitation," *Preussische Akademie der Wissenschaften, Sitzungsberichte*, vol. 1916 (part 1), pp. 688–696, 1916.

[2] A. Einstein, "Über Gravitationswellen," *Königlich Preußische Akademie der Wissenschaften (Berlin) Sitzungsberichte*, vol. 1918 (part 1), p. 154–167, 1918.

[3] L. Infeld and J. Plebanski, *Motion and Relativity*. Pergamon Press, 1960.

[4] J. Weisberg, J. Taylor, and L. Fowler, "Timing Measurements of the Relativistic Binary Pulsar psr b1913+16," *The Astrophysical Journal*, vol. 722, Issue 2, pp. 1030–1034, 2010.

[5] V. Fock, *The Theory of Space, Time and Gravitation*. Pergamon Press, 1959.

[6] L. B. T. Damour, "Radiative Gravitational Fields in General Relativity, I. General Structure of the Fields outside the Source," *Philosophical Transactions of the Royal Society of London, Series A Mathematical and Physical Sciences*, vol. 320, No. 1555, pp. 379–430, 1986.

[7] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*. W.H. Freeman & Co., 1973.

[8] M. Maggiore, *Gravitational Waves: Volume 1 Theory and Experiments*. Oxford University Press, 2014.

[9] T. Damour and B. R. Iyer, "Multipole analysis for electromagnetism and linearized gravity with irreducible cartesian tensors," *Phys. Rev. D*, vol. 43, pp. 3259–3272, May 1991.

[10] K. S. Thorne, "Multipole expansions of gravitational radiation," *Rev. Mod. Phys.*, vol. 52, pp. 299–339, Apr 1980.

[11] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. John Wiley & Sons, Inc., 1972.

[12] L. D. L. E. M. Lifshitz, *Course of Theoretical Physics. Vol. 2: the Classical Theory of Fields*. Butterworth Heinemann, 1994.

[13] L. Blanchet, "Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries," *Living Reviews in Relativity*, vol. 17, no. 1, 2014.

[14] L. B. . G. Faye, "General relativistic dynamics of compact binaries at the third post-Newtonian order," *Phys. Rev. D*, vol. 63, 2000.

[15] L. Blanchet, "Gravitational radiation from post-newtonian sources and inspiralling compact binaries," *Living Rev. Relativ.*, vol. 9, 2006.

[16] L. Blanchet, G. Faye, B. Iyer, and S. Sinha, "The third post-Newtonian gravitational wave polarisations and associated spherical harmonic modes for inspiralling compact binaries in quasi-circular orbits," *Class. Quantum Grav.*, vol. 25, 2008.

[17] L. Blanchet, G. Faye, and B. Ponsot, "Gravitational field and equations of motion of compact binaries to $5/2$ post-newtonian order," *Phys. Rev. D*, vol. 58, p. 124002, Oct 1998.

[18] H. Bondi, M. V. der Burg, and A. Metzner, "Gravitational waves in general relativity, vii. Waves from axi-symmetric isolated system," *Proceedings of the Royal Society of London. Series A*, vol. 269, 1962.

[19] R. K. Sachs, "Gravitational Waves in General Relativity. VIII. Waves in Asymptotically Flat Space-Time," *Proceedings of the Royal Society of London. Series A*, vol. 270, 1962.

[20] R. Penrose, "Asymptotic Properties of Fields and Space-Times," *Phys. Rev. Lett.*, vol. 10, 1963.

[21] R. Penrose, "Zero Rest-Mass Fields Including Gravitation: Asymptotic Behaviour," *Proceedings of the Royal Society of London. Series A*, vol. 284, 1965.

[22] L. Blanchet, "Radiative Gravitation Fields in General Relativity II. Asymptotic Behaviour at Future Null Infinity," *Proc. R. Soc. Lond. A*, vol. 409, 1987.

[23] J. Madore, “Gravitational radiation from a bounded source. I,” *Ann. Inst. Henri Poincare*, vol. 12, 1970.

[24] L. Blanchet and T. Damour, “Tail-transported temporal correlations in the dynamics of a gravitating system,” *Phys. Review D*, vol. 37, 1988. PhysRevD.37.1410.

[25] L. Blanchet, “Quadrupole-quadrupole gravitational waves,” *Classical and Quantum Gravity*, vol. 15, pp. 89–111, jan 1998.

[26] A. G. Wiseman and C. M. Will, “Christodoulou’s nonlinear gravitational-wave memory: Evaluation in the quadrupole approximation,” *Phys. Rev. D*, vol. 44, pp. R2945–R2949, Nov 1991.

[27] K. S. Thorne, “Gravitational-wave bursts with memory: The christodoulou effect,” *Phys. Rev. D*, vol. 45, pp. 520–524, Jan 1992.

Appendix A

TT GAUGE PROJECTOR

We can first begin the derivation of the full form of this TT gauge projection tensor by the definition of the Transverse projection tensor which simply acts on the spatial components of a 4-vector.

$$P^i_{\ j} = \delta^i_{\ j} - n^i n_j, \tag{A.1}$$

Which, when hitting a 3-vector v^j gives us

$$\begin{aligned} P^i_{\ j} v^j &= (\delta^i_{\ j} - n^i n_j) v^j \\ &= v^i - (n \cdot v) n^i \\ &= v^i_\perp . \end{aligned} \tag{A.2}$$

Where $v^i_\perp$ indicates the component of v^i which is transverse to n^i. We can now find the transverse part of $\overline{h}_{ij}$

$$\overline{h}^{\mathrm{Tr}}_{ij} = P^k_{\ i} P^l_{\ j} \overline{h}_{kl}. \tag{A.3}$$

Note that by virtue of the definitions of the quantities, we can check the transverse nature of this new quantity

$$\begin{aligned} \partial^j \overline{h}^{\mathrm{Tr}}_{ij} &= \partial^j P^k_{\ i} P^l_{\ j} \overline{h}_{kl} \\ &= n^j P^k_{\ i} P^l_{\ j} \overline{h}_{kl} \\ &= P^k_{\ i} \cancelto{0}{(n^i - n^i)} \overline{h}_{kl} \\ &= 0. \end{aligned} \tag{A.4}$$

One should note that this form is far from being traceless. We can remedy this by proposing a projection tensor which also includes a term which gets rid of the traces of the original tensor. It reads

$$\Lambda_{ij}^{\ \ kl} = P^k_{\ i} P^l_{\ j} - \frac{1}{2} P_{ij} P^{kl}. \tag{A.5}$$

Defining

$$\overline{h}_{ij}^{\mathrm{TT}} = \Lambda_{ij}{}^{kl}\overline{h}_{kl}, \tag{A.6}$$

The second term can be seen to be transverse to the propagation direction, so expression is clearly transverse in i and j. The traceless nature can, again, be verified from the fact that $\Lambda_{ij}{}^{kl}$ is traceless in the contraction of i and j.

Appendix B

SOLID ANGLE INTEGRATION OF ARBITRARY UNIT VECTORS

Consider the integral

$$\frac{1}{4\pi}\int n_{i_1}n_{i_2}\dots n_{i_k}d\Omega, \tag{B.1}$$

where n_i is the i^{th} component of some unit vector $\hat{n}$.

We might note that this integral should be the same under the parity transformation $\hat{n} \to -\hat{n}$ due to the nature of the integral being over all solid angles, but then this implies that for an odd number k, the integral can be written as the negative of itself;

$$\frac{1}{4\pi}\int n_{i_1}n_{i_2}\dots n_{i_k}d\Omega = -\frac{1}{4\pi}\int n_{i_1}n_{i_2}\dots n_{i_k}d\Omega = 0. \tag{B.2}$$

This implies that the only non-zero case can be for even k's.

In the even case, we now argue that since this integral will be symmetric over all the indices' exchange, it has to be proportional to all the indice combinations of Kronecker Deltas δ. As examples that are useful for our purposes, the $k = 2, 4$ cases would read

$$\begin{aligned}\frac{1}{4\pi}\int n_{i_1}n_{i_2}d\Omega &= A\delta_{i_1i_2},\\ \frac{1}{4\pi}\int n_{i_1}n_{i_2}n_{i_3}n_{i_4}d\Omega &= B\big(\delta_{i_1i_2}\delta_{i_3i_4} + \delta_{i_1i_3}\delta_{i_2i_4} + \delta_{i_1i_4}\delta_{i_2i_3}\big).\end{aligned} \tag{B.3}$$

Where A and B are constants that can be determined through contraction of all indices. For the $k = 2$ case, it is straightforward to see that contracting the single δ yields the number of dimensions over which the indices run ($\delta_{ii} = 3$ in our case). Contracting i_1 with i_2, with the knowledge that $|\hat{n}|^2 = n_in_i = 1$, we obtain

$$\begin{aligned}\frac{1}{4\pi}\int d\Omega &= A\delta_{i_1i_1},\\ 1 &= 3A.\\ \frac{1}{3} &= A.\end{aligned} \tag{B.4}$$

Following a similar treatment for $k = 4$, we contract i_1 with i_2 and i_3 with i_4, obtaining

$$\begin{aligned} \frac{1}{4\pi}\int d\Omega &= B\big(\delta_{i_1 i_1}\delta_{i_3 i_3} + \delta_{i_1 i_3}\delta_{i_1 i_3} + \delta_{i_1 i_3}\delta_{i_1 i_3}\big), \\ 1 &= B\big(9 + 3 + 3\big), \\ \frac{1}{15} &= B. \end{aligned} \tag{B.5}$$

So our $k = 2$ and $k = 4$ integrals becomes

$$\begin{aligned} \frac{1}{4\pi}\int n_{i_1} n_{i_2} d\Omega &= \frac{1}{3}\delta_{i_1 i_2}, \\ \frac{1}{4\pi}\int n_{i_1} n_{i_2} n_{i_3} n_{i_4} d\Omega &= \frac{1}{15}\big(\delta_{i_1 i_2}\delta_{i_3 i_4} + \delta_{i_1 i_3}\delta_{i_2 i_4} + \delta_{i_1 i_4}\delta_{i_2 i_3}\big). \end{aligned} \tag{B.6}$$

Appendix C

PALATINI FORMALISM AND DERIVATION OF THE VACUUM EINSTEIN FIELD EQUATION

As we have introduced in the Introduction of this thesis, I shall go over a similar approach to deriving the Einstein field equations. We shall adopt here the Palatini approach which, in my opinion, gives a satisfactory reasoning as to why the Levi-Civita connection has the specific form we've touched upon there. This formalism essentially implies treating the metric two form components $g_{\mu\nu}$ and the connection $\Gamma^{\rho}{}_{\mu\nu}$ as independent dynamical fields which are later related through the variation in those fields. I shall omit the matter contribution to the lagrangian, since it is essentially the same one we have found in the first chapter.

$$\mathcal{L}_H = \sqrt{-g}R, \tag{C.1}$$

Where $R = g^{\mu\nu}R_{\mu\nu}$ is the trace of the Ricci tensor $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$, which is itself the contraction of the Riemann tensor $R^{\mu}{}_{\nu\rho\sigma}$.

Taking the inverse metric ($g^{\mu\nu}$) and the affine connection $\Gamma^{\mu}{}_{\nu\rho}$ as our dynamical fields, the variation of the action (S) (c=1)

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma, g^{\mu\nu}). \tag{C.2}$$

With respect to those fields, individually, will give us the equations gouverning our calculations.

Remembering our Ricci tensor definition $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$, we first vary our action across $\Gamma^{\mu}{}_{\nu\rho}$

$$\begin{aligned}\delta S &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}(\Gamma)\\ &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}g^{\mu\nu}\big(\partial_\rho\delta\Gamma^{\rho}{}_{\mu\nu} - \partial_\nu\delta\Gamma^{\rho}{}_{\mu\rho}\\ &\qquad + \delta\Gamma^{\rho}{}_{\rho\sigma}\Gamma^{\sigma}{}_{\mu\nu} + \Gamma^{\rho}{}_{\rho\sigma}\delta\Gamma^{\sigma}{}_{\mu\nu} - \delta\Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\mu\rho} - \Gamma^{\rho}{}_{\nu\sigma}\delta\Gamma^{\sigma}{}_{\mu\rho}\big).\end{aligned} \tag{C.3}$$

Even though the connection is not a tensor, its variation is. We can thus simplify the above expression by introducing the *covariant derivative* as follows

$$\begin{aligned}\delta S &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}g^{\mu\nu}\big(\nabla_\rho\delta\Gamma^\rho{}_{\mu\nu} - \nabla_\nu\delta\Gamma^\rho{}_{\mu\rho}\big)\\ &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big\{\nabla_\rho\big(g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\sigma{}_{\sigma\mu}\big) + \delta\Gamma^\sigma{}_{\mu\sigma}\nabla_\rho g^{\mu\rho} - \delta\Gamma^\rho{}_{\mu\nu}\nabla_\rho g^{\mu\nu}\Big\}.\end{aligned} \tag{C.4}$$

Note that the full derivative part of the integrand can be evaluated at the boundaries through Stokes' theorem. The variation $\delta\Gamma^\mu{}_{\nu\rho}$, by construction, vanishes at those boundaries. Those parts will equivalently vanish. Putting the rest of the terms into order

$$\delta S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big(\delta^\nu{}_\rho\nabla_\sigma g^{\mu\sigma} - \nabla_\rho g^{\mu\nu}\Big)\delta\Gamma^\rho{}_{\mu\nu}. \tag{C.5}$$

We know that in a torsionless geometry, we expect $\Gamma^\rho{}_{\mu\nu} = \Gamma^\rho{}_{(\mu\nu)}$ to hold. Any antisymmetric term in $\mu \leftrightarrow \nu$ inside the paranthesis can thus be disregarded, since for an antisymmetric tensor $A^{\mu\nu} = A^{\{\mu\nu\}}$ contracted with a symmetric tensor $B_{\mu\nu} = B_{(\mu\nu)}$, we have

$$\begin{aligned}A^{\mu\nu}B_{\mu\nu} &= -A^{\nu\mu}B_{\mu\nu}\\ &= -A^{\mu\nu}B_{\nu\mu}\\ &= -A^{\mu\nu}B_{\mu\nu} = 0.\end{aligned} \tag{C.6}$$

So extracting the symmetric parts of the paranthesis term, the action variation becomes

$$\delta S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big(\frac{1}{2}\big(\delta^\nu{}_\rho\nabla_\sigma g^{\mu\sigma} + \delta^\mu{}_\rho\nabla_\sigma g^{\nu\sigma}\big) - \nabla_\rho g^{\mu\nu}\Big)\delta\Gamma^\rho{}_{\mu\nu}. \tag{C.7}$$

According to Hamilton's principle, the variation of the action vanishes throughout the motion ($\delta S = 0$). Inside the hypervolume over which the integration is carried, $\delta\Gamma^\rho{}_{\mu\nu}$ is completely arbitrary, thus the only way this principle is obeyed for any variation of $\Gamma^\rho{}_{\mu\nu}$ is when the integrand equivalently vanished.

$$\frac{1}{2}\big(\delta^\nu{}_\rho\nabla_\sigma g^{\mu\sigma} + \delta^\mu{}_\rho\nabla_\sigma g^{\nu\sigma}\big) - \nabla_\rho g^{\mu\nu} = 0. \tag{C.8}$$

One solution to this is by simply setting $\nabla_\rho g^{\mu\nu} = 0$, or $\nabla_\rho g_{\mu\nu} = 0$. A connection satisfying this relation with the metric is called *metric compatible* connection.

For the explicit relation between the metric $g_{\mu\nu}$ and the connection $\Gamma^{\rho}{}_{\mu\nu}$, we need to elaborate on the previous results

$$\begin{aligned} \nabla_\rho g_{\mu\nu} &= \partial_\rho g_{\mu\nu} - \Gamma^{\sigma}{}_{\rho\mu} g_{\sigma\nu} - \Gamma^{\sigma}{}_{\rho\nu} g_{\mu\sigma} = 0, \\ \partial_\rho g_{\mu\nu} &= \Gamma^{\sigma}{}_{\rho\mu} g_{\sigma\nu} + \Gamma^{\sigma}{}_{\rho\nu} g_{\mu\sigma}. \end{aligned} \tag{C.9}$$

Rewriting this last term with interchanged indices in two other ways

$$\begin{aligned} \partial_\mu g_{\rho\nu} &= \Gamma^{\sigma}{}_{\rho\mu} g_{\sigma\nu} + \Gamma^{\sigma}{}_{\mu\nu} g_{\rho\sigma}, \\ \partial_\nu g_{\rho\mu} &= \Gamma^{\sigma}{}_{\nu\mu} g_{\sigma\rho} + \Gamma^{\sigma}{}_{\rho\nu} g_{\mu\sigma}. \end{aligned} \tag{C.10}$$

And finally substracting the one found above from these two

$$\begin{aligned} \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} = {}& \cancelto{i}{\Gamma^{\sigma}{}_{\rho\mu} g_{\sigma\nu}} + \Gamma^{\sigma}{}_{\mu\nu} g_{\rho\sigma} + \Gamma^{\sigma}{}_{\nu\mu} g_{\sigma\rho} \\ & + \cancelto{ii}{\Gamma^{\sigma}{}_{\rho\nu} g_{\mu\sigma}} - \cancelto{i}{\Gamma^{\sigma}{}_{\rho\mu} g_{\sigma\nu}} - \cancelto{ii}{\Gamma^{\sigma}{}_{\rho\nu} g_{\mu\sigma}}. \end{aligned} \tag{C.11}$$

Thus we get the relation

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \big(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \big). \tag{C.12}$$

Let us now vary with respect to the inverse metric $\delta g^{\mu\nu}$

$$\delta S = \frac{1}{16\pi G} \int d^4x \big\{ g^{\mu\nu} R_{\mu\nu}(\Gamma) \delta\sqrt{-g} + \sqrt{-g} R_{\mu\nu}(\Gamma) \delta g^{\mu\nu} \big\}. \tag{C.13}$$

The variation term $\delta\sqrt{-g}$ can be found in the Introduction chapter, at equations (1.28) and (1.29)

$$\begin{aligned} \delta\sqrt{-g} &= \frac{1}{2} \frac{g g_{\mu\nu} \delta g^{\mu\nu}}{\sqrt{-g}} \\ &= -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \end{aligned} \tag{C.14}$$

Finally yielding the action

$$\delta S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \big\{ R_{\mu\nu}(\Gamma) - \frac{1}{2} g^{\rho\sigma} R_{\rho\sigma}(\Gamma) g_{\mu\nu} \big\} \delta g^{\mu\nu}. \tag{C.15}$$

Using Hamilton's principle once more, we deduce the integrand to be vanishing everywhere

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \tag{C.16}$$

This is the vacuum Einstein field equation. In the absence of matter, the metric describing spacetime in that vacuum should satisfy the relation above.