ABOUT THE DECOMPOSABILITY OF ALMOST COMPLETELY DECOMPOSABLE GROUPS

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ABSTRACT

ABOUT THE DECOMPOSABILITY OF ALMOST COMPLETELY DECOMPOSABLE GROUPS

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Almost completely decomposable groups are extensions of completely decomposable groups of finite index. They can be written as a sum of indecomposables. Every almost completely decomposable group is the direct sum of indecomposables but this decomposition is quite complicated and not unique. Almost completely decomposable groups can be represented by matrices, called coordinate matrices. If the coordinate matrix of a given almost completely decomposable group is decomposable then the group is decomposable. Under some restrictions, for example considering a weakening of isomorphism, called near-isomorphism some spacial classes of almost completely decomposable groups can be classified.

Keywords: Acd groups, Torsion free groups, Decomposability of acd groups.

ÖZ

HEMEN HEMEN AYRIŞAN GRUPLARIN PARÇALANMASI

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Hemen hemen ayrışan gruplar, ayrışan grupların sonlu indeksli genişlemeleridir. Bu gruplar ayrışamayan grupların direk toplamı olarak yazılabilir. Her hemen hemen ayrışan grup ayrışamayan grupların direk toplamı şeklinde yazılabilir fakat bu ayrışma oldukça karışıktır ve benzersiz değildir. Hemen hemen ayrışan gruplar koordinat matris adını verdiğimiz matrislerle temsil edilebilir. Verilen hemen hemen ayrışan bir grubun koordinat matrisi ayrışabilir ise grup da ayrışır. Bazı kısıtlamalar altında, örneğin izomorfizmanın daha hafifletilmiş bir hali olan yakın-izomorfizma düşünüldüğünde hemen hemen ayrışan grupların bazı özel sınıfları sınıflandırılabilir.

Anahtar Kelimeler: Acd gruplar, Torsiyonsuz gruplar, Acd grupların ayrışması.

To my family

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CHAPTER 1

INTRODUCTION

This thesis is about a class of torsion free abelian groups called almost completely decomposable group. Almost completely decomposable groups, briefly acd groups are torsion-free abelian groups which can be considered as extensions of completely decomposable groups. Lee Lady is the first mathematician who defined and studied acd groups and its related concepts. Abelian group theorists from different countries of the world are working with these groups. Acd groups are difficult to be specialized and also there is no general theory for the class of this special class of groups.

In this Master thesis we first present the basic properties of almost completely decomposable groups like regulator, types, characteristics, then we present the matrix representation of acd groups and finally we give the methods and the results about the decomposability of acd groups which is the main aim of this thesis.

Every almost completely decomposable group is the direct sum of indecomposables but this decomposition is quite complicated and not unique. Hence there is no hope to find isomorphisms classes of acd groups. But under some restrictions, for example considering a weakening of isomorphism, called near-isomorphism some spacial classes of acd groups can be classified.

Acd groups are represented by matrices and this matrices are formed by the bases of their regulator and regulator quotient. Hence it is possible to determine decomposability of some classes of acd groups by using linear algebra tools.

In Chapters 2,3 and 4 main definitions and the basic facts about acd groups are given. The definitions used in this work are taken from the books of A.Mader and L.Fuchs, see [3] and [2]. Chapter 5 contains material of Matrix Theory needed for the decomposition problem.

Chapter 6 is about the decomposability of acd groups by means of the corresponding coordinate matrices. The subsections describe permitted row and column transformations.

Chapter 7 shows how to get a kind of normal form of a coordinate matrix by mean of the transformations defined in Chapter 6.

Chapter 8 contains some examples of decomposable and indecomposable acd groups. Indecomposable groups have a big role in the decomposability problem of acd groups. Hence the main question about the decomposability of almost completely decomposable groups is the finiteness of the isomorphism types of indecomposable groups with the given critical typeset. The final chapter gives answer of this question for some subclasses of almost completely decomposable groups listing the bounded and unbounded cases.

The open problems in the list are waiting for curious readers to be solved.

CHAPTER 2

PRELIMINARIES

2.1 Basic Definitions

The groups with them we work in this thesis are all additive abelian groups. A torsion free group is a group whose elements except the identity have infinite order. Let G be a torsion free group. The $\mathbb{Q}$-vector space

$$\mathbb{Q}G := \langle qg \mid q \in \mathbb{Q}, g \in G \rangle$$

is called the *divisible hull* of G. Torsion free groups are subgroups of their divisible hulls, i.e., G can be considered as a pair $G \subseteq \mathbb{Q}\, G$. So we can embed G in a $\mathbb{Q}$-vector space, in its divisible hull $\mathbb{Q}G$. Hence the linear algebra tools are available for torsion free groups.

The dimension of the divisible hull of G is the *rank* of G denoted $\operatorname{rk} G$. The rank of G, $\operatorname{rk} G = \dim \mathbb{Q}G$, is the cardinality of a maximal independent subset of G. In this thesis we only consider groups which have a finite dimensional divisible hull. These are the groups having finite rank.

Rational groups are groups of rank 1, subgroups of $\mathbb{Q}$. There are uncountably many such as $\mathbb{Z},\ \mathbb{Q},\ \mathbb{Z}_p,\ \mathbb{Z}[p^{-1}]$ where p is prime.

Definition 1 *A torsion free group G is called* **completely decomposable** *if it can be written as a direct sum of rational groups and is called* **indecomposable** *if it has only trivial direct summand.*

A clear example of indecomposable groups are torsion free groups of rank 1.

Definition 2 *An* **almost completely decomposable group** *G is a torsion free group of finite rank which has a completely decomposable subgroup of finite index. Almost completely decomposable groups are written briefly as* **acd** *groups.*

Definition 3 *Let G a group and H be a subgroup of G. Then H is said to be* **pure***, denoted $H \subset_* G$, if*

$$nH = H \cap nG$$

for all $n \in \mathbb{Z}$. If G is torsion free then it follows that H is pure subgroup of G if and only if $h = ng$ where $g \in G$ and $h \in H$ implies $g \in H$. The group

$$H_*^G := \{x \in G \mid \exists_{n \in \mathbb{N}}\, nx \in U\} = G \cap \mathbb{Q}\, H$$

is said to be the purification *of H in G.*

Remark 1 *We list some useful facts about pure subgroups.*

1. *If H is a pure subgroup of G then $H_*^G = H$.*

2. *In torsion free groups, intersection of pure subgroups is pure again.*

3. *H_*^G is the intersection of all pure subgroups of G that contain H.*

4. *H_*^G is the smallest pure subgroup containing H.*

Lemma 2.1.1 *[4] Let G be an abelian group, $g \in G$ and H, L and M be subgroups of G. If*

- *$G = \langle H, g \rangle$,*
- *$H = L \oplus M$,*
- *M is a pure subgroup of G,*

- *$p^k g = l + m \in H$ with $l \in L$, $m \in M$ and*
- *the order of the coset $g + H$ is $p^k \neq 1$ where p is prime, $k \in \mathbb{N}$*

then $l \in L - p\,L$.

Proof 1 *Suppose that $l = p\,l'$ where $l' \in L$. Then*

$$m = p^k g - p\,l' = p(p^{k-1}g - l') \in pG.$$

Since $M \subset_* G$ we get $p^{k-1}g - l' \in M$. Let $m = pm'$ where $m' = p^{k-1}g - l' \in M$. Then $p^{k-1}g = l' + m' \in H$, but this contradicts the assumption that the order of $g + H$ is p^k.

CHAPTER 3

CHARACTERISTICS AND TYPES

Let G be a torsion free abelian group. In this section we first give the definition of a coefficient group $\mathbb{Q}_a^G$ where $a \in G$. Then we define characteristic of elements of G and type of G.

Definition 4 *Let a be an element of G. The* ***coefficient group*** *of a in G is defined by $\mathbb{Q}_a^G = \{s \in \mathbb{Q} \mid sa \in G\}$.*

Definition 5 *Let $a \in G$. For a prime p, the largest integer k such that $p^{-k}a$ is in G, i.e., $p^{-k} \in \mathbb{Q}_a^G$ is called the p-**height** of a denoted $h_p(a) = k$. If there is no such maximal integer, that is $p^{-k}a \in G$ for all $k \in \mathbb{N}$, then we say $h_p(a) = \infty$*

Definition 6 *Let $(p_1, p_2, \cdots, p_n, \cdots)$ be a sequence of prime numbers in an increasing order and let $a \in G$. Then the sequence of p-heights*

$$C_G(a) = (h_{p_1}(a), \dots, h_{p_n}(a), \dots)$$

is called the ***characteristic*** *of a.*

Remark 2 *If $C_1 = (k_1, \dots, k_n, \dots)$ and $C_2 = (l_1, \dots, l_n, \dots)$ are two characteristics of $a \in G$ then $C_1 \geq C_2$ means that $k_n \geq l_n$ for every n. The set of height sequences has a partial ordering. For any two characteristic we can define the following operations:*

$$\begin{aligned} (k_1, \dots, k_n, \dots) \cap (l_1, \dots, l_n, \dots) &:= (\min(k_1, l_1), \dots, \min(k_n, l_n), \dots) \\ (k_1, \dots, k_n, \dots) \cup (l_1, \dots, l_n, \dots) &:= (\max(k_1, l_1), \dots, \max(k_n, l_n), \dots) \end{aligned}$$

Lemma 3.0.1 *[2]*

1. *If $G = A \oplus B$ and $a \in A$, $b \in B$ then*

$$C(a+b) = C(a) \cap C(b).$$

2. *Let $f : G \to H$ be a homomorphism of groups and let $g \in G$, then*

$$C_G(g) \leq C_H(f(g)).$$

Proof 2 *(1) Suppose that p^x divides a and p^y divides b , where p is prime, $a \in A, b \in B$ and x, y are integers. Then $p^x a' = a$ and $p^y b' = b$ for some $a', b' \in G$. Let $\min(x, y) = r$. Then*

$$a + b = p^x a' + p^y b' = p^r(p^{x-r}a' + p^{y-r}b').$$

where $p^{x-r}a' + p^{y-r}b' \in G$. Hence we get $h_p(a+b) \geq \min(h_p(a), h_p(b))$ for all prime p, which shows $C(a+b) \geq C(a) \cap C(b)$.
Next show that $C(a+b) \leq C(a) \cap C(b)$. Suppose that $p^x z = a + b$ for some $z \in G$. Set $z = a' + b'$ where $a' \in A$,$b' \in B$. Then

$$a + b = p^x(a' + b') = p^x a' + p^x b'.$$

So we get $a - p^x a' = p^x b' - b \in A \cap B = \{0\}$, which implies $a = p^x a'$ and $p^x b' = b$. This finishes the proof.

(2) Let $g \in G$ and suppose that $p^x g' = g$ where $g' \in G$. Then $f(g) = f(p^x g') = p^x f(g')$. Thus $C_G(g) \leq C_H(f(g))$.

Definition 7 *Two characteristics $(k_1, \ldots, k_n, \ldots)$ and $(l_1, \ldots, l_n, \ldots)$ are said to be **equivalent** if they differ for finitely many components with finite entries and they have exactly the same ∞-components. An equivalence class of characteristics is called a **type**. The type of an element $x \in G$ is denoted by $\mathrm{t}(x)$.*

If t_1 and t_2 are two types then $\mathrm{t}_1 \leq \mathrm{t}_2$ means that there are characteristics C_1 in t_1 and C_2 in t_2 such that $C_1 \leq C_2$. If any two non-zero elements of G have the same type then G is called **homogenous.** If G has rank 1 then G is homogenous since if x and y are non-zero elements of G then $mx = ny$ for some non-zero integers m and n. Then

$$\mathrm{t}(x) = \mathrm{t}(mx) = \mathrm{t}(ny) = \mathrm{t}(y).$$

If G is of rank 1 then we set $\mathrm{t}(G)$ for the type of G. Note that every element x comes with its coefficient group $\mathbb{Q}_x^G$ and $\mathrm{t}(x) = \mathrm{t}(\mathbb{Q}_x^G)$.

Two torsion free groups G and H of rank 1 are isomorphic if and only if $\mathrm{t}(G) = \mathrm{t}(H)$, see [2, Theorem 85.1]. The isomorphism is given by the partial order on the set of the types of the groups.

Definition 8 *Let* t *be a type in* G. *Then*

$$G(t) := \{x \in G \mid \mathrm{t}(x) \geq t\},$$

$$G^*(t) := \langle x \in G \mid t(x) > t\rangle$$

and

$$G^{\#}(t) := G^*(t)_*^G = G \cap \mathbb{Q}\langle x \in G \mid \mathrm{t}(x) > t\rangle.$$

It is easy to show that $G^{\#}(t) \subseteq G(t) \subseteq G$. Note that if g is an element in $G(t)\setminus G^{\#}(t)$ then $\mathrm{t}(g) = t$. Since $G^{\#}(t)$ is the purification of $G^*(t)$ in G is of course pure in G. Now we will show that $G(t)$ is also pure in G.

Lemma 3.0.2 *The group* $G(t)$ *is a pure subgroup of* G.

Proof 3 *First we will show that* $G(t)$ *is a subgroup of* G. *Let* a *and* b *be in* $G(t)$. *By definition of* $G(t)$, $\mathrm{t}(a) \geq t$ *and* $\mathrm{t}(b) \geq t$. *Then*

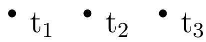
$$\mathrm{t}(a - b) = \mathrm{t}(a + (-b)) \geq \mathrm{t}(a) \cap \mathrm{t}(b) \geq t.$$

Hence $G(t)$ *is a subgroup of* G. *Now show that* $G(t)$ *is pure. For* $n \in \mathbb{N}$, $g \in G$ *and* $y \in G(t)$ *with* $y = ng$ *we have* $\mathrm{t}(y) = \mathrm{t}(ng) = \mathrm{t}(g) \geq t$. *This implies* $g \in G(t)$ *proving that* $G(t)$ *is pure in* G.

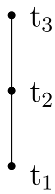
Definition 9 *The type* t *is called* ***critical*** *if* $G(t) \neq G^{\#}(t)$*. Denote the set of all critical types of* G *by* $T_c(G)$*.*

Let G be an almost completely decomposable group and T be the poset of the critical types of G. Two elements t_1 and t_2 of T are called *comparable* if either $\mathrm{t}_1 \leq \mathrm{t}_2$ or $\mathrm{t}_2 \leq \mathrm{t}_1$. Otherwise, they are said to be *incomparable*. The poset T can be represented by diagrams, known as Hasse diagrams. For instance, if the critical typeset of G has 3 elements t_1,t_2,t_3 then some of the possible Hasse diagrams can be listed as :

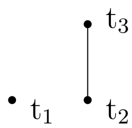
- If t_1,t_2,t_3 are pairwise not comparable then G is called a $\mathbf{(1,1,1)}$ **- Group** and the corresponding Hasse diagram of T is



- If the critical types are comparable, i.e., they are linearly ordered then G is called a **(3) - Group**. The corresponding Hasse diagram of T is



- If two of the critical types are comparable then G is called a $\mathbf{(1,2)}$ **- Group**. In this case t_1 is incomparable with $\mathrm{t}_2, \mathrm{t}_3$ and $\mathrm{t}_2 < \mathrm{t}_3$. There are two minimal and two maximal elements. The corresponding Hasse diagram of T is



We end up this section with a lemma which will be used in other chapters.

Lemma 3.0.3 *[3, Lemma 2.4.7] Let* t *be a type in G.*

1. $T_c(G(t)) = \{\tau \in T_c(G) \mid \tau \geq t\}$ *and* $T_c(G^{\#}(t)) = \{\tau \in T_c(G) \mid \tau > t\}$.

2. *If* H *is a subgroup of* G *such that* $mG \leq H \leq G$ *for some* $m \in \mathbb{Z}^{+}$, *then* $T_c(G) = T_c(H)$.

CHAPTER 4

REGULATOR AND REGULATOR QUOTIENT OF AN ACD GROUP

Every almost completely decomposable group G has a decomposition

$$G(t) = G_t \oplus G^{\#}(t)$$

where G_t is t-homogenous and completely decomposable. This decomposition is called *Butler decomposition.* The subgroup $\bigoplus_{t \in T_{c(G)}} G_t$ is called a **regulating subgroup** of G. Every regulating subgroup of an almost completely decomposable group is completely decomposable. The regulating subgroups of an acd group G are the completely decomposable subgroups of minimal index.

R.Burkhardt proved that the intersection of all regulating subgroups of an almost completely decomposable is again completely decomposable, fully invariant subgroup of finite index.

Definition 10 *Let G be an almost completely decomposable group. The intersection of all regulating subgroups of G is called the* ***regulator*** *G.*

The Regulator $R(G)$ is completely decomposable of finite index. If there is only one regulating subgroup of G then the regulator is called *regulating regulator*. If R is the regulator of an almost completely decomposable group G then the quotient group G/R is said to be the *regulator quotient*.

Remark 3 *Let G be an almost completely decomposable group and let R be its regulator.*

1. *G can be considered as an extension of R by its regulator quotient G/R, i.e., almost completely decomposable groups are torsion free finite extensions of completely decomposable groups of finite rank.*

2. *If m is the exponent of the regulator quotient G/R then $m(G/R) = R$ and for all $g \in G$ we have*

   $$m(g+R) = R \implies mg = r \text{ for some } r \in R \implies g = m^{-1}r \in m^{-1}R$$

   which shows that $R \subseteq G \subseteq m^{-1}R$.

3. *For almost completely decomposable groups the finite set of types of the direct summands of rank 1 of the regulator is called **critical typeset** and this is an invariant.*

A poset T is called an *inverted tree* or *V-free* if for each $t \in T$ the subset $\{\tau \mid \tau \geq t\}$ is a chain. In particular, anti-chains are V-free. The following two results are due to O.Mutzbauer.

Lemma 4.0.1 *[5] Let G be an almost completely decomposable group and $T_c(G)$ be its critical typeset. If $T_c(G)$ is an inverted forest then G has a regulating regulator.*

Lemma 4.0.2 *[5] Let G be an almost completely decomposable group with a critical typeset $T_c(G)$ that is V-free. Let R be a completely decomposable group of G of finite index. R is regulator of G if and only if $R(t) \subset_* G$ for all $t \in T_c(G)$.*

We now show how the regulator of an almost completely decomposable group of type $(1,2)$, called $(1,2)-$group, can be recognized. Recall that a $(1,2)-$group is an almost completely decomposable group with critical typeset $T_c = \{\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3\}$ where t_1 is incomparable with t_2, t_3 and $\mathrm{t}_2 < \mathrm{t}_3$.

Lemma 4.0.3 *Let G be an acd $(1,2)-$group. Let $R = R_1 \oplus R_2 \oplus R_3$ be a completely decomposable subgroup of G of finite index. Let t_1, t_2, t_3 be the types of R_1, R_2 and R_3, respectively. Then R is the Regulator of G if and only if $R_1, R_2 \oplus R_3 \subset_* G$.*

Proof 4 *Since* $\mathrm{t}(R_i) = \mathrm{t}_i$ *for* $i = 1, 2, 3$ *we have* $R(t_1) = R_1, R(t_2) = R_2 \oplus R_3$ *und* $R(t_3) = R_3$ *by the constellation of the given types.* $R(t_2) \subset_* G$ *implies* $R(t_3) \subset_* G$. *The result follows by Lemma 4.0.2.*

CHAPTER 5

MATRIX REPRESENTATION OF ALMOST COMPLETELY DECOMPOSABLE GROUPS

Let $R = \bigoplus_{i=1}^{n} M_i x_i$ be a completely decomposable group where M_i is the coefficient group of x_i in R. Then by definition of the coefficient group $M_i = \{m \in \mathbb{Q} \mid mx_i \in R\}$. The ordered set $(x_1, \dots, x_n)$ is a p- *basis* of R for a prime p when $x_i \in R$ for each i and $p^{-1} \notin M_i$. The purification of $\langle x_i \rangle$ in R is $\langle x_i \rangle_*^R = M_i x_i$, $1 \in M_i$, and $\mathrm{t}(x_i) = \mathrm{t}(M_i)$.

Let G be an almost completely decomposable group and let p be a prime. G is called *p-reduced* if G has no p-divisible subgroup. Let G be a p-reduced almost completely decomposable group of rank n and let R be a completely decomposable subgroup of G. Assume that the quotient group G/R is of exponent p^k for a prime p. Let $(v_1, \dots, v_n)$ is a p-basis of R and let $\mathbf{\bar{a}} := (\bar{a}_1, \dots, \bar{a}_r)$ be a basis of $G/R \subseteq p^{-k}R/R$. Then we can write $R = \bigoplus_{j=1}^{n} \langle v_j \rangle_*^R$. Let

$$\bar{\ } : R \to p^{-k}R/R, \ v \mapsto \bar{v} = p^{-k}v + R$$

be the natural epimorphism. Then the basis $\mathbf{\bar{v}} := (\bar{v}_1, \dots, \bar{v}_n)$ of the $\mathbb{Z}/p^k\mathbb{Z}$-module $p^{-k}R/R$ is called an induced decomposition basis, where $\bar{v}_j = p^{-k}v_j + R$.
The $\bar{a}_i$'s may be written as linear combinations of the induced decomposition basis

$$\bar{a}_i = \sum_{j=1}^{n} \alpha_{ij} \bar{v}_j \quad \text{for} \quad i = 1, \dots, r$$

where $\alpha_{ij} \in \mathbb{Z}_{p^k}$. This shows that almost completely decomposable groups can be described by matrices. The $r \times n$–matrix

$$M = (\alpha_{ij})_{\substack{i=1,\dots,r \\ j=1,\dots,n}} \in M_{r \times n}(\mathbb{Z}_{p^k})$$

is called the *coordinate matrix* of G over R relative to $\mathbf{\bar{a}}$ and $\mathbf{\bar{v}}$.

Lemma 5.0.1 *Let G be a p-reduced almost completely decomposable group and let $R = \bigoplus_{i=1}^{n} M_i v_i$ be a completely decomposable subgroup of G where $(v_1, \ldots, v_n)$ is a p-basis of R and M_i is the coefficient group. Let G/R be a finite group of exponent p^k with a basis $(g_1 + R, \ldots, g_r + R)$. If $\langle g_i + R \rangle \simeq \mathbb{Z}_{p^{k_i}}$ where p^{k_i} is the order of the g_i for $i = 1, \ldots, r$, then the representatives g_j can be written as*

$$g_j = p^{-k_j}\big(\alpha_{j,1}v_1 + \cdots + \alpha_{j,n}v_n\big),$$

for $1 \leq j \leq r$, where $\alpha_{j,i} \in \mathbb{Z}$. The entry $\alpha_{j,i}$ is unique modulo p^{k_j}.

Proof 5 *Set*

$$g_j = p^{-k_j}\big(\alpha_{j,1}v_1 + \cdots + \alpha_{j,n}v_n\big),$$

where $\alpha_{j,i}v_i \in M_i$ and M_i is the coefficient group of v_i in R. Then $\alpha_{j,i} = \beta_{j,i}/\gamma_{j,i}$ is a reduction fraction and the denominator $\gamma_{j,i}$ is relatively prime to p by the assumption. Let d be the least common multiple of all $\gamma_{j,1}, \ldots, \gamma_{j,n}$.
Then there exist integers a, b such that $ad = 1 + bp^{k_j}$ and so

$$g_j = (ad - bp^{k_j})g_j = \frac{1}{p^{k_j}}\big(ad\alpha_{j,1}v_1 + \cdots + ad\alpha_{j,n}v_n\big) - b\big(\alpha_{j,1}v_1 + \cdots + \alpha_{j,n}v_n\big).$$

Since $\alpha_{j,1}v_1 + \cdots + \alpha_{j,n}v_n \in R$ and since all coefficients $ad\alpha_{j,i} \in \mathbb{Z}$ the desired result is obtained.

5.1 Coordinate Matrices

Let G be an almost completely decomposable group and let p be a prime. Then G is called p-**local** if the regulator quotient is a group of exponent dividing p. Suppose that G is p-local and p-reduced. Let R be the regulator of G and assume that the regulator quotient $G/R \cong \bigoplus_{s=1}^{r} \mathbb{Z}_{p^{k_s}}$ where $k_i \in \mathbb{N}$. Let $S = \mathrm{diag}(p^{-k_i} \mid 1 \leq i \leq r)$ be a diagonal matrix corresponding to the isomorphism type of the regulator quotient. Moreover, let $(v_1, \ldots, v_n)$ be a p-basis of R and let $(g_1 + R, \ldots, g_r + R)$ be a basis G/R. Then by Lemma 5.0.1 there exist an $r \times n$ integer matrix $\alpha = (\alpha_{j,i})$. The matrix α is called *coordinate matrix* of G relative to the bases of the regulator and regulator quotient. A coordinate matrix δ of G is determined up to the congruence modulo p^k where k is the exponent of the regulator quotient, i.e., in coordinate matrix the entries are determined modulo p^k, and therefore can be considered to be a matrix with coefficients in $\mathbb{Z}/p^k\mathbb{Z}$.

Conversely, assume that $R = \bigoplus_{i=1}^{n} M_i v_i$ where $(v_1, \ldots, v_n)$ is a p-basis of R and M_i is the coefficient group. Let S be the diagonal matrix defined as above and let $\alpha = (\alpha_{i,j})$ be an $r \times n$ integer matrix with $r \leq n$. Then both the decomposition of R and α determine a unique group $G = \langle R, g_1, \ldots, g_r \rangle$ with $R \subset G \subset \mathbb{Q} R$, where

$$g_j = p^{-k_j}\big(\alpha_{j,1}v_1 + \cdots + \alpha_{j,n}v_n\big),$$

for $1 \leq j \leq r$.

Assume $k = k_1 \geq k_2 \geq \cdots \geq k_r$. By grouping the generators g_i of the regulator quotient of equal orders, the coordinate matrix α can be written in the form

$$\alpha = \left[\begin{array}{ccc} \alpha_{11} & \cdots & \alpha_{1n} \\ \hline \vdots & \cdots & \vdots \\ \hline \alpha_{r1} & \cdots & \alpha_{rn} \end{array}\right] \begin{array}{c} p^{-k_1} \\ \hline \vdots \\ \hline p^{-k_r} \end{array}$$

where $p^{k_i} = \operatorname{ord}(g_i + R)$ and $k_1 \geq k_2 \geq \cdots \geq k_r$.

Example 1 *Now we will show how to form the coordinate matrix of an almost completely decomposable group of type $(1,2)$, briefly a $(1,2)$ group and we list some of its properties. A $(1,2)$-group has three critical types $\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3$ such that t_1 is not comparable with t_2, t_3 and $\mathrm{t}_2 < \mathrm{t}_3$. Let G be a p-local group. Moreover, assume that G is* clipped *, i.e., G has no rank 1 summands.*
Let R be the regulator of G. We can write $R = R_1 \oplus R_2 \oplus R_3$, where R_i is homogeneous of rank r_i, and the types of the R_i are t_i, respectively. Let $\{u_1, \ldots, u_{r_1}\}$ be a p-basis of R_1, $\{v_1, \ldots, v_{r_2}\}$ be a p-basis of R_2 and $\{w_1, \ldots, w_{r_3}\}$ be a p-basis of of R_3. Since G is a p-local group, the regulator quotient G/R is of exponent p^k for some $k \in \mathbb{N}$. Assume that $G/R \cong \bigoplus_{s=1}^{r} \mathbb{Z}_{p^{k_s}}$ where $k_i \in \mathbb{N}$ and $k = k_1 \geq k_2 \geq \ldots \geq k_r$. Let $(g_j + R \mid 1 \leq i \leq r)$ be a basis of the regulator

quotient G/R and let $\operatorname{ord}(g_s + R) = p^{k_s}$. *Then*

$$g_j = p^{-k_j}\Big(\sum_{i=1}^{r_1} \alpha_{ji}x_i + \sum_{i=1}^{r_2} \beta_{ji}y_i + \sum_{i=1}^{r_3} \gamma_{ji}z_i\Big),$$

with respect to the given decomposition of G/R and

$$G = R + \sum_{j=1}^{r} \mathbb{Z}p^{-k_j}\Big(\sum_{i=1}^{r_1} \alpha_{ji}u_i + \sum_{i=1}^{r_2} \beta_{ji}v_i + \sum_{i=1}^{r_3} \gamma_{ji}w_i\Big).$$

By Lemma 5.0.1, the matrices $\alpha = (\alpha_{ji}), \beta = (\beta_{ji}), \gamma = (\gamma_{ji})$ can be considered as integer matrices. Let δ be the coordinate matrix of G. The choice of the p-basis of the regulator R allows us to divide δ in three blocks α,β and γ. Hence δ can be written as

$$\delta = \begin{array}{|ccccc||ccccc|ccccc|}
\hline
\alpha_{11} & \alpha_{12} & \cdots & \cdots & \alpha_{1r_1} & \beta_{11} & \beta_{12} & \cdots & \cdots & \beta_{1r_2} & \gamma_{11} & \gamma_{12} & \cdots & \cdots & \gamma_{1r_3} \\
\alpha_{21} & \alpha_{22} & \cdots & \cdots & \alpha_{2r_1} & \beta_{21} & \beta_{22} & \cdots & \cdots & \beta_{2r_2} & \gamma_{21} & \gamma_{22} & \cdots & \cdots & \gamma_{2r_3} \\
\vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots \\
\alpha_{r1} & \alpha_{r2} & \cdots & \cdots & \alpha_{rr_1} & \beta_{r1} & \beta_{r2} & \cdots & \cdots & \beta_{rr_2} & \gamma_{r1} & \gamma_{r2} & \cdots & \cdots & \gamma_{rr_3} \\
\hline
\end{array}$$

The coordinate matrices of acd groups with different type constellation can be written similarly.

Definition 11 *Let G be an almost completely decomposable group and let δ be its coordinate matrix. Then δ is said to be a* decomposable matrix *if there are permutation matrices P and Q such that $P\delta Q = \delta_1 \oplus \delta_2$.*

Almost completely decomposable groups can be represented by coordinate matrices and they are very helpful by determining decomposability of almost completely decomposable groups. The following lemma show us the relation between the decomposability of an acd group and its coordinate matrix.

Lemma 5.1.1 *Let G be an almost completely decomposable group and let δ be the coordinate matrix of G. Then G is decomposable if and only δ is decomposable.*

Proof 6 *First assume that δ is decomposable. By definition there exist permutation matrices P and Q such that $P\delta Q = \delta_1 \oplus \delta_2$. The coordinate matrix δ comes with a p-basis B of the regulator $R(G)$ and a basis of G/R. Since each column of δ corresponds to a basis element, we may write B as a union, $B = B_1 \cup B_2$. This partition leads us to write the regulator $R(G)$ as a direct sum of its two subgroups, i.e, $R = R_1 \oplus R_2$. Now considering the purification of the direct summands it is possible to write G as a direct sum of two groups G_i where $G_i = \langle R_i \rangle_*$. This show that G is decomposable.*

Conversely, suppose that G is decomposable. Then $G = G_1 \oplus G_2$. Let R_i be the regulator of G_i, respectively. Then $R(G)$ can be written as $R(G) = R_1 \oplus R_2$. Since the columns of the corresponding coordinate matrix of G are determined by the p-basis elements of R_1 and R_2, the coordinate matrix is the direct sum of the coordinate matrices of the direct summands G_1 and G_2.

CHAPTER 6

DECOMPOSABILITY OF ACD GROUPS

Almost completely decomposable groups are finite extensions of completely decomposable groups. However almost completely decomposable groups are quite different from the completely decomposable groups. Although the classification of almost completely decomposable groups up to isomorphism is not possible so far, a weaker form of isomorphism allows us to describe the isomorphism classes of some subclasses of acd groups. Let G and H be two torsion free groups of finite rank. If for every $n \in \mathbb{Z}^+$, there exists an integer m such that m and n are relatively prime and there are homomorphisms $f : G \mapsto H$ and $g : H \mapsto G$ such that $fg = m1_H$ and $gf = m1_G$ then G and H are called *nearly isomorphic*. If G and H are nearly isomorphic groups then $R(G) \simeq R(H)$ and $G/R(G) \simeq H/R(H)$, see [3].
By a well known theorem of Arnold , if G and H are two nearly isomorphic torsion free abelian groups then they have the same decomposition properties. Faticoni and Schultz proved that p-local almost completely decomposable groups are classified up to near-isomorphism once the near isomorphism of the indecomposable groups are known.

Definition 12 *Let G be an almost completely decomposable group and δ be its coordinate matrix. Rows and columns transformations of δ are called **permitted** if the transformed coordinate matrix is the coordinate matrix of G or another group H which is near isomorphic group to G. The permitted operations depend on the poset of the critical types and the isomorphism of G/R.*

Let G be a p-local almost completely decomposable group. The method used to determine the decomposability of G is as follows: To the group G there corresponds a coordinate matrix M. By Lemma 5.1.1, the group G is decomposable if and only if M is decomposable. Hence, applying certain *permitted row and column operations* which will be described later, if M can be transformed to a decomposable matrix M' that is a coordinate matrix of a nearly isomorphic group G' , due to the result of Arnold the group G is also decomposable.

6.1 Modified Gauss Algorithm

Let G a p-local, p-reduced almost completely decomposable group of rank n. Let R be the regulator and G/R be the regulator quotient. Let $(\zeta_1, \zeta_2, \ldots, \zeta_r)$ be a basis of G/R. Assume $G/R \cong \bigoplus_{s=1}^{h} \left(\mathbb{Z}_{p^{k_s}}\right)^{l_s}$, where $p^{k_i} = \operatorname{ord}(\zeta_i)$ and $r = l_1 + l_2 + \ldots + l_h$. Let $\delta \in M(r \times n, \mathbb{Z}_{p^k})$ be the coordinate matrix of G. Then we may write

$$\delta = \left[\begin{array}{c} \delta_1 \\ \hline \vdots \\ \hline \delta_h \end{array}\right] \begin{array}{c} p^{-k_1} \\ \hline \vdots \\ \hline p^{-k_h} \end{array}$$

where $k = k_1 > k_2 > \cdots > k_h$ and the block matrices $\delta_i \in M(l_i \times n, \mathbb{Z}_{p^{k_i}})$. The matrix δ is called the (k_i, l_i)- step matrix with the ith step δ_i.

Definition 13 *Let δ_1 und δ_2 be two (k_i, l_i)-step matrices. Then δ_1 und δ_2 are called* step equivalent *if δ_2 can be obtained form δ_1 by permitted row and column permutations.*

Definition 14 *An integer y is said to be a p-*unit *for a prime p if* $\gcd(p, y) = 1$. *If y is a p-unit then there exist an integer y' such that $yy' \equiv 1 \mod p^l$ for all $l > 0$.*
The units considered in this thesis are all p-units and whenever we write unit we mean a p-unit.

Remark 4 *Let δ be a (k_i, l_i)-step matrix. The following row operations are permitted:*

(a) Multiplying a row with a unit,

(b) Interchanging two rows in the same step,

(c) Adding a multiple of one row to another row in the same step,

(d) Let g_i be a row in δ_i and g_j be a row in δ_j. Adding a $p^{k_i-k_j}$ multiple of g_j to g_i for $i < j$.

(e) Let g_i be a row in δ_i and g_j be a row in δ_j. Adding any multiple of g_i to g_j for $i < j$.

By permitted transformations we try to get kind of a normal form that enables us to decide the decomposability of a given almost completely decomposable group. The row operations stated in Remark 4 can be done by multiplying the coordinate matrix δ from left with some elementary matrices. This operation is called the *modified Gauss elimination*.

Let I denote the identity matrix and let $I_{i,j}$ be the matrix whose (i, j)th entry is 1 and all the other entries are 0. The matrices corresponding to the permitted row operations defined in Remark 4 are given below:

(a) $I + (\lambda - 1)I_{i,j}$, where $\lambda \in \mathbb{Z}_{p^k}$,

(b) $I - I_{i,i} - I_{j,j} + I_{i,j} + I_{j,i}$,

(c) $I + \lambda I_{i,j}$, where $\lambda \in \mathbb{Z}_{p^k}$,

(d) $I + \lambda p^{k_i-k_j} I_{i,j}$, mit $\lambda \in \mathbb{Z}_{p^k}$,

(e) $I + \lambda I_{i,j}$, mit $\lambda \in \mathbb{Z}_{p^k}$.

Remark 5 *Let G and H be almost completely decomposable groups with regulator R. Let δ_1 and δ_2 be the coordinate matrices of G and H, respectively. Then G is nearly isomorphic to H if and only if $\delta_2 = M\delta_1 N$ where $M \in \mathrm{Aut}(G/R)$ and $N \in \mathrm{Aut}(R)$.*

Let G be a p-local p-reduced acd group and let $G/R \cong \bigoplus_{s=1}^{h} \left(\mathbb{Z}_{p^{k_s}}\right)^{l_s}$ be the regulator quotient of G. Then G/R is a finite group of exponent p^k for some $k \in \mathbb{N}$. Assume G/R has rank r and let $B = (\bar{g}_1, \ldots, \bar{g}_r)$ be a basis of G/R . Each automorphism of G/R describes a p-invertible integer matrix M by Remark 5. Let

$S = \operatorname{diag}(p^{-k_1}, \ldots, p^{-k_h})$, corresponding to the isomorphism type of G/R. Then M corresponds to an automorphism of G/R relative to B if and only if there is an integer matrix M' of size r such that $MS = SM'$, cf. [1, Section 3.11, Theorem 3.15]. The matrix M is said to be a *S-matrix*. A p-invertible matrix M of size r is a S-matrix if and only if all entries of $M = [m_{i,j}]$ are divisible by $p^{k_i - k_j}$ for $i \leq j$.

6.2 Permitted column operations

In this subsection we will describe the permitted column transformations and the corresponding matrices.

Lemma 6.2.1 *Let G be an almost completely decomposable group and let R be its regulator. Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two p-bases of R such that $\mathrm{t}(u_i) = \mathrm{t}(v_i)$ for all i. Suppose that $v_i = \sum_{j=1}^{n} \delta_{i,j} u_j$ where $\delta_{i,j} \in \mathbb{Q}$. If $\delta_{i,j} \neq 0$ then $\mathrm{t}(u_i) \leq \mathrm{t}(u_j)$.*

Proof 7 *Set $R = \bigoplus_{i=1}^{n} S_i u_i = \bigoplus_{i=1}^{n} T_i v_i$ where S_i and T_i are coefficient groups. By assumption and by the fact that $\mathrm{t}(u_i) = \mathrm{t}(S_i)$ and $\mathrm{t}(v_i) = \mathrm{t}(T_i)$, we get $S_i \cong T_i$ by* [2, Theorem 85.1]. *Using the properties of the types we can write*

$$\mathrm{t}(u_i) = \mathrm{t}(v_i) = \mathrm{t}(\sum_{j=1}^{n} \delta_{i,j} u_j) = \bigcap_{j=1}^{n} \mathrm{t}(\delta_{i,j} u_j)$$

Hence $\delta_{ij} = 0$ if $S_i \not\subset S_j$.

Remark 6 *Let G be an acd group. If $B_1 = (u_1, \ldots, u_n)$ and $B_2 = (v_1, \ldots, v_n)$ are two bases of the regulator $R(G)$ then by Lemma 4.0.2, the transition from B_1 to B_2 can be considered as an automorphism of $\mathbb{Q}\, R$ defined by $\delta(v_i) = \sum_{j=1}^{n} \delta_{i,j} u_j$. This can be done by the matrix N in Remark 5. Namely, multiplying a coordinate matrix of G from the right by N , the corresponding matrix of the automorphism δ, transforms the coordinate matrix to the coordinate matrix of another group which is nearly isomorphic to G.*

Lemma 6.2.2 [2, Proposition 85.4] *Let G and H be torsion free abelian groups of rank* 1. *Then* $\mathrm{Hom}(G, H) = 0$ *if* $\mathrm{t}(G) \not\leq \mathrm{t}(H)$.

Definition 15 *Let $\delta = (\delta_{ij})$ be a p-invertible matrix and let $T = (\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_n)$ be a sequence of critical types. δ is called T-*conformed *if $\delta_{ij} \neq 0$ then $\mathrm{t}_i \leq \mathrm{t}_j$.*

Example 2 *Let G be a $(1, 2)$-group . By* [2, Theorem 106.1] *and by Lemmas 4.0.2 and by Lemma 6.2.2, a T-conformed matrix N has the form*

$$N = \begin{pmatrix} N_1 & 0 & 0 \\ 0 & N_2 & N_3 \\ 0 & 0 & N_4 \end{pmatrix},$$

where the block matrices N_1, N_2, N_4 are p-invertible and N_3 is arbitrary.

Definition 16 *Let $S = \mathrm{diag}(p^{k_1}, \ldots, p^{k_r})$ and let $A = [a_{ij}]$, $A' = [a'_{ij}]$ be two integer matrices. Then A and A' are called congruent modulo S if $a_{ij} \equiv a'_{ij} \mod p^{k_i}$.*

Theorem 6.2.3 [12, Theorem 12] *Let G be a p-reduced, p-local, almost completely decomposable group with a regulating regulator R. Let $B_1 = (u_1, \ldots, u_n)$ be a p-basis of R and T be the critical typeset corresponding to B_1. Moreover, let $B'_1 = (g_1 + R, \ldots, g_r + R)$ be a basis of G/R where $\mathrm{ord}(g_i + R) = p^{k_i}$. Let $S = \mathrm{diag}\big(p^{k_1}, \ldots, p^{k_r}\big)$. Assume δ be the coordinate matrix relative to B_1 and B'_1.*

(a) Let $B_2 = (v_1, \ldots, v_n)$ be another p-basis of R and let $B'_2 = (h_1 + R, \ldots, h_r + R)$ be a basis of G/R with $\mathrm{ord}(h_i + R) = p^{k_i}$. Assume δ' be the coordinate matrix of G relative to the B_2 and B'_2. Moreover suppose that T is also the critical typeset of G corresponding to B_2. Then there is an S-matrix M and a T-conformed matrix N such that $M\delta N$ and δ' are congruent modulo S.

(b) Assume that M is S-matrix and N is a T-conformed matrix. Then there is a group H nearly isomorphic to G such that $M\delta N$ is the corresponding coordinate matrix of H.

CHAPTER 7

NORMAL FORM AND DECOMPOSABILITY

Let G be an almost completely decomposable group and R be a completely decomposable subgroup. The following Lemma shows us when R is the regulator of G.

Lemma 7.0.1 [12, Lemma 13] *Let G be a p-local almost completely decomposable group and R be a completely decomposable subgroup of G. Let δ be a $r \times n$-coordinate matrix formed with respect to R and let T be a critical typeset of G. Assume that δ' is the submatrix obtained from δ by taking the columns of δ belonging to types $\not\geq t$ where $t \in T$. Then R is the regulator of G if and only if δ' has rank r.*

Note that Lemma 7.0.1 can be specialized for different classes of almost completely decomposable subgroup.

Example 3 *Let G be a p-local $(1, n)$-group with a completely decomposable subgroup R and with a homocyclic G/R of rank r.*
Let $T = \{\mathrm{t}_1, \mathrm{t}_2 < \mathrm{t}_3 \ldots < \mathrm{t}_n\}$ be the critical typeset of G and let

$$\delta = [\delta_{\mathrm{t}_1} \mid\mid \delta_{\mathrm{t}_2} \mid \ldots \mid \delta_{\mathrm{t}_n}]$$

be the coordinate matrix of G where δ_{t_i} is the block matrix containing all t_i-columns. By Lemma 7.0.1 it follows that R is the regulator of G if δ_{t_1} and $[\delta_{\mathrm{t}_2} \mid \ldots \mid \delta_{\mathrm{t}_n}]$ both have rank r.

Example 4 *Let G be a p-local $(2, 2)$-group with a critical typeset*

$$T = \{\mathrm{t}_1 < \mathrm{t}_2, \mathrm{t}_3 < \mathrm{t}_4\}.$$

Let $R = R_1 \oplus R_2 \oplus R_3 \oplus R_4$ where $R_i = R(t_i)$ be a completely decomposable subgroup. Assume G/R has rank r. Let $\delta = [\delta_{\mathrm{t}_1} \mid \delta_{\mathrm{t}_2} \mid\mid \delta_{\mathrm{t}_3} \mid \delta_{\mathrm{t}_4}]$ where δ_{t_i} is the block matrix containing all t_i-columns. By Lemma 7.0.1, R is the regulator of G if $[\delta_{\mathrm{t}_1} \mid \delta_{\mathrm{t}_2}]$ and $[\delta_{\mathrm{t}_3} \mid \delta_{\mathrm{t}_4}]$ both have rank r. This is the case if and only if $R_1 \oplus R_2$ and $R_3 \oplus R_4$ are both pure. By Lemma 1 this implies there is a unit in each row of $R_1 \oplus R_2$ and of $R_3 \oplus R_4$.

Lemma 7.0.2 *[4] Let G be a torsion free abelian group and H be a subgroup of G such that G/H is torsion. Then* $\operatorname{rank}(G) = \operatorname{rank}(H)$.

Proof 8 *Since H is a subgroup of G, $\operatorname{rank}(H) \leq \operatorname{rank}(G)$. Let $g \in G \setminus H$. Since G/H is torsion, $g + H$ has finite order, i.e., there is a positive integer n such that $ng \in H$. Let $B = \{k_i \mid 1 \leq i \leq \operatorname{rank} H\}$ be a maximal linearly independent set in H. Then $\{ng\} \cup B_1$ is linearly independent in H and hence*

$$\operatorname{rank}(G) = \dim_Q(\mathbb{Q}\, G) \leq \dim_Q(\mathbb{Q}\, H) = \operatorname{rank}(H).$$

Lemma 7.0.3 *Let $G = H_1u_1 + H_2u_2 + H_3u_3 + \mathbb{Z}p^{-k}(\alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3)$ be an almost completely decomposable group with $\mathbb{Z} \subseteq H_i \subseteq \mathbb{Q}_p$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$. Assume also that $\alpha_3 \in p^a\mathbb{Q}_p \setminus p^{a+1}\mathbb{Q}_p$ where $a \in \mathbb{N}$. Then*

$$\langle u_1, u_2\rangle_*^G = H_1u_1 + H_2u_2 + \mathbb{Z}p^{-a}(\alpha_1u_1 + \alpha_2u_2).$$

In particular, $\langle u_1, u_2\rangle_*^G = H_1u_1 \oplus H_2u_2$ if and only if α_3 is a unit.

Proof 9 *By definition $\langle u_1, u_2\rangle_*^G = (\mathbb{Q}u_1 \oplus \mathbb{Q}u_2) \cap G$. It is easy to see that $\langle H_1u_1 + H_2u_2, p^{-a}(\alpha_1u_1 + \alpha_2u_2)\rangle \subset \langle u_1, u_2\rangle_*^G$. Let $y \in \langle u_1, u_2\rangle_*^G$. Then $y = \lambda_1u_1 + \lambda_2u_2 \in (\mathbb{Q}u_1 \oplus \mathbb{Q}u_2) \cap G$ where $\lambda_1, \lambda_2 \in \mathbb{Q}$. So,*

$$y = \epsilon_1u_1 + \epsilon_2u_2 + \epsilon_3u_3 + \lambda p^{-k}(\alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3)$$

where $\epsilon_i \in H_i$ and $\lambda \in \mathbb{Z}$.

Set $\alpha_3 = p^a\alpha_3'$, where α_3' is a unit. Then equating the coefficients of the equations of y above we get

$$\epsilon_3 + \lambda p^{-k}\alpha_3 = \epsilon_3 + \lambda p^{-k}p^a\alpha_3' = 0.$$

By setting $\lambda = p^{k-a}\lambda'$ *where* $k > a$ *and* $\lambda' \in \mathbb{Z}$ *we get*

$$\begin{aligned}\lambda_1 u_1 + \lambda_2 u_2 &= \epsilon_1 u_1 + \epsilon_2 u_2 + \epsilon_3 u_3 + \lambda p^{-k}(\alpha_1 u_1 + \alpha_2 u_2 + p^a \alpha_3' u_3),\\ &= \epsilon_1 u_1 + \epsilon_2 u_2 + \epsilon_3 u_3 + \lambda p^{-k}(\alpha_1 u_1 + \alpha_2 u_2) + \lambda p^{-k} p^a \alpha_3' u_3,\\ &= \epsilon_1 u_1 + \epsilon_2 u_2 + p^{k-a}\lambda' p^{-k}(\alpha_1 u_1 + \alpha_2 u_2) + (\lambda p^{-k+a}\alpha_3' + \epsilon_3)u_3.\end{aligned}$$

Since $\lambda p^{-k+a}\alpha_3' + \epsilon_3 = 0$, *we get* $y \in \langle H_1 u_1 + H_2 u_2, p^{-a}(\alpha_1 u_1 + \alpha_2 u_2)\rangle$. *Therefore* $\langle u_1, u_2\rangle_*^G = H_1 u_1 + H_2 u_2 + \mathbb{Z}p^{-a}(\alpha_1 u_1 + \alpha_2 u_2)$.

Our aim is to determine whether a given almost completely decomposable group is decomposable or not. To do this we make use of the coordinate matrix. Since we know by a well-known result of Arnold that the decomposability of the coordinate matrix implies the decomposability of the group to which it belongs, by applying certain row and column transformations on the coordinate matrix we try to get a kind of Normal form of the matrix. If the transformed matrix is decomposable then we say that the group is decomposable and we can read off the direct summands from the normal form of the matrix.
Now we will show how this method can be applied to some classes of almost completely decomposable groups.

Example 5 *Let* G *be a clipped* p*-local* $(1,2)$*-group with a critical typeset* $T = \{\mathrm{t}_1, \mathrm{t}_2 < \mathrm{t}_3\}$ *and regulator* $R = R_1 \oplus R_2 \oplus R_3$ *where* $R_i = R(\mathrm{t}_i)$ *are* t_i*-homogenous completely decomposable groups. Let* $G/R \cong \bigoplus_{j=1}^{r} \mathbb{Z}_{p^{k_j}}$ *where* $k_1 \geq k_2 \geq \ldots \geq k_r > 0$. *Then* G *can be written in the form:*

$$G = \Big(\underbrace{\sum_{i=1}^{r_1} H_1 x_i}_{=R_1} \oplus \underbrace{\sum_{j=1}^{r_2} H_2 y_j}_{=R_2} \oplus \underbrace{\sum_{l=1}^{r_3} H_3 z_l}_{=R_3}\Big) + \sum_{s=1}^{r} \mathbb{Z}p^{-k_s}\Big(\sum_{i=1}^{r_1}\alpha_{si}x_i + \sum_{j=1}^{r_2}\beta_{sj}y_j + \sum_{l=1}^{r_3}\gamma_{sl}z_l\Big),$$

where H_i *are coefficient groups and* $\{x_1, \ldots, x_{r_1}; y_1, \ldots, y_{r_2}; z_1, \ldots, z_{r_3}\}$ *is a* p*-basis of* $R = R_1 \oplus R_2 \oplus R_3$. *Let* $\delta = (\epsilon \mid\mid \zeta \mid \eta)$ *be a coordinate matrix of* G. *Then* δ *can be written in the form with respect to regulator and regulator quotient:*

$$\left(\epsilon \,\middle\|\, \zeta \,\middle|\, \eta \right) = \left|\begin{array}{ccccc||ccccc|ccccc}
\epsilon_{11} & \epsilon_{12} & \cdots & \cdots & \epsilon_{1r_1} & \zeta_{11} & \zeta_{12} & \cdots & \cdots & \zeta_{1r_2} & \eta_{11} & \eta_{12} & \cdots & \cdots & \eta_{1r_3} \\
\epsilon_{21} & \epsilon_{22} & \cdots & \cdots & \epsilon_{2r_1} & \zeta_{21} & \zeta_{22} & \cdots & \cdots & \zeta_{2r_2} & \eta_{21} & \eta_{22} & \cdots & \cdots & \eta_{2r_3} \\
\vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots \\
\epsilon_{r1} & \epsilon_{r2} & \cdots & \cdots & \epsilon_{rr_1} & \zeta_{r1} & \zeta_{r2} & \cdots & \cdots & \zeta_{rr_2} & \eta_{r1} & \eta_{r2} & \cdots & \cdots & \eta_{rr_3}
\end{array}\right|$$

Since $R = R_1 \oplus R_2 \oplus R_3$ is the regulator, by Lemma 7.0.1 both R_1 and $R_2 \oplus R_3$ have rank r. Hence there is a unit in each row of ϵ and $(\zeta \mid \eta)$. Since G is clipped, i.e, δ has no 0-column, ϵ can be transformed to the identity matrix. Hence δ has the form

$$\begin{array}{|c||c|c|l}
\hline
 & \zeta_1 & & p^{-k_1} \\
\cline{2-2}
 & \zeta_2 & & p^{-k_2} \\
\cline{2-2}
I_r & \vdots & \eta & \vdots \\
\cline{2-2}
 & \zeta_r & & p^{-k_r} \\
\hline
\end{array}$$

where ζ_i are block matrices. If there is unit in ζ_1, then this unit can be permuted to the $(1,1)$th position. By modified Gauss algorithm and by permitted column operations all the entries below this unit and right to it can be annihilated. Then the first block $[\zeta_1 \mid \eta_1]$ is of the form

$$[\zeta_1, \eta_1] = \left[\begin{array}{cc|c} I & 0 & 0 \\ \hline 0 & p\zeta_1' & \eta_1' \end{array}\right]$$

Since there is a unit in each row $[\zeta_1 \mid \eta_1]$, the submatrix η_1' can be transformed to $[I \mid 0]$ and we get

$$[\zeta_1, \eta_1] = \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ \hline 0 & p\zeta_1' & I & 0 \end{array}\right]$$

We apply the same method to the block $[\zeta_2, \eta_2]$. By identity matrices in ζ_1 and η_1 we can annihilate all the entries below in δ. The permitted row and column trans-

formations change ϵ. But by applying permitted column transformations, ϵ can be transformed to the identity matrix again. Hence the first two blocks of δ is of the form

$$\left(\begin{array}{c||c|c} \epsilon_1 & \zeta_1 & \eta_1 \\ \hline \epsilon_2 & \zeta_2 & \eta_2 \end{array}\right) =$$

<table>
<tr><td rowspan="4">I</td><td>I</td><td colspan="2">0</td><td colspan="3">0</td></tr>
<tr><td rowspan="3">0</td><td colspan="2">$p\zeta_1$</td><td>I</td><td colspan="2">0</td></tr>
<tr><td>I</td><td>0</td><td rowspan="2">0</td><td colspan="2">0</td></tr>
<tr><td>0</td><td>$p\zeta_2$</td><td>I</td><td>0</td></tr>
</table>

We apply the same method on all the block submatrices and get a normal form and then read the direct summands if the group is decomposable.

CHAPTER 8

SOME EXAMPLES OF DECOMPOSABLE AND INDECOMPOSABLE ACD GROUPS

In the following proposition we will prove the decomposability of some class of almost completely decomposable groups of small rank. After stating and proving the proposition we will give some examples. The following results can be found in [8], [11] and [14].

PROPOSITION 8.0.1 *The following almost completely decomposable groups with given typeset T, regulator quotient G/R and coordinate matrix are indecomposable.*

(a) The $(1,2)$-group G_1 with regulator quotient $G_1/R \simeq \mathbb{Z}_{p^3}$ and with coordinate matrix $\delta_1 = [1||p|1]$ is indecomposable of rank 3.

(b) The $(1,3)$-group G_2 with regulator quotient $G_2/R \simeq (\mathbb{Z}_{p^3})^2$ and with coordinate matrix $\delta_2 = \left[\begin{array}{cc|c|c|c} 1 & 0 & p & 1 & 0 \\ 0 & 1 & 0 & p & 1 \end{array}\right]$ is indecomposable of rank 5.

(c) The $(2,2)$-group G_3 with regulator quotient $G_3/R \simeq \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ and with coordinate matrix $\delta_3 = \left[\begin{array}{c|c|c|c} 1 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & 0 \end{array}\right]$ is indecomposable of rank 4.

Proof 10 *By Lemma 5.1.1 a group G is indecomposable if and only if its coordinate matrix δ is indecomposable. To show that whether a matrix is decomposable or not we apply permitted row and column transformations on δ. This counts multiplying δ by S-matrices from left and by T-conformed matrices from right. If this operations change δ to a decomposable form then we say that G is decomposable.*

For (2) it is sufficient to see that modulo p^3 the transformed matrix has no 0-row. Let $\delta_2 = [\epsilon_2 \mid\mid \zeta_2 \mid \eta_2 \mid \gamma_2]$. For decomposability it is enough to check $[\zeta_2 \mid \eta_2]$

because by row and column transformations the block matrices ϵ_2 and γ_2 can always be transformed to the identity matrix again, namely the permitted row and column transformations may cause changes in ϵ_2 and γ_2 but by column transformations we can transform them to I again. Denote the p-units by 1. Let $M = \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix}$ be a S-matrix and let $N = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$ be T-conformed. Then $[\zeta_2'|\eta_2']$ modulo p^3 is of the form

$$[\zeta_2'|\eta_2'] = M[\zeta_2|\eta_2]N = \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix} \begin{bmatrix} p & 1 \\ 0 & p \end{bmatrix} \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \equiv \left[\begin{array}{c|c} p & zp + xp + 1 \\ yp & yzp + p + y \end{array}\right]$$

It is easy to see that none of the entries in the first row is 0 modulo p^3. For the second row we check the possibility to make it zero. For this, assume that $y = p^2k$ for some $k \in \mathbb{Z}$. But then the second entry in the second row is never 0 modulo p^3. Hence G is indecomposable.

(3) The proof is similar to the proof above. But in this case the regulator quotient is not homocyclic. Let $\delta_3 = [\epsilon_3 \mid \zeta_3 \mid\mid \eta_3 \mid \gamma_3]$. For the decomposability it is enough to check the $[\eta_3 \mid \gamma_3]$-part of the coordinate matrix δ_3 by Lemma 4.5 in [11]. We multiply $[\eta_3 \mid \gamma_3]$ from left with a S-matrix M and from right with a T-conformed matrix N. The form of M and N with respect to the given regulator quotient are $M = \begin{bmatrix} 1 & px \\ 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$. Hence we get

$$[\eta_3' \mid \gamma_3'] = M[\eta_3 \mid \gamma_3]N = \begin{bmatrix} 1 & px \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + px & y + pxy + 1 \\ 1 & y \end{bmatrix}$$

The first row is never 0 modulo p^2 because of the first entry and also it is clear that the second row is not 0 modulo p. Hence G is indecomposable.

Example 6 *Let p,q,r and s be prime numbers.*

(a) Let G be a clipped $(1,2)$-group. Let $R = \mathbb{Z}[q^{-1}]x_1 \oplus \mathbb{Z}[q^{-1}]x_2 \oplus \mathbb{Z}[r^{-1}]y_1 \oplus$

$\mathbb{Z}[(rs)^{-1}]z_1 \oplus \mathbb{Z}[(rs)^{-1}]z_2$ be the regulator of G. Assume that the regulator quotient G/R is of exponent p^5.

Let $G = \langle R, g_1, g_2 \rangle$, $g_1 = p^{-5}(x_1 + p^3 y_1 + z_1)$, $g_2 = p^{-3}(x_2 + p^2 y_1 + z_2)$. Then the coordinate matrix δ modulo p^5 is of the form

$$\delta = \begin{bmatrix} 1 & 0 & | & p^3 & | & 1 & 0 \\ 0 & 1 & | & p^2 & | & 0 & 1 \end{bmatrix} \begin{matrix} p^5 \\ \hline p^3 \end{matrix}$$

δ is indecomposable implying that G is indecomposable.

Now we consider another group G' with the same regulator. Let $G' = \langle R, g_1', g_2' \rangle$, $g_1' = p^{-5}(x_1 + p^3 y_1 + z_1)$, $g_2' = p^{-3}(x_2 + p y_1 + z_2)$. Then the coordinate matrix δ is of the form

$$\delta = \begin{bmatrix} 1 & 0 & | & p^3 & | & 1 & 0 \\ 0 & 1 & | & p & | & 0 & 1 \end{bmatrix} \begin{matrix} p^5 \\ \hline p^3 \end{matrix}.$$

We can write $g_2' = p^{-3}(x_2 + p y_1 + z_2) = p^{-5}(p^2 x_2 + p^3 y_1 + p^2 z_2)$. Then by modified Gauss algorithm we can annihilate p^3 by the p below it, that is,

$$-g_2' + g_1' = p^{-5}(-p^2 x_2 - p^3 y_1 - p^2 z_2 + x_1 + p^3 y_1 + z_1) = p^{-5}(x_1 - p^2 x_2 + z_1 - p^2 z_2)$$

Then δ' is transformed to

$$\delta' = \begin{bmatrix} 1 & -p^2 & | & 0 & | & 1 & -p^2 \\ 0 & 1 & | & p & | & 0 & 1 \end{bmatrix} \begin{matrix} p^5 \\ \hline p^3 \end{matrix}$$

Set $x_1' = x_1 - p^2 x_2$, $x_2' = x_2$, $y_1' = y_1$ and $z_1' = z_1 - p^2 z_2$, $z_2' = z_2$. Then we get

$$\delta' = \begin{bmatrix} 1 & 0 & | & 0 & | & 1 & 0 \\ 0 & 1 & | & p & | & 0 & 1 \end{bmatrix} \begin{matrix} p^5 \\ \hline p^3 \end{matrix}$$

Hence $\langle x_1', z_1' \rangle_*$ is a direct summand of rank 2 and $\langle x_2', y_1', z_2' \rangle_*$ is a direct summand of rank 3 by Lemma 7.0.3 and so G is decomposable.

(b) *Let G be a clipped $(1,3)$-group with a regulator quotient of exponent p^4. Let $R = \mathbb{Z}[q^{-1}]x_1 \oplus \mathbb{Z}[q^{-1}]x_2 \oplus \mathbb{Z}[r^{-1}]y_1 \oplus \mathbb{Z}[(rs)^{-1}]z_1 \oplus \mathbb{Z}[(rst)^{-1}]w_1$ be the regulator of G.*

Let $G = \langle R, g_1, g_2\rangle$, $g_1 = p^{-4}(x_1 + p^2y_1 + z_1)$, $g_2 = p^{-4}(x_2 + pz_1 + w_1)$. Then the coordinate matrix δ modulo p^4 is of the form

$$\delta = \begin{bmatrix} 1 & 0 & | & p^2 & | & 1 & | & 0 \\ 0 & 1 & | & 0 & | & p & | & 1 \end{bmatrix} \begin{matrix} p^4 \\ \hline p^4 \end{matrix}$$

δ is indecomposable implying that G is indecomposable.
Consider another group G' with the same regulator. Let $G' = \langle R, g_1', g_2'\rangle$, $g_1' = p^{-4}(x_1 + py_1 + p^2z_1 + w_1)$, $g_2' = p^{-4}(x_2 + p^2y_1 + z_1)$. Then the coordinate matrix δ is of the form

$$\delta = \begin{bmatrix} 1 & 0 & | & p & | & p^2 & | & 1 \\ 0 & 1 & | & p^2 & | & 1 & | & 0 \end{bmatrix} \begin{matrix} p^4 \\ \hline p^4 \end{matrix}.$$

By modified Gauss algorithm we can annihilate p^2 in the first row by the unit below it, that is,

$$-p^2g_2' + g_1' = p^{-4}(-p^2x_2 - p^2z_1 + x_1 + py_1 + p^2z_1 + w_1) = p^{-4}(x_1 - p^2x_2 + py_1 + w_1)$$

Choose a new basis for the regulator. Set $x_1' = x_1 - p^2x_2$, $x_2' = x_2$, $y_1' = y_1$ and $z_1' = z_1$. Then δ' is transformed to

$$\delta' = \begin{bmatrix} 1 & 0 & | & p & | & 0 & | & 1 \\ 0 & 1 & | & p^2 & | & 1 & | & 0 \end{bmatrix} \begin{matrix} p^4 \\ \hline p^4 \end{matrix}$$

The entry p^2 in the second row can be annihilated by p above it. We get

$$\delta' = \begin{bmatrix} 1 & 0 & | & p & | & 0 & | & 1 \\ p & 1 & | & 0 & | & 1 & | & p \end{bmatrix} \begin{matrix} p^4 \\ \hline p^4 \end{matrix}$$

We set $x_1' = x_1$, $x_2' = px_1 + x_2$, $y_1' = y_1$ and $z_1' = z_1$, $w_1' = w_1$ to get

$$\delta' = \begin{bmatrix} 1 & 0 & | & p & | & 0 & | & 1 \\ 0 & 1 & | & 0 & | & 1 & | & 0 \end{bmatrix} \begin{matrix} p^4 \\ \hline p^4 \end{matrix}$$

Hence $\langle x_2', x_4'\rangle_*$ is a direct summand of rank 2 by Lemma 7.0.3 and G is decomposable.

(c) Let G be a clipped $(2,2)$-group with a regulator quotient of exponent 7^3. Let $R = \mathbb{Z}[3^{-1}]x_1 \oplus \mathbb{Z}[(3.5)^{-1}]x_2 \oplus \mathbb{Z}[(11)^{-1}]y_1 \oplus \mathbb{Z}[(11.13)^{-1}]z_1$ be the regulator of G. Let $G = \langle R, g_1, g_2 \rangle$, $g_1 = 7^{-3}(x_2 + 7^2 y_1 + z_1)$, $g_2 = 7^{-2}(x_1 + 7y_1)$. Then the coordinate matrix δ modulo 7^3 is of the form

$$\delta = \begin{bmatrix} 0 & | & 1 & | & 7^2 & | & 1 \\ 1 & | & 0 & | & 7 & | & 0 \end{bmatrix} \begin{matrix} 7^3 \\ \hline 7^2 \end{matrix}$$

δ is indecomposable implying that G is indecomposable.
Consider another group G' with the same regulator. Let $G' = \langle R, g_1', g_2' \rangle$, $g_1' = 7^{-3}(x_1 + 7y_1 + z_1)$, $g_2' = 7^{-2}(x_2 + y_1)$. Then the coordinate matrix δ is of the form

$$\delta = \begin{bmatrix} 1 & | & 0 & | & 7 & | & 1 \\ 0 & | & 1 & | & 1 & | & 0 \end{bmatrix} \begin{matrix} 7^3 \\ \hline 7^2 \end{matrix}.$$

By modified Gauss algorithm we can annihilate 7 in the first row by the unit below it, that is,

$$-g_2' + g_1' = 7^{-3}(-7x_2 - 7y_1 + x_1 + 7y_1 + z_1) = 7^{-3}(x_1 - 7x_2 + z_1)$$

Choose a new basis for the regulator. Set $x_1' = x_1 - 7x_2$, $x_2' = x_2$, $y_1' = y_1$ and $z_1' = z_1$. Then δ' is transformed to

$$\delta' = \begin{bmatrix} 1 & | & 0 & | & 0 & | & 1 \\ 0 & | & 1 & | & 1 & | & 0 \end{bmatrix} \begin{matrix} 7^3 \\ \hline 7^2 \end{matrix}$$

Hence $\langle x_1', z_1' \rangle_*$ is a direct summand of rank 2 and G' is decomposable.

CHAPTER 9

LIST OF ACD GROUPS WITH BOUNDED AND UNBOUNDED REPRESENTATION TYPES

In this section we will give a summary about the decomposability of some classes of almost completely decomposable groups discussed in this thesis. The groups on the list are all have V-free critical typeset, i.e., the typeset is an inverted tree. By Lemma 5.1.1, this groups have regulating regulator. Some of these groups have bounded representation type, that is they have finitely many isomorphism types and some of them have unbounded representation type meaning that they have infinitely many isomorphism types.

Decomposability of $(1,2)$-Groups

(a) The class of $(1,2)$-p-groups has bounded representation type. There are no indecomposables, cf.[13].

(b) The class of $(1,2)$-p^2-groups has bounded representation type. There is only one near-isomorphism class and it is of rank 3 , cf. [13] and [14].

(c) The class of $(1,2)$-p^3-groups has bounded representation type. There are precisely 3 near-isomorphism classes, all of rank ≤ 4 , cf. [13] and [14].

(d) The class of $(1,2)$-p^4-groups has bounded representation type. There are precisely 8 near-isomorphism classes, all of rank ≤ 5 , cf. [14].

(e) The class of $(1,2)$-p^m-hc-groups for $m \geq 2$ has bounded representation type. There are precisely $m-1$ near-isomorphism classes, all of rank 3, cf. [14].

(f) The class of $(1,2)$-p^m-groups for $m \geq 6$ has unbounded representation type , cf. [7] .

(g) The decomposability question for $(1,2)$-p^5-groups is not resolved. The conjecture by E.Solak and O.Mutzbauer is that this class is bounded.

Decomposability of $(1,3)$-Groups

(a) The class of $(1,3)$-p-groups has bounded representation type. There are no indecomposables , cf. [8].

(b) The class of $(1,3)$-p^2-groups has bounded representation type. There are no indecomposables , cf. [8].

(c) The class of $(1,3)$-p^3-groups has bounded representation type. There are precisely 6 near-isomorphism classes, all of rank ≤ 5 , cf. [8].

(d) The class of $(1,3)$-p^4-hc-groups has bounded representation type. There are precisely 9 near-isomorphism classes, all of rank ≤ 6 , cf. [10].

(e) The class of $(1,3)$-p^5-hc-groups has bounded representation type.

(f) The class of $(1,3)$-p^m-groups for $m \geq 4$ has unbounded representation type , cf. [7].

Decomposability of $(2,2)$-Groups

(a) The class of $(2,2)$-p-groups has bounded representation type. There are no indecomposables, cf. [11].

(b) The class of $(2,2)$-p^2-groups has bounded representation type. There are precisely 4 near-isomorphism classes, all of rank 4, cf.[11].

(c) The class of $(2,2)$-p^3-hc groups has bounded representation type. There are precisely 16 near-isomorphism classes, all of rank ≤ 8, cf.[9].

(d) The class of $(2,2)$-p^m-groups for $m \geq 3$ has unbounded representation type, cf. [7].

(e) The class of $(2,2)$-p^m-hc-groups for $m \geq 5$ has unbounded representation type.

(f) The class question for $(2,2)$-p^4-hc-groups is not resolved.

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