

PHILOSOPHICAL IMPLICATIONS OF CANTOR'S SET THEORY

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## **ABSTRACT**

### **PHILOSOPHICAL IMPLICATIONS OF CANTOR'S SET THEORY**

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This thesis is devoted to examining Georg Cantor's understanding of infinity and his philosophy of mathematics. Even though Aristotle differentiated the concept of infinity as potential infinite and actual infinite, he argued against the existence of actual infinity and accepted only the existence of potential infinity. With the effect of this distinction, the impossibility of actual infinity was regarded as the fundamental principle in the history of the concept of infinity. Cantor was the first thinker to attempt to refute Aristotle's arguments by introducing a new understanding of infinity that has one of the greatest impacts on its development in mathematics. Cantor mathematically demonstrated that there would not be any one-to-one correspondence between the set of natural numbers and the set of real numbers. This result implies that there must be at least two different sizes of infinite sets, namely the set of real numbers and the set of natural numbers. Based on the concept of a well-ordered set, Cantor not only showed the way how to count infinite sets but also assigned numbers to differentiate the different sizes of infinite sets. Thus, transfinite

numbers and their arithmetic are introduced into mathematics. After examining the distinction between potential infinite and actual infinite in both Aristotle's framework and Cantor's framework, the existence of mathematical objects in the Cantorian framework will be shown.

**Keywords:** Potential infinity, actual infinity, well-ordered sets, transfinite numbers, transfinite arithmetic.

## ÖZ

### CANTOR'UN KÜMELER KURAMININ FELSEFİ SONUÇLARI

ŞAHİN, Şafak

Yüksek Lisans, Felsefe Bölümü

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Bu tez, Georg Cantor'un sonsuzluk anlayışını ve matematik felsefesini incelemeye adanmıştır. Aristoteles sonsuzluk kavramını potansiyel sonsuz ve aktüel sonsuz olarak ayırt etmesine rağmen aktüel sonsuzluğun varlığını reddederek yalnızca potansiyel sonsuzluğun varlığını kabul etmiştir. Bu ayrımın etkisi ile, gerçek sonsuzluğun imkansızlığı, sonsuzluk kavramının tarihinde temel ilke olarak kabul edilmiştir. Cantor, matematikteki gelişimi üzerinde en büyük etkilerden birine sahip olan yeni bir sonsuzluk anlayışı ortaya atarak Aristoteles'in argümanlarını çürütmeye çalışan ilk düşünür olmuştur. Cantor matematiksel olarak doğal sayılar kümesi ile reel sayılar kümesi arasında bire bir eşleşmenin olmayacağını göstermiştir. Bu sonuç, en az iki farklı boyutta sonsuz kümenin, yani reel sayılar kümesi ve doğal sayılar kümesi, olması gerektiği anlamına gelir. İyi sıralı küme kavramına istinaden, Cantor sonsuz kümelerin nasıl sayılacağını göstermekle kalmadı, aynı zamanda sonsuz kümelerin farklı boyutlarını ayırt etmek için onlara sayılar attı. Böylece sonlu ötesi sayılar ve sonlu ötesi aritmetik matematiğe tanıtıldı. Potansiyel sonsuz ve aktüel



sonsuz ayrımını Aristoteles'in sisteminde ve Cantor'un sisteminde incelendikten sonra, Cantor'un sistemindeki matematiksel nesnelerin varlığı gösterilecektir.

**Anahtar Kelimeler:** Potansiyel sonsuzluk, aktüel sonsuzluk, iyi sıralı kümeler, sonlu ötesi sayılar, sonlu ötesi aritmetik.

*To My Brother*

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## **CHAPTER 1**

### **INTRODUCTION**

The main purpose of this thesis is to investigate the Cantorian theory of transfinite numbers, which had an undeniably crucial impact on the development of set theory. In doing so, my study will answer three questions. First, what are the underlying reasons for Aristotle's rejection of actual infinity? In the case of this question, I will examine Aristotle's perspective of infinity with the distinction between potential infinite and actual infinite. Then, I will discuss his mathematical arguments that led people to assume the impossibility of infinite numbers. And the second question: did Georg Cantor's counter-arguments succeed to respond to Aristotle's arguments against the existence of actual infinity? To answer this question, I will first analyze Georg Cantor's theory of infinity with the new numbers that he introduced and called transfinite numbers, and then, I will examine the counter-arguments that Cantor proposed against Aristotle's mathematical rejection of infinite numbers. The final and the third question is how Cantor managed to prove the existence of actual infinity through both mathematical and philosophical arguments in his framework? While providing an extensive analysis of Cantor's theory, I will discuss his philosophical and mathematical framework behind the existence of transfinite numbers.

There is no doubt that the concept of infinity is one of the most controversial concepts of Western Philosophy. In the history of the concept of infinity, many philosophers and mathematicians have tried to understand both the nature of the concept and its mathematical implications. Some of these are Aristotle, Spinoza, Leibniz, and Georg Cantor among many others. They all provided differing presentations of the concept of infinity and, for this reason, I will briefly discuss the historical evolution of the concept in section 2.1. It is a fact that the distinction between potential infinite and actual infinite proposed by Aristotle dominated the intellectual landscape without even being questioned for a long time. We will reveal how the idea of potential infinite was considered as the only acceptable way of understanding infinity in the literature. Then, in chapter 2.2., I will examine Aristotle's theory of infinity. He differentiated the concept as actual infinite and potential infinite. The idea of infinity has taken its place in his framework as an unending process that cannot be completed. For Aristotle, one of the obvious reasons for the rejection of actual infinity is its inconsistency; an entity being actual and being infinite is a contradiction in terms. Since the completion of the process of infinity is impossible, the infinite cannot reveal itself as an actually existing entity. Thus, by only allowing the existence of potential infinite, Aristotle completely rejected the idea of actual infinity for several reasons that will be discussed.

In the history of the concept of infinity, the mathematical implication of potential infinite gives rise to many paradoxical results. Although both philosophers and mathematicians tried to explain those results, no one has been provided a successful explanation until Georg Cantor. Up to the late 19<sup>th</sup> century, no one has considered the endless sequence of natural numbers as a completed set, but Cantor

ascribed this feature to the sequence. And this reasoning brought him the idea of transfinite numbers. Throughout his career, Cantor suggested that the idea of actual infinity might be a subject for mathematics and dedicated his work to clarify the concept of actual infinity. As we will analyze in chapter 2.3., he not only proved that counting infinite numbers is mathematically possible, but he also provided two counter-examples against Aristotle's rejection of actual infinity.

Furthermore, Cantor asserted that mathematics is free to create its objects on the grounds of internal consistency because its objects have two sides of reality, namely the transient reality and the immanent reality. At the beginning of the third chapter, I will reflect on the result of these two sides in Cantor's ontological framework for transfinite numbers. Then, in chapter 3.3, I would like to remark on the relation between sets and numbers in order to understand what Cantor presented with the distinction between multiplicities. Cantor not only showed the way how to count infinite sets by abstracting the properties of their elements but also assigned numbers to mathematically differentiate them and introduced transfinite arithmetic into mathematics. As a result, in chapter 3.4., transfinite arithmetic will be examined.



## CHAPTER 2

### THE DISTINCTION BETWEEN POTENTIAL INFINITE AND ACTUAL INFINITE

#### 2.1. History of the concept of infinity

In this section of my thesis, I examine the distinction between potential infinity and actual infinity in terms of the Aristotelian rejection of actual infinity, and I demonstrate the philosophical importance of this distinction in the Cantorian framework.

To accomplish these, firstly, I examine the historical evolution of the concept of infinity. It is generally accepted that the discussion about the concept of infinity had begun with Anaximander. He, as an Ancient Greek philosopher, conceptualized infinity as the principle that governs nature. He identified this principle with the word “*apeiron*”, which literally means unlimited. The ultimate source of all things cannot be subjected to any kind of limitations both in space and in time because it must be boundless and indestructible by its nature. It generates the four primary elements in the way that they all need an underlying changeless primary substance as the source of all things. The opposites in nature (e.g. fire and water) are generated by

apeiron, then the transformations and interactions of four elements bring the existence of objects around us. Nevertheless, when the objects in nature are destroyed, they return back to their first cause, i.e. apeiron. Even though apeiron, as the eternal movement, generates the four elements, it exists independently from them. This makes apeiron indestructible and infinite substance for Anaximander. Hence, the idea of infinity, for Anaximander, was emerged from the idea of apeiron as the originating principle.

Many other Ancient Greek philosophers followed Anaximander's footsteps and conceptualized infinite as *ipso facto* a "principle" to explain other things as the fundamental principle. Accordingly, the idea of infinity has emerged with their desire to comprehend the most basic principle of nature. Since their first aim was to understand the nature, the idea of infinity was associated with nature itself. As a result, the concept was neither a mathematical notion nor a metaphysical notion, rather it was only considered for explaining the unlimitedness of nature.

After the pre-Socratic philosophers, Aristotle's distinction between potential infinity and actual infinity dominated the intellectual landscape. To put it briefly, Aristotle considered the notion of infinity as a continuously growing process that cannot have a limitation. This is the idea of potential infinite. On the other hand, he argued against the idea of a complete infinity or so-called actual infinity for several reasons. One of the reasons is that human understanding cannot comprehend the existence of actual infinite because such an entity transcends the human mind. Actual infinite, if exists, must be a complete entity, but a complete entity cannot have parts that are also infinite by themselves. This seems impossible because if the parts are infinite, then the whole becomes indeterminate. Thus, actual infinite either does not

exist or exists as an indeterminate entity. And since indeterminacy is not acceptable, actual infinite does not exist according to Aristotle.

Another reason why the idea of actual infinite seems to be unacceptable for Aristotle is that what is actual must be complete whole that its parts must be present “all at once”. Nevertheless, the parts of infinite present themselves successively, not independent presence. The reason is that, in the case of infinitude, both infinite and its parts are not infinite yet; rather a process that is never-ending. Either by addition or by division the process proceeds indefinitely, and it cannot be completed at any point in time. Hence, the idea of actual infinite seems to be incoherent for Aristotle. For several other reasons that will be examined in the following chapter, he argued against the idea of a complete infinity, which is considered as actual infinity.

Similar to Aristotle, many philosophers have argued against the existence of actual infinity. Each of those philosophers presented differing reasons for this claim. So, the history of the concept of infinity gives us many different arguments for rejecting the existence of actual infinity. The finitude of human understanding is perhaps the most common one. The argument basically claims that the human mind is finite, and its capacity to comprehend the actual infinite seems to be impossible for the fact that something finite could not understand the nature of infinite with its own restricted understanding capacity. This idea is one of the most fundamental problems of the notion of infinity because many philosophers and mathematicians analyze the concept of infinity from this standpoint. Galileo Galilei (1564-1642), as an important scholar for the impossibility of infinite numbers, defends a very similar view. In *Dialogues Concerning Two New Sciences*, Galileo presented a paradoxical example to show how mathematically problematic the idea of infinity is. The example is as

follows: it is always possible to demonstrate the one-to-one correspondence between the set of natural numbers and the set of their squares because, for every natural number  $n$ , there is always a corresponding squared number  $n^2$ :

|    |    |    |     |     |     |       |     |
|----|----|----|-----|-----|-----|-------|-----|
| 1, | 2, | 3, | 4,  | 5,  | ... | n,    | ... |
| ↕  | ↕  | ↕  | ↕   | ↕   | ... | ↕     | ... |
| 1, | 4, | 9, | 16, | 25, | ... | $n^2$ | ... |

The one-to-one corresponding demonstrates that the set of natural numbers and the set of even numbers are the same sizes because it is possible to pair off their elements. The problem, however, is the fact that the set of natural numbers contains both squares and non-squares, which should have made the set of all natural numbers mathematically greater than the set of squares. Therefore, these two sets are in a sense equal; the set of natural numbers has as many elements as the set of squares, and in a sense unequal; there are more numbers in the set of natural numbers. This is clearly paradoxical. Accordingly, he defends that it is impossible to understand the properties of infinity with a finite mind because the concept of infinity transcends human understanding<sup>1</sup>. As his own words, “But let us remember that we are dealing with infinites and indivisibles, both of which transcend our finite understanding...” (1638, p. 26). The paradoxical results of infinity arise because of human understanding. It uses finite understanding to conceptualize the concept of infinity because it can only understand finite quantities, not infinite quantities. Hence, Galileo concludes that infinite quantities, that has different characteristics from finite quantities, cannot be comprehensible in human understanding. He wrote:

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<sup>1</sup> For more detail, see Knobloch (1999).

This is one of the difficulties which arises when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another. (1638, p.31)

Another general assumption about infinity is its indeterminacy. As clearly stated by both Baruch Spinoza (1632-1677) and Gottfried Wilhelm Leibniz (1646-1716), the concept of infinite cannot be the subject of any determination. They would be the first people to come across after Aristotle's effect on the concept, accepting the actual infinity to a certain level. For Spinoza, as one of the great rationalists of the 17<sup>th</sup> century, the concept of infinity has two sides. On the one side, similar to the idea of potential infinity, he assigned the property of unlimitedness to the concept of infinity. Since it does not have any kind of limitation either by addition or by division, it must be mathematically indefinite and indeterminate.

On the other side, there is the concept of infinite that has infinitely many parts that are also infinite themselves. Spinoza indicates that human understanding cannot have the capacity to comprehend this kind of entity. In a letter to Louis Meyer<sup>2</sup>, Spinoza wrote, "Finally, there are things that can be called infinite, or if you prefer, indefinite, because they cannot be accurately expressed by any number, while yet being conceivable as greater or less" (2002, p.790). Since infinite cannot have a maximum or minimum in terms of magnitude, it thus follows that assigning a

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<sup>2</sup> The letter is also known as "Letter on the Infinite"(Letter XII). The purpose of the letter is to explain the questions of infinity regarding its controversy and problems. For Spinoza, the uncertainty on different types of infinity is the main reason why the concept of infinity is controversial. He represented his account of the concept of infinity by differentiating the types of infinities in this letter. For more detail, Morgan (2002), pp. 787-792.

number to infinite quantities becomes impossible. Nevertheless, he accepts the possibility of different sizes of infinities by refuting their mathematical applications<sup>3</sup>.

On the other hand, Leibniz's conception of infinity is more related to nature. For him, there is infinite in nature in the way that there are infinitely many individuals in nature. In a letter to Foucher in 1693, Leibniz wrote:

I am so much in favor of an actual infinite that instead of admitting that nature abhors it, as is commonly said, I hold that it affects nature everywhere in order to indicate the perfections of its Author. So, I believe that every part of matter is, I do not say divisible, but actually divided, and consequently, the smallest particle should be considered as a world full of an infinity of creatures. (1951, p.99)

As it seems, he accepted the idea of actual infinity in nature, but he argued against the existence of actual infinity in mathematics. According to Leibniz, it is impossible to think of an infinite number without contradictory results. Any number should be definite and determined so that it has a definite place in arithmetic. However, in the case of infinite numbers, this is impossible because the notion of infinite numbers transcends the idea of numbers. After all, it is impossible to determine the mathematical value of an infinite number. Consequently, infinite numbers, for Leibniz, becomes indeterminate numbers, that cannot take place in mathematics. To this respect, he derived two conclusions. The first one is that all numbers must be inherently finite, and the second one is that the human mind cannot have the capacity to understand the concept of infinite numbers. Even though he accepts the existence of actual infinite in nature, he argues against the existence of actual infinities in mathematics<sup>4</sup>.

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<sup>3</sup> For more detail, see Bussotti and Tapp (2009) and Newstead (1975).

<sup>4</sup> For more detail, see Knobloch (1999).

After a long time, Georg Cantor (1845-1918) was the first man ever tried to justify the existence of different kinds of infinities in mathematics. Before Cantor's revolutionary ideas, the concept of infinity had always been mathematically unclear and counter-intuitive. Many philosophers and mathematicians interested in researching the concept of infinity, but, before Georg Cantor, no one had ever claimed that numbers can be assigned to infinite quantities and their arithmetic can be well defined like finite arithmetic<sup>5</sup>. One of the most remarkable contributions of Cantor's theory, which I will analyze with all the details, is to suggest that an infinite sequence can be mathematically determinate as much as finite numbers despite his colleagues who were against this view. For this reason, I examine the distinction between potential infinite and actual infinite in terms of Aristotelian understanding and Cantorian understanding, then show the philosophical importance of this distinction in the light of transfinite numbers.

## **2.2. Aristotle's theory of infinity**

In *Physics*, Aristotle properly formulated the idea of infinite in two categories as potential infinite and actual infinite to clarify the paradoxical problems of the concept of infinity. What he meant by potential infinity is a continuous process that has the potency to proceed indefinitely. It is something limitless and boundless. In addition, since it cannot have any limit or bound, its existence, for Aristotle, must

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<sup>5</sup> It should be mentioned that even though Emmanuel Maignan (1601 – 1676), as a French physicist, argued that one can compare different sizes of infinities as being the one greater or less than or equal, he never asserted that numbers can be assigned to infinite quantities. For more detail, see Mancosu (2009).

always be incomplete and indeterminate. If the proceeding process had somehow been terminated, then it would have an actuality. But this is impossible because there will always be a possible division or addition, so infinite cannot present itself as a completed entity. Therefore, potentially infinite sequences can only present themselves as a continuous process that never reaches completion.

He also differentiated the idea of potential infinite into two categories to better understand the notion, namely infinite by addition and infinite by division. The idea of infinite by addition can be comprehended as the unending sequence of natural numbers. The sequence consists of numbers that are constructed by the successive addition of units. For a given number, it is always possible to construct a larger number by the addition of units; consequently, there would be no greatest number among them. Each number in the sequence has a definite magnitude and distinct features, but the sequence can proceed towards infinity potentially. Since the process never ends, the sequence is always incomplete; there would always be a greater number that is not contained in the list. Consequently, the sequence itself cannot correspond to an actually existing entity. As Aristotle puts it:

Hence this infinite is potential, never actual: the number of parts that can be taken always surpasses any assigned number. But this number is not separable from the process of bisection, and its infinity is not a permanent actuality but consists in a process of coming to be, like time and the number of time. (*Physics*, III.7 207b10-15)

If the process of adding one more unit has an ending, then it would have an actuality, but this seems impossible. For the fact that it is impossible to simplify all the elements, this guarantees the potential infinity of the sequence of natural numbers for Aristotle.



On the other hand, infinite by division can be seen as the infinitude of a straight line<sup>6</sup>. It is always possible to reveal different parts by dividing a finite straight line; it can be divided into two parts, then each part after the division can also be divided into two again and so on. Even though parts in the line get smaller and smaller, there will always be another part that can be divided. Since there would always be possible division left out, the divisibility of a straight line can be held indefinitely. According to Aristotle, for the fact that the division of parts never ends, a finite mind cannot actually comprehend the indefinite divisibility of a straight line. It is only comprehensible as possible divisions that can be performed. So, dividing a finite straight line can be a potentially infinite process for Aristotle. In the same way, this reasoning also guarantees the infinite divisibility of a physical body for Aristotle. Even though any physical body appears to be materially finite because it necessarily has a finite surface, dividing any physical body into infinitely many parts seems to be potentially possible. But this does not mean that body has actually infinite parts, instead, dividing it into infinitely many parts is only potentially possible in thinking. As Aristotle puts it, “For the fact that the process of dividing never comes to an end ensures that this activity exists potentially, but not that the infinite exists separately” (*Metaphysics*, IX.6, 1048b14-17). Since the process of division is never-ending, it follows that infinite by division is only possible in a potential sense. Therefore, for Aristotle, potential infinite can manifest itself either infinite by addition or infinite by division.

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<sup>6</sup> Thales was the first man to suggest the process of dividing a straight line into infinitely many parts is possible, which eventually led Zeno to come up with many paradoxes about the infinity.

As I briefly mentioned before, Aristotle argues against the existence of actual infinity because the idea has some problematic features. According to Aristotle, actual infinite is the infinite type that should present itself complete and definite. However, it is obvious that this is impossible in the case of infinitude. Being complete necessarily requires limitation, but infinite, by definition, cannot have limitations. Therefore, completeness cannot be a characteristic of any infinitude. Hence, the concept of actual infinite, for Aristotle, becomes contradiction in terms: to be complete and to be infinite cannot be the properties of the same thing at the same time. Aristotle wrote, “Whole and complete are either altogether the same or of a similar nature. Nothing is complete which has no end, and the end is a limit” (*Physics*, III.6, 206b33-207a15). Even though a completed infinite must be determined by definition, in the case of infinity, determinateness cannot be a feature. The complete infinite, then, becomes incoherent and unknowable. As Aristotle stated:

It is in fact the matter of the completeness which belongs to size, and what is potentially a whole, though not in the full sense. It is divisible both in the direction of reduction and of the inverse addition. It is a whole and limited; not, however, in virtue of its own nature, but in virtue of what is other than it. It does not contain, but, in so far as it is infinite, is contained. Consequently, also, it is unknowable, qua infinite; for the matter has no form. (*Physics*, III.6, 207a22-27)

Another reason for the impossibility of actual infinity lies in nature. Like many other Ancient Greek philosophers, Aristotle conceptualized infinity as the fundamental substance that governs everything in nature. For him, the important question concerning infinity was whether there is infinite in nature or not. Aristotle himself wrote, “The study of nature is concerned with extension, motion and time; and since each one of these must be either limited or unlimited..., it follows that the

student of Nature must consider the question of the unlimited, with a view to determining whether it exists at all, and, if so, what is its nature” (*Physics*, III.4, 202b 30-35). Therefore, he first considered the concept of quantity. According to Aristotle, quantity must be defined as that of which is divisible into parts, and each part would necessarily be countable or measurable (*Metaphysics*, 1020a7-10). This definition alone, for Aristotle, highlights the fact that there could not be an infinite quantity in nature. The idea of infinite quantity is a contradiction in terms for Aristotle. The reason is that if there is an actually infinite quantity in nature, then this would mean that it must be divisible into parts in which those parts must also be infinite themselves, which is unacceptable for Aristotle. Infinite quantities cannot have parts that are also infinite by themselves and this necessitates the idea that infinite quantities must be indivisible. Nevertheless, this is also self-contradictory because any quantity whether finite or infinite must be divisible. Aristotle wrote:

It is impossible that the infinite should be a thing which is in itself infinite, separable from sensible objects. If the infinite is neither a magnitude nor an aggregate, but is itself a substance and not an accident, it will be indivisible; for the divisible must be either a magnitude or an aggregate. But if indivisible, then not infinite, except in the way in which the voice is invisible. (*Physics*, III.5 204a8-14)

It thus follows that there would not be an infinite quantity in nature. By only allowing the existence of potential infinity, Aristotle concludes that actual infinity does not exist.

In summary, what is most evident, for Aristotle, is that actual infinity or completed infinite cannot exist as an actual entity. For the fact that it is impossible to comprehend its actuality all at once, the idea of actual infinity only exists as a matter of speaking. Therefore, he argues against the existence of actual infinity and states that the concept of infinite is only comprehensible as either infinite by addition or

infinite by division, both of which are only potentially existing. In the following section, I examine Aristotle's mathematical arguments against the possibility of infinite numbers.

### **2.2.1. Mathematical arguments against the existence of actual infinity**

In addition to the philosophical arguments against the idea of actual infinity, Aristotle also argued against the existence of actual infinity for mathematical purposes. Aristotle offered two reasons why the idea of actual infinity seems to be mathematically impossible. It should be noted that he did not provide these reasons in formal argument forms, especially in the second argument. Aristotle, firstly, held that counting cannot generate infinite numbers for the fact that the successive addition of units ensures that all numbers are finite in number formation. For Aristotle, this reasoning guarantees that the counting procedure is only applicable to finite numbers. Secondly, he argued that there would be an infinite substance in nature. If there was such an element, it would cause the destruction of other elements. Georg Cantor interpreted these two arguments directly on numbers as the mathematical arguments against the existence of actual infinity and indicated that these arguments formed the traditional understanding of the concept of infinity.

The first argument clarifies that the main function of numbers is counting, and only finite numbers are countable. For Aristotle, a number can only be constituted by counting and what is uncountable cannot be a number. He wrote, "Nor can number taken in abstraction be infinite, for number or that which has number is numerable. If then the numerable can be numbered, it would also be possible to go

through the infinite” (*Physics*, III.5 204b8-10). All numbers in the unending sequence of natural numbers are constructed in relation to the previous number, which is already finite, each number exceeding the previous one would inherently be finite by the process of counting. Although the sequence of natural numbers would have infinitely many elements, all of its elements should be a finite number. In his words, “For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different” (*Physics*, III.6, 206a26-29). In other words, all the elements in the sequence are different from one another, but they are all finite numbers. Furthermore, for Aristotle, mathematical objects are the entities that we abstracted their reality from things in the physical world. Numbers are the results of this abstraction and rely on quantities found in nature. Nevertheless, as we analyzed, infinite quantities are necessarily indeterminate quantities in Aristotle’s philosophy. Since indeterminacy is not acceptable in mathematics, there would not be any specific number for the enumeration of infinite quantities. Thus, Aristotle concludes that the process of counting is only applicable to finite numbers and it cannot generate an infinite number<sup>7</sup>.

The second mathematical objection against the existence of actual infinity is the annihilation of finite numbers. He did not present this argument directly on numbers, but he indicated that if there was an infinite element in nature, it would destroy finite elements in nature. The four primary elements (water, air, fire, and

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<sup>7</sup> By considering Aristotle’s arguments in *Metaphysics* Book XI, Cantor represents “...only counting procedures with respect to finite aggregates (sets) were known to him” (1976, p.75).

earth) must be finite to coexist. If any of them were infinite, then it necessarily destroys opposites elements. As Aristotle puts it:

The infinite can neither be composite nor simple. For it cannot be a composite body if the elements are limited in number; for the contraries must be equal, and no one of them must be infinite; for if the potency of one of the two corporeal elements is in any way inferior, the finite element will be destroyed by the infinite. (*Metaphysics*, II., 1066b27-30)

For instance, a finite amount of fire cannot maintain its presence with an infinite amount of water in nature, instead, it would be annihilated by water. As Aristotle own words, “Nor can fire or any other of the elements be infinite. For generally, and apart from the question of how any of them could be infinite, the All, even if it were limited, cannot either be or become one of them...” (*Physics*, III.5, 205a1-4).

Another reason why any of those elements cannot be infinite is the fact that the physical body is generated from those elements. Since all bodies must be extended finite space, they cannot contain any infinite element. He wrote:

It is impossible, however, that there should be such a body; not because it is infinite on that point a general proof can be given which applies equally to all, air, water, or anything else-but simply because there is, as a matter of fact, no such sensible body, alongside the so-called elements. Everything can be resolved into the elements of which it is composed. Hence the body in question would have been present in our world here, alongside air and fire and earth and water: but nothing of the kind is observed. (*Physics*, III.5, 204b29-35)

The non-existence of an infinite element in nature can be interpreted mathematically as following; finite numbers would necessarily be annihilated by infinite numbers when the calculation is applied to infinite numbers<sup>8</sup>. The unending sequence of infinite numbers, assuming that they exist, must always be incomplete

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<sup>8</sup> Cantor represented his interpretation of Aristotle’s argument, “...the finite would be dissolved into the infinite and destroyed, if the latter existed, since the finite number allegedly is annulled by an infinite number” (1976, p.75).

by definition, and destroy whatever it is added. Since infinite numbers contain every number, they absorb what is contained in themselves. In this sense, when finite numbers and infinite numbers are considered together in a mathematical operation, the permanence of finite numbers would become impossible. As Aristotle stated, “It is absurd and impossible to suppose that the unknowable and indeterminate should contain and determine” (*Physics*, III.6, 207a31-32). When finite numbers and infinite numbers are considered together in a mathematical operation, what we have from this operation would not be different from the unending sequence of infinite number. The infinitude of the sequence must destroy the finite number in its sequence. As it appears, even if we assume that it is possible to count infinite numbers, Aristotle had a good reason to believe that the arithmetic of infinite numbers is not possible because applying any mathematical operation to infinite numbers is impossible<sup>9</sup>.

Indeed, for Aristotle, these two mathematical arguments guarantee that there would not be an infinite number, neither in nature nor in mathematics. Thus, by only accepting the existence of potential infinite, he concludes that there are only finite numbers for the reason that the counting cannot be considered in infinite numbers.

### **2.3. Georg Cantor’s theory of infinity**

The traditional understanding of infinity relies on the idea that “*Infinitum actu non datur*” which literally means that actual infinite does not exist.

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<sup>9</sup> He states, “The unlimited, then, is the open possibility of taking more, however much you have already taken; that of which there is nothing more to take is not unlimited, but whole or completed” (*Physics*, III.6, 207a 7-9).

Accordingly, the impossibility of actual infinity was regarded as the fundamental principle in the history of the concept of infinity. Any attempt to demonstrate the reality of actual infinity had failed. Even if some scholars accept the idea of actual infinity, such as Nicholas of Cusa (1401-1464) and Giordano Bruno (1548-1600), Aristotle's arguments have not been refuted successfully in the history of the concept of infinity. Nevertheless, Georg Cantor was the first thinker who attempted to refute Aristotle's arguments against the existence of actual infinity.

He introduced his theory mainly in two articles; "*Grundlagen einer Allgemeinen Mannigfaltigkeitslehre*"<sup>10</sup> and "Beiträge zur Begründung der transfiniten Mengenlehre"<sup>11</sup>. The first article, *Grundlagen*, is dedicated to constructing transfinite ordinal numbers while analyzing the arguments against the existence of actual infinity. It should be noted that I will refer to the article as *Grundlagen* in the rest of this thesis. The second article, on the other hand, is devoted to the theory of transfinite numbers (including both ordinals and cardinals); the rules of transfinite arithmetic are comprehensively provided. For this thesis, the first article, *Grundlagen*, has an essential role in representing Cantor's mathematical and philosophical arguments about the existence of actual infinity.

In 1883, when Cantor published *Grundlagen*, the main purpose of the article was to justify the existence of actual infinity in the light of transfinite numbers. He used both philosophical arguments and mathematical arguments in defending his theory and subtitled the article as "A Mathematical - Philosophical Study in the

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<sup>10</sup> Translated as "Foundations of a General Theory of Manifolds".

<sup>11</sup> Translated as "Contributions to the Founding of the Theory of Transfinite Numbers".



Theory of the Infinite”. One of the most crucial points of the article is to defend that it is not possible to understand the nature of infinity with finitist reasoning; in fact, finitist reasoning necessarily leads to contradictory results in infinite quantities.

To give an illustration of finitist reasoning let me use John Locke’s (1632-1704) reasoning in his *An Essay Concerning Human Understanding*. He conceptualized infinity by assuming there is only one infinity and evaluated the mathematical applications of infinity as absurd. Locke wrote: “If a man had a positive idea of infinite, either duration or space, he could add two infinities together: nay, make one infinite infinitely bigger than another, absurdities too gross to be confuted” (as cited in Suber, 1998, p.29). In this sense, it is indeed impossible to speak of infinite quantities with finitist understanding because of the paradoxical results. However, in another sense, most scholars (including John Locke) only assumed the existence of the properties of finite quantities. In other words, infinite sets were regarded as if they have the same properties as finite sets. For Cantor, using finitist reasoning to understand the concept of infinity was an undeniable mistake, and this is the reason why the concept had been so problematic in the history of mathematics. The possibility that infinite quantities might have different characteristics was ignored by those great scholars<sup>12</sup>. It was this reasoning that led most people to presuppose the impossibility of infinite numbers in the first place.

At the beginning of his career, Cantor, as many other scholars, accepted the idea that actual infinite has nothing to do with rigorous mathematics because he believed that the concept of actual infinite was hard to consistently formulate in

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<sup>12</sup> It should be noted that some scholars, such as Galileo and Blaise Pascal (1623-1662), suggested that infinite quantities have different properties than finite quantities, but they also held that human reasoning cannot understand those properties with its own restricted capacity.

mathematical notions (Dauben, 1983, p.122). However, in 1874, Cantor came across the existence of different infinite sets for the first time in the article called “*On a Property of the Collection of All Real Algebraic Numbers*”. He mathematically demonstrated the fact that the list of all real numbers is incomplete in this article. Given the list of all real numbers in a closed interval (such as all the real numbers which are  $\geq 0$  and  $\leq 1$ ), it is always possible to construct a new number that would not be in the list of real numbers (1874, p.258). Since this number would not be on the list, it is impossible to list all the real numbers. Consequently, Cantor states that there would not be any one-to-one correspondence between the set of natural numbers and the set of real numbers. Then, he derived two conclusions from what he discovered. The first conclusion is the fact that the set of real numbers is non-denumerable or uncountable<sup>13</sup>. The second one is the fact that that there must be at least two different sizes of infinite sets, namely the set of real numbers and the set of natural numbers. The reason for both conclusions is that the set of real numbers must have more elements than the set of natural numbers. Thus, Cantor suggested the idea that some infinite sets must have a higher degree of infinity than other infinite sets.

He was surprised by the result, and, in a letter to Richard Dedekind (1831-1916) in 1877, Cantor wrote “I see it, but I don't believe it!” (1991, p.44). By mathematically proving that there are different sizes of infinities, he introduced the concept of infinite sets into mathematical studies; in fact, there are infinitely many

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<sup>13</sup> Cantor differentiates denumerable sets and non-denumerable sets on the basis of the one-to-one correspondence principle. Any set that can be paired with the set of natural numbers is identified as denumerable and non-denumerable sets become the sets that cannot be paired off with the set of natural numbers.

different sizes of infinite sets. But he was already aware that he had been dealing with something that no one had achieved:

The preceding exposition of my research in the theory of manifolds<sup>14</sup> has come to a point when its further development depends on an extension of the notion of true integer number beyond previous boundaries, an extension which goes in a direction that, to my knowledge, nobody has tried yet. My dependence on this extension of the notion of number is so great that without it, it would be almost impossible for me to make freely the least step further in set theory. (Cantor, 1976, p.70)

Cantor's views about the concept of infinity were radically different from the great majority of his colleagues. The reason the Cantorian theory of infinity was criticized so much at the time it appeared was that the general understanding about the concept of infinity contradicts with what Cantor proposed. For this reason, Cantor preferred to use the words “proper (or genuine)” and “improper (or non-genuine)” instead of actual and potential while mentioning infinity<sup>15</sup>. What he intended to explain by the word “improper infinite” is the concept of potential infinite in exactly the same way Aristotle proposed. It is a variable that can be increased indefinitely. Cantor defined as follows, “...the mathematical infinite has principally occurred in the meaning of a variable magnitude, either growing beyond all limits or diminishing to an arbitrary smallness, always, however remaining finite. I call this infinite the non-genuine infinite” (1976, p.70).

The successive addition of units reveals that the process of constructing numbers in the unending sequence of natural numbers is obviously never-ending. Even though the sequence can proceed towards infinity, for Cantor, this does not

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<sup>14</sup> Used for “sets” in his writings.

<sup>15</sup> Even he sometimes misleads his articles to avoid prejudices, such as “*On a Property of the Collection of All Real Algebraic Numbers*”. For more detail see Dauben (1989).

entail the argument that it is infinite. For the fact that each number in the sequence is necessarily constructed with relation to the previous number, the sequence would only consist of finite numbers. Cantor wrote, “Whereas the potential infinite means nothing more than an indeterminate, always finite, variable magnitude taking values which become either as small as we please or larger than any arbitrary finite bound...” (1887, p. 409). To put it differently, all numbers must be finite because each number is constructed from another finite number. For this reason, the concept of potential infinite (or improper infinite) is not regarded as truly infinite in the Cantorian framework.

On the other hand, proper infinite, as distinct from improper infinite, is the infinite type that is completed and definite. While improper infinite remains finite in terms of magnitude, proper infinite exceeds all the finite magnitudes and it is indeed infinite. In the Cantorian framework, the idea of proper infinite bases upon the idea of the complete collection of natural numbers as an actually existing entity<sup>16</sup>. As Cantor described:

...the actual infinite refers to a fixed in itself, constant quantum which is larger than any finite magnitude of the same kind. Thus, for example, a variable magnitude  $x$  successively taking the different finite whole number values  $1, 2, 3, \dots, v, \dots$  represents a potential infinite, while the set  $(v)$  of all whole finite numbers, conceptually determined in full by a conceptual law, offers the simplest example of an actual infinite quantum. (1887, p. 409)

For Cantor, “the set  $(v)$  of all whole finite numbers” represents an entity that is an actually infinite quantity. He assigned a number to the entity and called it the first transfinite ordinal number. Then, he applied the usual process of counting and

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<sup>16</sup> As his own words, “The infinity of this sequence  $[1, 2, 3, \dots, v]$  affords the proof that the totality of all finite numbers, considered as a thing in itself, is an actually infinite set, a transfinite” (Cantor 1887, p. 419).

constructed the rest of the transfinite numbers<sup>17</sup>. As it appears, the idea of actual infinity is reflected with the idea of transfinite numbers in the Cantorian framework. According to Cantor, transfinite numbers are not becoming infinite because any kind of limitation cannot be applied to them. Hence, these new numbers are actually infinite themselves. In the following section, Cantor's two counter-arguments against Aristotle's mathematical rejection of the actual infinite are examined.

### **2.3.1. Rejection of Aristotelian arguments against the existence of actual infinity**

The first argument which is defended by Aristotle relies on the assumption that only finite numbers exist because the counting procedure can only be applied to finite numbers. For Aristotle, since the main purpose of numbers is counting, all numbers must only be countable by finite numbers. Then, he asserted that infinite numbers do not exist since they cannot be counted. However, as Cantor stated, this argumentation involves a *petitio principii*:

If one considers the arguments which Aristotle presented against the real existence of the infinite (vid. his *Metaphysics*, Book XI, Chap. 10), it will be found that they refer back to an assumption, which involves a *petitio principii*, the assumption, namely, that there are only finite numbers, from which he concluded that to him only enumerations of finite sets were recognizable. (1976, p.75)

According to Aristotle, countability is only accessible on finite numbers. The reason is that every number can only be numbered as finite numbers. By doing so, he eliminated the possibility of infinite numbers in the first place and concluded that infinite numbers do not exist. Nonetheless, for Cantor, showing the logical fallacy in

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<sup>17</sup> The construction method for transfinite ordinal numbers will be examined in chapter 2.3.1.

Aristotle's arguments would not be sufficient to refute the arguments against the existence of actual infinite. For this reason, he proposed two counter-arguments.

Before going into the details of Cantor's arguments, it is noteworthy that the new approach to the number concept is produced by the concept of a well-ordered set, which has an essential role in the construction of transfinite numbers. A well-ordered set is defined as follows; firstly, there must be a first element in the set in terms of the order, not of the multitude. Secondly, every element must be followed by another element as a successor unless it is the last element of the succession. Lastly, for any finite or infinite set of elements, there must exist a determinate immediate successor which is known as the limit ordinal. For instance, the natural ordering of natural numbers is a well-ordered set; every number in the sequence is defined as the next number then the previous one by the successor operation, i.e. a well-ordering of natural numbers is  $\{1, 2, 3, \dots\}$ . Then consider the ordering of the sequence as  $\{1, 3, 5, \dots, 2, 4, 6, \dots\}$ . This is an unusual ordering, but it is also a well-ordering of the set of natural numbers.

Cantor believed that infinite sequences can also be numbered by using the concepts of a well-ordered set. The reason is that the order of the last member of any sequence signifies the order type of sequence. For finite sets, different orderings of elements cannot change the order of the last member. In order to illustrate, consider two different ordered sets  $\{1, 3, 5, 7\}$  and  $\{7, 5, 3, 1\}$ . Both have their last element as the fourth element and they paired off with the sequence  $\{1, 2, 3, 4\}$  to be numbered. Even though they have different orderings, they have the same order type because of the order of their own last element. Nevertheless, as we will examine in the second-counter argument, it is possible to differentiate infinite sets from each other in the

way that, for infinite sets, different orderings can change their order types. Through different orderings of elements, Cantor assigned numbers to infinite sets. This is the reason why he called these new numbers transfinite ordinal numbers.

The first counter-argument, defended by Cantor, implies that infinite sets can also be counted like finite sets. While finite numbers are dependent on finite sets, which are enumerated by their own units, the enumeration of infinite sets can only be held by their limiting elements. In this manner, Cantor demonstrated the new method to count infinite sets in contrast to what Aristotle represented. To count an infinite set, the size of the set is abstracted from its members because what we need is the size of the set, not the members of the set. Then, Cantor showed the way the first transfinite ordinal defined. “ $\omega$ ” (omega) is constructed as the limit to the sequence of natural numbers by the second number generating principle. As Cantor’s words,

If any definite succession of defined integers is put forward of which no greatest exists, a new number is created by means of this second principle of generation, which is thought of as the *limit* of those numbers; that is, it is defined as the next number greater than all of them. (1976, p.87)

The principle basically allows us to conceive the sequence of natural numbers as a completed entity, then a new number, that is the limiting number of the sequence as the next number greater than all finite numbers, is assigned to the complete sequence of natural numbers. The new number exceeds all finite numbers as the limiting element; consequently, this makes it greater than any finite number. And since the number cannot take its place in the sequence, it cannot be a finite number, but a transfinite number.

Furthermore, Cantor showed the mathematical conditions to construct the rest of the transfinite numbers. The new infinite numbers, after the first transfinite, are constructed by the first number generating principle, i.e. the successive addition of

units. By adding 1 to the first transfinite number, the second transfinite ( $\omega+1$ ) is constructed as a distinguishable number. Again, by adding 1 to the second number, the third one is appeared and so on. Thus, the new number sequence, namely the sequence of transfinite ordinal numbers, turns out to be  $\omega, \omega+1, \omega+2\dots$ . Thus, contrary to Aristotle represented in his arguments, counting infinite quantities is indeed mathematically possible<sup>18</sup> and this implication alone, for Cantor, shows that infinite numbers cannot be considered from a finitist point of view. As he stated:

There we only made use of the first principle of generation (the principle of counting) and consequently stepping out of the series (the sequence of natural numbers) was impossible. The second generation principle, however, not only had to lead beyond the number field given up to now, but indeed proves itself to be a means which, in conjunction with the first principle of generation, provides the capacity to break through every boundary in the concept formation of the real whole numbers. (1976, p.88)

The second counter-argument, on the other hand, is against the annihilation of finite numbers. The argument proposed by Aristotle to refute the existence of actual infinity implies that even if infinite numbers exist, finite numbers would be annihilated by the sequence of infinite numbers when the calculation is applied<sup>19</sup>. However, for Cantor, it is possible to apply mathematical operations to transfinite numbers contrary to what Aristotle presented. It should be noted that transfinite ordinal numbers are constructed as well-ordered sets and mathematical operations have applied them by concatenating their order types. In the operation of addition,

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<sup>18</sup> According to Cantor, “Rather the number of elements of an infinite aggregate is an infinite whole number co-determined by the law of counting; in this and in this alone lies the essential distinction between the finite and the infinite, which has its basis in nature itself, and thus can never be removed” (1976, p.75).

<sup>19</sup> On finite arithmetic, for any finite numbers  $m$  and  $n$ ,  $m + n > m$  and  $m + n > n$ . But it was accepted that if we considered an infinite number, then the equations would turn out to be  $m + \infty = \infty$  or  $n + \infty = \infty$ . For the fact that any infinite quantity must be greater than all finite quantities, finite numbers must necessarily be destroyed.



for instance, two ordinals  $\alpha$  and  $\beta$  are positioned one after another by keeping the ordering they already have, and the sum  $\alpha + \beta$  (containing indexed order types of these ordinals) is obtained as the sequence of  $\alpha$  followed by the sequence of  $\beta$ . For the separation of distinct order types of ordinals, a semicolon is used in all mathematical operations.

In *Grundlagen*, Cantor asserted that if any transfinite number is adjoined to a finite number, the finite number would be dissolved in the sequence of transfinite number as Aristotle did in his argument. But this does not mean that the annihilation argument is always applicable in transfinite numbers. On the contrary, if any finite number is added to a transfinite number, what the result of the operation would be a new transfinite number according to Cantor. The main reason is that the finite number would take its place with no immediate predecessor after the whole sequence. Consequently, the new sequence would be different from the former one.

Cantor wrote:

To an infinite number (if it is thought of as determinate and complete) a finite number can indeed be adjoined and united without effecting the dissolution of the latter (the finite number)-the infinite number is itself modified by such an adjunction of a finite number. (1976, p.75)

As can be seen from the quote that adjoining a finite number to a transfinite number would give us a different sequence, which corresponds to a different number with its new order type. This is the main reason why Cantor emphasized the order of finite numbers to avoid the annihilation of finite numbers in mathematical operations.

Let me give an example,  $1 + \omega$  is equal to  $\omega$  because of the following reasons;  $1 + \omega$  has the order type of  $\{1; 1, 2, 3, \dots\}$  while  $\omega$  has the order type of  $\{1, 2, 3, \dots\}$ . Both do not have the last element and have the same order type. Therefore,  $1 + \omega = \omega$ . However, on the other hand,  $\omega + 1$ , which is the immediate successor of  $\omega$ , has a

different order type. It has the order type of  $\{1, 2, 3, \dots; 1\}$  and it is not ordered isomorphic to  $\omega$ ; the last element has no immediate predecessor and it is positioned as the  $\omega^{\text{th}}$  element in the sequence. Therefore,  $\omega + 1$  is a different number from both  $\omega$  and  $1 + \omega$ . The order of the finite number in mathematical operations, as can be seen above examples, is significant to avoid annihilation. This is because different orderings can correspond to different numbers in the enumeration of infinite sets. The analysis has sufficiently shown that the commutative law for addition does not hold for transfinite numbers.

In conclusion, it is noteworthy that the number generating principles, especially the second number generating principle, provide the mathematical conditions that lead to the construction of transfinite numbers. The first counter-argument provides the method to count infinite sequences by assigning a new number to the unending sequence of natural numbers. As Cantor stated:

However, I believe I have proved above, and it will be shown even more clearly in the rest of this work, that definite counting can be effected both on finite and on infinite sets, assuming that one gives a definite law according to which they become well-ordered sets. That without such a lawlike succession of the elements of a set no counting with it can be affected lies in the nature of the concept counting. (1976, p.75)

The second counter-argument demonstrates that the order of finite numbers is significant in arithmetic operations to avoid the annihilation of finite numbers. When the finite number takes its place after the sequence of transfinite number, the annulment of finite numbers has not appeared. Hence, arithmetical operations can also be applied to infinite sets with the proper ordering of elements of the operation. For the fact that the results of those enumerations differ from one another, we cannot disregard the characteristics of transfinite numbers; they behave differently in their arithmetic operations. Thus, the annihilation argument, defended by Aristotle, is not

always applicable to transfinite numbers. We may, therefore, assert that the mathematical arguments against the existence of actual infinity were answered and ruled out by these two counter-arguments.

## **2.4. Conclusion**

For Cantor, to put it briefly, all anti-infinitistic arguments rely on two assumptions. The first assumption is that all numbers are necessarily finite by the counting method, and the second assumption is that an infinite number cannot be subjected to any kind of determination. Nevertheless, for Cantor, the first assumption is disputable because there are transfinite numbers, which have allowed us to exceed the domain of finite magnitudes. The argument that infinite numbers do not exist relies on the impossibility of enumeration of infinite sets, but Cantor proposed the new way of enumerating infinite sets by their limiting numbers and constructed new arithmetic for transfinite numbers. The second assumption, on the other hand, is also disputable because transfinite numbers are constructed by their well-ordered sets that make them mathematically as determinate as finite numbers. The rules of transfinite arithmetic show us that transfinite numbers are grounded upon the objective reality of finite numbers. Hence, we cannot disregard the determinateness of these new numbers. About these two assumptions, Cantor wrote:

All so-called proofs against the possibility of actually infinite numbers are faulty, as can be demonstrated in every particular case, and as can be concluded on general grounds as well... From the outset they expect or even impose all the properties of finite numbers upon the numbers in question, while on the other hand the infinite numbers, if they are to be considered in any form at all, must (in their contrast to the finite numbers) constitute an entirely new kind of number, whose nature is entirely dependent upon the

nature of things and is an object of research, but not of our arbitrariness or prejudices. (as cited in Dauben, 1991, p.125)

Based on the enumeration of the unending sequence of natural numbers, Cantor not only constructed the new kind of mathematical system but also showed that paradoxical results in potentially infinite sequences have arisen from the finitist reasoning; which is actually originated from the assumption that infinite sets have the same characteristics as finite sets.

Apparently, if transfinite numbers are as legitimate as finite numbers, then the Aristotelian rejection of actual infinity would confront a logical error, i.e. *petitio principii*. Let me remind you that Aristotle asserted that the idea of actual infinite is impossible by rejecting the existence of infinite numbers in the first place. However, the assumption that only finite numbers can be counted does not entail the idea that infinite numbers cannot be counted. Instead, it only implies that infinite numbers cannot be counted as exactly the same way finite numbers are counted. The problem of Aristotle's arguments is that he defends the idea that infinite numbers cannot exist based on the assumption that all numbers are inherently finite. Accordingly, he answered the question of whether infinite numbers exist or not by begging the question and eliminated the possible existence of infinite numbers. Hence, they are subjected to a logical fallacy, i.e. a *petitio principii*.

Setting that aside, in addition to *petitio principii*, there is another problem in Aristotle's arguments. In Aristotle's philosophy, being is a process that is formed by the actualization of potentials, and existence is regarded as the process of becoming in which potentials are actualized and completeness acquired<sup>20</sup>. For Aristotle, change

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<sup>20</sup> For more detail, see *Physics* Book III.

in nature has emerged with the actualization of potentials, all things by their nature are in a constant process of becoming; they change their potentials to reach their ultimate actuality. Those potentials which are actualized made things actual and complete. For instance, bricks have the potential to construct a house and when the house is built, the potentiality of bricks has become actualized. They are no longer some bricks, instead, they collectively constitute a house. Since their potentiality became their complete actuality by constructing the house, they no longer have their potentials. As long as there are potentials that remain unactualized, the entity must be incomplete. But, firstly, to have potentials, there must be an entity whose potentials are waiting to be actualized. Aristotle mentioned this in *Physics*. He wrote:

Then again, there must be something to initiate the process of the change or its cessation when the process is completed, such as the act of a voluntary agent (of the smith, for instance), or the father who begets a child; or more generally the prime, conscious or unconscious, agent that produces the effect and starts the material on its way to the product, changing it from what it was to what it is to be. (*Physics*, II.3, 194b 29-33)

Apparently, the Aristotelian ontology seems to presume the existence of an entity that has the potentiality to become something other than itself. Accordingly, there would not be any potentiality without an actual entity in the first place. However, this assumption is not required in the course of infinitude. As Aristotle's own words:

The infinite, then, exists in no other way, but in this way it does exist, potentially and by reduction. It exists fully in the sense in which we say 'it is day' or 'it is the games'; and potentially as matter exists, not independently as what is finite does. (*Physics*, III.6, 206b13-16)

By rejecting the relation between potential infinite and actual infinite, he concludes that the distinction between potential infinite and actual infinite is "*sui generis*" which means something unique by its own characteristics. This seems to be a quite

ad-hock attempt in his theory regarding the characterizations of potential infinity and actual infinity.

In the Cantorian framework, this is a problem; the concepts of potential infinite and actual infinite are not as separate as Aristotle presented. For Cantor, potential infinite is meaningless without actual infinite because potentials have only occurred if there is an underlying reality. As his words, “In truth the potentially infinite has only a borrowed reality, insofar as a potentially infinite concept always points towards a logically prior actually infinite concept whose existence it depends on” (as cited in Rucker, p.3). This would mean that, at least for Cantor, actual infinite has ontological superiority to potential infinity because potential infinite presupposes an actual infinite to exist in the first place. Therefore, this reasoning alone necessitates the existence of actual infinity for Cantor in order to speak of the existence of potential infinity.

On the one hand, Cantor’s strong commitment to the existence of transfinite numbers provided him a way to refute the traditional understanding of infinity. Nevertheless, on the other hand, he has also fallen into the same position with Aristotle; Cantor also begged the question and answered the question of whether infinite numbers exist or not by showing transfinite numbers. He used these numbers to show the fact that infinite numbers can also be enumerated as finite numbers after mathematically constructing them. In this regard, claiming that these numbers correspond to the existence of actual infinities seems to be an overstatement. One can, for instance, argue against the existence of these numbers and defend that they are not essentially different from potentially infinite sequences. As an example, Anne Newstead wrote,

However, finitists would argue that Cantor’s transfinite numbers are too determinate, too similar to finite numbers, to be truly infinite. Finitists agree

with Aristotle that the proper conception of infinity is that of something that is endless and essentially incomplete and indeterminate. (1975, p.10)

Probably, the answer that Cantor would offer is the differences between transfinite numbers and finite numbers. Then, the question that would be examined turns out to be what makes transfinite numbers different from finite numbers.

First of all, according to Cantor, a new criterion is required for the arithmetization of different sizes of infinite sets. The criterion is the one-to-one correspondence between infinite sets and their proper subsets<sup>21</sup>. When this criterion is applied to finite sets, the pairing would be impossible for the fact that the members of any finite set must have more in quantity than its proper subsets in all conditions. Consequently, it is impossible to construct such a pairing between a finite set and its proper subsets. Nonetheless, this kind of correspondence can be constructed in infinite sets. Accordingly, the first distinction between finite sets and infinite sets has emerged in this way: infinite sets become the sets that can be paired off with their own proper subsets. Even though this implication seems counterintuitive, it was exactly the point Cantor emphasized the different characteristics between infinite sets and finite sets.

Secondly, although it is also possible to construct an unlimited amount of transfinite number by using the same principles, having unlimited elements does not make these numbers incomplete for Cantor. On the contrary, they are the instances of complete infinite. The set of natural numbers itself contains all of its numbers all at once. Numbers in this set do not manifest themselves as variables that proceed indefinitely, instead, they collectively constitute a whole that Cantor identified as a

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<sup>21</sup> As it is known, Richard Dedekind (1831-1916) firstly mentioned this criterion as the definition of infinite sets: infinite sets are the sets that have the same cardinality with their proper subsets.

genuine constant. This constant as the limiting number really belongs to actual infinite because the limiting number corresponds to all the numbers in the sequence simultaneously. In this manner, the sequence whose process is not yet completed is regarded as completed. On this subject manner, David Hilbert (1862- 1943) states:

We meet the true infinite when we regard the totality of numbers 1, 2, 3, 4, ... itself as a completed unity, or when we regard the points of an interval as a totality of things which exists all at once. This kind of infinity is known as actual infinity. (1926, p.188)

Thirdly, the laws that are applied to the arithmetical operations of transfinite numbers are different from the arithmetical operations of finite numbers. For finite numbers, the commutative law for addition always holds. Nevertheless, in the course of transfinite numbers, the order of numbers is significant to clarify the result of the operation because infinite sets would have different enumerations with different orderings<sup>22</sup>. For the same reason, the commutative law for multiplication does not hold in transfinite arithmetic. Furthermore, since transfinite ordinals do not have a predecessor unlike finite numbers, it is impossible to clarify the previous number before transfinite ordinals. It thus follows that the subtraction operation and the division operation cannot be applied for all transfinite ordinals<sup>23</sup>. As it seems, even though transfinite numbers are subjected to have different rules, their arithmetic is well-defined as finite arithmetic. We may, therefore, claim that these new numbers are mathematically different from finite numbers.

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<sup>22</sup> See the examples I have given in 2.3.1.

<sup>23</sup> Given any transfinite numbers  $\alpha$  and  $\beta$ , subtraction operation is possible as following: assuming that we have an equation as  $\alpha + \gamma = \beta$ , where  $\gamma$  can be either finite or transfinite, it is possible to derive  $\beta - \alpha$ , that is equal to  $\gamma$ . But if we rewrite the equation as  $\gamma + \alpha = \beta$ , then we cannot always derive  $\beta - \alpha$  for the non-commutativity of transfinite operations.



I would like to conclude that, for Cantor, Aristotle's arguments against the existence of actual infinity were the source of the scholastic position towards the concept<sup>24</sup>. But the proper way of understanding the concept of infinity cannot be potential infinity as Aristotle did in his opposition. Cantor introduced a new understanding of the concept of infinity that has one of the greatest impacts on its development in mathematics:

Once the actual infinite in the form of actually infinite sets had in this way asserted its citizenship in mathematics, then the development of the actually infinite number concept became inevitable, through appropriate, natural abstractions, just as the finite number concept, the material of arithmetic hitherto, had been achieved through abstraction from finite sets. (Cantor, 1887, p. 411)

Cantor not only provided an account of how to count infinite sets but also, introduced transfinite numbers and their arithmetic to stand against the traditional understanding of infinity. In this respect, I intend to claim that the Cantorian Set Theory which is against the dogmas of finitist mathematics is an outstanding response what most people thought after Aristotle's ideas. The arithmetization of infinite sets, which Cantor successfully demonstrated, leads us to conclude that the Cantorian Set Theory is a revolution in the history of the concept of infinity. So far, I have dealt with the Aristotelian rejection of actual infinity within the framework of Cantorian transfinite numbers. Now, I proceed to the next chapter where the existence of mathematical objects in the Cantorian framework is examined.

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<sup>24</sup> In the article called "Cantor's Transfinite Numbers and Traditional Objections to Actual Infinity", Jean Rioux states "Cantor saw Aristotle as the source of the Scholastic position on infinity, and in the *Grundlagen*, he addressed the basic error involved in all 'finitist' reasoning" (2000, p.101).

## CHAPTER 3

### THE EXISTENCE OF MATHEMATICAL OBJECTS IN CANTORIAN FRAMEWORK

#### 3.1. Introduction

Cantor formulated his theory that has intertwined with different distinctions (including the distinction between the two sides of reality and the distinction between multiplicities), and it turns out that all the distinctions he made are proposed to prove the existence of actual infinity. My intention in this chapter is to examine these distinctions in Cantor's theory and to show their importance as the ontological framework of actual infinity in the light of transfinite numbers.

In *Grundlagen*, when Cantor dealt with the foundation of transfinite numbers, he was building the theory of transfinite numbers based on the order type of different infinite sets. Cantor's strategy was to defend transfinite numbers as the legitimate extensions of finite numbers in which the relations between transfinite numbers and finite numbers provide the objective reality of transfinite numbers in mathematics. The reason for this strategy is the freedom of mathematics. Cantor believed that new concepts for mathematics can be introduced through already existing definitions and

relations. As we will see in section 3.2., when the construction of transfinite numbers considered, the number generating principles have a significant role regarding the correlation between natural numbers and transfinite numbers.

Another significant aspect of Cantor's theory is that the objective reality of transfinite numbers actually depends on the two aspects of reality, namely the immanent reality and transient reality. I will describe these two sides in more detail in chapter 3.2.1., for now, let me summarize the distinction briefly. While the immanent reality of concepts corresponds to a possible idea in the human mind, the transient reality of concepts is an object that is corresponding to images of physical phenomena independent from the human mind. The importance of this distinction particularly lies in mathematics. Mathematical objects are abstract entities; they do not have an existence as physical objects around us. For the fact that transfinite numbers are required justification for their usage in mathematics, Cantor considered these two sides. For him, the duality between immanent reality and transient reality makes transfinite numbers actually infinite numbers, rather than just symbols of infinity.

### **3.2. Free mathematics**

Mathematical objects, for Cantor, are the concepts that we abstract, and their reality as mathematical objects rely on definitions and relations. If these definitions and relations are established without any contradictory result, then one can easily accept the existence of new objects based on earlier concepts within the consistent arithmetic system. The idea behind this reasoning is the freedom of mathematics. For

Cantor, mathematic is entirely free to create new concepts on the grounds of intellectual consistency because the freedom is the essence of mathematics. As Cantor puts it:

Mathematics is in its development entirely free and is only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, established by definitions, to those concepts which have previously been introduced and are already at hand and established. (1976, p.79)

According to this criterion, as long as new mathematical objects stand in a certain relationship among others, they must be regarded as real as other objects because mathematics itself guarantees their reality via definitions and relations. Cantor states the argument as follows:

In particular, in the introduction of new numbers, it is only obligated to give definitions of them which will bestow such a determinacy and, in certain circumstances, such a relationship to the older numbers that they can in any given instance be precisely distinguished. As soon as a number satisfies all these conditions it can and must be regarded in mathematics as existent and real. (1976, p.79)

Since mathematics is free to generate its own objects, Cantor used this freedom to establish transfinite numbers on the basis of the number generating principles, namely the first generation principle and the second generation principle. The first principle of generation, which I examined above in section 2.3.1, provides a way to define the immediate successor of any number by the repeated addition of units. This is the usual process of counting<sup>25</sup>. We have the unending sequence of finite numbers with no greatest among them. But Cantor's main interest, of course, was not to validate the unending sequence of the finite numbers, instead, his intention here was

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<sup>25</sup> This is also known as the principle of induction or mathematical induction in contemporary mathematics. Given an infinite sequence, if a proposition  $P(1)$  is true, and by assuming that  $P(x)$  is also true, then it can be shown that the proposition is also true for  $x + 1$ , then the proposition is true for all natural numbers.

to comprehend the sequence as a complete entity to construct infinite numbers. When the second principle of generation is considered, the properties and orders of elements are abstracted from the sequence and a completed entity, i.e.  $\{1, 2, 3, \dots\}$ , is acquired. Then, a new number<sup>26</sup> is assigned to the entity as the limit of previous numbers:

If any definite succession of defined integers is put forward of which no greatest exists, a new number is created by means of this second principle of generation, which is thought of as the *limit* of those numbers; that is, it is defined as the next number greater than all of them. (Cantor, 1976, p.87)

The new number is called the first transfinite ordinal number “ $\omega$ ” and it is the limiting element of the previous number sequence, i.e. natural numbers. Being the limit makes the number greater than all elements of the sequence and, at the same time, the smallest one after the whole sequence. Since it exceeds all finite numbers in terms of size, it cannot take its place in the sequence and, more importantly, it cannot be a finite number for the same reason.

Since the freedom of mathematics ensures the existence of new concepts with regard to definitions and relations, Cantor constructed the new numbers based on the sequence of natural numbers by way of the number generating principles to justify their mathematical existence. As long as the mathematical system, which includes both transfinite numbers and finite numbers, maintains its consistency and coherency, then transfinite numbers must be acknowledged as a legitimate extension of finite numbers. As William Walker Tait stated in his article “Cantor’s Grundlagen

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<sup>26</sup> After defining the first limit ordinal by the second principle of generation, Cantor introduced the principle of transfinite induction. While the principle of mathematical induction only works for finite ordinals, transfinite induction is used for limit ordinals. Given a well-ordered set  $A$ , if a proposition  $P(0)$  is true, then it is possible to assume that  $P(\beta)$  is true for all  $\beta < \alpha$ . Then, by assuming  $P(\beta)$ , it is possible to prove  $P(\alpha)$  for all  $\alpha \in A$ . After verifying these, then it is possible to prove that for any limit ordinal  $\gamma$ ,  $P(\gamma)$  is also true for all  $\beta < \gamma$ . The principle of transfinite induction implies that the proposition  $P$  is true for all ordinals.

and the Paradoxes of Set Theory”, “We are justified in regarding the numbers as real in so far as the system of transfinite numbers has been consistently defined and integrated with the finite numbers” (2000, p.258).

All things considered; it is not unreasonable to claim that transfinite numbers have a conceptual base in our understanding with the number generating principles.

Cantor wrote:

What I declare and believe to have demonstrated in this work as well as in earlier papers is that following the finite there is a transfinite (transfinitum) that is, there is an unlimited ascending ladder of modes, which in its nature is not finite but infinite, but which can be determined as can the finite by determinate, well-defined and distinguishable numbers. (1976, p.76)

Each number generating principles evidently has a significant role in the relations between transfinite numbers and other mathematical objects. Therefore, the freedom of mathematics and the number generating principles lead us to conclude that these new numbers are mathematically as justifiable as finite numbers. The following section is devoted to examining the two sides of reality which underlies the argument for the freedom of mathematics.

### **3.2.1. Two sides of reality from Cantor’s perspective**

Under the influence of Plato’s two-world doctrine<sup>27</sup>, many philosophers believed that there are two sides of reality in which we live in as the physical

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<sup>27</sup>Plato indicates that the world composed of two distinct realms; the material world and the world of ideas. The world of ideas is the realm that is perfect itself and contains every perfect form as ideas. On the other hand, the material world is just a reflection of the world of ideas in the sense that entities in the material world are imperfect copies of those perfect forms. The importance of this distinction grounds the argument that it is impossible to think of the reflection of any form without the underlying perfect form in the world of ideas. For more detail see Ross (1951), pp. 22-37.

manifestation of concepts and the ideal side as the source of knowledge. Of course, this duality was reflected in different ways by the people who believed this duality, but the underlying reason is generally the same. Georg Cantor also was one of them.

He stated:

For this (any secure knowledge) can be obtained only from concepts and ideas that are stimulated by external experience, and are essentially formed by inner induction and deduction as something that, as it were, was already in us and is merely awakened and brought to consciousness. (1976, p.95)

In this regard, the distinction between the two sides of reality, namely the immanent reality and the transient reality, has a significant role in his philosophical framework for the justification of transfinite numbers.

He describes immanent or intrasubjective reality as “in a connective sense to modify the object of thought” (1976, p.79). The immanent side of concepts is concerned with the relations with the already well-defined objects. This means the internal consistency between old concepts and new concepts will guarantee the legitimacy of new concepts in mathematics. For Cantor, as long as the consistency of the system is held, then any new concept will have an existence in the immanent side as possibly exist ideas. In a sense, every consistent and coherent idea in the Cantorian framework can eventually correspond to an actuality as a possible being, but this does not always mean that they will exist as an actual entity in the physical world. Instead, he wrote:

I call the being concerned a ‘possible’ being. By this is not meant that the being somewhere, somehow and sometime exists, since that depends on further factors, but only that it can exist. Thus, for me, the two concepts ‘suited for existence, i.e., for being created’ and ‘possibility’ coincide. (as cited in Hallett, 1986, p.20)

On the other side, he describes transient or transsubjective reality as “expressions or images in the physical world” (1976, p.79). The transient reality is

the reality side that concepts have their physical reality independent from the human mind. To put it differently, transient reality of an object is concerned with the physical manifestation in the natural world. According to Cantor, when mathematical objects are considered, these two sides cannot be differentiated from each other; instead, they always coexist together. As he puts it:

Given the thoroughly realist foundations of my investigations, there is no doubt in my mind that these two types of reality will also be found together, in the sense that a concept to be regarded as existent in the first respect (immanently real) will always in certain, even in infinitely many ways, possess a transient reality as well. (1976, p.79)

Accordingly, any well-defined mathematical idea, which is immanently real in mathematics, it would have a corresponding reality in the transient side as well because the idea always exists as a possible idea. However, for the fact that mathematical objects are abstract entities, their manifestation can only occur in definitions and relations with former objects. This is the reason why Cantor suggested the argument that mathematics must only concern with the immanent reality of concepts because its objects have relational existence. In Cantor's words, "... mathematics in the shaping of its conceptual material need take into account solely and uniquely the immanent reality of its concepts and thus is under no obligation whatsoever to also test these concepts with respect to their transient reality<sup>28</sup>" (1976, p.79).

Furthermore, for Cantor, so long as any new mathematical concept is constructed based on former concepts without any contradiction, the immanent reality of this concept guarantees its place as a distinct object of thought by the free

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<sup>28</sup> On the determination of transient reality of mathematical objects, Cantor infers the following, "Admittedly, the determination of this reality generally is among the most troublesome and difficult tasks of metaphysics..."(1976, p.79).



act of our construction ability. In this regard, the construction of transfinite numbers on the grounds of the number generating principles justifies that these new numbers have their existence as distinguishable numbers because the integration and consistency with finite numbers assure their place in our understanding:

First, we may regard the whole numbers as real in so far as, on the basis of definitions, they occupy an entirely determinate place in our understanding, are well distinguished from all other parts of our thought and stand to them in determinate relationships, and thus modify the substance of our minds in a determinate way. (Cantor, 1976, p.79)

This argument is also provided the reason why Cantor has suggested the freedom of mathematics on the grounds of internal consistency. Accordingly, on the authenticity of mathematics, he infers the following, “Because of this distinguished position, which differentiates mathematics from all other sciences..., it quite specifically deserves the name of free mathematics, a designation to which, if I had the choice, I would give preference over the now customary ‘pure’ mathematics” (1976, p.79).

In conclusion, Cantor’s arguments for the freedom of mathematics and the objective reality of transfinite numbers are based on the distinction between the two aspects of reality, namely the immanent reality and transient reality and what he proposed with this distinction is an interesting framework to demonstrate the existence of these new numbers. Cantor believed the consistency of the mathematical system, in which all transfinite ordinal numbers are obtained through abstraction from the set of natural numbers by the second number generating principles, would demonstrate the mathematical reality of transfinite numbers based on the well-ordered sets. Cantor infers the following:

Then again we can ascribe reality to numbers insofar as they must be regarded as an expression or image of occurrences and relationships in the external world confronting the intellect, further insofar as the different

number classes (I), (II), (III) <sup>29</sup>, and so on represent powers, which in fact occur in corporeal and mental nature. (1976, p.79)

Each number generating principles has a significant role regarding the correlation between natural numbers and transfinite numbers, and this is the way the mathematical consistency of these new numbers is maintained, and their immanent reality is shown in the Cantorian framework. Therefore, it can be concluded that the immanent side of transfinite numbers assures that the new numbers, which are constructed in a similar way to finite numbers, are as legitimate as finite numbers based on the two aspects of reality.

### 3.3. The notion of set in Cantorian framework

Before going into the arithmetization of transfinite numbers, the concept of a set must be analyzed to understand Cantor's transfinite theory; it should not be forgotten that the idea of transfinite numbers is emerged from the set of natural numbers by considering it as a completed set. And this is the reason why he puts the concept of a set at the center of his theory. For this reason, the following section of my thesis is dedicated to analyzing the concept of set in the Cantorian framework.

First of all, it is noteworthy that Bernard Bolzano (1781-1848), as an Austrian mathematician, had an undeniable impact in Cantor's work both in the concept of set

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<sup>29</sup> Cantor not only mathematically constructed infinitely many number sequences which enumerate infinite sets, but also distinguished them into number classes. At this point, he introduced the third number generating principle (also called as limiting principle), which reveals the different number classes. The first number class is the set of natural numbers whose cardinality is equal to  $\aleph_0$ . The unending series of transfinite ordinals ( $\omega, \omega + 1, \dots, \omega \cdot 2, \omega \cdot 2 + 1, \dots, \omega \cdot 3, \dots, \omega^2, \dots, \omega^3, \dots, \omega^\omega, \dots, \omega^{\omega^\omega}, \dots$ ) is named as the second number class which is the next higher cardinality ( $\aleph_1$ ) after than the first number class.

and in the concept of actual infinity. Bolzano studied the concepts of set in his works and the way he defined it contributed to the formation of this concept in modern mathematics. In fact, he added four different attributes to the concept (Felgner, 2010). To put it briefly, the first attribute is that the totality of elements determines sets. In other words, the collection of objects constitutes a new object which we defined as a set. The second one is that, for Bolzano, sets can consist of different kinds of entities; any collection of objects can form a set. The third is that the existence of sets does not need to be definable. And the last attribute is that sets have mind-independent existence for Bolzano. These four attributes not only improved the formation of the modern concept of a set but also affected Cantor's way of the conception of sets.

Furthermore, in *Paradoxes of the Infinite*, Bolzano questioned the nature of infinity to solve paradoxical results, especially Galileo's paradoxical examples. He was the first person to claim that the idea of actual infinite can be introduced in mathematics as a legitimate object after properly clarifying its mathematical nature. Contrary to the traditional understanding, the idea of actual infinity was indispensable for Bolzano. Since mathematic deals with abstract sets, infinite sequences can also be constructed as mathematical sets with the true criteria. The subset criteria, for Bolzano, were the true criteria for comparing infinite sets in sizes. Any subset whether finite or infinite must always be smaller than the set itself in terms of numerosity. If this criterion is applied properly, then it is possible to show that one infinite set can be greater than another infinite set. Even though Bolzano failed in arithmeticity of infinite quantities, the subset criteria influenced Cantor

about infinite sets; different infinite sets can be obtained according to their order type.

For a long time, it is assumed that enumerating an infinite sequence is impossible, but Cantor realized that if the infinite sequences are demonstrated as well-ordered sets, then the enumeration of them can be mathematically obtained. In fact, Cantor believes that every set can be turned into a well-ordered set. This is also known as the well-ordering theorem<sup>30</sup>. He wrote:

The concept of a well-ordered set is fundamental for the whole theory of manifolds. It is a basic law of thought, rich in consequences and particularly remarkable for its general validity, that it is always possible to bring any well-defined set into the form of a well-ordered set. (1976, p.72)

With the concept of a well-ordered set, Cantor not only indicated that infinite sets are countable as finite sets but also, he argued that infinite sets would certainly generate different implications in mathematics with their different order types. Consequently, it is not unreasonable to argue that transfinite numbers secure their place in mathematics with regard to well-ordered sets they depend on.

In *Grundlagen*, Cantor defined a set as following, “By an aggregate, I generally understand every multiplicity which can be thought of as one, i.e. any totality of definite elements which by means of a law can be bound up into a whole” (1976, p.93). The problematic aspect of this definition is that, as many people has

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<sup>30</sup> It should be emphasized here that the well-ordering theorem and the well-ordering principle are different from each other. The well-ordering principle implies that, in every non-empty set of positive integers, there is always a least element, that makes them well-ordered sets. The well-ordering theorem is reflected as the Axiom of Choice in ZFC (Zermelo-Fraenkel Set Theory with the Axiom of Choice). The Axiom of Choice is in fact formally equivalent to Cantor’s well-ordering theorem. The axiom implies that it is possible to demonstrate one set which its elements are chosen from infinite collection of sets one by one. Since it does not indicate the choice function obviously, i.e. a function  $f$  such that every non-empty subset  $Y \in X, f(Y) \in Y$ , it leads to many controversial debates in set theory. Even some mathematicians reject the axiom and prefer to use ZF over ZFC (note that ‘C’ indicates the Axiom of Choice). For more detail see Gillman (2002).

argued, the unrestricted comprehension principle<sup>31</sup> can be applied to his conception of a set. The principle, which gives rise to paradoxical sets in the set-theoretical universe, has no rule to determine a set; every collection of objects can determine a set e.g. the collection of all natural numbers between 10 and 35 or the collection of all sets in the set-theoretical universe. Nonetheless, it is possible to think of a set that leads to a contradiction in the system, so-called Russell's paradoxical set is such an example. The set is defined as “the set of all sets which are not members of themselves”. But since it is a self-referential set<sup>32</sup>, it contradicts its definition and leads to problematic results in the system.

However, Cantor realized the problematic situation of some collections beforehand. The idea that a set is a member of itself is substantially problematic when the concept of well-ordered set is considered. For this reason, Cantor believed that any consistent set cannot consist of itself as a member. Accordingly, Cantor did not accept the unrestricted comprehension principle in his framework and, even before Russell's paradox, already recognized that there must be a distinction between multiplicities for the fact that some multiplicities cannot constitute mathematically consistent sets.

So, in what sense Cantor differentiate multiplicities and considered some of them as sets and some not? To find out which multiplicities can determine a mathematical set, he made a distinction between multiplicities, namely consistent multiplicities, that are regarded as mathematical sets, and inconsistent multiplicities.

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<sup>31</sup> The unrestricted comprehension is the statement that, given any condition expressible by a formula  $\phi(x)$ , it is possible to form the set of all sets  $x$  meeting that condition, denoted  $\{x \mid \phi(x)\}$ .

<sup>32</sup> Let me call this set  $R$ . If  $R$  is a member of  $R$ , then  $R$  is not a member of  $R$  by the definition of the set. And if  $R$  is not a member of  $R$ , then  $R$  is a member of  $R$ . Thus, the contradiction occurs by the definition of the set. Accordingly, it is concluded that no set can be a member of itself.

On the one hand, consistent sets<sup>33</sup> are the multiplicities that have no contradiction while mathematically constructing them. In a letter to David Hilbert dated 2 October 1897, Cantor wrote:

One must only understand the expression ‘finished’ correctly. I say of a set that it can be thought as finished (and call such a set, if it contains infinitely many elements, ‘transfinite’ or ‘super-finite’) if it is possible without contradiction (as can be done with finite sets) to think of all its elements as existing together, and so to think of the set itself as a compounded thing for itself; or ( in other words) if it is possible to imagine the set as actually existing with the totality of its elements. (1991, p.390)

In other words, the elements of such multiplicities can be thought as distinct from each other and, at the same time, it is possible to comprehend the totality as a mathematical collection. The elements and the totality as a separate entity must be coexisted to determine a mathematical set in the Cantorian framework. For instance, the set of natural numbers is a consistent multiplicity; each number has distinct properties and we can conceptualize the whole set as a totality “ $\omega$ ”. Hence, multiplicity itself and its elements must be particularly distinct to have a definite place in our understanding while constituting a mathematical set.

On the other hand, inconsistent multiplicities cannot be mathematically conceivable. Either they are too large<sup>34</sup> to mathematically construct, or the totality of their elements as distinct units is mathematically impossible. Cantor wrote:

For a multiplicity can be such that the assumption that all of its elements "are together" leads to a contradiction, so that it is impossible to conceive of the

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<sup>33</sup> In his writings, Cantor also used the term “finished set” to identify consistent multiplicities. A finished set is not the set that has a finite number of elements, instead if it is possible to think of ‘all of its elements as existing together’ and the set as ‘a compounded thing for itself’ without any contradiction, only then it becomes a set in the Cantorian framework.

<sup>34</sup> Von Neumann also considered some sets as “too big” and identified them as classes to resolve the paradoxical results they lead. For example, there is no such a thing as the set of all sets, instead, it appears as the class of all sets. In his hierarchical universe, classes are ranked above sets. For more detail, see Hallett (1986).

multiplicity as a unity, as "one finished thing". Such multiplicities I call absolutely infinite or inconsistent multiplicities. (1991, p.407)

To be more precise, such a multiplicity can correspond to an idea in thinking, but not a mathematically well-defined multiplicity; its totality as a whole cannot be mathematically formed<sup>35</sup>. Such multiplicities would never be considered as mathematically definite sets in the Cantorian framework<sup>36</sup>. To give an illustration of what he meant, consider Russell's paradoxical set. The property that collects its members turns out to be contradictory to its definition. Thus, some multiplicities (including Russell's set, the set of all sets, and the set of all ordinals) became mathematically indeterminate since they cannot contain themselves as a member.

Let us examine the problem of one of these. We can think of a set  $S$  represents the set of all sets, when we apply the Cantor's theorem<sup>37</sup>, what we would have is that the cardinality of the power set of  $S$  is greater than the cardinality of the set  $S$ . Since the power set of any set must be greater than the set itself, the set  $S$  should have smaller cardinality than its power set. However, this is impossible because the set  $S$  is defined as the set of all sets (including all possible sets). At the same time, we would also have that the cardinality of the set  $S$  must be greater than

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<sup>35</sup> In *the Fundamental Ideas and Axioms of Mathematics*, Bernard Russell (1872-1970), with similar reasoning, wrote, "This arises most simply from applying the idea of a totality to numbers. There is, and is not, a number of numbers. This and causality are the only antinomies known to me. This one is more all-pervading.... No existing metaphysics avoids this antinomy" (1899a, p.267).

<sup>36</sup> As Cantor puts it, "Only complete things can be taken as elements of a multiplicity, only sets, but not inconsistent multiplicities, in whose nature it lies, that they can never be conceived as complete and actually existing" (as cited in Lavine, 1994, p.99).

<sup>37</sup> The theorem implies that given any set, the cardinality of a set must always smaller than the cardinality of its power set, which consist of all of its subsets. In other words, the cardinality of power set of a set has always bigger than the cardinality of the set itself. Given any set with  $n$  elements, its power set must always include  $2^n$  elements. Mathematical symbolization for this theorem is, for every set  $A$ ,  $|A| < |P(A)|$  where  $|A|$  represents cardinality of the set  $A$ .

the cardinality of the power set of the set  $S$  by definition. But this is also impossible because the power set of a set cannot be smaller than the set itself. This is clearly a contradiction. Consequently, the set of all sets cannot be comprehended without any contradictory results. The property of containing itself as a member prevents the multiplicity from being a consistent set. For the same reason discussed here, some multiplicities, such as the set of all ordinals and the set of all alephs or the whole set-theoretic universe, cannot be constructed as a single set in the Cantorian framework.

Also, another reason why such multiplicities are left out is the impossibility of their enumeration. Every well-ordered set must have an ordinal number, which corresponds to its order type. However, these multiplicities cannot have an ordinal number because the elements of such sets do not satisfy the condition the set itself signifies in the first place. They become mathematically indeterminate and cannot be ordered. Since it is impossible to turn them into the form of well-ordered sets, they cannot be mathematically enumerated; consequently, they cannot be mathematical sets. This is exactly the emphasis on Cantor saying mathematically constructing them. In fact, the difference between inconsistent multiplicity and consistent multiplicity is as simple as Ignacio Jane stated in his article "The Role of the Absolute Infinite in Cantor's Conception of Set", "No possible collection can encompass all sets" (1995, p.400). Since inconsistent multiplicities cannot constitute a mathematically legitimate set in the first place, Cantor excluded them from his theory.



Many scholars, even today, identify the Cantorian Set Theory as a naïve set theory by ignoring the distinction between multiplicities<sup>38</sup>, but, in different passages, there are good reasons to believe that Cantor was already aware of the paradoxical situation of the unrestricted comprehension principle. In the correspondence with Gottlob Frege (1848-1925), in 1885, Cantor accused him to use the unrestricted comprehension principle, which turns out to be a failure in Frege's project, with too much confidence<sup>39</sup>. Another passage is about the Burali-Forti paradox, which came up on 28 March 1897. The paradox asserts that the set of all ordinal numbers leads to a contradiction<sup>40</sup>. However, Cantor wrote:

I expressly say that I only call multiplicities 'sets' if they can be conceived without contradiction as unities, that is, as things...What Burali-Forti has put forward is utterly foolish. If you will look at his paper in the *Circolo Mathematico* you will see that he has not even correctly understood the concept of a well-ordered set (Moore and Garciadiego, 1981, p.342)

For the fact that an ordinal would always be left out, the set of all ordinals cannot consist of all ordinals. Consequently, the set turns out to be an inconsistent one. Similarly, the set of all alephs cannot also indicate a mathematical set in the Cantorian framework. In a letter to Hilbert, dated back 26 September 1897, Cantor puts it:

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<sup>38</sup> As Ignacio Jane (2010) presented in his article, *"Idealist and Realist Elements in Cantor's Approach to Set Theory"*, people identify his definition with the unrestricted comprehension principle, but what Cantor has presented is not the same idea with this principle.

<sup>39</sup> In the review of Frege's *Grundlagen*, Cantor warned Frege on the usage of notion "extension", which eventually become the foundation of his Basic Law V that leads to Russell's paradox. See Cantor on Frege's *Foundations of Arithmetic* (1885) for more information.

<sup>40</sup> When the set of all ordinals " $\Omega$ " is considered by the relation  $<$  on ordinals, another ordinal number must be assigned to the set as a successor ordinal (the second condition for well-ordered sets). But this is impossible because the ordinal that is assigned to the set would not be in the set. Consequently, " $\Omega$ " cannot consist of all the ordinals. This is obviously a contradiction. See for more detail Heijenoort (1967, pp. 104-113).

For the totality of all alephs is one that cannot be conceived as a determinate, well-defined, finished set. If this were the case, then this totality would be followed in size by a determinate aleph, which would therefore both belong to this totality (as an element) and not belong, which would be a contradiction. (1991, p.388)

Inconsistent multiplicities are not regarded as mathematical sets in the Cantorian framework. In a letter to Hilbert, in 1897, Cantor wrote, "Totalities that cannot be regarded as sets, I have already many years ago called absolute infinite totalities<sup>41</sup>, which I sharply distinguish from infinite sets" (1991, p.389). Thus, contrary to common conception, both the Russell's paradox and the Burali-Forti paradox do not appear in the Cantorian Set Theory.

From all these, what is most evident is that Cantor differentiated his theory from the naïve set theory<sup>42</sup> by restricting multiplicities as consistent and inconsistent on the basis of well-ordered sets. Accordingly, I would like to pay attention to the fact that Cantorian Set Theory does not give rise to any antinomies or paradoxical sets unlike most of his contemporary colleagues think because only well-ordered multiplicities are considered as sets. Therefore, the analysis has sufficiently shown that the Cantorian Set Theory is neither naïve nor paradoxical.

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<sup>41</sup> The word "absolute infinities" used in two different meanings in Cantor's works. The first one is for inconsistent multiplicities which cannot coincide with a determinate multiplicity as we analyzed. The second usage is for the Absolute, which he defined as the unity of All. Cantor wrote, "The true infinite or absolute, which is in God, admits no kind of determination" (1976, p.76). This side of the notion is rather a metaphysical one and it is not the subject of this thesis. For more detail see Jané (1995), pp. 383- 388.

<sup>42</sup> In naïve set theory, any collection of objects can correspond to a set without restriction. It generally uses natural language to describe its objects, rather than formal language of mathematics. But these are not necessary conditions to declare theories as naïve set theory. Some theories which are proven to be inconsistent are also considered as naïve set theory. The obvious example is Frege's project on reducing mathematics into pure logic. It is well-known fact that Frege's Basic Law of V (also known as the axiom schema of unrestricted comprehension), which allows to create paradoxical sets - Russell's paradoxical set is the example, lead to failure in the system.

### 3.4. Transfinite arithmetic

With the paradoxes of infinity, it was assumed that the sizes of infinite quantities cannot be determined as a legitimate object of mathematical study. However, as the first man ever tried to do, Cantor demonstrated that it is mathematically possible to differentiate infinite sets and clarified their sizes by the concept of cardinal number. According to their cardinality, any two sets whether finite or infinite can be compared with the one-to-one correspondence principle. The principle implies that any two sets are the same size (or having the same cardinality) if it is possible to demonstrate a one-to-one correspondence between the members of sets. Each element in one set is matched with an element of another set to compare their size. The obvious example is that the elements of the set of natural numbers and the elements of the set of even natural numbers can be paired off with each other. Even though we intuitively inclined to claim that the set of all natural numbers must be greater than its subsets (including the set of even and odd numbers) because the size of natural numbers is twice the size of even numbers (and also odd numbers), there are as many natural numbers as even numbers. This implication led to the assumption that all infinite sets are the same size.

However, as I have mentioned before, Cantor showed that there would not be a one-to-one correspondence between the set of real numbers and the set of natural numbers in his article "*On a Property of the Collection of All Real Algebraic Numbers*" in 1874. Accordingly, not all infinite sets are the same size. In 1891, Cantor provided a much simpler proof for the non-denumerability of the set of real

numbers<sup>43</sup>. In the article, he showed the fact that, for any given set A, the cardinality of the power set of A has a greater cardinality than the cardinality of the set of A. For instance, let S be {1, 2, 3}, then  $P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ . When this theorem is considered in infinite sets, it is possible to construct different sizes of infinite sets because every infinite set has at least one larger set as its own power set. Cantor wrote:

This proof is remarkable not only because of its great simplicity, but more importantly because the principle followed therein can be extended immediately to the general theorem that the powers of well-defined manifolds have no maximum, or, what is the same thing, that for any given manifold L we can produce a manifold M whose power is greater than that of L. (1991, pp. 921-922)

In fact, this theorem also led Cantor to defend the argument that different sizes of infinite sets can be constructed unlimitedly:

$$|\mathbb{N}| < |\mathbb{P}(\mathbb{N})| < |\mathbb{P}(\mathbb{P}(\mathbb{N}))| < |\mathbb{P}(\mathbb{P}(\mathbb{P}(\mathbb{N})))| \dots$$

As it appears, there is not just one size of infinity, but there are infinitely many different sizes of infinity, which necessarily requires the fact that there would not be the largest infinite in size.

Cantor used the Greek letter  $\omega$  (omega) for the symbolization of transfinite ordinal numbers and the Hebrew letter  $\aleph$  (aleph) for the symbolization of transfinite cardinal numbers. An ordinal number<sup>44</sup> describes the position of a member in a well-ordered sequence. Given any set, the properties of its elements are abstracted from

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<sup>43</sup> The name of the article is “*On an Elementary Question in the Theory of Manifolds*”. It is also marked Cantor’s diagonal argument, which is known nowadays as the diagonalization method. The method proves the existence of non-denumerable (or uncountable) sets, such as the set of real numbers. For more detail see Dauben (1991), pp. 165-168.

<sup>44</sup> Given any set M, then the ordinal number or its order type is denoted as  $M^*$ .

the set and what we have is the order of elements as the first element, as the second element, etc. This is the first abstraction on well-ordered sets to be able to clarify the ordinal number of any set. For instance, the order type of the empty set is 0 because it does not have any member to be ordered and let  $S$  be  $\{a, b, c\}$ , in this case, the ordinal number of the element 'b' would be 2. After all, 'b' is the second element in the order of the set. For all sets whether finite or transfinite, the last member, which pair off with the corresponding sequence, signifies the order type. Consider the  $S$  again, the ordinal number of the set would be 3 because the last element 'c' is the 3<sup>rd</sup> element for the set and the set is paired off with its correspondence, i.e.  $\{1, 2, 3\}$ . For this reason, as I mentioned before, different orderings on finite sets do not change the order type because the order of the last element will always be the same.

Up to this point, there is no problem with clarifying the order type of finite sets. But when infinite sets are considered, the situation becomes complicated. Consider the usual ordering of natural numbers that is  $\{1, 2, 3, \dots\}$ . The sequence has no last element, but to identify its order type the first transfinite ordinal " $\omega$ " is assigned as the limit to the sequence. In the same way, even though the sequence  $\{1, 3, 5, \dots\}$  has different ordering, it has also the order type of  $\omega$ . Then, consider the following sequence;  $\{1, 2, 3, \dots; 1, 2, 3, \dots\}$ . It is obvious that the order type of it differs from previous examples; in fact, it is the order type of  $\omega + \omega$  (also equal to  $\omega \cdot 2$ ). If this is so, it is possible to add one more unit. The new sequence will be  $\{1, 2, 3, \dots; 1, 2, 3, \dots; 1\}$  which has the order type of  $\omega \cdot 2 + 1$ . By the first principle of generation and the second principle of generation, we may, therefore, claim that the formation of transfinite ordinal numbers is limitless with the proper ordering<sup>45</sup>.

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<sup>45</sup> Consider the example  $\omega + 1 \neq 1 + \omega$  I have given in section 2.3.1.

On the other hand, the concept of cardinal number is defined as the generalization of the number concept for infinite sets. Given an infinite set, the orders of its elements are abstracted and what we have is the size of the set as its cardinal number<sup>46</sup>. This is the second abstraction to clarify the size of a set. As Cantor's own words;

We will call by the name "power" or "cardinal number" of  $M$  the general concept which, by means of our active faculty of thought, arises from the aggregate  $M$  when we make abstraction of the nature of its various elements  $m$  and of the order in which they are given. (1895, p. 86)

The cardinality of a set is the number of elements that are contained in the set. One way to clarify the cardinality of a set is to construct one-to-one correspondence. Given any two sets,  $A$  and  $B$ , they have the same cardinality if there exists a one-to-one correspondence between them, mathematically denoted as  $|A| = |B|$ .

For finite sets, cardinality corresponds to the number of the elements of the set, i.e. how many elements the set has, and this must be a finite number; all natural numbers are particularly finite sets in which their numbers correspond to their cardinal numbers. For instance, the cardinal number of the empty set is 0 because it has no elements. And let  $S$  be  $\{a, b, c\}$ , in this case, the cardinal number of  $S$ , denoted as  $|S|$ , would be 3 because it has 3 elements - 3 is the ordinal that has the order type of  $\{1, 2, 3\}$ , which paired off with the set. Two conclusions have been derived here. The first one is that cardinal number for a finite set is the same as its ordinal number. And the second one is that ordering in finite sets can be changed, but their corresponding cardinality would always be the same. For infinite sets, on the

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<sup>46</sup> Given any set  $M$ , then the cardinal number of  $M$  is denoted as  $M^{**}$ .

other hand, the cardinality cannot correspond to a finite number because there are infinitely many elements in the set; instead, it must be a transfinite cardinal. For instance, Cantor assigned  $\aleph_0$  (aleph-null or aleph-zero) to the cardinality of the set of natural numbers<sup>47</sup>, and  $\aleph_1$  to the cardinality of the set of all countable ordinal numbers<sup>48</sup>.

Cantor introduced transfinite arithmetic (including ordinal arithmetic and cardinal arithmetic separately), but firstly the construction of natural numbers<sup>49</sup> must be examined to understand the similarity with the construction of transfinite numbers. All natural numbers are constructed from the empty set and each one after that is constructed based on previous numbers;

$$0 = \emptyset, \text{ that is the empty set}$$

$$1 = \{0\} = \{\emptyset\} = \emptyset \cup \{\emptyset\} = 0 \cup \{0\}$$

$$2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\} = \{\emptyset\} \cup \{\{\emptyset\}\} = 1 \cup \{1\}$$

$$3 = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = 2 \cup \{2\} \dots$$

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<sup>47</sup> It is also denoted as  $|N| = \text{Card}(N) = \aleph_0$ .

<sup>48</sup> Cantor defended that there is no cardinality between the cardinality of set of real numbers ( $2^{\aleph_0}$ ) and the cardinality of set of natural numbers ( $\aleph_0$ ) and identified  $2^{\aleph_0}$  with the  $\aleph_1$  as the next and smallest cardinal number after  $\aleph_0$ . This is known as the Continuum Hypothesis, that is listed as one of the 23 most important unanswered mathematical question by David Hilbert. Nevertheless, neither Cantor nor anyone else has been able to prove this hypothesis. In fact, Kurt Gödel (1906-1978), in 1940, proved that the hypothesis cannot be disproved in ZFC. Then, in 1963, Paul Cohen (1934-2007) proved that the hypothesis also cannot be proved in ZFC. It is accepted that the hypothesis is independent from ZFC. For more detail see Dauben (1991), pp. 268-270.

<sup>49</sup> Although there are some other ways to construct natural numbers, the way John von Neumann's construction of natural numbers is commonly used in set theory related works.

As the same way, for a given ordinal  $\alpha$ , the next one always is defined as  $\alpha + 1$  (or  $\alpha \cup \{\alpha\}$ )<sup>50</sup>. We have here infinitely many finite ordinals. Then, the second number generating principle is considered and Cantor introduced the first transfinite ordinal as the set of all finite ordinals, i.e.  $\omega = \{1, 2, 3, \dots\}$ . As Cantor puts it:

As contradictory as it would be, therefore, to speak of a greatest number of class (I), there is, on the other hand, nothing objectionable in conceiving of a new number – we shall call it  $\omega$  - which is intended to be the expression for the fact that the totality (I) as a whole be given in its natural and lawful succession. (1976, p.87)

After constructing the first transfinite ordinal, Cantor indicated that there are also other numbers just like the first transfinite ordinal. He called them limiting ordinals. By the first number generating principle, a new number sequence is constructed as follows;

$$\omega + 1 := \omega \cup \{\omega\}$$

$$\omega + 2 := (\omega + 1) \cup \{\omega + 1\} \dots$$

We have here infinitely many transfinite ordinals, i.e.  $\omega$ ,  $\omega + 1$ ,  $\omega + 2$ , and so on. Since there would not be the greatest element, Cantor applied the second number generating principle again, and  $\omega + \omega$  (equal to  $\omega \cdot 2$ ) is constructed as a limiting number to the sequence. By applying the same rules one by one, it is possible to construct the unending sequence of transfinite ordinals;

$$0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots, \omega + \omega = \omega \cdot 2, \omega \cdot 2 + 1, \omega \cdot 2 + 2, \dots, \omega \cdot 3, \dots,$$

$$\omega \cdot 4, \dots, \omega \cdot 5, \dots, \omega \cdot \omega = \omega^2, \dots, \omega^3, \dots, \omega^\omega, \dots, \omega^{\omega^\omega}, \dots, \omega^{\omega^\omega}, \dots \text{ and so on.}$$

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<sup>50</sup> The mathematical notation is following; for every ordinal  $\alpha$ , there is a succeeding ordinal  $\beta$  such that  $\alpha < \beta$  and there is no ordinal in between. So,  $\beta = \alpha + 1$  (every element must be followed by another element as a successor unless it is the last element of the succession).



Since every single element must have a successor, what we have then is infinitely many new number sequences that each of its elements has a definite magnitude and all of them are different from one another with respect to their orderings. As Cantor puts it:

As one sees, there is no end to the formation of new numbers. By following *both* principles of generation, one obtains again and again new numbers and number sequences which have a fully definite succession. Thus, it first *appears* as if in this way of building new definitely infinite whole numbers we should lose ourselves in the *limitless*. (1976, p.88)

Now, it is necessary to analyze the arithmetic of transfinite ordinal numbers.

The addition operation in ordinal arithmetic is as follows;

Let  $\alpha$  and  $\beta$  be ordinal numbers. Then,  $\alpha + \beta = \text{Ord} (\alpha; \beta)$ .

$$2 + 3 = \text{Ord} (\{0, 1\}; \{0, 1, 2\}) = \text{Ord} \{0, 1; 0, 1, 2\} = 5$$

$$\omega + 1 = \text{Ord} (\{0, 1, 2, \dots\}; \{0\}) = \text{Ord} \{0, 1, 2, \dots; 1\} = \omega + 1$$

$$1 + \omega = \text{Ord} (\{0\}; \{0, 1, 2, \dots\}) = \text{Ord} \{1; 0, 1, 2, \dots\} = \omega$$

The second and the third examples show that commutativity does not hold in the transfinite ordinal arithmetic for addition. It can be easily seen that the finite number, in the case of  $1 + \omega$ , is annulled in the sequence, and the result would be equal to  $\omega$ . However, in the case of  $\omega + 1$ , its order type is different from  $1 + \omega$ . What we have then is the sequence of  $\omega$  and additionally one more elements, which is positioned as the  $\omega^{\text{th}}$  element. In fact, the last element also makes  $\omega + 1$  is the immediate successor of  $\omega$ . Accordingly, the most important conclusion regarding the Aristotle's rejection manifest itself:  $1 + \omega = \omega$ , but  $\omega + 1 \neq \omega$ .

The multiplication operation in ordinal arithmetic is as follows;

Let  $\alpha$  and  $\beta$  be ordinal numbers. Then,  $\alpha \cdot \beta = \text{Ord} (\alpha \times \beta)$ .

$$2 \cdot 3 = \text{Ord} (\{0, 1\} \times \{0, 1, 2\}) = \text{Ord} \{(0, 0), (1, 0), (0, 1), (1, 1), (0, 2), (1, 2)\} = 6$$

$$\omega \cdot 2 = \text{Ord} (\{0, 1, 2, \dots\} \times \{0, 1\}) = \text{Ord} \{(0, 0), (1, 0), (2, 0), \dots; (0, 1), (1, 1), (2, 1), \dots\} = \omega \cdot 2 = \omega + \omega$$

$$2 \cdot \omega = \text{Ord} (\{0, 1\} \times \{0, 1, 2, \dots\}) = \text{Ord} \{(0, 0), (1, 0), (0, 1), (1, 1), (0, 2), (1, 2), \dots\} = \omega$$

The second and the third examples demonstrate that commutativity for multiplication is not held in the ordinal arithmetic.

In addition to ordinal arithmetic, Cantor also introduced the cardinal arithmetic. It is not my intention here to analyze cardinal arithmetic in a detailed manner for the fact that transfinite ordinal arithmetic is alone the proof for the mathematical applications of transfinite numbers. For this reason, I will briefly mention the rules of cardinal arithmetic. Just like finite arithmetic and ordinal arithmetic, the concepts of being the one greater or less than or equal to another are the same in cardinal arithmetic. For any given two cardinal numbers  $m$  and  $n$ , there are only three possibilities;  $m = n$ ,  $m < n$ , or  $m > n$ . However, there are some operational differences in cardinal arithmetic.

For a given natural number  $k$ ;

$$\aleph_0 + k = k + \aleph_0 = \aleph_0 \text{ and } \aleph_0 \cdot k = k \cdot \aleph_0 = \aleph_0 .$$

The addition of a finite number does not correspond to a new cardinal, unlike in the case of ordinal arithmetic. In fact, the addition of the same cardinal number also does not generate a new cardinal;

$$\aleph_0 + \aleph_0 = \aleph_0 \text{ and } \aleph_0 \cdot \aleph_0 = \aleph_0 .$$

Consequently, it is possible to infer that  $\aleph_0^k = \aleph_0$  because multiplication does not generate a different number in cardinal arithmetic if we use the same cardinal number for the multiplication operation. Furthermore, if there are two different cardinalities in the operation, the larger one specifies the result of the operation;

$$\aleph_1 + \aleph_0 = \aleph_1 \text{ and } \aleph_1 \cdot \aleph_0 = \aleph_1^{51}.$$

We may, therefore, claim that differentiating infinite sets from each other in terms of sizes is the first step for constructing transfinite numbers. Then, Cantor proposed the method on how to count them systematically. Cantor's approach to multiplicities was to comprehend them as units composed of distinct elements. The properties of elements and their order are abstracted from the multiplicity and what we would have “one” entity. Then, the entity will be enumerated with the corresponding number to clarify its size. So, there are different sizes of infinities and all of them are countable as he argued throughout his works:

The (infinite) 'cardinalities' represent the only and necessary generalization of the finite 'cardinal numbers', they are nothing else than the actual infinitely large cardinal numbers, and they have the same reality and definiteness as the former, save that the laws between them, i.e. the arithmetic in respect to them, is partly different from that in the domain of the finite. (Cantor, 1892, p. 280)

As a result, there is no difference between finite numbers and transfinite numbers in terms of mathematical legitimacy; the foundation of transfinite numbers grounded upon the sequence of finite numbers by the number generating principles and the concept of well-ordered set. The above analysis has sufficiently shown that while the ordinal number of a set is related to the ordering, the cardinal number is related to the size. Transfinite arithmetic with the particular rules represents the same

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<sup>51</sup> While  $\aleph_0$  represents the cardinality of set of natural numbers,  $\aleph_1$  represents the cardinality of set of all countable ordinal numbers.

determinateness of finite arithmetic. We cannot disregard the mathematical applications of transfinite numbers for their mathematical justification because transfinite arithmetic is a systematic and coherent theory of actual infinities. Thus, contrary to common conception, Cantor's arguments about the existence of actual infinite are both mathematically and philosophically justified in his theory.

## CHAPTER 4

### CONCLUSION

For Aristotle, one of the most compelling reasons for the impossibility of infinite numbers is that no one could reach an infinite number by the successive addition of units. In counting, any number would still be a finite number no matter how big the number is. Since all numbers have been constructed from a specific set in which the number of objects corresponds to its number, the enumeration of infinite sets would become impossible. Hence, any counting would necessarily be limited to finite numbers. Nevertheless, Cantor showed otherwise. By applying certain rules (the number generating principles), firstly, Cantor suggested that a definite number can be assigned to the unending sequence of finite numbers as the first transfinite ordinal. Then, he suggested that differentiating the sizes of infinite sets is possible by the orderings of their elements. Thus, Cantor not only showed that counting can also be held in infinite sets, but also, he challenged Aristotle's rejection of actual infinity.

The sequence of natural numbers which is considered as a completed infinite provides Cantor a basis to create a new understanding of infinity. Even though we could not identify all elements of infinite sets, as we analyzed in the previous chapters, it is possible to mathematically evaluate transfinite numbers as being

greater or smaller or equal by one-to-one correspondence principle<sup>52</sup>. Besides differentiating different sizes of infinite sets, transfinite arithmetic which is subjected to different arithmetical rules reveals that transfinite numbers are mathematically as legitimate as finite numbers. In a letter to Wilhelm Wundt (1832-1920) dated back 5 October 1883, Cantor clarifies his belief on the existence of transfinite numbers:

I claim that my infinite number concepts are free from any arbitrariness and that they arise by abstraction from reality with the same necessity as the usual finite whole numbers, which so far have been used as the unique source of all other mathematical conceptual constructions. The transfinite numbers are not in any way, as you say, mere 'fictions' or 'logical postulates', as the geometrical spaces of  $n$  dimensions are, but they have the same character of reality as the old numbers: 1, 2, 3, etc. (1991, p.136)

In this sense, he defended the idea that transfinite numbers are not just symbols for infinity; rather, they are “concrete numbers having a real meaning” (1976, p.71). Accordingly, transfinite numbers represent a revolution in the history of mathematics; Cantor was the first man ever tried to refute the Aristotelian arguments against the existence of actual infinity and he successfully managed to stand against the traditional understanding of infinity. Thus, in my opinion, Cantor’s arguments about the existence of actual infinite are both mathematically and philosophically justified.

In the Cantorian framework, as long as mathematical objects ground some certain mathematical principles, it is, therefore, possible to claim that they have an objective reality based on two sides of reality. Cantor constructed new numbers as the extension of natural numbers that is already assumed to be autonomous and systematic. In his own words:

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<sup>52</sup> According to Hans Niels Jahnke, “Cantor was the first to use the concept of pairwise correspondence to distinguish meaningfully and systematically between the sizes of infinite sets” (2001, p.178).

...even for finite multiplicities a proof of their 'consistency' cannot be given. The fact of the 'consistency' of finite multiplicities is a simple, undemonstrable truth, it is 'the axiom of arithmetic'. And likewise, the 'consistency' of multiplicities to which I assign the alephs as cardinal numbers is 'the axiom of the extended, of the transfinite arithmetic. (1991, p.412)

By extending the number concept into infinities, Cantor established a new kind of arithmetical system with particular rules in mathematical operations. In his article "Beiträge zur Begründung der transfiniten Mengenlehre", he provided every detail of transfinite arithmetic based on the concept of a well-ordered set. In this regard, Cantor indicated that transfinite numbers must be regarded as existent as finite numbers thanks to the new arithmetic system based on infinite sets. As a matter of fact, the hierarchy of infinite sets can be demonstrated both in terms of cardinality and ordinality. Hence, transfinite numbers gain their mathematical reality by the number generating principles. Additionally, Cantor also defended that one cannot accept the existence of the irrational numbers and, at the same time, denies the existence of transfinite numbers because both of them are defined via infinite sets.

Cantor wrote:

The transfinite numbers are in a certain sense themselves new irrationalities and in fact, in my opinion, the best method of defining the finite irrational numbers is wholly dissimilar to, and I might even say in principle the same as, my method described above of introducing transfinite numbers. One can say unconditionally: the transfinite numbers stand or fall with the finite irrational numbers; they are like each other in their innermost being; for the former like the latter are definite delimited forms or modifications of the actual infinite. (1887, pp. 395-396)

Although Cantor's account of multiplicities provides him stronger and more comprehensive theory to maintain the self-consistency of the system, the distinction can be regarded as "ad hoc". The reason is that the distinction was constructed to avoid possible paradoxical outcomes and protect the theory from troublesome sets. It

is true that Cantor did not provide any distinction in multiplicities at first, but he realized that there must be a distinction on multiplicities to prevent paradoxical results. In fact, the emergence of paradoxes required the necessity to impose a restriction on the definition of a set. I would like to highlight the fact that ZFC, which is perhaps the most well-known axiomatic set theory, also restricts the definition of a set by its own axioms. The axiom of selection, which implies that subsets of a given set are also sets, allows constructing only subsets of existing sets. It avoids the formation of “too large” sets. Furthermore, the axiom of regularity which implies that no set is an element of itself entails that universal sets cannot be derivable in the system. For the fact that their sizes are undefined in the system, some sets (including the set of all ordinals or the set of all sets) do not appear in ZFC.

All things considered, my analysis has sufficiently shown that Cantor’s definition of a set is not arbitrary as many scholars called; rather, some multiplicities lack the property of being a mathematical set. The distinction between multiplicities indicates that Cantor’s system does not allow the emergence of paradoxical sets. Hence, we cannot disregard the fact that Cantor eliminated paradoxical sets even before they appeared in the system. Thus, I intend to emphasize that the Cantorian Set Theory is neither naïve nor paradoxical.



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## APPENDICES

### A. TURKISH SUMMARY / TÜRKE ÖZET

Bu tez Georg Cantor'un aktüel sonsuzluk düşüncesinin matematiksel olarak formalize edilmesi ve bu sayede ortaya çıkan sonlu ötesi sayılar teorisini konu edinmektedir. Kendisinden daha önceki dönemlerde sonsuzluk düşüncesinin kavramsallaştırılmasının belirli problemler içerdiğini iddia eden Cantor'un teorisini incelemek adına öncelikle sonsuzluk kavramının tarihsel gelişimi ele alınmalıdır. Sonsuzluk düşüncesi ile ilgili tartışmanın Eski Yunan'da başladığını varsayabiliriz. O zamanlarda birçok düşünürün öncelikli hedefi doğayı ve nesnelerin en temelinde yatan prensibi anlamak olduğundan, sonsuzluk fikri doğanın kendi temelinde yatan sınırsızlığı açıklamak için ortaya çıkmıştır. Bu nedenle, Sokrates öncesi dönemde sonsuzluk düşüncesi matematiksel ya da metafiziksel bir kavram değildir.

Sokrates sonrası döneme bakıldığında ise Aristoteles'in yaptığı sonsuzluk ayırımının tarihsel etkisinin ne kadar önemli olduğu yadsınamaz. Aristoteles sonsuzluk düşüncesini potansiyel sonsuzluk ve aktüel sonsuzluk olarak ikiye ayırmıştır. Sonsuzluk düşüncesi herhangi bir sınırlamaya bağlı olmadan devam eden bir süreç olarak tanımlanabilir. Başka bir deyişle, sonsuzluk potansiyel olarak sonsuza kadar devam edebilecek bir süreçtir ve sadece potansiyel olarak var olabilir. İnsan zihni sonsuzluk düşüncesini sadece potansiyel olarak kavrayabilir çünkü herhangi bir sınırlamaya sahip olamayacağı için bu süreç her zaman eksik ve belirsiz

olmak zorundadır. Bu Aristoteles'e göre potansiyel sonsuz fikridir. Buna ek olarak potansiyel sonsuzluk kendisini iki şekilde ortaya çıkartabilir: toplamayla yoluyla sonsuz ve bölme yoluyla sonsuz.

Aristoteles'e göre toplama yoluyla sonsuz fikri doğal sayıların sonsuz serisi olarak anlaşılabilir. Doğal sayıların her biri birimlerin art arda eklenmesiyle oluşturulan sayılardan oluşmaktadır, yani belirli bir doğal sayıya 1 sayısını ekleyerek daha büyük bir doğal sayı oluşturmak her zaman mümkündür. Bu durumda bütün doğal sayıların listesi her zaman eksiktir ve her zaman listede bulunmayan daha büyük bir sayı olacaktır. Başka bir deyişle, elimizdeki sayıya 1 ekleyerek elde edebileceğimiz yeni sayılar ile bu toplama süreci hiçbir zaman bitirilemeyeceği için doğal sayılar serisi potansiyel olarak sonsuza kadar devam edebilir. Bu sayı oluşturma işlemi sırasında en büyük sayı olarak adlandırılacak bir sayı olamayacaktır çünkü oluşturulan sayı ne kadar büyük olursa olsun onun da ardında olan bir başka sayı daima olacaktır. Bu durum, Aristoteles için, doğal sayıların serisinin potansiyel bir sonsuzluk oluşturduğunu garanti eder.

Diğer taraftan bölme yoluyla sonsuz, düz bir çizginin sonsuzluğu olarak görülebilir. Aristoteles'e göre, bir doğru parçasını bölerek sonsuz sayıda farklı parça ortaya çıkarmak mümkündür. Bu doğru parçası sonlu olsa dahi bölme işlemi potansiyel olarak sonsuzdur. Örneğin, ilk başta ikiye böldüğümüz bir doğru parçası için bölümden sonraki her bir parça da tekrar ikiye bölünebilir. Daha sonra aynı şekilde her bir parça da tekrar ikiye bölünebilir ve bu bölme işlemi sonsuza kadar devam edebilir. Çünkü parçaların boyutları küçülse dahi, her zaman bölünebilecek başka bir parça olacaktır. Olası bir bölme işlemi her zaman dışarıda bırakılacağından,

bir doğru parçasının bölünebilirliği sınırsız olarak gerçekleştirilebilir. Dolayısıyla, Aristoteles'e göre, sonlu bir doğruyu bölmek potansiyel olarak sonsuzluğa gidebilecek bir süreçtir. Burada vurgulanması gereken şey şudur ki; parçaların sonsuz bölünmesi yalnızca potansiyel gerçekleştirilebilecek olası bölünmeler olarak kavranabilir ve bu bölme işleminin de sadece düşünme ile mümkün olduğu anlamına gelir. Bölünme olasılığı hiç bitmediğinden bölünmeyle sonsuzluk potansiyel anlamda mümkündür. Bu nedenle, potansiyel sonsuzluk kendisini ya toplamayla yoluyla sonsuz ya da bölme yoluyla sonsuz olarak tezahür edebilir.

Öte yandan, Aristoteles'e göre aktüel sonsuzluk tanımı gereği kendisini eksiksiz ve tamamlanmış olarak sunması gereken sonsuz türüdür. Fakat çeşitli nedenlerden dolayı bu sonsuzluk mümkün değildir ve insan zihni tarafından kavranamazdır. Aristoteles'e göre aktüel sonsuzluğun bir varlık olarak var olamayacağının sebeplerden biri ve en önemlisi sonsuz olma sıfatının tamamlanmış olma sıfatı ile çelişmesidir. Tam olmak ya da tamamlanmış olmak zorunlu olarak belirli bir sınırlama gerektirir, ancak sonsuzluk tanımı gereği sınırlamalara sahip olamaz. Bu nedenle tamamlanmış olmak herhangi bir sonsuzluğun özelliği olamaz. Dolayısıyla sonsuzluk fikri ele alındığında bu durumun imkânsız olduğu açıktır ve Aristoteles'e göre aktüel sonsuzluk kavramı kendi içerisinde çelişkili bir kavram haline gelir: tam olmak ve sonsuz olmak aynı zamanda herhangi bir varlığın özellikleri olamaz. Söz konusu kavram Aristoteles felsefesinde tutarsız ve bilinemez bir kavram haline dönüşür.

Aktüel sonsuzluğun imkansızlığının bir başka nedeni ise doğada yatmaktadır. Aristoteles için doğada aktüel olarak sonsuz bir nicelik olamaz. Eğer doğada aktüel



anlamda sonsuz bir nicelik varsa, bu nicelik sonsuz sayıda parçaya sahip olan ve aynı zamanda parçaları da sonsuz olan bir varlık olmalıdır. Fakat Aristoteles için bu kabul edilebilir değildir çünkü bir bütünün parçaları kendi başlarına da sonsuz olan parçalara sahip olamazlar. Bu imkânsız gibi görünüyor çünkü parçaların sonsuzluğu bütünü belirsiz bir varlık haline dönüştürür. Dolayısıyla, belirsizlik doğada kabul edilebilir olmadığından, Aristoteles'e göre aktüel anlamda sonsuz bir nicelik var olamaz. Özetle, Aristoteles için sonsuzluk düşüncesinde açık olan bir şey vardır ki o da aktüel sonsuzluk fikrinin gerçek bir varlık olarak doğada var olamayacağıdır. Bu da aynı zamanda aktüel sonsuzluğun yalnızca sözün gelişi olarak var olduğunun göstergesidir.

Aktüel sonsuzluk fikrine karşı felsefî argümanlara ek olarak, Aristoteles aktüel sonsuzluğun imkansızlığına dair iki matematiksel argüman daha ortaya atmıştır. İlk argüman temelde sayma işleminin sonsuz sayı üretemeyeceğini savunmuştur. Doğal sayılar birimlerin art arda eklenmesi sayesinde oluşturulur ve bu yöntem oluşturulan bütün sayıların sonlu sayılar olmasını zorunlu kılar. Bunun nedeni ise her bir sayının başka bir sonlu sayı üzerinden oluşturulması ve olası bütün sayıların temelini başka bir sonlu sayı olmasıdır. Bir başka deyişle, sayma sürecinde doğal sayılar serisi potansiyel olarak sonsuz sayıda öğeye sahip olabilir, fakat sayma metodumuza göre tüm öğeler sonlu birer sayı olmalıdır. Dolayısıyla sayma işlemi yalnızca sonlu sayılara uygulanabilir ve sayma prosedürü ile sonsuz bir sayı oluşturulamaz.

Aristoteles'in aktüel sonsuzluğun varlığına karşı sunduğu ikinci matematiksel argüman ise sonlu sayıların matematiksel işlemlerde yok edilmesidir. Burada

vurgulanması gereken şey şudur ki Aristoteles bu argümanı doğrudan sayılar üzerinden sunmamıştır. Aristoteles'e göre doğada sonsuz bir madde varsa, bu madde doğadaki sonlu maddeleri yok etmelidir. Bunun temel nedeni ise dört ana elementin (su, hava, ateş ve toprak) bir arada var olabilmesi için sonlu olması gerekliliğidir. Aralarından herhangi biri sonsuz olsaydı, o zaman zorunlu olarak karşıt element sonsuz element tarafından yok edilirdi. Bu noktada doğada sonsuz bir maddenin bulunmaması matematiksel olarak şu şekilde yorumlanabilir; sonsuz sayılar ile sonlu sayılar birlikte düşünülüp matematiksel işlem uygulandığında sonsuz sayılar sonlu sayıları yok etmelidir ve sonlu sayıların kalıcılığı imkânsız hale gelir. Bu işlemde elde ettiğimiz sonuç sonsuz sayının kendisinden farklı olmayacaktır çünkü sonsuz sayı kendi içerisinde sonlu sayıyı ortadan kaldırmak zorundadır. Aristoteles için bu durum şu anlama gelir; sonsuz sayıları saymanın mümkün olduğunu varsaysak bile sonsuz sayılara herhangi bir matematiksel işlem uygulamak imkansızdır. Sonuç olarak, Aristoteles için bu iki matematiksel argümanı ne doğada sonsuz bir maddenin ne de matematikte sonsuz bir sayının var olmayacağını garanti eder. Bu nedenlerden dolayı, Aristoteles yalnızca potansiyel sonsuzluğun varlığını kabul ederek, aktüel sonsuzluğun varlığını reddetmiştir.

Aristoteles'in yaptığı potansiyel ve aktüel sonsuzluk ayrımının etkisiyle tarihsel süreçte aktüel sonsuzluğun imkansızlığı reddedilemez bir argüman olarak görülmüş ve Cantor'a kadar hiçbir düşünür bu argümanları başarılı bir şekilde çürütememiştir. Cantor ise üzerinde çalıştığı bir makale sırasında reel sayılar kümesinin doğal sayılar kümesinden daha yüksek bir sonsuzluk derecesine sahip olması gerektiğini fark etmiştir. Bu durumun nedeni ise bu iki küme arasında

herhangi bir şekilde bire bir eşleşme söz konusu olamamasıdır. Cantor bu argümandan yola çıkarak farklı sonsuz kümeler olduğunu iddia etmiştir. Hatta bu sonsuz kümelere sayılar atfedilebileceğini ve bu sayılarında yeni bir matematik sistemi oluşturacağını savunmuştur. Fakat bunun için alışılmış sayı kavramının genişletilmesi gerektiğini belirterek yeni bir yöntem ortaya atmıştır. Sonlu ötesi sayılar ve sonlu ötesi aritmetik bu şekilde ortaya çıkmıştır. Bu noktada sayı kavramına yeni yaklaşımın temeli iyi sıralı küme kavramıdır. İyi sıralı küme kavramı ile sonlu ötesi sayıların oluşturulması mümkün hale gelmiştir çünkü sayılar artık büyüklükleri üzerinden değil sıralanış biçimleri üzerinden numaralandırılırlar. Örneğin, iki farklı sıralı kümeyi  $\{1, 3, 5, 7\}$  ve  $\{7, 5, 3, 1\}$  düşünelim. Her ikisinin de son ögesi kümedeki dördüncü öge olduğu için bu iki küme de 4 sayısı ile numaralandırılırlar. İyi sıralı küme kavramı ile Cantor farklı sıralanış biçimlerine sahip olan sonsuz kümelerin numaralandırılabilceğini ve bu numaraların farklı büyüklükteki sonsuzluklara matematiksel olarak denk geleceğini iddia etmiştir. Bu yeni sayılara sonlu ötesi ordinal sayılar demesinin nedeni temelde budur.

Bu sayıların matematiksel olarak kurulumunu sağlayan iki prensip vardır ve Cantor bu prensiplere sayı üretme prensipleri olarak adlandırmıştır. Birinci prensip temel olarak alışılmış sayma yöntemine denk gelmektedir, yani verilen herhangi bir sayıya 1 eklenerek bir sonraki sayıyı tanımlamamızı sağlar. Bu prensip ile doğal sayılar kümesi potansiyel olarak sonsuz elemana sahiptir ve bütün elemanları belirlemek mümkün değildir. İkinci prensip ise doğal sayıların dizisini tamamlanmış bir küme olarak algılamamıza izin verir ve ardından yeni bir sayı “ $\omega$ ” (omega) doğal sayılar kümesinin tamamını numaralandırmak için atanır. Bunu yapabilmemizi

sağlayan şey bu kümenin elemanlarının her birinin kendinden önce gelen sayı üzerinden tanımlanarak bütün kümenin belirli bir sıraya sahip olmasıdır. Bu yeni sayı bütün doğal sayılardan daha büyüktür çünkü tüm doğal sayıların sayısı olarak bütün sonlu sayıları aşmaktadır. Başka bir deyişle, bu yeni sayı sıralama olarak bütün doğal sayılardan daha sonraki bir sırada yer alır ve bu onu herhangi bir sonlu sayıdan daha büyük yapar. Aynı nedenle bu sayı sonlu bir sayı olamaz, aksine sonlu ötesi bir sayıdır. Sonlu ötesi sayıların her biri farklı sıralanışa sahip olan sonsuz kümeleri numaralandırmak için kullanılırlar ve bu sayılar ile sonsuz kümeleri saymak mümkün hale gelmiştir. Aslında “ $\omega$ ” en küçük sonlu ötesi ordinal sayıdır çünkü bu iki prensip kullanılarak sonsuz sayıda sonlu ötesi üretmek mümkündür:  $\omega, \omega + 1, \omega + 2, \dots, \omega + \omega = \omega \cdot 2, \omega \cdot 2 + 1, \omega \cdot 2 + 2, \dots, \omega \cdot 3, \dots, \omega \cdot 4, \dots, \omega \cdot 5, \dots, \omega \cdot \omega = \omega^2, \dots, \omega^3, \dots, \omega^\omega, \dots, \omega^{\omega^\omega}, \dots$

Daha önce belirttiğim gibi Aristoteles’in yaptığı potansiyel sonsuzluk ve aktüel sonsuzluk ayrımı sonsuzluk düşüncesini temelden etkilemiş ve birçok düşünür benzer nedenlere dayanarak aktüel sonsuzluğun varlığını reddetmiştir. Dolayısıyla Cantor, Aristoteles’in argümanlarının ne kadar büyük bir öneme sahip olduğunu bilerek kendi teorisini ortaya atmış ve aynı zamanda Aristoteles’in sunduğu her bir argümana karşı bir argüman getirmiştir. Bu argümanları incelemeyen önce kısaca bahsetmek isterim ki, Cantor'a göre Aristoteles'in aktüel sonsuzluğun varlığına karşı argümanları mantıksal bir yanılığa tabidir. Bunun nedeni, Aristoteles'in tüm sayıların sonlu sayılarla sayılabilir olması gerektiği varsayımına dayanarak sonsuz sayıların olmadığı fikrini savunmasıdır. Cantor burada Aristoteles'in argümanlarındaki mantıksal hatayı göstermekle kalmayıp aynı zamanda bu

argümanların her birine karşıt bir argüman ortaya sunmuştur. Bunu yapabilmesini sağlayan durum ise sonlu ötesi sayılar ve sonlu ötesi aritmetiğin varlığıdır.

İlk karşıt argüman olarak Cantor, Aristoteles'in savunduğunun aksine, sonsuz elemana sahip olan setlerin de numaralandırılabilceğini ve bu numaraların sonsuz ya da sonlu ötesi sayılar olduğunu savunmuştur. Bu noktada farklı bir sayma metodunun kullanıldığı unutulmamalıdır.  $\omega$  doğal sayılar kümesinin sayısıdır  $\{1, 2, 3, \dots\}$  ve ilk sonlu ötesi sayıdır. Ayrıca  $\omega$  kendisinden sonra gelen bir başka sonlu ötesi sayıya sahiptir, yani  $\omega + 1$ . Cantor matematiksel olarak sonlu ötesi sayıların aynı anda denk geldikleri kümelerdeki bütün sayılara karşılık geldiğini ve bu sayede sonsuz kümelerin numaralandırılması için kullanılacaklarını iddia etmiştir. Bu şekilde Aristoteles'in argümanlarında savunulunun aksine sonsuz kümeleri saymanın matematiksel olarak mümkün olduğunu göstermekle kalmayıp, bunu yapabilmek için iyi sıralı kümeler kavramı ile bu sayıların matematiksel olarak sağlam bir temele dayandığını kanıtlamıştır.

Aristoteles sunduğu ikinci argümanda sonsuz sayıların varlığını kabul etmiş olsak bile sonsuz sayıları ve sonlu sayıları birlikte matematiksel işleme tabi tutamayacağımızı çünkü sonlu sayıların sonsuz sayılar tarafından yok edileceğini savunmuştur. Bunun aksine Cantor, sonlu sayıların yok oluşu olmaksızın sonlu ötesi sayılar ile matematiksel işlemlere tabi olabileceğini matematiksel olarak göstermiştir. Bu aritmetikte sonlu sayılar aritmetiğinden farklı kurallar geçerlidir çünkü sonlu ötesi sayılar sıralamanın önem arz ettiği iyi sıralı kümeler kavramından yola çıkarak oluşturulmuştur. Örneğin,  $\omega + 1$  sayısı ile  $1 + \omega$  sayısı birbirlerine eşit değillerdir. Bu iki farklı durumun nedeni bu işlemde karşımıza çıkan 1 sayısının  $\omega$  sayısının

sıralanışında farklı bir pozisyon alması ve iki farklı sonsuz küme oluşturarak, farklı bir şekilde numaralandırılmaya neden olmasıdır. Başka bir deyişle  $\omega + 1$  sayısındaki 1 kendisini sonsuz elemana sahip olan kümenin sonuncu sırasındaki eleman olarak konumlandığı için  $\omega + 1$  ile  $1 + \omega$  birbirine eşit değildir. Aynı zamanda  $1 + \omega$  sayısı matematiksel olarak  $\omega$  sayısına eşittir çünkü ikisi de numaralandırılmayı sağlayan son elemana sahip değildir ve 1 sayısı aldığı pozisyonundan dolayı  $\omega$  sayısı tarafından yok edilmektedir. Bu sayılar arasındaki ayrım sonlu sayıların mutlaka sonlu sayılar tarafından yok edilmeyeceğini garanti eder. Sonlu bir sayıya herhangi bir sonlu ötesi sayı eklenirse sonlu yok edilecektir. Ancak, sonlu bir sayı bir sonlu sayıya eklenirse, işlemin sonucu yeni bir sonlu ötesi sayıya neden olacaktır. Dolayısıyla, Aristoteles'in savunduğunun aksine aritmetik işlemler uygun sıralama ile sonlu ötesi sayılara da uygulanabilirler ve bu aynı zamanda sonlu ötesi aritmetiğin sonlu aritmetikten ne kadar farklı kurallara sahip olduğunu göstermektedir.

Sonlu aritmetiğin kuralları bize sonlu ötesi sayıların sonlu sayılar kadar belirgin nesnel bir gerçekliğe dayandığının kanıtıdır. Bu noktada, Cantor'un sonlu ötesi sayıların varlığına olan güçlü bağlılığı ona geleneksel sonsuzluk anlayışını çürütmek için bir yol sağlamıştır. Cantor'un sonsuzluk felsefesini incelediğimizde, potansiyel sonsuzluk kavramının tam olarak Aristoteles'in savunduğu gibi olduğu görülebilir. Fakat bu iki düşünürü birbirinden ayıran farklılık ise Cantor'un matematiğe tanıttığı yeni sistem içerisindeki sonlu ötesi sayılar aktüel sonsuzluğun örnekleridir. Potansiyel sonsuzluk kavramı üzerinden baktığımızda doğal sayılar kümesindeki her bir eleman sonlu birer sayıdır ve bu sayıların oluşturdukları seri potansiyel olarak sonsuz bir seridir. Fakat sonlu ötesi sayılar kendisini oluşturan tüm

sayıları aynı anda içerir. Sonlu ötesi sayılar kendilerini sonsuza kadar devam eden değişkenler olarak göstermezler, bunun yerine bir bütünlük oluştururlar ve bu bütünlük Cantor'un sonsuzluk felsefesinde aktüel sonsuzluk olarak adlandırılmıştır. Bu açıdan, sonlu matematiğin dogmalarına karşı olan ve Cantor'un başarıyla gösterdiği sonsuz kümelerin matematiğe kazandırılması sonsuzluk kavramı tarihinde bir devrim niteliği taşımaktadır.

Tezimin ikinci kısmı ise Cantor'un sonsuzluk felsefesi içerisinde matematiksel nesnelerin varlığını doğrulamak adına savunduğu felsefi ve matematiksel arka plana adanmıştır. Cantor'un sonlu ötesi sayıları savunma stratejisi matematiğin özgürlüğü prensibine dayanmaktadır. Bu prensibe göre matematik kendi nesnelerini oluşturma noktasında özgürdür çünkü matematik için yeni nesneler halihazırda var olan daha önceki başka matematiksel nesneler üzerinden tanıtılabilir. Yeni nesneler ile eski nesneler arasındaki ilişkiler ve tanımlar birbiri ile çelişmediği sürece yeni nesnelerin varlığı kolayca kabul edilebilir. Dolayısıyla, Cantor'a göre, tutarlı bir aritmetik sistem içerisinde matematik yeni nesneler yaratmakta özgürdür ve bu özgürlük matematiği diğer bütün bilimlerden ayrı bir noktaya koyar. Matematiğin özgürlüğü, tanımlar ve ilişkiler açısından yeni kavramların varlığını sağladığı için sonlu ötesi sayılar sayı üretme prensipleri üzerinden oluşturulmuştur. Bu noktada her bir sayı üretme prensibi sonlu sayılar ile sonlu ötesi sayılar arasındaki ilişkide önemli bir role sahiptir. Cantor yeni sayıların matematiksel varoluşlarını haklı çıkarmak adına yeni sayıları doğal sayıların kümesi üzerinden oluşturmuştur. Bu sayede hem sonlu sayıları hem de sonlu ötesi sayıları içeren yeni

matematikselsistemintutarlılıđı sayı üretme prensipleri ile korunur ve bu iki farklı sayı çeşidi arasındaki matematiksel ilişki tutarlı bir şekilde gösterilmiştir.

Cantor'un teorisinin bir başka önemli yönü de içkin gerçeklik ile bireyin zihinden bağımsız olarak var olan gerçekliği birbirinden ayırmasıdır. Kısaca bahsetmek gerekirse, kavramların içkin gerçekliği daha önceden tanımlanmış nesnelere olan ilişkileri ile ilgilidir. Cantor'a göre eski kavramlar ile yeni kavramlar arasındaki iç tutarlılık yeni kavramın olası bir varlık olarak bir gerçekliğe karşılık geldiğini iddia etmiştir. Ancak bu gerçekliğin her zaman fiziksel yansıması olacağı anlamına gelmez. Aksine muhtemel bir fikir olarak her zaman içkin tarafta bir varlığa sahip olacaktır. Bireyin zihinden bağımsız olarak var olan gerçeklik ise bir nesnenin doğal dünyadaki fiziksel tezahürü ile ilgilidir. Cantor'a göre matematiksel nesnelere düşünülduğünde bu iki taraf birbirinden ayırt edilemez çünkü matematiksel nesnelere soyut varlıklardır ve tezahürleri ancak tanımlarda ve diğer nesnelere olan ilişkilerde ortaya çıkabilir. Dolayısıyla, Cantor'un matematiğinin yalnızca kavramların içkin gerçekliği ile ilgilenmesi gerektiği düşüncesi bu argümana dayanmaktadır. Bunun nedeni matematikte içkin olarak var olan herhangi bir fikir, Cantor için her zaman olası bir fikir olarak gerçektir ve insan zihninden bağımsız olarak da bir gerçekliğe sahip olacaktır. Bu bağlamda sonlu ötesi sayıların sayı üretme prensipleri temelinde inşa edilmesi ve kendilerine ait bir aritmetik sistem oluşturmaları bu yeni sayıların varlığını Cantor'un felsefesi içerisinde haklı çıkarmaktadır. Çünkü önemli olan nokta bu iki farklı sayı çeşidi arasındaki bütünleşme ve tutarlılık sayesinde yeni sayılar zihnimizde yerini temin etmektedir.



Cantor'un kümeler teorisi küme kavramına bir sınırlama getirmediği için çeşitli matematikçiler ve filozoflar tarafından eleştirilmiştir. Fakat sanılanın aksine Cantor çoklukları birbirinden ayırarak, hangi çoklukların matematiksel bir küme olarak ifade edilebileceğini ve hangi çoklukların matematiksel bir küme oluşturamayacağını sınıflandırmasını yapmıştır. Bu sınıflandırmanın temeli iyi sıralı küme kavramı içerisinde yatmaktadır. Cantor'a göre çokluklar ikiye ayrılır: matematiksel kümeler olarak karşımıza çıkan tutarlı çokluklar ve tutarsız çokluklar. Tutarlı çokluklar matematiksel olarak gösterilebilen ve çelişkiye sahip olmayan çokluklardır. Başka bir deyişle, bu tür çoklukların öğeleri birbirinden ayrı düşünülebilir ve aynı zamanda bu öğelerin oluşturduğu bütünlüğü matematiksel olarak kavramak mümkündür. Örneğin, doğal sayılar kümesi tutarlı bir çokluktur; kümenin bütün öğeleri farklı özelliklere sahiptir ve aynı zamanda matematiksel bir bütünlük " $\omega$ " oluştururlar.

Tam tersine, tutarsız çokluklar matematiksel bir bütünlük oluşturamazlar çünkü buldukları ayrı öğelerin bütünlüğünü matematiksel olarak formalize etmek imkansızdır. Bu tarz çokluklar düşünce temelinde bir fikre karşılık gelebilirler, fakat matematiksel olarak bir küme oluşturamazlar. Bu durumun iki nedeni vardır. İlk olarak tutarlı çokluklar kendilerini aynı zamanda bir eleman olarak içeremezler çünkü aksi durumda elemanlarını birbirine bağlayan özellik kendi içerisinde çelişki ortaya çıkartarak bütünlüğün matematiksel olarak formalize edilmesine engel olur. Bir diğer neden ise bu tür çoklukların numaralandırılmalarının imkansızlığıdır. Her iyi sıralı küme bir ordinal sayıya karşılık gelmek zorundadır. Tutarsız çokluklar herhangi bir ordinale karşılık gelemezler çünkü çokluğun kendisinin ilk başta ifade

ettiği koşul öğeleri tarafından karşılanamaz ve matematiksel olarak belirsiz hale gelirler. Dolayısıyla bu tarz çoklukları numaralandırmak imkânsız hale gelir çünkü onları iyi sıralı kümeler biçimine dönüştürmek imkânsızlaşır ve matematiksel kümeler oluşturamazlar. Örneğin, Burali-Forti tarafından ortaya atılan paradoksa göre bütün ordinallerin kümesi de bir ordinal olmak zorundadır. Fakat bu ordinal bütün ordinals kümesinin içerisinde yer alamaz çünkü kümenin kendisine atanan ordinal kümenin içerisinde yer alamaz. Tam olarak aynı nedenden dolayı, Cantor'un yaptığı ayrımaya göre bu tutarsız bir çokluktur ve matematiksel olarak formüle edilemez. Cantor'un küme teorisinde paradoksal kümelerin ortaya çıkması yapılan bu ayrım ile engellenmiştir ve yalnızca iyi sıralı kümeler haline dönüştürülebilen çokluklar teorisinde matematiksel kümeler olarak kabul edilirler. Sonuç olarak Cantor'un küme teorisi ne sezgisel ne de paradoksaldır. Tutarlı çokluklar ve tutarsız çokluklar ayrımı içerisinde savunulan bütün argümanlar alıntılar eşliğinde bölüm 3.3. içerisinde yer almaktadır.

Sonlu ötesi sayılar düşünüldüğünde, Cantor sayı kavramını sonsuz kümeleri de kapsayacak şekilde genişleterek kendine özgü belirli kuralları olan yeni bir tür aritmetik sistem inşa etmiştir. Örneğin, bu aritmetikte toplama işleminin ve çarpma işleminin değişme özelliği yoktur çünkü sonsuz kümeler farklı sıralamalara sahip olduklarında farklı şekilde numaralandırılırlar. Bu nedenle, sonlu ötesi sayıların aritmetiğinde matematiksel işlemin sonucunu sayıların sıralamaları belirler. Fakat burada vurgulanması gereken şey şudur ki; sonsuz kümelerin hiyerarşisi hem büyüklük olarak hem de sıralanış olarak farklılık göstermektedir. Cantor'un bu iki açıdan farklı matematiksel kurallar etrafında ortaya koyduğu aritmetik sistemleri

bölüm 3.4. içerisinde benzerlikleri ve farklılıkları ile detaylıca anlatılmaktadır. Söz edilen bölümdeki analiz yeterince göstermiştir ki, bir kümenin büyüklüğü sahip olduğu eleman sayısı ile ilgili iken, bir kümenin ordinal sayısı kümedeki elemanların sıralamasıyla ilişkilidir. Bunun sonucu olarak ortaya çıkan sonlu ötesi sayıların, yani bir anlamda aktüel sonsuzlukların, matematiksel uygulamaları ve tutarlı teorisi hem matematiksel hem de felsefi olarak göz ardı edilemez.

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