GENERALIZED KERR-SCHILD METRICS IN EINSTEIN'S GRAVITY

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ABSTRACT

GENERALIZED KERR-SCHILD METRICS IN EINSTEIN‘S GRAVITY

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Several aspects of the Kerr-Schild ansatz; including the generalized ansatz which has a generic background metric instead of a flat metric, Kerr-Schild groups and Classical Double Copy, are investigated. It is shown that, in vacuum, not all generalized Kerr-Schild spacetimes are algebraically special, whereas the Kerr-Schild spacetimes are always algebraically special. We also touch upon Kerr-Schild-Kundt spacetimes and further discuss their universality property before we continue with the Kerr-Schild groups in the next chapter where we work on the coordinate transformations leading to the generalized Kerr-Schild ansatz. Finally, we study the classical double copy proposition which relates some solutions of gravity theories with the solutions of gauge theories. To this end, we examine the correspondence between solutions of the Kerr-Schild form and the Yang-Mills solutions.

Keywords: Null Congruence, Kerr-Schild Metrics, Kerr-Schild-Kundt Metrics, Algebraic Classification of Spacetimes, Kerr-Schild Groups, Classical Double Copy

ÖZ

EİNSTEİN KÜTLEÇEKİM TEORİSİNDE KERR-SCHİLD METRİKLERİ

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Düz bir uzayzaman yerine herhangi bir uzayzaman ile oluşturulan genelleştirilmiş yaklaşım-formu, Kerr-Schild grupları ve Klasik Çift Nüsha da dâhil olmak üzere Kerr-Schild yaklaşık-formun birçok yönü araştırıldı. Kerr-Schild uzayzamanların cebirsel olarak özel olduğu gösterilirken, bütün genelleştirilmiş Kerr-Schild uzayzamanların cebirsel olarak özel olmadığı gösterildi. Ayrıca, Kerr-Schild-Kundt uzayzaman sınıfına da değinildi ve evrensellik özelliğinden bahsedildi. Daha sonra ise genelleştirilmiş Kerr-Schild metriklerinin koordinat dönüşümü sonucu olarak ortaya çıkabileceği sebebiyle Kerr-Schild grupları çalışıldı. Son olarak ise bazı kütleçekim teorileri çözümleri ile ölçü teorileri çözümleri arasında bir bağlantı öne sürmüş olan Klasik Çift Nüsha hesaplarını çalıştık. Bu nedenle Kerr-Schild formundaki çözümler ile Yang-Mills çözümleri arasındaki bağlantıyı araştırdık.

Anahtar Kelimeler: Işık-gibi Eğriler Topluluğu, Kerr-Schild Metrikleri, Kerr-Schild-Kundt Metrikleri, Uzayzamanların Cebirsel Sınıflandırılması, Kerr-Schild grupları,

Klasik Çift Nüsha

To my family

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GR	General Relativity
KS	Kerr-Schild
KSK	Kerr-Schild-Kundt
PND	Principal Null Direction
KSVF	Kerr-Schild Vector Field
LHS	Left Hand Side
RHS	Right Hand Side

CHAPTER 1

INTRODUCTION

As the field equations of general theory of relativity had been found, the quest for finding solutions began. Almost a month later, Karl Schwarzschild discovered the first nontrivial solution which describes the exterior gravitational field of spherically symmetric mass with no charge. A couple of years later, the solution for a spherically symmetric mass with charge, called the Reissner-Nordström solution, was found. Later, for nearly 50 years, a metric that describes the asymptotically vanishing gravitational field of an axially symmetric rotating stationary mass with no charge had been sought for, but to no avail until it was found by Kerr in 1963[1][2]. This discovery was made possible by a method of classification of spacetimes by studying the algebraic character of the Weyl tensor, developed by Petrov in 1954[3]. A few years after this development, starting with Pirani and then by many other physicists, including Bel, Debever, Penrose and Sachs, this classification was studied and formulated in equivalent ways which turned out to be quite useful to find new exact solutions of the field equations. Two years after the discovery of the Kerr metric, its charged version, namely the Kerr-Newman metric, was found and the fact that all of these very important solutions could be written in a simple form, called Kerr-Schild(KS) form[4], was realized. As a matter of fact, this form of the metric was already given for the Schwarzschild metric in 1924[5], but the importance of the underlying form went unnoticed. The fact that KS metrics lead pseudo energy-momentum tensors to vanish and linearize the field equations, not necessarily in vacuum, was shown by Gürses and Gürsey[6]. The KS ansatz was generalized to have a generic background spacetime instead of flat spacetime by Xanthopoulos[7],[8] and the generalized ansatz was also studied in details by Taub[9].

In this thesis, we start off with a discussion of the properties of the the KS metrics and then proceed to calculate the transformations of the the Christoffel symbols and the curvature tensors. We then delve into the field equations in vacuum for the generalized KS ansatz, which give restrictions on the profile function and the null vector of the ansatz. Next, we consider the general situation of nonvanishing energy-momentum tensor and choose a different ansatz with a null geodesic vector that leads to the linearization of the Ricci tensor. Intersection of the Kundt class of metrics and the KS metrics, called the Kerr-Schild-Kundt(KSK) metrics, is also investigated and the chapter ends with a discussion of KS forms of the abovementioned solutions of utmost importance. In the next chapter, we give a treatment of the KS ansatz from a mathematical point of view[10][11] without the restrictions of the field equations. We investigate the KS symmetries of a given metric in a similar way that the isometries and conformal transformations are studied and to this end, we consider the group of transformations that change the metric tensor in a way such that the transformed metric is the generalized KS metric. In the last chapter, the proposed double copy correspondence between a solution of the field equations of general relativity in the KS form and a solution of Abelian gauge theory is discussed at the classical level[12][13][14]. We show that a procedure similar to BCJ duality[15] yields the covariant Maxwell's equations from the field equations of general theory of relativity. Gauge theory solution is seen as a single copy of gravity solution whereas the biadjoint scalar field theory solution is seen as the zeroth copy.

The conventions used in this work are the following

$$\begin{aligned}
&\eta = diag(-1,1,1,1)\\
&R^{\mu}{}_{\nu\sigma\rho} = \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\sigma\beta}\Gamma^{\beta}{}_{\nu\rho} - (\sigma \leftrightarrow \rho)\\
&R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}\\
&G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}
\end{aligned} \tag{1.1}$$

Lastly, greek indices are used for spacetime coordinates whilst the latin indices are used for spatial coordinates.

CHAPTER 2

GENERALIZED KERR-SCHILD METRICS

Although the KS form of the Schwarzschild metric was already written down in 1924, its importance was realized with the discovery of the Kerr metric which turns out to be manifestly in the KS form in the derivation. The derivation starts with considering a metric written in an orthonormal frame

$$g_{ab} = \eta_{ab}\sigma^a \otimes \sigma^b \tag{2.1}$$

and defining $\{k^a, l^a, m^a, \bar{m}^a\}$, where k and l are null real vectors and $m^a, \bar{m}^a$ are null complex vectors, by

$$\begin{aligned} k \cdot dx &= \frac{\sigma^0 + \sigma^1}{\sqrt{2}}, \quad l \cdot dx = \frac{\sigma^0 - \sigma^1}{\sqrt{2}} \\ m \cdot dx &= \frac{\sigma^2 + i\sigma^3}{\sqrt{2}}, \quad \bar{m} \cdot dx = \frac{\sigma^2 - i\sigma^3}{\sqrt{2}} \end{aligned} \tag{2.2}$$

Then, one obtains the following complex null tetrad

$$\begin{aligned} &g_{ab} = -k_a l_b - k_b l_a + m_a \bar{m}_b + m_b \bar{m}_a \\ &k^2 = l^2 = m^2 = \bar{m}^2 = 0 = k \cdot m = k \cdot \bar{m} = l \cdot m = l \cdot \bar{m} \\ &k \cdot l = -1 = -m \cdot \bar{m} \end{aligned} \tag{2.3}$$

Here k and l are null geodesic vectors; furthermore, these are the outgoing and in-going null rays. Essentially, by using the optical properties of the null rays and the Killing vectors, one can show that the Kerr metric has the KS form which is given by

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + 2V\lambda_\mu\lambda_\nu \quad where \\ \eta_{\mu\nu}\lambda^\mu\lambda^\nu &= g_{\mu\nu}\lambda^\mu\lambda^\nu = \lambda^2 = 0 \end{aligned} \tag{2.4}$$

The total metric, or the full metric, $g_{\mu\nu}$ is formed by adding a term $2V\lambda_\mu\lambda_\nu$ to a flat background metric $\eta_{\mu\nu}$ where V is a scalar function of spacetime coordinates and λ

is a null vector with respect to both metrics. Inverse metric is equal to

$$g^{\mu\nu} = \eta^{\mu\nu} - 2V\lambda^{\mu}\lambda^{\nu} \tag{2.5}$$

The metric $g_{\mu\nu}$ is exact in the sense that if we introduce $h_{\mu\nu} \equiv 2V\lambda_{\mu}\lambda_{\nu}$ then the higher order terms vanish due to the nullity of λ. It is also an algebraically special metric in vacuum by the Goldberg-Sachs theorem. If the energy-momentum tensor is not zero, then one can define λ to be a null geodesic vector in the ansatz and the metric $g_{\mu\nu}$ becomes algebraically special[6]. This also leads to the linearization of the field equations and vanishing of the pseudo energy-momentum tensor.

It is quite interesting that almost all fundamental solutions; including the Schwarzschild, Reissner-Nordström, Kerr, Kerr-Newman solutions, of the field equations of general theory of relativity can be written in such a simple form of the metric. In the next section, we will consider a generalized KS ansatz and the results for the KS ansatz can be seen as a special case $\bar{g} = \eta$.

2.1 Vacuum Field Equations

One way to generalize the KS ansatz is to replace the flat background spacetime with a generic background spacetime:

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + 2V\lambda_{\mu}\lambda_{\nu} \quad where \\ \bar{g}_{\mu\nu}\lambda^{\mu}\lambda^{\nu} &= g_{\mu\nu}\lambda^{\mu}\lambda^{\nu} = \lambda^2 = 0 \end{aligned} \tag{2.6}$$

The inverse metric for a generic vector, not necessarily null, can be calculated as follows

$$\begin{aligned} g^{\mu\nu} = \bar{g}^{\mu\nu} - \kappa\lambda^{\mu}\lambda^{\nu} \quad &\Rightarrow \quad g_{\mu\nu}g^{\nu\alpha} = \bar{g}_{\mu\nu}\bar{g}^{\nu\alpha} + 2V\lambda_{\mu}\lambda^{\alpha} - \kappa\lambda_{\mu}\lambda^{\alpha} - 2V\kappa\lambda^{\nu}\lambda^{\alpha}\lambda_{\mu}\lambda_{\nu} \\ &\Rightarrow \quad \kappa\lambda_{\mu}\lambda^{\alpha}(1 + 2V\lambda_{\nu}\lambda^{\nu}) = 2V\lambda_{\mu}\lambda^{\alpha} \end{aligned} \tag{2.7}$$

Then the inverse metric is

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \frac{2V\lambda^{\mu}\lambda^{\nu}}{1 + 2V\lambda_{\nu}\lambda^{\nu}} \tag{2.8}$$

An intriguing result follows when λ is a null vector,

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - 2V\lambda^{\mu}\lambda^{\nu} \tag{2.9}$$

that is, the inverse metric is also linear in V. Now, transformation of the Christoffel Symbols can be obtained starting from

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\nu g_{\rho\alpha} + \partial_\rho g_{\nu\alpha} - \partial_\alpha g_{\nu\rho}\right) \tag{2.10}$$

We have

$$\begin{aligned} \partial_\nu g_{\rho\alpha} - \bar{\Gamma}^{\sigma}{}_{\nu\rho} g_{\sigma\alpha} - \bar{\Gamma}^{\sigma}{}_{\nu\alpha} g_{\rho\sigma} &= \bar{\nabla}_\nu g_{\rho\alpha} \\ &= \bar{\nabla}_\nu \left(\bar{g}_{\rho\alpha} + 2V\lambda_\rho\lambda_\alpha\right) \\ &= 2\bar{\nabla}_\nu \left(V\lambda_\rho\lambda_\alpha\right) \end{aligned} \tag{2.11}$$

Note that there is no distinction between $\bar{\partial}$ and ∂ since there is no transformation of coordinates. Then,

$$\begin{aligned} \Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \Big[& 2\bar{\nabla}_\nu \left(V\lambda_\rho\lambda_\alpha\right) + \bar{\Gamma}^{\sigma}{}_{\nu\rho} g_{\sigma\alpha} + \bar{\Gamma}^{\sigma}{}_{\nu\alpha} g_{\rho\sigma} + 2\bar{\nabla}_\rho \left(V\lambda_\nu\lambda_\alpha\right) + \bar{\Gamma}^{\sigma}{}_{\rho\nu} g_{\sigma\alpha} \\ & + \bar{\Gamma}^{\sigma}{}_{\rho\alpha} g_{\nu\sigma} - 2\bar{\nabla}_\alpha \left(V\lambda_\nu\lambda_\rho\right) - \bar{\Gamma}^{\sigma}{}_{\alpha\nu} g_{\sigma\rho} - \bar{\Gamma}^{\sigma}{}_{\alpha\rho} g_{\sigma\nu} \Big] \end{aligned} \tag{2.12}$$

which, after cancellations, gives

$$\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + g^{\mu\alpha} \Delta_{\alpha\nu\rho} \tag{2.13}$$

Here, $\Delta_{\alpha\nu\rho}$ is defined as

$$\Delta_{\alpha\nu\rho} \equiv \bar{\nabla}_\nu (V\lambda_\rho\lambda_\alpha) + \bar{\nabla}_\rho (V\lambda_\nu\lambda_\alpha) - \bar{\nabla}_\alpha (V\lambda_\nu\lambda_\rho) \tag{2.14}$$

Transformation of the Riemann tensor can be obtained from the above result,

$$\begin{aligned} R^{\mu}{}_{\nu\sigma\rho} &= \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\sigma\beta} \Gamma^{\beta}{}_{\nu\rho} - (\sigma \leftrightarrow \rho) \\ &= \partial_\sigma \bar{\Gamma}^{\mu}{}_{\nu\rho} + \partial_\sigma (g^{\mu\alpha} \Delta_{\alpha\nu\rho}) + \bar{\Gamma}^{\mu}{}_{\sigma\beta} \bar{\Gamma}^{\beta}{}_{\nu\rho} + g^{\mu\alpha} \Delta_{\alpha\sigma\beta} \bar{\Gamma}^{\beta}{}_{\nu\rho} + g^{\beta\alpha} \Delta_{\alpha\nu\rho} \bar{\Gamma}^{\mu}{}_{\sigma\beta} \\ &\qquad + g^{\mu\alpha} g^{\beta\gamma} \Delta_{\alpha\sigma\beta} \Delta_{\gamma\nu\rho} - (\sigma \leftrightarrow \rho) \\ &= \bar{R}^{\mu}{}_{\nu\sigma\rho} + \Big[\partial_\sigma (g^{\mu\alpha} \Delta_{\alpha\nu\rho}) + g^{\mu\alpha} \Delta_{\alpha\sigma\beta} \bar{\Gamma}^{\beta}{}_{\nu\rho} + g^{\beta\alpha} \Delta_{\alpha\nu\rho} \bar{\Gamma}^{\mu}{}_{\sigma\beta} \\ &\qquad + g^{\mu\alpha} g^{\beta\gamma} \Delta_{\alpha\sigma\beta} \Delta_{\gamma\nu\rho} - (\sigma \leftrightarrow \rho) \Big] \end{aligned} \tag{2.15}$$

From this we can calculate the transformation of the Ricci tensor,

$$\begin{aligned} R_{\nu\rho} = \bar{R}_{\nu\rho} + \Big[& \partial_\sigma (g^{\sigma\alpha} \Delta_{\alpha\nu\rho}) - \partial_\rho (g^{\sigma\alpha} \Delta_{\alpha\nu\sigma}) + g^{\sigma\alpha} \Delta_{\alpha\sigma\beta} \bar{\Gamma}^{\beta}{}_{\nu\rho} - g^{\sigma\alpha} \Delta_{\alpha\rho\beta} \bar{\Gamma}^{\beta}{}_{\nu\sigma} \\ & + g^{\beta\alpha} \Delta_{\alpha\nu\rho} \bar{\Gamma}^{\sigma}{}_{\sigma\beta} - g^{\beta\alpha} \Delta_{\alpha\nu\sigma} \bar{\Gamma}^{\sigma}{}_{\rho\beta} + g^{\sigma\alpha} g^{\beta\gamma} (\Delta_{\alpha\sigma\beta} \Delta_{\gamma\nu\rho} - \Delta_{\alpha\rho\beta} \Delta_{\gamma\nu\sigma}) \Big] \end{aligned} \tag{2.16}$$

We can readily notice that

$$\begin{aligned} g^{\sigma\alpha} \Delta_{\alpha\nu\sigma} &= (\bar{g}^{\sigma\alpha} - 2V\lambda^\sigma\lambda^\alpha) \Delta_{\alpha\nu\sigma} = 0 \\ g^{\sigma\alpha} \Delta_{\alpha\sigma\beta} &= (\bar{g}^{\sigma\alpha} - 2V\lambda^\sigma\lambda^\alpha) \Delta_{\alpha\sigma\beta} = 0 \end{aligned} \tag{2.17}$$

where

$$\bar{g}^{\sigma\alpha}\Delta_{\alpha\nu\sigma} = 0 \quad since \quad \lambda_\alpha\lambda^\alpha = 0\,, \quad and$$
$$\lambda^\sigma\lambda^\alpha\Delta_{\alpha\nu\sigma} = 0 \quad since \quad \lambda^\alpha\bar{\nabla}_\nu(V\lambda_\alpha\lambda_\sigma) = V\lambda_\sigma\frac{1}{2}\bar{\nabla}_\nu(\lambda_\alpha\lambda^\alpha) = 0$$

Essentially, all the simplifications boil down to the nullity of λ. Using these results, we find

$$\begin{aligned} R_{\nu\rho} =& \bar{R}_{\nu\rho} + \partial_\sigma(g^{\sigma\alpha}\Delta_{\alpha\nu\rho}) - g^{\sigma\alpha}\Delta_{\alpha\rho\beta}\bar{\Gamma}^\beta{}_{\nu\sigma} + g^{\beta\alpha}\Delta_{\alpha\nu\rho}\bar{\Gamma}^\sigma{}_{\sigma\beta} - g^{\beta\alpha}\Delta_{\alpha\nu\sigma}\bar{\Gamma}^\sigma{}_{\rho\beta} \\ & - g^{\sigma\alpha}g^{\beta\gamma}\Delta_{\alpha\rho\beta}\Delta_{\gamma\nu\sigma} \end{aligned} \tag{2.18}$$

This can be further simplified by calculating the following terms;

- $$\begin{aligned} g^{\sigma\alpha}\Delta_{\alpha\nu\rho} &= \bar{g}^{\sigma\alpha}\Delta_{\alpha\nu\rho} - 2V\lambda^\sigma\lambda^\alpha\Delta_{\alpha\nu\rho} \\ &= \Delta^\sigma{}_{\nu\rho} + 2V\lambda^\sigma\lambda^\alpha\bar{\nabla}_\alpha(V\lambda_\nu\lambda_\rho) \end{aligned} \tag{2.19}$$

where we used $\lambda^\alpha\Delta_{\alpha\nu\rho} = -\lambda^\alpha\bar{\nabla}_\alpha(V\lambda_\nu\lambda_\rho)$. Next, using this result, we calculate

- $$\begin{aligned} g^{\sigma\alpha}g^{\beta\gamma}\Delta_{\alpha\rho\beta}\Delta_{\gamma\nu\sigma} &= \left(\Delta^\sigma{}_{\rho\beta} + 2V\lambda^\sigma\lambda^\alpha\bar{\nabla}_\alpha(V\lambda_\rho\lambda_\beta)\right)\left(\Delta^\beta{}_{\nu\sigma} + 2V\lambda^\beta\lambda^\gamma\bar{\nabla}_\gamma(V\lambda_\nu\lambda_\sigma)\right) \\ &= \Delta^\sigma{}_{\rho\beta}\Delta^\beta{}_{\nu\sigma} + 2V(\lambda^\alpha\bar{\nabla}_\alpha(V\lambda_\rho\lambda_\beta))\lambda^\sigma\Delta^\beta{}_{\nu\sigma} \\ &\quad + 2V(\lambda^\gamma\bar{\nabla}_\gamma(V\lambda_\nu\lambda_\sigma))\lambda^\beta\Delta^\sigma{}_{\rho\beta} \end{aligned}$$

where the term that is quartic in V vanishes since $\lambda^\sigma\bar{\nabla}_\gamma(V\lambda_\nu\lambda_\sigma) = 0$. Now, we also have

$$\lambda^\sigma\Delta^\beta{}_{\nu\sigma} = \lambda^\sigma\bar{\nabla}_\sigma\left(V\lambda_\nu\lambda^\beta\right) \quad and$$
$$\bar{\nabla}_\alpha\left(V\lambda_\rho\lambda_\beta\right)\bar{\nabla}_\sigma\left(V\lambda_\nu\lambda^\beta\right) = V^2\lambda_\nu\lambda_\rho(\bar{\nabla}_\alpha\lambda_\beta)(\bar{\nabla}_\sigma\lambda^\beta)$$

from which it follows that

$$g^{\sigma\alpha}g^{\beta\gamma}\Delta_{\alpha\rho\beta}\Delta_{\gamma\nu\sigma} = \Delta^\sigma{}_{\rho\beta}\Delta^\beta{}_{\nu\sigma} + 4V^3\lambda_\rho\lambda_\nu\lambda^\alpha\lambda^\sigma(\bar{\nabla}_\alpha\lambda_\beta)(\bar{\nabla}_\sigma\lambda^\beta)^1 \tag{2.20}$$

The quadratic term $\Delta^\sigma{}_{\rho\beta}\Delta^\beta{}_{\nu\sigma}$ simplifies as follows

$$\begin{aligned} \Delta^\sigma{}_{\rho\beta}\Delta^\beta{}_{\nu\sigma} =& \Big[\bar{\nabla}_\rho(V\lambda_\beta\lambda^\sigma) + \bar{\nabla}_\beta(V\lambda_\rho\lambda^\sigma) - \bar{\nabla}^\sigma(V\lambda_\rho\lambda_\beta)\Big]\Big[\bar{\nabla}_\nu(V\lambda_\sigma\lambda^\beta) + \bar{\nabla}_\sigma(V\lambda_\nu\lambda^\beta) \\ & - \bar{\nabla}^\beta(V\lambda_\nu\lambda_\sigma)\Big] \\ =& V^2\lambda^\sigma\lambda_\nu\bar{\nabla}_\rho\lambda_\beta\bar{\nabla}_\sigma\lambda^\beta - V^2\lambda_\beta\lambda_\nu\bar{\nabla}_\rho\lambda^\sigma\bar{\nabla}^\beta\lambda_\sigma + V^2\lambda^\beta\lambda_\rho\bar{\nabla}_\beta\lambda^\sigma\bar{\nabla}_\nu\lambda_\sigma \\ & \bar{\nabla}_\beta(V\lambda_\rho\lambda^\sigma)\bar{\nabla}_\sigma(V\lambda_\nu\lambda^\beta) - V^2\lambda_\nu\lambda_\rho\bar{\nabla}_\beta\lambda^\sigma\bar{\nabla}^\beta\lambda_\sigma - V^2\lambda_\rho\lambda_\sigma\bar{\nabla}^\sigma\lambda_\beta\bar{\nabla}_\nu\lambda^\beta \\ & - V^2\lambda_\rho\lambda_\nu\bar{\nabla}^\sigma\lambda_\beta\bar{\nabla}_\sigma\lambda^\beta + \bar{\nabla}^\sigma(V\lambda_\rho\lambda_\beta)\bar{\nabla}^\beta(V\lambda_\nu\lambda_\sigma) \\ =& 2\bar{\nabla}_\beta(V\lambda_\rho\lambda^\sigma)\bar{\nabla}_\sigma(V\lambda_\nu\lambda^\beta) - 2V^2\lambda_\nu\lambda_\rho\bar{\nabla}_\beta\lambda^\sigma\bar{\nabla}^\beta\lambda_\sigma \end{aligned} \tag{2.21}$$

[1] Henceforth, parantheses will not be used for products of two derivative terms unless there is a total derivative term.

Lastly, note that

- $$\begin{aligned}\partial_\sigma(g^{\sigma\alpha}\Delta_{\alpha\nu\rho}) &= (\partial_\sigma g^{\sigma\alpha})\Delta_{\alpha\nu\rho} + g^{\sigma\alpha}\partial_\sigma\Delta_{\alpha\nu\rho} \\ &= \left[\bar{\nabla}_\sigma g^{\sigma\alpha} - \bar{\Gamma}^\sigma{}_{\sigma\beta}g^{\beta\alpha} - \bar{\Gamma}^\alpha{}_{\sigma\beta}g^{\sigma\beta}\right]\Delta_{\alpha\nu\rho} \\ &\qquad + g^{\sigma\alpha}\left[\bar{\nabla}_\sigma\Delta_{\alpha\nu\rho} + \Delta_{\beta\nu\rho}\bar{\Gamma}^\beta{}_{\sigma\alpha} + \Delta_{\alpha\beta\rho}\bar{\Gamma}^\beta{}_{\sigma\nu} + \Delta_{\alpha\nu\beta}\bar{\Gamma}^\beta{}_{\sigma\rho}\right] \\ &= -2\bar{\nabla}_\sigma(V\lambda^\sigma\lambda^\alpha) - g^{\beta\alpha}\Delta_{\alpha\nu\rho}\bar{\Gamma}^\sigma{}_{\sigma\beta} + \bar{\nabla}^\alpha\Delta_{\alpha\nu\rho} - 2V\lambda^\sigma\lambda^\alpha\bar{\nabla}_\sigma\Delta_{\alpha\nu\rho} \\ &\quad + g^{\sigma\alpha}\Delta_{\alpha\beta\rho}\bar{\Gamma}^\beta_{\sigma\nu} + g^{\sigma\alpha}\Delta_{\alpha\nu\beta}\bar{\Gamma}^\beta_{\sigma\rho}\end{aligned} \tag{2.22}$$

Hence, after the cancellation of connection coefficients, we obtain

$$\begin{aligned}R_{\nu\rho} &= \bar{R}_{\nu\rho} + \bar{\nabla}_\alpha\Delta^\alpha{}_{\nu\rho} - 2V\lambda^\sigma\lambda^\alpha\bar{\nabla}_\sigma\Delta_{\alpha\nu\rho} - 2\bar{\nabla}_\sigma(V\lambda^\sigma\lambda^\alpha)\Delta_{\alpha\nu\rho} - \Delta^\sigma{}_{\rho\beta}\Delta^\beta{}_{\nu\sigma} \\ &\quad - 4V^3\lambda_\rho\lambda_\nu\lambda^\alpha\lambda^\sigma(\bar{\nabla}_\alpha\lambda_\beta)(\bar{\nabla}_\sigma\lambda^\beta) \\ &= \bar{R}_{\nu\rho} + \bar{\nabla}_\alpha\Delta^\alpha{}_{\nu\rho} - 2\bar{\nabla}_\sigma(V\lambda^\sigma\lambda^\alpha\Delta_{\alpha\nu\rho}) - 2\bar{\nabla}_\beta(V\lambda_\rho\lambda^\sigma)\bar{\nabla}_\sigma(V\lambda_\nu\lambda^\beta) \\ &\quad 2V^2\lambda_\nu\lambda_\rho\bar{\nabla}_\beta\lambda^\sigma\bar{\nabla}^\beta\lambda_\sigma - 4V^3\lambda_\rho\lambda_\nu\lambda^\alpha\lambda^\sigma(\bar{\nabla}_\alpha\lambda_\beta)(\bar{\nabla}_\sigma\lambda^\beta)\end{aligned} \tag{2.23}$$

For the class of solutions which have the mass term present in the scalar function V one can write the vacuum field equations, $R_{\nu\rho} = 0$, order by order;

- (V^2) $\quad \bar{\nabla}_\sigma(V\lambda^\sigma\lambda^\alpha\Delta_{\alpha\nu\rho}) + \bar{\nabla}_\beta(V\lambda_\rho\lambda^\sigma)\bar{\nabla}_\sigma(V\lambda_\nu\lambda^\beta) - V^2\lambda_\nu\lambda_\rho\bar{\nabla}_\beta\lambda^\sigma\bar{\nabla}^\beta\lambda_\sigma = 0$
- (V^3) $\quad V^3\lambda_\rho\lambda_\nu\lambda^\alpha\lambda^\sigma(\bar{\nabla}_\alpha\lambda_\beta)(\bar{\nabla}_\sigma\lambda^\beta) = 0$

(2.24)

Now since the mass of the background spacetime might be nonzero, we cannot claim that the terms of order V^0 and V^1 must be zero. Instead, we will take the background spacetimes to be vacuum solutions, $\bar{R}_{\nu\rho} = 0$, and then for order V^1 we will have

- (V^1) $\quad \bar{\nabla}_\alpha\Delta^\alpha{}_{\nu\rho} = 0$

Next, order V^3 gives

$$\lambda^\alpha(\bar{\nabla}_\alpha\lambda_\beta)\lambda^\sigma(\bar{\nabla}_\sigma\lambda^\beta) = 0 \quad \Longrightarrow \quad \lambda^\mu\bar{\nabla}_\mu\lambda^\nu \quad is\ a\ null\ vector\ field \tag{2.25}$$

Note that we have

$$\lambda_\nu\lambda^\mu\bar{\nabla}_\mu\lambda^\nu = 0 \tag{2.26}$$

which is the orthogonality of λ and $\bar{\nabla}_\lambda\lambda$. The fact that $\bar{\nabla}_\lambda\lambda$ is null and orthogonal to λ implies that it must be a multiple of λ; that is,

$$\lambda^\mu\bar{\nabla}_\mu\lambda^\nu = \tilde{c}\lambda^\nu \quad for\ some\ constant\ \tilde{c} \tag{2.27}$$

The above equation describes a geodesic without an affine parameter, but it can be reparametrized to have the RHS equal to zero. Therefore, the null vector in the ansatz is also a geodesic vector for vacuum solutions. Next, instead of attempting to directly solve order V^1, which is a rather complicated set of equations to solve for V and λ, one can contract these equations and obtain some simpler results:

Contracting $\bar{\nabla}_\alpha \Delta^\alpha_{\ \nu\rho} = 0$; with $\bar{g}^{\nu\rho}$ gives

$$(i) : \bar{\nabla}_\alpha(\lambda^\alpha \bar{\nabla}_\nu(V\lambda^\nu) + \tilde{c}V\lambda^\alpha) = 0$$

and with λ^ν gives

$$\begin{aligned}(ii) : \bar{\nabla}_\alpha(\lambda^\alpha\lambda_\rho\lambda^\nu\bar{\nabla}_\nu V + 2V\tilde{c}\lambda^\alpha\lambda_\rho) &= (\bar{\nabla}_\alpha\lambda^\nu)\big[-V\lambda_\rho\bar{\nabla}^\alpha\lambda_\nu + \bar{\nabla}_\nu(V\lambda^\alpha\lambda_\rho) \\ &\qquad + V\lambda^\alpha\bar{\nabla}_\rho\lambda_\nu\big] \\ &= V\lambda_\rho(\bar{\nabla}_\alpha\lambda^\nu)(-\bar{\nabla}^\alpha\lambda_\nu + \bar{\nabla}_\nu\lambda^\alpha) \\ &\quad + \lambda^\alpha(\bar{\nabla}_\alpha\lambda^\nu)\bar{\nabla}_\nu(V\lambda_\rho) + \tilde{c}V\lambda^\nu\bar{\nabla}_\rho\lambda_\nu \\ &= V\lambda_\rho(\bar{\nabla}_\alpha\lambda^\nu)(\bar{\nabla}_\nu\lambda^\alpha - \bar{\nabla}^\alpha\lambda_\nu) + \tilde{c}\lambda^\nu\bar{\nabla}_\nu(V\lambda_\rho)\end{aligned} \tag{2.28}$$

Corollary (ii) can be further simplified by calculating the LHS

$$\begin{aligned}&\lambda_\rho\lambda^\nu\bar{\nabla}_\alpha(\lambda^\alpha\bar{\nabla}_\nu V) + 2\tilde{c}\lambda_\rho\lambda^\nu\bar{\nabla}_\nu V \\ &\quad + 2\tilde{c}\lambda_\rho\bar{\nabla}_\alpha(V\lambda^\alpha) + 2V\tilde{c}^2\lambda_\rho = V\lambda_\rho(\bar{\nabla}_\alpha\lambda^\nu)(\bar{\nabla}_\nu\lambda^\alpha - \bar{\nabla}\lambda_\nu) + \tilde{c}\lambda_\rho\lambda^\nu\bar{\nabla}_\nu V + \tilde{c}^2 V\lambda_\rho\end{aligned} \tag{2.29}$$

which yields

$$\lambda^\nu\bar{\nabla}_\alpha(\lambda^\alpha\bar{\nabla}_\nu V) + \tilde{c}\lambda^\nu\bar{\nabla}_\nu V + 2\tilde{c}\bar{\nabla}_\alpha(V\lambda^\alpha) + V\tilde{c}^2 = V\bar{\nabla}_\alpha\lambda^\nu(\bar{\nabla}_\nu\lambda^\alpha - \bar{\nabla}^\alpha\lambda_\nu) \tag{2.30}$$

For the V^2 order we have

$$\bar{\nabla}_\sigma(V\lambda^\sigma\lambda^\alpha\Delta_{\alpha\nu\rho}) + \bar{\nabla}_\beta(V\lambda_\rho\lambda^\sigma)\bar{\nabla}_\sigma(V\lambda_\nu\lambda^\beta) - V^2\lambda_\nu\lambda_\rho\bar{\nabla}_\beta\lambda^\sigma\bar{\nabla}^\beta\lambda_\sigma = 0 \tag{2.31}$$

Note that

- $$\begin{aligned}\bar{\nabla}_\sigma\Big(V\lambda^\sigma\lambda^\alpha\bar{\nabla}_\alpha(V\lambda_\nu\lambda_\rho)\Big) &= \bar{\nabla}_\sigma\Big(V\lambda^\sigma\lambda_\nu\lambda_\rho(\lambda^\alpha\bar{\nabla}_\alpha V + 2\tilde{c}V)\Big) \\ &= \lambda_\nu\lambda_\rho\left(\lambda^\sigma\bar{\nabla}_\sigma V + V\bar{\nabla}_\sigma\lambda^\sigma + 2\tilde{c}V\right)\left(\lambda^\alpha\bar{\nabla}_\alpha V + 2\tilde{c}V\right) \\ &\quad + V\lambda_\nu\lambda_\rho\left(\tilde{c}\lambda^\alpha\bar{\nabla}_\alpha V + \lambda^\sigma\lambda^\alpha\bar{\nabla}_\sigma\bar{\nabla}_\alpha V + 2\tilde{c}\lambda^\sigma\bar{\nabla}_\sigma V\right) \\ &= \lambda_\nu\lambda_\rho\Big(\lambda^\sigma\lambda^\alpha\bar{\nabla}_\sigma V\bar{\nabla}_\alpha V + 7\tilde{c}V\lambda^\sigma\bar{\nabla}_\sigma V + V\lambda^\alpha\bar{\nabla}_\alpha V\bar{\nabla}_\sigma\lambda^\sigma \\ &\qquad + 2\tilde{c}V^2\bar{\nabla}_\sigma\lambda^\sigma + 4\tilde{c}^2V^2 + V\lambda^\sigma\lambda^\alpha\bar{\nabla}_\sigma\bar{\nabla}_\alpha V\Big)\end{aligned} \tag{2.32}$$

and

$$\bullet\ \bar\nabla_\beta(V\lambda_\rho\lambda^\sigma)\bar\nabla_\sigma(V\lambda_\nu\lambda^\beta) = \Big[\lambda_\rho\lambda^\sigma\bar\nabla_\beta V + V\bar\nabla_\beta(\lambda_\rho\lambda^\sigma)\Big]\Big[(\lambda_\nu\lambda^\beta\bar\nabla_\sigma V + V\bar\nabla_\sigma(\lambda_\nu\lambda^\beta)\Big]$$
$$= \lambda_\rho\lambda_\nu\Big(\lambda^\sigma\lambda^\beta\bar\nabla_\sigma V\bar\nabla_\beta V + 4\tilde c V\lambda^\beta\bar\nabla_\beta V + 3\tilde c^2 V^2 + V^2\bar\nabla_\beta\lambda^\sigma\bar\nabla_\sigma\lambda^\beta\Big) \tag{2.33}$$

Then, the equations for the V^2 order become

$$\lambda_\nu\lambda_\rho\Big[3\tilde c V\lambda^\sigma\bar\nabla_\sigma V + V\lambda^\alpha\bar\nabla_\alpha V\bar\nabla_\sigma\lambda^\sigma + 2\tilde c V^2\bar\nabla_\sigma\lambda^\sigma + \tilde c^2 V^2 + V\lambda^\sigma\lambda^\alpha\bar\nabla_\sigma\bar\nabla_\alpha V + V^2\bar\nabla_\beta\lambda^\sigma\bar\nabla^\beta\lambda_\sigma - V^2\bar\nabla_\beta\lambda^\sigma\bar\nabla_\sigma\lambda^\beta\Big] = 0 \tag{2.34}$$

Setting the above paranthesis to 0 to get nontrivial solutions and rearranging the terms yields

$$V\left[\bar\nabla_\beta\lambda^\sigma\bar\nabla_\sigma\lambda^\beta - \bar\nabla_\beta\lambda^\sigma\bar\nabla^\beta\lambda_\sigma\right] = 3\tilde c\lambda^\sigma\bar\nabla_\sigma V + \lambda^\alpha\bar\nabla_\sigma(\lambda^\sigma\bar\nabla_\alpha V) + 2\tilde c V\bar\nabla_\sigma\lambda^\sigma + \tilde c^2 V \tag{2.35}$$

which is the same equation as the corollary (ii) equation of the V^1 order. We see that not all equations are independent and this can also be seen from $R^\mu_{\ \nu} = 0$, where there is no V^3 equation. As we will see in the next section, if λ^μ is a null geodesic, then $R^\mu_{\ \nu}$ is linear in V and there is no V^2 equation. Therefore, the equations of order V^1 determine all properties of λ^μ completely; that is, they provide the most general restrictions on the null vector λ. However, note also that although all the equations involving λ^μ and V can be written in a compact manner as $\bar\nabla_\alpha\Delta^\alpha_{\ \mu\nu} = 0$, it is still helpful to write down the contracted equations, which are less complicated, out of these.

One may employ corollary (i) to write the equation for order V^2 in terms of $\bar\nabla V$ instead of $\bar\nabla\bar\nabla V$;

$$\begin{aligned}
&\left(\bar\nabla_\alpha\lambda^\alpha\right)\bar\nabla_\nu\left(V\lambda^\nu\right) + \lambda^\alpha\bar\nabla_\alpha\bar\nabla_\nu\left(V\lambda^\nu\right) + \tilde c\bar\nabla_\alpha\left(V\lambda^\alpha\right) = 0\\
\Rightarrow\quad & V(\bar\nabla_\alpha\lambda^\alpha)^2 + (\bar\nabla_\alpha\lambda^\alpha)\lambda^\nu\bar\nabla_\nu V + \lambda^\alpha\bar\nabla_\alpha\left(\lambda^\nu\bar\nabla_\nu V + V\bar\nabla_\nu\lambda^\nu\right) + \tilde c\bar\nabla_\alpha(V\lambda^\alpha) = 0\\
\Rightarrow\quad & V(\bar\nabla_\alpha\lambda^\alpha)^2 + \lambda^\nu\bar\nabla_\alpha(\lambda^\alpha\bar\nabla_\nu V) + \tilde c\lambda^\nu\bar\nabla_\nu V + \lambda^\alpha\bar\nabla_\alpha(V\bar\nabla_\nu\lambda^\nu) + \tilde c\bar\nabla_\alpha(V\lambda^\alpha) = 0
\end{aligned} \tag{2.36}$$

Substituting the expression for $\lambda^\nu\bar\nabla_\alpha(\lambda^\alpha\bar\nabla_\nu V)$ into the equation for order V^2, we get

$$V\left[\bar\nabla_\beta\lambda^\sigma\bar\nabla^\beta\lambda_\sigma - \bar\nabla_\beta\lambda^\sigma\bar\nabla_\sigma\lambda^\beta\right] = V(\bar\nabla_\alpha\lambda^\alpha)^2 - \tilde c\lambda^\sigma\bar\nabla_\sigma V - \tilde c V\bar\nabla_\sigma\lambda^\sigma - \tilde c^2 V + \lambda^\alpha\bar\nabla_\alpha(V\bar\nabla_\nu\lambda^\nu) \tag{2.37}$$

Note that

$$
\begin{aligned}
\lambda^\alpha \bar{\nabla}_\alpha \bar{\nabla}_\nu \lambda^\nu &= \lambda^\alpha \left(\left[\bar{\nabla}_\alpha, \bar{\nabla}_\nu \right] + \bar{\nabla}_\nu \bar{\nabla}_\alpha \right) \lambda^\nu \\
&= \lambda^\alpha \bar{R}^\nu{}_{\sigma\alpha\nu} \lambda^\sigma + \lambda^\alpha \bar{\nabla}_\nu \bar{\nabla}_\alpha \lambda^\nu \\
&= -\lambda^\alpha \lambda^\sigma \bar{R}_{\sigma\alpha} + \bar{\nabla}_\nu (\lambda^\alpha \bar{\nabla}_\alpha \lambda^\nu) - (\bar{\nabla}_\nu \lambda^\alpha)(\bar{\nabla}_\alpha \lambda^\nu) \\
&= 0 + \tilde{c} V \bar{\nabla}_\nu \lambda^\nu - (\bar{\nabla}_\nu \lambda^\alpha)(\bar{\nabla}_\alpha \lambda^\nu)
\end{aligned}
\tag{2.38}
$$

Then, for the V^2 order equation we finally have

$$
V \bar{\nabla}_\beta \lambda^\sigma \bar{\nabla}^\beta \lambda_\sigma = V (\bar{\nabla}_\alpha \lambda^\alpha)^2 - \tilde{c} \lambda^\sigma \bar{\nabla}_\sigma V - \tilde{c}^2 V + \lambda^\alpha \bar{\nabla}_\alpha V \bar{\nabla}_\nu \lambda^\nu \tag{2.39}
$$

When the null vector field λ^μ is also tangent to the geodesics, $\tilde{c} = 0$, the above equation becomes:

$$
\bar{\nabla}_\beta \lambda^\sigma \bar{\nabla}^\beta \lambda_\sigma = (\bar{\nabla}_\alpha \lambda^\alpha)^2 + \frac{1}{V} \lambda^\alpha \bar{\nabla}_\alpha V \bar{\nabla}_\nu \lambda^\nu \tag{2.40}
$$

which resembles the shearlessness condition of λ;

$$
\bar{\nabla}_\beta \lambda^\sigma \bar{\nabla}^\beta \lambda_\sigma + \bar{\nabla}_\beta \lambda^\sigma \bar{\nabla}_\sigma \lambda^\beta = (\bar{\nabla}_\alpha \lambda^\alpha)^2 \tag{2.41}
$$

The shear of a vector field is one of the optical scalars which are defined for null vector fields as follows:

Given a null geodesic vector k^μ, one can decompose its covariant derivative into three parts; a divergence term, a shear term and a rotation (curl) term. The corresponding scalar quantities are

- The expansion of k is $\theta = \frac{1}{2} \nabla_\mu k^\mu$
- The shear of k is σ: $\sigma^2 = \frac{1}{2} \nabla_{(\mu} k_{\nu)} \nabla^\mu k^\nu - \frac{1}{4} (\nabla_\mu k^\mu)^2$
- The twist of k is ω: $\omega^2 = \frac{1}{2} \nabla_{[\mu} k_{\nu]} \nabla^\mu k^\mu$

In order to check whether the field equations impose the shearlessness condition on the null vector λ or not, one needs to use the order V^1 equations which encapsulate all properties of λ and relate V, λ and their derivatives.

Alternatively one may opt to use the Goldberg-Sachs theorem[16] which states that in vacuum, a spacetime has an algebraically special Weyl tensor if and only if it admits a shearless null geodesic congruence with the tangent vector field satisfying

$$
\lambda_{[\beta} C_{\xi]\nu\sigma\rho} \lambda^\nu \lambda^\sigma = 0
$$

At this point, an interlude of discussion of the algebraically special metrics is in order.

Algebraic Classification of the Weyl Tensor

There are two equivalent methods; one way is to solve the characteristic equation of the eigenvalue problem for the Weyl tensor $C^{\mu\nu}{}_{\rho\sigma}A^{\rho\sigma} = \Lambda A^{\mu\nu}$ and the other method is to use some certain null directions which are called the principal null directions(PNDs). The spacetimes are then classified by the multiplicities of the roots of the characteristic equation or the principal null directions. The result is that there are 6 types of spacetimes:

• Type I, No PNDs are repeated.

• Type II, Two PNDs are repeated, the remaining two are distinct from each other and the repeated ones.

• Type D, Two PNDs are repeated, the remaining two are also repeated but distinct from the other repeated ones.

• Type III, Three PNDs are repeated, the other one is distinct.

• Type N, All four are repeated.

• Type O, $C_{\mu\nu\rho\sigma} = 0$

Type I is called algebraically general whilst the other types are called algebraically special. Given a null vector k^μ, it is a PND if $k_{[\mu}C_{\alpha\beta\rho[\sigma}k_{\nu]}k^\beta k^\rho = 0$ and the Weyl tensor is; type II if

$C_{\mu\nu\rho[\sigma}k_{\alpha]}k^\nu k^\rho = 0$

type D if

$C_{\mu\nu\rho[\sigma}k_{\alpha]}k^\nu k^\rho = 0$ and $C_{\mu\nu\rho[\sigma}l_{\alpha]}l^\nu l^\rho = 0$ where l is another null vector,

type III if

$C_{\mu\nu\rho[\sigma}k_{\alpha]}k^\rho = 0$

type N if

$C_{\mu\nu\rho\sigma}k^\sigma = 0$

Clearly, $C_{\mu\nu\rho\sigma}$ is type I if none of the PNDs satisfy type II, III, N, O equations.

Now, returning to our discussion, since we study vacuum solutions, the Riemann tensor reduces to the Weyl tensor $R_{\mu\nu\sigma\rho} = C_{\mu\nu\sigma\rho}$. Then we have

$$C_{\xi\nu\sigma\rho}\lambda^\nu\lambda^\sigma = R_{\xi\nu\sigma\rho}\lambda^\nu\lambda^\sigma = g_{\mu\xi}R^{\mu}{}_{\nu\sigma\rho}\lambda^\nu\lambda^\sigma \tag{2.42}$$

and substituting the expression for $R^{\mu}{}_{\nu\sigma\rho}$;

$$R^{\mu}{}_{\nu\sigma\rho} = \bar{R}^{\mu}{}_{\nu\sigma\rho} + \Big[\partial_\sigma(g^{\mu\alpha}\Delta_{\alpha\nu\rho}) + g^{\mu\alpha}\Delta_{\alpha\sigma\beta}\bar{\Gamma}^{\beta}{}_{\nu\rho} + g^{\beta\alpha}\Delta_{\alpha\nu\rho}\bar{\Gamma}^{\mu}{}_{\sigma\beta} \\ + g^{\mu\alpha}g^{\beta\gamma}\Delta_{\alpha\sigma\beta}\Delta_{\gamma\nu\rho} - (\sigma \leftrightarrow \rho)\Big]$$

we obtain

$$\begin{aligned} C_{\xi\nu\sigma\rho}\lambda^\nu\lambda^\sigma &= g_{\mu\xi}\bar{R}^{\mu}{}_{\nu\sigma\rho}\lambda^\nu\lambda^\sigma + g_{\mu\xi}\lambda^\nu\lambda^\sigma\Big(\partial_\sigma(g^{\mu\alpha}\Delta_{\alpha\nu\rho}) - \partial_\rho(g^{\mu\alpha}\Delta_{\alpha\nu\sigma})\Big) \\ &\quad + \lambda^\nu\lambda^\sigma\left(\Delta_{\xi\sigma\beta}\bar{\Gamma}^{\beta}{}_{\rho\sigma} - \Delta_{\xi\rho\beta}\bar{\Gamma}^{\beta}{}_{\nu\sigma}\right) + \lambda^\nu\lambda^\sigma g_{\mu\xi}g^{\beta\alpha}\left(\Delta_{\alpha\nu\rho}\bar{\Gamma}^{\mu}{}_{\sigma\beta} - \Delta_{\alpha\nu\sigma}\bar{\Gamma}^{\mu}{}_{\rho\beta}\right) \\ &\quad + \lambda^\nu\lambda^\rho g^{\beta\alpha}\big(\Delta_{\xi\sigma\beta}\Delta_{\gamma\nu\rho} - \Delta_{\xi\rho\beta}\Delta_{\gamma\nu\sigma}\big) \end{aligned} \tag{2.43}$$

This can be simplified calculating the following terms

- $$\begin{aligned} \partial_\sigma(g^{\mu\alpha}\Delta_{\alpha\nu\rho}) &= (\partial_\sigma g^{\mu\alpha})\Delta_{\alpha\nu\rho} + g^{\mu\alpha}\partial_\sigma\Delta_{\alpha\nu\rho} \\ &= \left[\bar{\nabla}_\sigma g^{\mu\alpha} - \bar{\Gamma}^{\mu}{}_{\sigma\beta}g^{\mu\beta} - \bar{\Gamma}^{\alpha}{}_{\sigma\beta}g^{\mu\beta}\right]\Delta_{\alpha\nu\rho} \\ &\quad + g^{\mu\alpha}\left[\bar{\nabla}_\sigma\Delta_{\alpha\nu\rho} + \bar{\Gamma}^{\beta}{}_{\sigma\alpha}\Delta_{\beta\nu\rho} + \bar{\Gamma}^{\beta}{}_{\sigma\nu}\Delta_{\beta\nu\rho} + \bar{\Gamma}^{\beta}{}_{\sigma\rho}\Delta_{\alpha\nu\beta}\right] \end{aligned} \tag{2.44}$$

- $$\begin{aligned} \partial_\rho(g^{\mu\alpha}\Delta_{\alpha\nu\rho}) &= \left[\bar{\nabla}_\rho g^{\mu\alpha} - \bar{\Gamma}^{\mu}{}_{\rho\beta}g^{\beta\alpha} - \bar{\Gamma}^{\alpha}{}_{\rho\beta}g^{\mu\beta}\right]\Delta_{\alpha\nu\sigma} \\ &\quad + g^{\mu\alpha}\left[\bar{\nabla}_\rho\Delta_{\alpha\nu\sigma} + \bar{\Gamma}^{\beta}_{\rho\alpha}\Delta_{\beta\nu\sigma} + \bar{\Gamma}^{\beta}{}_{\rho\nu}\Delta_{\alpha\beta\sigma} + \bar{\Gamma}^{\beta}{}_{\rho\sigma}\Delta_{\alpha\nu\beta}\right] \end{aligned} \tag{2.45}$$

and noting that $\lambda^\nu\lambda^\sigma\Delta_{\alpha\nu\sigma} = 0$. Then we have

$$\begin{aligned} &g_{\mu\xi}\lambda^\nu\lambda^\sigma\Big(\partial_\sigma(g^{\mu\alpha}\Delta_{\alpha\nu\rho}) - \partial_\rho(g^{\mu\alpha}\Delta_{\alpha\nu\sigma})\Big) \\ &\qquad = g_{\mu\xi}\lambda^\nu\lambda^\sigma\Delta_{\alpha\nu\rho}\left[-2\bar{\nabla}_\sigma(V\lambda^\mu\lambda^\alpha) - \bar{\Gamma}^{\mu}{}_{\sigma\beta}g^{\beta\alpha} - \bar{\Gamma}^{\alpha}{}_{\sigma\beta}g^{\mu\beta}\right] \\ &\qquad\quad + \lambda^\nu\lambda^\sigma\Big[\bar{\nabla}_\sigma\Delta_{\xi\nu\rho} + \bar{\Gamma}^{\beta}{}_{\sigma\xi}\Delta_{\beta\nu\rho} + \bar{\Gamma}^{\beta}{}_{\sigma\nu}\Delta_{\xi\beta\rho} - \bar{\nabla}_\rho\Delta_{\xi\nu\sigma} - \bar{\Gamma}^{\beta}{}_{\rho\nu}\Delta_{\xi\beta\sigma}\Big] \end{aligned} \tag{2.46}$$

Inserting these results into the expression for $C_{\xi\nu\sigma\rho}\lambda^\nu\lambda^\sigma$ gives

$$\begin{aligned} C_{\xi\nu\sigma\rho}\lambda^\nu\lambda^\sigma &= g_{\mu\xi}\bar{R}^{\mu}{}_{\nu\sigma\rho}\lambda^\nu\lambda^\sigma + g_{\mu\xi}\lambda^\nu\lambda^\sigma\Delta_{\alpha\nu\rho}\left[-2\bar{\nabla}_\sigma(V\lambda^\mu\lambda^\alpha) - \bar{\Gamma}^{\alpha}{}_{\sigma\beta}g^{\mu\beta}\right] \\ &\quad + \lambda^\nu\lambda^\sigma\Big[\bar{\nabla}_\sigma\Delta_{\xi\nu\rho} + \bar{\Gamma}^{\beta}{}_{\sigma\xi}\Delta_{\beta\nu\rho} - \bar{\nabla}_\rho\Delta_{\xi\nu\sigma}\Big] + \lambda^\nu\lambda^\sigma\Delta_{\xi\sigma}{}^{\beta}\Delta_{\beta\nu\rho} \\ &= g_{\mu\xi}\bar{R}^{\mu}{}_{\nu\sigma\rho}\lambda^\nu\lambda^\sigma - 2\lambda^\nu\lambda^\sigma\Delta_{\alpha\nu\rho}\bar{\nabla}_\sigma(V\lambda_\xi\lambda^\alpha) + \lambda^\nu\lambda^\sigma\left(\bar{\nabla}_\sigma\Delta_{\xi\nu\rho} - \bar{\nabla}_\rho\Delta_{\xi\nu\sigma}\right) \\ &\quad + \lambda^\nu\Delta_{\beta\nu\rho}\lambda^\sigma\bar{\nabla}_\sigma(V\lambda_\xi\lambda^\beta) \end{aligned} \tag{2.47}$$

Hence, the final result is given by

$$
\begin{aligned}
C_{\xi\nu\sigma\rho}\lambda^\nu\lambda^\sigma = {} & g_{\mu\xi}\bar{R}^\mu{}_{\nu\sigma\rho}\lambda^\nu\lambda^\sigma - \lambda^\nu\bar{\nabla}_\nu(V\lambda_\alpha\lambda_\rho)\lambda^\sigma\bar{\nabla}_\sigma(V\lambda_\xi\lambda^\alpha) \\
& + \Big[\lambda^\sigma\bar{\nabla}_\sigma\big(\lambda^\nu\bar{\nabla}_\nu(V\lambda_\xi\lambda_\rho)\big) - (\lambda^\sigma\bar{\nabla}_\sigma\lambda^\nu)\Delta_{\xi\nu\rho} + \bar{\nabla}_\rho(\lambda^\nu\lambda^\sigma)\Delta_{\xi\nu\sigma}\Big]
\end{aligned}
\tag{2.48}
$$

Now since we have:

- $\lambda^\nu\bar{\nabla}_\nu(V\lambda_\alpha\lambda_\rho)\lambda^\sigma\bar{\nabla}_\sigma(V\lambda_\xi\lambda^\alpha) = V^2\lambda_\rho\lambda_\xi\bar{\nabla}_\nu\lambda_\alpha\bar{\nabla}_\sigma\lambda^\alpha$
- $\begin{aligned}\lambda^\sigma\bar{\nabla}_\sigma(\lambda^\nu\bar{\nabla}_\nu(V\lambda_\xi\lambda_\rho)) &= \lambda^\sigma\bar{\nabla}_\sigma(\tilde{c}\lambda_\xi\bar{\nabla}_\nu(V\lambda_\rho) + \lambda_\xi\lambda^\nu\bar{\nabla}_\nu(V\lambda_\rho)) \\ &= \tilde{c}^2\lambda_\xi\bar{\nabla}_\nu(V\lambda_\rho) + \tilde{c}\lambda_\xi\bar{\nabla}_\sigma\bar{\nabla}_\nu(V\lambda_\rho) + \tilde{c}\lambda_\xi\lambda^\nu\bar{\nabla}_\nu(V\lambda_\rho) \\ &\quad + \lambda_\xi\lambda^\sigma\bar{\nabla}_\sigma(\lambda^\nu\bar{\nabla}_\nu(V\lambda_\rho))\end{aligned}$
- $\begin{aligned}\bar{\nabla}_\rho(\lambda^\nu\lambda^\sigma)\Delta_{\xi\nu\sigma} &= 2(\bar{\nabla}_\rho\lambda^\nu)\lambda^\sigma\bar{\nabla}_\sigma(V\lambda_\xi\lambda_\nu) \\ &= 2V\lambda_\xi\lambda^\sigma\bar{\nabla}_\rho\lambda^\nu\bar{\nabla}_\sigma\lambda_\nu\end{aligned}$
- $(\lambda^\sigma\bar{\nabla}_\sigma\lambda^\nu)\Delta_{\xi\nu\rho} = \tilde{c}\lambda^\nu\bar{\nabla}_\nu(V\lambda_\xi\lambda_\rho) = \tilde{c}^2\lambda_\xi V\lambda_\rho + \tilde{c}\lambda_\xi\lambda^\nu\bar{\nabla}_\nu(V\lambda_\rho)$

(2.49)

these terms do not contribute to $\lambda_{[\beta}C_{\xi]\nu\sigma\rho}\lambda^\nu\lambda^\sigma$. Hence, we obtain

$$
\begin{aligned}
\lambda_{[\beta}C_{\xi]\nu\sigma\rho}\lambda^\nu\lambda^\sigma &= (\lambda_\beta g_{\mu\xi} - \lambda_\xi g_{\mu\beta})\,\bar{R}^\mu{}_{\nu\sigma\rho}\lambda^\nu\lambda^\sigma \\
&= (\lambda_\beta\bar{R}_{\xi\nu\sigma\rho} - \lambda_\xi\bar{R}_{\beta\nu\sigma\rho})\lambda^\nu\lambda^\sigma \\
&\qquad + 2V\lambda_\mu\,(\lambda_\beta\lambda_\xi - \lambda_\xi\lambda_\beta)\,\bar{R}^\mu{}_{\nu\sigma\rho}\lambda^\nu\lambda_\sigma \\
&= \lambda_{[\beta}\bar{R}_{\xi]\nu\sigma\rho}\lambda^\nu\lambda^\sigma \\
&\neq 0 \quad \textit{in general.}
\end{aligned}
\tag{2.50}
$$

Then the generalized Kerr-Schild ansatz metric, $g_{\mu\nu}$, is not guaranteed to admit a shearless null geodesic congruence in general. An instance when it does is the case of a maximally symmetric spacetime,

$$
\bar{R}_{\mu\nu\rho\sigma} = \frac{\bar{R}}{12}\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{2.51}
$$

which gives

$$
\lambda_{[\alpha}R_{\mu]\nu\rho\sigma}\lambda^\nu\lambda^\rho = 0 = \lambda_{[\alpha}R_{\mu]\nu\rho[\sigma}\lambda_{\beta]}\lambda^\nu\lambda^\rho \tag{2.52}
$$

Therefore, the null vector λ is a repeated principle null direction and the metric is algebraically special. By the Goldberg-Sachs theorem λ is geodesic and shearless in vacuum.

Now suppose that $g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu$ is a solution to the field equations in vacuum

and $\lambda_{[\alpha}C_{\mu]\nu\rho\sigma}\lambda^\nu\lambda^\rho = 0$, i.e. $g_{\mu\nu}$ is an algebraically special metric. Then, the null vector λ^μ is also a shearless geodesic vector by the Goldberg-Sachs theorem.
Comparing the order V^2 field equation (in the case of null geodesic λ^μ) with the shearless condition given by

$$\bar{\nabla}_\mu\lambda^\nu\bar{\nabla}^\mu\lambda_\nu + \bar{\nabla}_\mu\lambda^\nu\bar{\nabla}_\nu\lambda^\mu = (\bar{\nabla}_\mu\lambda^\mu)^2 \tag{2.53}$$

we arrive at the following, rather simple, equation relating V and λ,

$$\bar{\nabla}_\mu\lambda^\nu\bar{\nabla}_\nu\lambda^\mu = -\frac{1}{V}\lambda^\mu\bar{\nabla}_\mu V\bar{\nabla}_\nu\lambda^\nu \tag{2.54}$$

The LHS can be rewritten as

$$\begin{aligned}\bar{\nabla}_\mu\lambda^\nu\bar{\nabla}_\nu\lambda^\mu &= \bar{\nabla}_\mu(\lambda^\nu\bar{\nabla}_\nu\lambda^\mu) - \lambda^\nu\bar{\nabla}_\mu\bar{\nabla}_\nu\lambda^\mu \\ &= 0 - \lambda^\nu\left([\bar{\nabla}_\mu,\bar{\nabla}_\nu] + \bar{\nabla}_\nu\bar{\nabla}_\mu\right)\lambda^\mu \\ &= -\lambda^\nu\bar{R}^\mu{}_{\sigma\mu\nu}\lambda^\sigma - \lambda^\nu\bar{\nabla}_\nu\bar{\nabla}_\mu\lambda^\mu \\ &= -\lambda^\nu\bar{\nabla}_\nu\bar{\nabla}_\mu\lambda^\mu\end{aligned} \tag{2.55}$$

where we used $\lambda^\nu\bar{\nabla}_\nu\lambda^\mu = 0 = \bar{R}^\mu{}_{\sigma\mu\nu} = \bar{R}_{\sigma\nu}$. Then, we get the following, corollary (iii),

$$\lambda^\nu\bar{\nabla}_\nu\bar{\nabla}_\mu\lambda^\mu = \frac{1}{V}\lambda^\mu\bar{\nabla}_\mu V\bar{\nabla}_\nu\lambda^\nu \tag{2.56}$$

which can be used to construct the KS form of some metrics. Consider the Schwarzschild in the standard coordinates and suppose $\bar{g} = \eta$ and $\lambda^\mu = (1,1,0,0)$ in spherical coordinates which is manifestly spherically symmetric, null, geodesic and shearless. We have $\bar{\nabla}_\mu = \partial_\mu$ and hence,

$$\begin{aligned}\bar{\nabla}_\mu\lambda^\mu = \partial_\mu\lambda^\mu &= \partial_0(1) + \vec{\nabla}\cdot\vec{\lambda} \\ &= 0 + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cdot 1) \\ &= \frac{2}{r}\end{aligned} \tag{2.57}$$

Using corollary (iii) we get

$$\lambda^\nu\partial_\nu\frac{2}{r} = \frac{1}{V}\left(\lambda^\mu\partial_\mu V\right)\frac{2}{r} \tag{2.58}$$

Now, since the solution is static and spherically symmetric, we take $V(t,r,\theta,\phi) = V(r)$ and the above equation becomes

$$\frac{d}{dr}\frac{2}{r} = \frac{1}{V}\frac{dV}{dr}\frac{2}{r} \quad \Rightarrow \quad -\frac{V}{r} = \frac{dV}{dr} \tag{2.59}$$

which yields $rV = constant = M$. Therefore, as we will see in the last section of this chapter, the KS form of the Schwarzschild metric is given as

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\frac{M}{r}\lambda_\mu\lambda_\nu \quad with \quad \lambda_\mu = (-1, 1, 0, 0) \tag{2.60}$$

In the case of the Kerr metric in advanced Eddington-Finkelstein coordinates, we have

$$\begin{aligned} ds^2 = &- \left[1 - \frac{2mr}{\Sigma}\right](du + a\sin^2\theta d\phi)^2 + 2\left(du + a\sin^2\theta d\phi\right)(dr + a\sin^2\theta d\phi) \\ &+ \Sigma\left(d\theta^2 + \sin^2\theta d\phi^2\right) \end{aligned} \tag{2.61}$$

where $u = t + r$, $\Sigma = r^2 + a^2\cos^2\theta$, $\lambda_\mu = (1, 0, 0, a\sin^2\theta)$ and $\lambda^\mu = (0, 1, 0, 0)$. Here, λ^μ is a null geodesic shearless vector and its divergence is

$$\bar{\nabla}_\mu\lambda^\mu = \frac{2r}{\Sigma} \tag{2.62}$$

Then, corollary (iii) gives

$$\lambda^\mu\bar{\nabla}_\mu\frac{2r}{\Sigma} = \frac{1}{V}(\lambda^\mu\bar{\nabla}_\mu V)\frac{2r}{\Sigma} \quad \Rightarrow \quad \frac{1}{\frac{2r}{\Sigma}}\partial_r\frac{2r}{\Sigma} = \frac{1}{V}\partial_r V \tag{2.63}$$

KS form is given by $\lambda^\mu = (0, 1, 0, 0)$ and $V = \frac{2r}{r^2+a^2\cos^2\theta}m$.

As for the electrovacuum solutions of the field equations, corollary (iii) does not hold in general. The field equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} = 8\pi(F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}) \tag{2.64}$$

The energy-momentum tensor for an electromagnetic field is traceless, $g^{\mu\nu}T_{\mu\nu} = 0$ and thus the trace of the LHS, $-R$, must also vanish. We have $R = 0$ and the field equations become

$$R_{\mu\nu} = 8\pi(F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}) \tag{2.65}$$

When the background spacetime is a vacuum solution, the equation of the transformation of R becomes

$$0 = R = 2\bar{\nabla}_\mu\bar{\nabla}_\nu(V\lambda^\mu\lambda^\nu) + 2V^2\lambda^\mu\lambda^\nu\bar{\nabla}_\mu\lambda^\sigma\bar{\nabla}_\nu\lambda_\sigma \tag{2.66}$$

which does not imply that the null vector λ^μ is a geodesic, unlike the vacuum solutions. If we further assume that λ^μ is a geodesic, we obtain

$$0 = \bar{\nabla}_\mu\bar{\nabla}_\nu(V\lambda^\mu\lambda^\nu) = \bar{\nabla}_\mu(\lambda^\mu\bar{\nabla}_\nu(V\lambda^\nu)) \tag{2.67}$$

As an application of this result, consider the Reissner-Nordström metric for which we will choose, similar to the Schwarzschild metric, $\bar{g} = \eta$ and $\lambda^\mu = (1, 1, 0, 0)$. Then,

$$\bar{\nabla}_\nu (V\lambda^\nu) = \vec{\nabla} \cdot (V\vec{\lambda}) = \frac{1}{r^2}\frac{d}{dr}(r^2 V) \tag{2.68}$$

whence we obtain

$$\partial_\mu \left(\lambda^\mu \frac{1}{r^2}\frac{d}{dr}(r^2 V) \right) = 0 \;\; \Rightarrow \frac{1}{r^2}\frac{d}{dr}\left[\frac{d}{dr}(r^2 V)\right] = 0 \tag{2.69}$$

Hence, $\frac{d}{dr}(r^2 V) = A$ and integration yields $r^2 V = Ar + B$ where A and B are constants.

Therefore, KS form of the Reissner-Nordström metric in spherical coordinates is given by

$$\bar{g} = \eta, \quad \lambda^\mu = (1, 1, 0, 0), \quad V(r) = \frac{A}{r} + \frac{B}{r^2} \tag{2.70}$$

where A and B are related to mass and charge respectively.

In general, there is no compelling reason to assume that the mass term is present only in the scalar function and the equations can be written order-by-order. Instead, the properties of λ will be determined by contracting the complete expression for $R_{\nu\rho}$ with $\lambda^\nu \lambda^\rho$ and $g^{\nu\rho}$. We have

$$\begin{aligned}
\lambda^\nu \lambda^\rho R_{\nu\rho} &= \lambda^\nu \lambda^\rho \bar{R}_{\nu\rho} + \lambda^\nu \lambda^\rho \bar{\nabla}_\alpha \Delta^\alpha{}_{\nu\rho} - 2\lambda^\nu \lambda^\rho \bar{\nabla}_\sigma \big(V\lambda^\sigma \lambda^\alpha \Delta_{\alpha\nu\rho}\big) \\
&= \lambda^\nu \lambda^\rho \bar{R}_{\nu\rho} + \big[\bar{\nabla}_\alpha (\lambda^\nu \lambda^\rho \Delta^\alpha{}_{\nu\rho}) - \Delta^\alpha{}_{\nu\rho} \bar{\nabla}_\alpha (\lambda^\nu \lambda^\rho)\big] \\
&\quad - 2\left[\bar{\nabla}_\sigma (V\lambda^\nu \lambda^\rho \lambda^\sigma \lambda^\alpha \Delta_{\alpha\nu\rho}) - V\lambda^\sigma \lambda^\alpha \Delta_{\alpha\nu\rho} \bar{\nabla}_\sigma (\lambda^\nu \lambda^\rho)\right] \\
&= \lambda^\nu \lambda^\rho \bar{R}_{\nu\rho} - 2\lambda^\rho \Delta^\alpha{}_{\nu\rho} \bar{\nabla}_\alpha \lambda^\nu + 4V\lambda^\sigma \lambda^\alpha \Delta_{\alpha\nu\rho} \lambda^\rho \bar{\nabla}_\sigma \lambda^\nu \\
&= \lambda^\nu \lambda^\rho \bar{R}_{\nu\rho} - 2V\lambda^\alpha \lambda^\rho \bar{\nabla}_\rho \lambda_\nu \bar{\nabla}_\alpha \lambda^\nu
\end{aligned} \tag{2.71}$$

where the simplifications are due to the fact that

$$\lambda^2 = \lambda^\mu \bar{\nabla}_\mu \lambda^\nu = \lambda^\nu \lambda^\rho \Delta_{\alpha\nu\rho} = 0 \tag{2.72}$$

Similarly

$$\begin{aligned}
R &= (\bar{R} - 2V\lambda^\nu \lambda^\rho \bar{R}_{\nu\rho}) + \big[\bar{\nabla}_\alpha \Delta^\alpha{}_\nu{}^\nu - 2V\lambda^\nu \lambda^\rho \bar{\nabla}_\alpha \Delta^\alpha{}_{\nu\rho}\big] \\
&\quad - 2\left[\bar{\nabla}_\sigma (V\lambda^\sigma \lambda^\alpha \Delta_{\alpha\nu}{}^\nu) - 2V\lambda^\nu \lambda^\rho \bar{\nabla}_\sigma (V\lambda^\sigma \lambda^\alpha \Delta_{\alpha\nu\rho})\right] \\
&\quad - 2V^2 \lambda^\sigma \lambda^\beta \bar{\nabla}_\beta \lambda^\nu \bar{\nabla}_\sigma \lambda_\nu \\
&= (\bar{R} - 2V\lambda^\nu \lambda^\rho \bar{R}_{\nu\rho}) + \big[2\bar{\nabla}_\alpha \bar{\nabla}_\nu (V\lambda^\nu \lambda^\alpha) + 4V^2 \lambda^\alpha \lambda^\rho \bar{\nabla}_\alpha \lambda^\nu \bar{\nabla}_\rho \lambda_\nu\big] \\
&\quad - 2V^2 \lambda^\sigma \lambda^\beta \bar{\nabla}_\beta \lambda^\nu \bar{\nabla}_\sigma \lambda_\nu \\
&= \big(\bar{R} - 2V\lambda^\nu \lambda^\rho \bar{R}_{\nu\rho}\big) + 2\bar{\nabla}_\alpha \bar{\nabla}_\nu (V\lambda^\alpha \lambda^\nu) + 2V^2 \lambda^\alpha \lambda^\rho \bar{\nabla}_\alpha \lambda^\nu \bar{\nabla}_\rho \lambda_\nu
\end{aligned} \tag{2.73}$$

where we substituted the simplified expression for $\lambda^\nu\lambda^\rho\bar{\nabla}_\alpha\Delta^\alpha{}_{\nu\rho}$ from the calculations of $\lambda^\nu\lambda^\rho R_{\nu\rho}$ and also used the fact that $\lambda^\alpha\Delta_{\alpha\nu}{}^\nu = 0 = \lambda^\nu\lambda^\rho\bar{\nabla}_\sigma(V\lambda^\sigma\lambda^\alpha\Delta_{\alpha\nu\rho})$. The latter can easily be seen by taking one of the null vectors inside the paranthesis and subtracting the extra term added in this way, both of which are zero. Hence, from the $\lambda^\nu\lambda^\rho R_{\nu\rho}$ result we observe that if $\lambda^\nu\lambda^\rho\bar{R}_{\nu\rho} = 0$, then $\lambda^\nu\lambda^\rho R_{\nu\rho} = 0$ implies that $\lambda^\rho\lambda^\alpha\bar{\nabla}_\rho\lambda_\nu\bar{\nabla}_\alpha\lambda^\nu = 0$; that is, $\lambda^\mu\bar{\nabla}_\mu\lambda^\nu = \tilde{c}\lambda^\nu$, which after reparametrization, can be turned into a geodesic equation with an affine parameter. Therefore, λ is a null geodesic vector.

If $\bar{R}_{\nu\rho} = 0$, then from $R = 0$; we obtain

$$\bar{\nabla}_\alpha\bar{\nabla}_\nu(V\lambda^\alpha\lambda^\nu) = 0 \quad and \quad \lambda^\mu\bar{\nabla}_\mu\lambda^\nu = \tilde{c}\lambda^\nu \tag{2.74}$$

which are corollary (i) of the previous discussion and the geodesic equation.

Lastly, we calculate $\lambda^\nu R_{\nu\rho}$:

$$\lambda^\nu R_{\nu\rho} = \lambda^\nu\bar{R}_{\nu\rho} + \lambda^\nu\bar{\nabla}_\alpha\Delta^\alpha_{\nu\rho} - 2V\lambda^\nu\bar{\nabla}_\sigma(V\lambda^\sigma\lambda^\alpha\Delta_{\alpha\nu\rho}) \tag{2.75}$$

For vacuum solutions, $R_{\nu\rho} = 0 = \bar{R}_{\nu\rho}$, we get

$$\begin{aligned}\lambda^\nu\bar{\nabla}_\alpha\Delta^\alpha_{\nu\rho} &= 2V\lambda^\nu\bar{\nabla}_\sigma(V\lambda^\sigma\lambda^\alpha\Delta_{\alpha\nu\rho}) \\ &= -2V^2(\bar{\nabla}_\sigma\lambda^\nu)\lambda^\sigma\lambda^\alpha\Delta_{\alpha\nu\rho} \\ &= -2V^2\tilde{c}\lambda^\nu\lambda^\alpha\Delta_{\alpha\nu\rho} \\ &= 0\end{aligned} \tag{2.76}$$

which is corollary (ii) of the previous section. Since our previous discussions depend only on corollary (i), (ii) and the geodesic property, evidently they are valid in general.

2.2 Ansatz with Null Geodesics

Having been motivated by the simplifications in the calculations brought forth by the ansatz of the preceding section, we begin with a similar ansatz in this section and consider the field equations when the energy-momentum tensor does not vanish identically. We have

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu \quad and \quad g^{\mu\nu} = \bar{g}^{\mu\nu} - 2V\lambda^\mu\lambda^\nu \quad as\ solutions\ to$$
$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad and \quad \bar{R}_{\mu\nu} - \frac{\bar{R}}{2}\bar{g}_{\mu\nu} = 8\pi G\bar{T}_{\mu\nu} \tag{2.77}$$

where V is a scalar function and λ is a null geodesic vector with respect to both metrics;

$$\bar{g}_{\mu\nu}\lambda^\mu\lambda^\nu = 0 = g_{\mu\nu}\lambda^\mu\lambda^\nu \quad \lambda^\mu\nabla_\mu\lambda^\nu = 0 = \lambda^\mu\bar{\nabla}_\mu\lambda_\nu \tag{2.78}$$

Transformation of the Christoffel Symbols is given by

$$\Gamma^\mu{}_{\nu\rho} = \bar{\Gamma}^\mu{}_{\nu\rho} + g^{\mu\alpha}\Delta_{\alpha\nu\rho} = \bar{\Gamma}^\mu{}_{\nu\rho} + \Delta^\mu{}_{\nu\rho} + 2V\lambda^\mu\lambda^\alpha\bar{\nabla}_\alpha(V\lambda_\nu\lambda_\rho) \tag{2.79}$$

Following the notation of [17] we introduce $\delta^\mu{}_{\nu\rho} \equiv 2V\lambda^\mu\lambda_\nu\lambda_\rho\lambda^\alpha\bar{\nabla}_\alpha V$ as the quadratic term in V and obtain

$$\Gamma^\mu{}_{\nu\rho} = \bar{\Gamma}^\mu{}_{\nu\rho} + \Delta^\mu{}_{\nu\rho} + \delta^\mu{}_{\nu\rho} \tag{2.80}$$

Then

$$\begin{aligned} R^\mu{}_{\nu\sigma\rho} &= \left(\partial_\sigma\Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\sigma\alpha}\Gamma^\alpha{}_{\nu\rho}\right) - (\sigma \leftrightarrow \rho) \\ &= \Big[\partial_\sigma\bar{\Gamma}^\mu{}_{\nu\rho} + \partial_\sigma\Delta^\mu{}_{\nu\rho} + \partial_\sigma\delta^\mu{}_{\nu\rho} + \bar{\Gamma}^\mu{}_{\sigma\alpha}\bar{\Gamma}^\alpha{}_{\nu\rho} + \bar{\Gamma}^\mu{}_{\sigma\alpha}\Delta^\alpha{}_{\nu\rho} \\ &\quad + \bar{\Gamma}^\alpha{}_{\sigma\alpha}\delta^\alpha{}_{\nu\rho} + \Delta^\mu{}_{\sigma\alpha}\bar{\Gamma}^\alpha{}_{\nu\rho} + \Delta^\alpha{}_{\sigma\alpha}\Delta^\alpha{}_{\nu\rho} + \Delta^\mu{}_{\sigma\alpha}\delta^\alpha{}_{\nu\rho} \\ &\quad + \delta^\mu{}_{\sigma\alpha}\bar{\Gamma}^\alpha{}_{\nu\rho} + \delta^\mu{}_{\sigma\alpha}\Delta^\alpha{}_{\nu\rho} + \delta^\mu{}_{\sigma\alpha}\delta^\alpha{}_{\nu\rho}\Big] - (\sigma \leftrightarrow \rho) \end{aligned} \tag{2.81}$$

Splitting terms in orders of V and noting that $\delta^\mu{}_{\sigma\alpha}\delta^\alpha{}_{\nu\rho} = 0$, we get

$$\begin{aligned} R^\mu{}_{\nu\sigma\rho} = \bar{R}^\mu{}_{\nu\sigma\rho} &+ \left[\partial_\sigma\Delta^\mu{}_{\nu\rho} + \bar{\Gamma}^\mu{}_{\sigma\alpha}\Delta^\alpha{}_{\nu\rho} + \bar{\Gamma}^\alpha{}_{\nu\rho}\Delta^\mu{}_{\sigma\alpha} - (\sigma \leftrightarrow \rho)\right] \\ &+ \left[\partial_\sigma\delta^\mu{}_{\nu\rho} + \bar{\Gamma}^\mu{}_{\sigma\alpha}\delta^\alpha{}_{\nu\rho} + \bar{\Gamma}^\alpha{}_{\nu\rho}\delta^\mu{}_{\sigma\alpha} + \Delta^\mu{}_{\sigma\alpha}\Delta^\alpha{}_{\nu\rho} - (\sigma \leftrightarrow \rho)\right] \\ &+ \left(\Delta^\mu{}_{\sigma\alpha}\delta^\alpha{}_{\nu\rho} + \Delta^\alpha{}_{\nu\rho}\delta^\mu{}_{\sigma\alpha} - (\sigma \leftrightarrow \rho)\right) \end{aligned} \tag{2.82}$$

Now, using the following results

- $$\partial_\sigma\Delta^\mu{}_{\nu\rho} = \bar{\nabla}_\sigma\Delta^\mu{}_{\nu\rho} - \bar{\Gamma}^\mu{}_{\sigma\alpha}\Delta^\alpha{}_{\nu\rho} + \bar{\Gamma}^\alpha{}_{\sigma\nu}\Delta^\mu{}_{\alpha\rho} + \bar{\Gamma}^\alpha{}_{\sigma\rho}\Delta^\mu{}_{\nu\alpha}$$

- $$\partial_\sigma\delta^\mu{}_{\nu\rho} = \bar{\nabla}_\sigma\delta^\mu{}_{\nu\rho} - \bar{\Gamma}^\mu{}_{\sigma\alpha}\delta^\alpha{}_{\nu\rho} + \bar{\Gamma}^\alpha{}_{\sigma\nu}\delta^\mu{}_{\alpha\rho} + \bar{\Gamma}^\alpha{}_{\sigma\rho}\delta^\mu{}_{\nu\alpha}$$

- $$\Delta^\mu{}_{\sigma\alpha}\delta^\alpha{}_{\nu\rho} = 2V\lambda_\nu\lambda_\rho\lambda^\mu\lambda_\sigma\lambda^\alpha\lambda^\beta\bar{\nabla}_\beta V\bar{\nabla}_\alpha V$$

- $$\Delta^{\alpha}_{\ \nu\rho}\delta^{\mu}_{\ \sigma\alpha} = -2V\lambda^{\mu}\lambda_{\sigma}\lambda_{\nu}\lambda_{\rho}\lambda^{\alpha}\lambda^{\beta}\bar{\nabla}_{\beta}V\bar{\nabla}_{\alpha}V$$

and noting that the terms that are symmetric in $\sigma \leftrightarrow \rho$ vanish, transformation of the Riemann tensor is given by

$$\begin{aligned}
R^{\mu}_{\ \nu\sigma\rho} &= \bar{R}^{\mu}_{\ \nu\sigma\rho} + \left[\bar{\nabla}_{\sigma}\Delta^{\mu}_{\ \nu\rho} + \bar{\Gamma}^{\alpha}_{\ \nu\rho}\Delta^{\mu}_{\ \sigma\alpha} + \bar{\Gamma}^{\alpha}_{\ \sigma\nu}\Delta^{\mu}_{\ \alpha\rho} - (\sigma \leftrightarrow \rho)\right] \\
&\qquad + \left[\bar{\nabla}_{\sigma}\delta^{\mu}_{\ \nu\rho} + \bar{\Gamma}^{\alpha}_{\ \nu\rho}\delta^{\mu}_{\ \sigma\alpha} + \bar{\Gamma}^{\alpha}_{\ \sigma\nu}\delta^{\mu}_{\ \alpha\rho} + \Delta^{\mu}_{\ \sigma\alpha}\Delta^{\alpha}_{\ \nu\rho} - (\sigma \leftrightarrow \rho)\right] \\
&= \bar{R}^{\mu}_{\ \nu\sigma\rho} + \left[\bar{\nabla}_{\sigma}\Delta^{\mu}_{\ \nu\rho} - \bar{\nabla}_{\rho}\Delta^{\mu}_{\ \nu\sigma}\right] + \left[\bar{\nabla}_{\sigma}\delta^{\mu}_{\ \nu\rho} - \bar{\nabla}_{\rho}\delta^{\mu}_{\ \nu\sigma} + \Delta^{\mu}_{\ \sigma\alpha}\Delta^{\alpha}_{\ \nu\rho} - \Delta^{\mu}_{\ \rho\alpha}\Delta^{\alpha}_{\ \nu\sigma}\right] \\
&\equiv \bar{R}^{\mu}_{\ \nu\sigma\rho} + R^{\mu}_{(L)\ \nu\sigma\rho} + R^{\mu}_{(Q)\ \nu\sigma\rho}
\end{aligned} \tag{2.83}$$

where

$$\begin{aligned}
R^{\mu}_{(L)\ \nu\rho\sigma} &= \bar{\nabla}_{\rho}\Delta^{\mu}_{\ \nu\sigma} - \bar{\nabla}_{\sigma}\Delta^{\mu}_{\ \nu\rho} \\
R^{\mu}_{(Q)\ \nu\rho\sigma} &= \bar{\nabla}_{\rho}\delta^{\mu}_{\ \nu\sigma} - \bar{\nabla}_{\sigma}\delta^{\mu}_{\ \nu\rho} + \Delta^{\mu}_{\ \rho\alpha}\Delta^{\alpha}_{\ \nu\sigma} - \Delta^{\mu}_{\ \sigma\alpha}\Delta^{\alpha}_{\ \nu\rho}
\end{aligned} \tag{2.84}$$

are the linear and the quadratic terms in V, respectively. Note that there are no cubic or quartic terms in V as one would expect from a generic transformation of the Christoffel Symbols with linear and quadratic terms. Next, we find the transformation of the Ricci tensor

$$R_{\nu\sigma} = \bar{R}_{\nu\sigma} + R^{\mu}_{(L)\ \nu\mu\sigma} + R^{\mu}_{(Q)\ \nu\mu\sigma} \tag{2.85}$$

Using the calculations below

$$\rightarrow \quad R^{\mu}_{(L)\ \nu\mu\sigma} = \bar{\nabla}_{\mu}\Delta^{\mu}_{\ \nu\sigma}\,, \quad since \quad \Delta^{\mu}_{\ \nu\mu} = 0$$

$$\rightarrow \quad R^{\mu}_{(Q)\ \nu\mu\sigma} = \bar{\nabla}_{\mu}\delta^{\mu}_{\ \nu\sigma} - \Delta^{\mu}_{\ \sigma\alpha}\Delta^{\alpha}_{\ \nu\mu}\,, \quad since \quad \Delta^{\mu}_{\ \nu\mu} = 0 = \delta^{\mu}_{\ \nu\mu}$$

$$\rightarrow \quad \bar{\nabla}_{\mu}\delta^{\mu}_{\ \nu\sigma} = \bar{\nabla}_{\mu}(2V\lambda^{\mu}\lambda_{\nu}\lambda_{\sigma}\lambda^{\alpha}\bar{\nabla}_{\alpha}V) = 2\lambda_{\nu}\lambda_{\sigma}\lambda^{\alpha}\bar{\nabla}_{\mu}(V\lambda^{\mu}\bar{\nabla}_{\alpha}V)$$

$$\begin{aligned}
\rightarrow \Delta^{\mu}_{\ \sigma\alpha}\Delta^{\alpha}_{\ \nu\mu} &= \left[\bar{\nabla}_{\sigma}(V\lambda^{\mu}\lambda_{\alpha}) + \bar{\nabla}_{\alpha}(V\lambda^{\mu}\lambda_{\sigma}) - \bar{\nabla}^{\mu}(V\lambda_{\sigma}\lambda_{\alpha})\right]\Big[\bar{\nabla}_{\nu}(V\lambda^{\alpha}\lambda_{\mu}) \\
&\qquad + \bar{\nabla}_{\mu}(V\lambda^{\alpha}\lambda_{\nu}) - \bar{\nabla}^{\alpha}(V\lambda_{\nu}\lambda_{\mu})\Big] \\
&= \Big[\lambda^{\mu}\lambda_{\sigma}\lambda^{\alpha}\lambda_{\nu}\bar{\nabla}_{\alpha}V\bar{\nabla}_{\mu}V + V^{2}\lambda_{\sigma}\lambda_{\nu}\bar{\nabla}_{\alpha}\lambda^{\mu}\bar{\nabla}_{\mu}\lambda^{\alpha} - V^{2}\lambda_{\sigma}\lambda_{\nu}\bar{\nabla}_{\alpha}\lambda^{\mu} \\
&\qquad + \lambda_{\sigma}\lambda_{\alpha}\lambda_{\nu}\lambda_{\mu}\bar{\nabla}^{\mu}V\bar{\nabla}^{\alpha}V + V^{2}\lambda_{\sigma}\lambda_{\nu}\bar{\nabla}^{\mu}\lambda_{\alpha}\bar{\nabla}^{\alpha}\lambda_{\mu}\Big] \\
&= 2\lambda_{\sigma}\lambda_{\nu}\left[\lambda^{\mu}\lambda^{\alpha}\bar{\nabla}_{\mu}V\bar{\nabla}_{\alpha}V + V^{2}\bar{\nabla}_{\alpha}\lambda^{\mu}\bar{\nabla}_{\mu}\lambda^{\alpha} - V^{2}\bar{\nabla}_{\alpha}\lambda^{\mu}\bar{\nabla}^{\alpha}\lambda_{\mu}\right]
\end{aligned} \tag{2.86}$$

we obtain

$$R_{(Q)\ \nu\sigma} \equiv R^{\mu}_{(Q)\ \nu\mu\sigma} = 2V\lambda_\nu\lambda_\sigma\Big[\lambda^\alpha\bar{\nabla}_\mu(\lambda^\mu\bar{\nabla}_\alpha V) - V\bar{\nabla}_\alpha\lambda^\mu\bar{\nabla}_\mu\lambda^\alpha V\bar{\nabla}_\alpha\lambda^\mu\bar{\nabla}^\alpha\lambda_\mu\Big] \tag{2.87}$$

Hence, transformation of the Ricci tensor in its covariant form is given by

$$R_{\nu\sigma} = \bar{R}_{\nu\sigma} + \bar{\nabla}_\mu\Delta^\mu{}_{\nu\sigma} + R_{(Q)\ \nu\sigma} \tag{2.88}$$

which involves both the linear and the quadratic terms. Transformation of the Ricci tensor in its mixed form is an even more interesting result and is given by

$$R^\alpha_{\ \sigma} = g^{\alpha\nu}R_{\nu\sigma} = \bar{R}^\alpha_{\ \sigma} - 2V\lambda^\alpha\lambda^\nu\bar{R}_{\nu\sigma} + \bar{\nabla}_\mu\Delta^{\mu\alpha}{}_\sigma - 2V\lambda^\alpha\lambda^\nu\bar{\nabla}_\mu\Delta^\mu{}_{\nu\sigma} + \bar{g}^{\alpha\nu}R_{(Q)\ \nu\sigma} \tag{2.89}$$

Now, we have

$$\begin{aligned}
\lambda^\nu\bar{\nabla}_\mu\Delta^\mu{}_{\nu\sigma} &= \bar{\nabla}_\mu(\lambda^\nu\Delta^\mu{}_{\nu\sigma}) - (\bar{\nabla}_\mu\lambda^\nu)\Delta^\mu{}_{\nu\sigma} \\
&= \bar{\nabla}_\mu\left(\lambda^\nu\bar{\nabla}_\nu(V\lambda^\mu\lambda_\sigma)\right) - \bar{\nabla}_\mu\lambda^\nu\left(-V\lambda_\sigma\bar{\nabla}^\mu\lambda_\nu + V\lambda_\sigma\bar{\nabla}_\nu\lambda^\mu\right) \\
&= \bar{\nabla}_\mu\left(\lambda^\nu\lambda^\mu\lambda_\sigma\bar{\nabla}_\nu V\right) - V\lambda_\sigma(\bar{\nabla}_\mu\lambda^\nu\bar{\nabla}_\nu\lambda^\mu - \bar{\nabla}_\mu\lambda^\nu\bar{\nabla}^\mu\lambda_\nu) \\
&= \lambda_\sigma\Big[\lambda^\nu\bar{\nabla}_\mu(\lambda^\mu\bar{\nabla}_\nu V) - V\bar{\nabla}_\mu\lambda^\nu\bar{\nabla}_\nu\lambda^\mu + V\bar{\nabla}_\mu\lambda^\nu\bar{\nabla}^\mu\lambda_\nu\Big]
\end{aligned} \tag{2.90}$$

which yields

$$-2V\lambda^\alpha\lambda^\nu\bar{\nabla}^\mu{}_{\nu\sigma} = -\bar{g}^{\alpha\nu}R_{(Q)\ \nu\sigma} \tag{2.91}$$

Therefore, transformation of $R^\alpha_{\ \sigma}$ becomes

$$R^\alpha_{\ \sigma} = \bar{R}^\alpha_{\ \sigma} - 2V\lambda^\alpha\lambda^\nu\bar{R}_{\nu\sigma} + \bar{\nabla}_\mu\Delta^{\mu\alpha}{}_\sigma \tag{2.92}$$

where there are no quadratic or higher order terms. The Ricci scalar curvature transforms as follows

$$\begin{aligned}
R &= \bar{R} - 2V\lambda^\alpha\lambda^\nu\bar{R}_{\alpha\nu} + \bar{\nabla}_\mu\Delta^{\mu\alpha}{}_\alpha \\
&= \bar{R} - 2V\lambda^\alpha\lambda^\nu\bar{R}_{\alpha\nu} + 2\bar{\nabla}_\mu\left(\lambda^\mu\bar{\nabla}_\alpha(V\lambda^\alpha)\right)
\end{aligned} \tag{2.93}$$

From these one can calculate the transformation of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, which in turn yields the transformation of the energy-momentum tensor $T_{\mu\nu}$.

2.3 Kerr-Schild-Kundt Metrics

Kundt class of spacetimes are defined by the existence of a null congruence such that the tangent vector field has no divergence, no shear and no twist.[18] If we start with the generalized Kerr-Schild metric,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu$$

where λ is null.

We see that divergence of λ is not necessarily zero. However, as in [19]-[20], if we introduce another vector ξ such that

$$(i)\ \nabla_\mu\lambda_\nu = \xi_{(\mu}\lambda_{\nu)} \qquad (ii)\ \xi_\mu\lambda^\mu = 0 \qquad (iii)\ \lambda^\mu\partial_\mu V = 0$$

then (i) and (ii) imply that λ is a geodesic vector with respect to both $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$.

$$\lambda^\mu\nabla_\mu\lambda_\nu = \lambda^\mu\xi_\mu\lambda_\nu + \lambda^\mu\xi_\nu\lambda_\mu = 0 \tag{2.94}$$

since λ is null and orthogonal to ξ. Similarly,

$$\begin{aligned}\lambda^\mu\bar{\nabla}_\mu\lambda_\nu &= \lambda^\mu\nabla_\mu\lambda_\nu + \left(\Gamma^\sigma{}_{\mu\nu} - \bar{\Gamma}^\sigma{}_{\mu\nu}\right)\lambda_\sigma \\ &= g^{\sigma\alpha}\Delta_{\alpha\mu\nu}\lambda_\sigma \\ &= -\lambda^\alpha\bar{\nabla}_\alpha(V\lambda_\mu\lambda_\nu) \\ &= -\lambda_\mu\lambda_\nu\lambda^\alpha\bar{\nabla}_\alpha V \\ &= 0\end{aligned} \tag{2.95}$$

where the last equality follows from the defining property (iii) and the equality before that follows from the geodesic property of λ. In addition to being a null geodesic vector with no-divergence, λ is also non-twisting. This result immediately follows from the scalar definition of twist: Twist of λ is:

$$\omega = \frac{1}{2}\nabla^\mu\lambda^\nu\nabla_{[\mu}\lambda_{\nu]} = 0 \tag{2.96}$$

Since $\nabla_\mu\lambda_\nu$ is symmetric by definition (in the Kerr-Schild-Kundt ansatz) and thus antisymmetric term $\nabla_{[\mu}\lambda_{\nu]}$ is 0.

Alternatively, one can reach the same result rom the vectoral definition of twist. Twist vector of λ is given by:

$$\begin{aligned}\omega^\mu &= \epsilon^{\mu\nu\rho\sigma}\lambda_\nu\nabla_\rho\lambda_\sigma \\ &= \epsilon^{\mu\nu\rho\sigma}\lambda_\nu\xi_{(\rho}\lambda_{\sigma)} \\ &= 0\end{aligned} \tag{2.97}$$

Hence, we see that the symmetric definition of $\nabla_\mu \lambda_\nu = \xi_{(\mu}\lambda_{\nu)}$ in the KSK ansatz leads λ to be non-twisting.

Another consequence of properties (i) and (ii) is the shearlessness of λ:

$$\nabla_\mu \lambda^\nu \nabla_\mu \lambda^\nu \nabla_\nu \lambda^\mu - (\nabla_\mu \lambda^\mu)^2 = 0 \tag{2.98}$$

We have already shown that λ has no divergence, $\nabla_\mu \lambda^\mu = 0$. From

$$\begin{aligned} &\bullet\ \nabla_\mu \lambda^\nu = \xi_\mu \lambda^\nu + \xi^\nu \lambda_\mu \ \ and \\ &\bullet\ \nabla^\mu \lambda_\nu = \nabla_\nu \lambda^\mu = \xi_\nu \lambda^\mu + \xi^\mu \lambda_\nu \end{aligned} \tag{2.99}$$

it can also be seen that

$$\nabla_\mu \lambda^\nu \nabla^\mu \lambda_\nu = 0 = \nabla_\mu \lambda^\nu \nabla_\nu \lambda^\mu \tag{2.100}$$

since λ is null and orthogonal to ξ, $\lambda^\mu \lambda_\mu = 0$, $\lambda^\mu \xi_\mu = 0$. Lastly, it is evident from property (ii) that the index of the ξ can be raised or lowered with respect to both metrics.

These results hold in the background spacetime as well. From:

$$\begin{aligned} \nabla_\mu \lambda^\nu &= \bar{\nabla}_\mu \lambda^\nu + \left(\Gamma^\nu_{\ \mu\sigma} - \bar{\Gamma}^\nu_{\ \mu\sigma}\right)\lambda^\sigma \\ &= \bar{\nabla}_\mu \lambda^\nu + g^{\nu\alpha}\Delta_{\alpha\mu\sigma}\lambda^\sigma \\ &= \bar{\nabla}_\mu \lambda^\nu + \left(\bar{g}^{\nu\alpha} - 2V\lambda^\nu\lambda^\alpha\right)\lambda^\sigma \bar{\nabla}_\sigma (V\lambda_\alpha\lambda_\mu) \\ &= \bar{\nabla}_\mu \lambda^\nu + \lambda^\sigma \bar{\nabla}_\sigma (V\lambda^\nu\lambda_\mu) \\ &= \bar{\nabla}_\mu \lambda^\nu + \lambda^\nu\lambda_\mu\lambda^\sigma \bar{\nabla}_\sigma V \end{aligned} \tag{2.101}$$

it is obvious that λ is nondiverging;

$$\nabla_\mu \lambda^\mu = \bar{\nabla}_\mu \lambda^\mu = 0 \tag{2.102}$$

nontwisting;

$$\begin{aligned} \nabla^\mu \lambda^\nu \nabla_{[\mu}\lambda_{\nu]} &= \left(\bar{\nabla}^\mu \lambda^\nu + \lambda^\mu\lambda^\nu\lambda^\sigma \bar{\nabla}_\sigma V\right)\nabla_{[\mu}\lambda_{\nu]} \\ &= \bar{\nabla}^\mu \lambda^\nu \nabla_{[\mu}\lambda_{\nu]} \\ &= \frac{1}{2}\bar{\nabla}^\mu \lambda^\nu \left(\bar{\nabla}_\mu \lambda_\nu + \lambda_\nu\lambda_\mu\lambda^\sigma \bar{\nabla}_\sigma V - (\mu \leftrightarrow \nu)\right) \\ &= \bar{\nabla}^\mu \lambda^\nu \bar{\nabla}_{[\mu}\lambda_{\nu]} \end{aligned} \tag{2.103}$$

$$\Rightarrow \quad \bar{\nabla}^\mu \lambda^\nu \bar{\nabla}_{[\mu}\lambda_{\nu]} = 0 \tag{2.104}$$

and shearless,

$$
\begin{aligned}
& 2\nabla^\mu\lambda^\nu\nabla_{(\mu}\lambda_{\nu)} - (\nabla_\mu\lambda^\mu)^2 = 0 \\
\Rightarrow\quad & 2\left(\bar{\nabla}^\mu\lambda^\nu + \lambda^\mu\lambda^\nu\lambda^\sigma\bar{\nabla}_\sigma V\right)\nabla_{(\mu}\lambda_{\nu)} - \left(\bar{\nabla}_\mu\lambda^\mu\right)^2 = 0 \\
\Rightarrow\quad & 2\bar{\nabla}^\mu\lambda^\nu\left(\bar{\nabla}_{(\mu}\lambda_{\nu)} + \lambda_{(\mu}\lambda_{\nu)}\lambda^\sigma\bar{\nabla}_\sigma V\right) - \left(\bar{\nabla}_\mu\lambda^\mu\right)^2 = 0 \\
\Rightarrow\quad & 2\bar{\nabla}^\mu\lambda^\nu\bar{\nabla}_{(\mu}\lambda_{\nu)} - \left(\bar{\nabla}_\mu\lambda^\mu\right)^2 = 0
\end{aligned}
\tag{2.105}
$$

with respect to $\bar{g}_{\mu\nu}$. Note that the calculations here follow from the fact that $\lambda^\mu\lambda^\nu\nabla_\mu\lambda_\nu = \lambda^\mu\lambda^\nu\bar{\nabla}_\mu\lambda_\nu = 0$ which in turn is a consequence of λ being a null geodesic vector with respect to both metrics.

Now, looking at the KSK ansatz, one can realize that there are many other different ansätze that may be used to describe the Kundt spacetimes. In other words, starting with

$$
g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu, \qquad \lambda^\mu\lambda_\mu = 0 \tag{2.106}
$$

The following ansätze

$$
\begin{aligned}
\rightarrow\quad & \nabla_\mu\lambda_\nu = \xi_{[\mu}\lambda_{\nu]}\ , \quad \xi^\mu\lambda_\mu = 0 \\
\rightarrow\quad & \nabla_\mu\lambda_\nu = \xi_\mu\lambda_\nu\ , \quad \xi^\mu\lambda_\mu = 0
\end{aligned}
\tag{2.107}
$$

also make λ a null geodesic vector field that is non-diverging, non-twisting and shearless.

$$
\begin{aligned}
& \bullet\ \lambda^\mu\nabla_\mu\lambda^\nu = 0\ , \quad \nabla_\mu\lambda^\mu = 0 \\
& \bullet\ \nabla^\mu\lambda^\nu\nabla_\mu\lambda_\nu = \nabla^\mu\lambda^\nu\nabla_\nu\lambda_\mu = 0 = \nabla^\mu\lambda^\nu\nabla_{[\mu}\lambda_{\nu]} = \nabla^\mu\lambda^\nu\nabla_{(\mu}\lambda_{\nu)}
\end{aligned}
\tag{2.108}
$$

For the KSK ansatz $\nabla_\mu\lambda_\nu = \xi_{[\mu}\lambda_{\nu]}$. We also see that $\nabla_\mu\lambda_\nu + \nabla_\nu\lambda_\mu = 0$ and hence λ is also a Killing vector field.

The KSK ansatz can be used to find exact solutions to the quadratic gravity theories and further, these are universal spacetimes; to wit, these spacetimes solve all generic gravity theories constructed from the Riemann tensor and its covariant derivatives. Lastly, by using the defining properties one can simplify the expressions for the transformations of the Riemann tensor and the Ricci tensor of the previous section for the KSK metrics, as done in [21]. In this case, the Riemann tensor itself turns out to be linear in V.

2.4 Kerr-Schild Forms of Some Exact Solutions

In this final section of this chapter, we will derive KS forms of some well-known exact solutions of GR.

2.4.1 The Schwarzschild Metric

The Schwarzschild metric in its standard form is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad (2.109)$$

where $f(r) = 1 - \frac{2M}{r}$. The above metric does not admit a KS form, $g_{\mu\nu} = \eta_{\mu\nu} + 2V\lambda_\mu\lambda_\nu$, and this can be observed as follows;

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 + (1-f)dt^2 + \frac{1-f}{f}dr^2 \\ &= \eta_{\mu\nu}dx^\mu dx^\nu + (1-f)dt^2 + \frac{1-f}{f}dr^2 \end{aligned} \qquad (2.110)$$

Note that

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}dx^\mu dx^\nu + 2V\lambda_\mu\lambda_\nu dx^\mu dx^\nu \qquad (2.111)$$

Then there exists a KS form if

$$2V\lambda_\mu\lambda_\nu dx^\mu dx^\nu = (1-f)\left[dt^2 + \frac{1}{f}dr^2\right] \qquad (2.112)$$

However, since the paranthesis is not a square, there are no such V and λ in these coordinates. In order to complete the RHS to a square, we make the following change of coordinates:

$$t' = \tilde{t} = t - A(r), \qquad r' = r, \qquad \theta' = \theta, \qquad \phi' = \phi \qquad (2.113)$$

which will give the necessary cross term $dtdr$ in the paranthesis, with

$$dt^2 = d\tilde{t}^2 + 2A'(r)d\tilde{A}dr + (A'(r))^2 dr^2 \qquad (2.114)$$

We have

$$ds^2 = -fd\tilde{t}^2 + \frac{1}{f}dr^2 + r^2 d\Omega^2 - 2fA'(r)d\tilde{t}dr - fA'^2(r)dr^2 \qquad (2.115)$$

Then,

$$\begin{aligned} g_{\mu\nu}(\tilde{t}, r, \theta, \phi)dx^\mu dx^\nu &= \eta_{\mu\nu}dx^\mu dx^\nu + (1-f)d\tilde{t}^2 + \left(\frac{1-f}{f} - fA'^2(r)\right)dr^2 \\ &\quad - 2fA'(r)d\tilde{t}dr \\ &= \eta_{\mu\nu}dx^\mu dx^\nu + (1-f)\Bigg[d\tilde{t}^2 - 2\frac{f}{1-f}A'(r)d\tilde{t}dr \\ &\qquad + \left(\frac{1}{f} - \frac{f}{1-f}A'^2(r)\right)dr^2\Bigg] \end{aligned} \tag{2.116}$$

For the expression in the paranthesis to be a square, it must be that

$$\left(\frac{f}{1-f}A'(r)\right)^2 = \frac{1}{f} - \frac{f}{1-f}A'^2(r) \tag{2.117}$$

which yields

$$\begin{aligned} A'^2(r)\frac{f}{1-f}\left(\frac{f}{1-f}+1\right) &= \frac{1}{f} \\ \Rightarrow \quad A'^2(r) &= \frac{(1-f)^2}{f^2} \\ \Rightarrow \quad A(r) &= \pm\int \frac{\frac{2M}{r}}{1-\frac{2M}{r}}dr \\ \Rightarrow \quad A(r) &= \pm 2M\int\frac{dr}{r-2M} \\ \Rightarrow \quad A(r) &= \pm 2M\ln(r-2M) + constant \end{aligned} \tag{2.118}$$

It follows that

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= \eta_{\mu\nu}dx^\mu dx^\nu + \frac{2M}{r}\left[d\tilde{t}^2 \mp 2d\tilde{t}dr + dr^2\right] \\ &= \eta_{\mu\nu}dx^\mu dx^\nu + \frac{2M}{r}(d\tilde{t} \mp dr)^2 \end{aligned} \tag{2.119}$$

and thus we have

$$2V\lambda_\mu\lambda_\nu dx^\mu dx^\nu = \frac{2M}{r}(d\tilde{t} \mp dr)^2 \tag{2.120}$$

which is now solvable for V and λ.

$$V = \frac{M}{r} \quad and \quad \lambda_\mu dx^\mu = \pm(d\tilde{t} \mp dr) \tag{2.121}$$

For $\lambda_\mu dx^\mu = \pm(d\tilde{t} - dr)$,

$$\begin{aligned} &\Rightarrow \ \lambda_\mu = (1,-1,0,0) \ \ or \ \ \lambda_\mu = (-1,1,0,0) \\ &\Rightarrow \ \lambda^\mu = \eta^{\mu\nu}\lambda_\nu = (-1,-1,0,0) \ \ or \ \ \lambda^\mu = (1,1,0,0) \end{aligned} \tag{2.122}$$

and for $\lambda_\mu dx^\mu = \pm(d\tilde{t} + dr)$

$$\begin{aligned} &\Rightarrow \;\; \lambda_\mu = (1,1,0,0) \;\; or \;\; \lambda_\mu = (-1,-1,0,0) \\ &\Rightarrow \;\; \lambda^\mu = \eta^{\mu\nu}\lambda_\nu = (-1,1,0,0) \;\; or \;\; \lambda^\mu = (1,-1,0,0) \end{aligned} \tag{2.123}$$

2.4.2 The Reissner-Nordström Metric

KS form of the Reissner-Nordström solution can be found in a similar way. Now we have

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{2.124}$$

Then, we get

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}dx^\mu dx^\nu + (1-f)[d\tilde{t}^2 \mp 2d\tilde{t}dr + dr^2] \\ &= \eta_{\mu\nu}dx^\mu dx^\nu + \left(\frac{2M}{r} - \frac{Q^2}{r^2}\right)(d\tilde{t} \mp dr)^2 \end{aligned} \tag{2.125}$$

Therefore, the null vector is the same as that of the Schwarzschild solution whereas the profile function V is given by

$$V_{RN} = \frac{M}{r} - \frac{Q^2}{2r^2} \tag{2.126}$$

2.4.3 The Kerr Metric

In Kerr's coordinates, the Kerr metric is given by

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right)(du + a\sin^2\theta d\phi)^2 + 2(du + a\sin^2\theta d\phi)(dr + a\sin^2\theta d\phi) \\ &+ (r^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned} \tag{2.127}$$

where r is not the radius coordinate but instead defined implicitly by

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \tag{2.128}$$

and $u = t + r$. The coordinates might also be written as,

$$\begin{aligned} x &= (r\cos\phi + a\sin\phi)\sin\theta \\ y &= (r\sin\phi - a\cos\phi)\sin\theta \\ z &= r\cos\theta \end{aligned} \tag{2.129}$$

Similar to the arguments in the Schwarzschild case, we expect the mass term to be proportional to a complete square and remaining terms to give the flat spacetime. We have

$$
\begin{aligned}
d\bar{s}^2 = ds^2|_{m=0} &= -\left(du + a\sin^2\theta d\phi\right)^2 + 2\left(du + a\sin^2\theta d\phi\right)\left(dr + a\sin^2\theta d\phi\right) \\
&\quad + \left(r^2 + a^2\cos^2\theta\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\
&= -du^2 + (r^2 + a^2)\sin^2 d\phi^2 + 2dudr + 2a\sin^2\theta d\phi dr \\
&\quad + (r^2 + a^2\cos^2\theta)d\theta^2 \\
&= -dt^2 + dr^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + 2a\sin^2\theta d\phi dr \\
&\quad + (r^2 + a^2\cos^2\theta)d\theta^2
\end{aligned}
\tag{2.130}
$$

With the above coordinate transformations, one can show that $d\bar{s}^2$ is the flat Minkoswki spacetime. We then have

$$
ds^2 = d\bar{s}^2 + \frac{2Mr}{r^2 + a^2\cos^2\theta}(du + a\sin^2\theta d\phi)^2 \tag{2.131}
$$

Hence, KS form is written with

$$
\bar{g} = \eta, \quad V = \frac{Mr}{r^2 + a^2\cos^2\theta}, \quad \lambda_\mu dx^\mu = du + a\sin^2\theta d\phi \tag{2.132}
$$

whence one can obtain the shearless null geodesic vector λ;

$$
\lambda_\mu = (1, 0, 0, a\sin^2\theta) \quad and \quad \lambda^\mu = (0, 1, 0, 0) \tag{2.133}
$$

Applying the coordinate transformations given above, the metric can be written in Cartesian coordinates.

$$
ds^2 = d\bar{s}^2 + \frac{2mr^3}{r^4 + a^2z^2}\left[dt + \frac{r(xdx + ydy)}{r^2 + a^2} + \frac{a(ydx - xdy)}{r^2 + a^2} + \frac{z}{r}dz\right]^2 \tag{2.134}
$$

where $d\bar{s}^2 = -dt^2 + dx^2 + dy^2 + dz^2$ and $x^2 + y^2 + z^2 = r^2 + a^2\sin^2\theta = r^2 + a^2(1 - \frac{z^2}{r^2})$ Therefore, we have

$$
V = \frac{Mr^3}{r^4 + a^2z^2}, \quad \lambda_\mu = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right) \tag{2.135}
$$

Similar to the relation between the Schwarzschild and the Reissner-Nordström solutions, where we found that the null vectors are the same but the profile function V changes in the KS form of the charged solution, the Kerr-Newman metric(charged rotating solution) can be obtained by choosing the same null vector and the following smooth function

$$V_{KN} = \frac{r^2}{r^4 + a^2 z^2}\left(Mr - \frac{Q}{2}\right) \tag{2.136}$$

Note that without rotation this reduces to $V_{RN} = \frac{1}{r^2}\left(Mr - \frac{Q}{2}\right)$ and the null vector is $\lambda_\mu = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$. Without charge, V_{RN} in turn reduces to $V_S = \frac{M}{r}$ whereas the null vector remains the same.

CHAPTER 3

KERR-SCHILD TRANSFORMATIONS

3.1 Motivations

KS ansatz is about the transformation of the metric, and the coordinates are not transformed.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu \tag{3.1}$$

At this point, one may ask for the transformations of coordinates leading to the generalized KS ansatz;

$$\bar{g}'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\bar{g}_{\alpha\beta}(x) = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu \tag{3.2}$$

This idea can be better grasped if we consider isometries and conformal transformations. For isometries we have

$$x^\mu \to x'^\mu = \phi(x^\mu),\ g_{\mu\nu} \xrightarrow{\phi^*} g'_{\mu\nu} = g_{\mu\nu} \tag{3.3}$$

where ϕ is a diffeomorphism that leaves the metric invariant. Now, a Weyl transformation is;

$$g_{\mu\nu} \to g'_{\mu\nu} = e^{2\sigma(x)}g_{\mu\nu} \tag{3.4}$$

which simply rescales the metric. On the other hand, we know that a conformal transformation is a particular coordinate transformation such that

$$x^\mu \to x'^\mu = \phi(x^\mu),\ \ g_{\mu\nu} \xrightarrow{\phi^*} g'_{\mu\nu} = e^{2\sigma(x)}g_{\mu\nu} \tag{3.5}$$

where the diffeomorphism ϕ is an angle-preserving smooth map. Hence, we see that a transformation of metric (Weyl transformation) is the result of a coordinate transformation(conformal transformation) and in this chapter we will study the diffeomorphisms leading to generalized KS ansatz and the related subjects.

3.2 Kerr-Schild Coordinate Transformations

Expression for the KS ansatz with null geodesics, without coordinates, is given by,

$$g = n + 2V\lambda \otimes \lambda \tag{3.6}$$

where $\lambda^\sharp = \eta^{-1}(\lambda, \cdot)$ is a null vector with respect to both metrics. The Generalized KS ansatz is,

$$\begin{aligned} g &= \bar{g} + 2V\lambda \otimes \lambda, \quad where \\ \bar{g}^{-1}(\lambda, \lambda) &= 0 = g^{-1}(\lambda, \lambda) \end{aligned} \tag{3.7}$$

Here, $V : M \to \mathbb{R}$ is a smooth function of spacetime coordinates, $V \in C^\infty(M)$[1], and M is the smooth manifold representing the spacetime. The one-form $\lambda \in \Lambda^1(M)$ is null and $\lambda^\sharp = \bar{g}^{-1}(\lambda, \cdot)$. The inverse metric is given as

$$g^{-1} = \bar{g}^{-1} - 2V\lambda^\sharp \otimes \lambda^\sharp \tag{3.8}$$

Any two metrics in the generalized KS ansatz are referred to as KS related metrics. Then, a diffeomorphism ϕ is called a Kerr-Schild transformation ($\phi : M \to M$, and hence an automorphism) if $g = \phi^*\bar{g}$ that is, the pullback of the background metric gives the full metric. Next, one can look for the one-parameter group of these transformations which is given by $(\Phi, \circ)$, with

$$\Phi = \{\phi_s \in Diff(M) : s \in \mathbb{R},\ \phi_s^*\bar{g} = \bar{g} + 2V_s\lambda \otimes \lambda\}$$

being the set of KS transformations and map composition being the group operation. The identity element is given by $Id_M = \phi_0,\ \ \phi_0 \circ \phi_s = \phi_s \circ \phi_0 = \phi_s$ and the inverse element of a given element ϕ_s is given by

$$\begin{aligned} \phi_s^{-1} = \phi_{-s} \quad \Rightarrow \quad & \phi_s \circ \phi_s^{-1} = \phi_s \circ \phi_{-s} = \phi_o \quad and \\ & \phi_s^{-1} \circ \phi_s = \phi_{-s} \circ \phi_s = \phi_o \end{aligned} \tag{3.9}$$

[1] In this chapter, definitions will be written in this way for the sake of brevity, eventhough they are local. To write these rigorously we can use open sets of the manifold $U \subset M$

Associativity property follows from the fact that map composition operation is associative. Then, lastly we need to look at the closure property. Now, since the flow ϕ is defined as a group action by

$$\begin{aligned} &\phi : \mathbb{R}\times M \to M, \\ &\phi(0,p) = p \quad and \quad \phi(s,\phi(t,p)) = \phi(s+t,p) \end{aligned} \tag{3.10}$$

where $s,t \in \mathbb{R}, p \in M$. Equivalently we can write

$$\phi_o(p) = p \;\; and \;\; (\phi_s \circ \phi_t)(p) = \phi_{s+t}(p) \tag{3.11}$$

As $p \in M$ is arbitrary, we must have

$$\phi_s \circ \phi_t = \phi_{s+t} \tag{3.12}$$

Thus, $G = (\phi, \circ)$ is indeed a one-parameter group of transformations if $\phi_s \circ \phi_t = \phi_{s+t}$

$$\begin{aligned} (\phi_s \circ \phi_t)^* \bar{g} &= \phi_t^* \circ \phi_s^* \bar{g} \\ &= \phi_t^*(\bar{g} + 2V_s \lambda \otimes \lambda) \\ &= \bar{g} + 2V_t \lambda \otimes \lambda + 2(\phi_t^* V_s)(\phi_t^* \lambda) \otimes (\phi_t^* \lambda) \end{aligned} \tag{3.13}$$

which is equal to

$$\phi_{s+t}^* \bar{g} = \bar{g} + 2V_{s+t} \lambda \otimes \lambda \tag{3.14}$$

if $\phi_t^* \lambda = U_t \lambda$, for some $U_t \in C^\infty(M)$, and $V_t + (\phi_t^* V_s) U_t^2 = V_{s+t}$. Hence, the KS group on M, $(\Phi, \circ)$, can be defined as the one-parameter group of KS transformations from M to M;

$$\Phi = \{\phi_s \in Diff(M) |\; s \in \mathbb{R}, \;\; \phi_s^* \bar{g} = \bar{g} + 2V_s \lambda \otimes \lambda, \;\; \phi_s^* \lambda = U_s \lambda\} \tag{3.15}$$

where $V_s, U_s \in C^\infty(M), V_o = 0 \;\; and \;\; V_t + (\phi_t^* V_s) U_t^2 = V_{s+t}$.

Now, let ξ be the infinitesimal generator of a KS group(ξ generates the flow ϕ). We have

$$\begin{aligned} \mathcal{L}_\xi \bar{g} = \lim_{s\to 0} \frac{\phi_s^*(\bar{g}_{\phi_s(P)}) - \bar{g}_p}{s} &= \lim_{s\to 0} \frac{(\phi_s^* \bar{g})_p - \bar{g}_P}{s} \\ &= \frac{d}{ds} (\phi_s^* \bar{g})_P |_{s=0} \end{aligned} \tag{3.16}$$

Hence,

$$
\begin{aligned}
\mathcal{L}_\xi \bar{g} &= 2\frac{dV_s}{ds}\Big|_{s=0} \lambda \otimes \lambda \\
&\equiv 2p\lambda \otimes \lambda \quad and \\
\mathcal{L}_\xi \lambda &= \frac{d}{ds}(\phi_s^* \lambda)\Big|_{s=0} = \frac{dU_s}{ds}\Big|_{s=0} \lambda \\
&\equiv \; m\lambda
\end{aligned}
\tag{3.17}
$$

These equations are called the KS equations for the metric and the transformation null direction, respectively, and in coordinates they are given as

$$
\begin{aligned}
\bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu = 2p\lambda_\mu \lambda_\nu \quad and \\
\xi^\mu \bar{\nabla}_\mu \lambda_\nu + (\bar{\nabla}_\nu \xi^\mu)\lambda_\mu = m\lambda_\nu
\end{aligned}
\tag{3.18}
$$

Solutions of the KS equations are called the Kerr-Schild vector fields (KSVFs) with respect to the transformation null direction λ[♯2]. One can show that a KSVF ξ has no divergence by taking trace of the KS metric equation.

Note that p and λ are not uniquely determined in the KS-metric equation:

$$
\begin{aligned}
&\mathcal{L}_\xi \bar{g} = 2p\lambda \otimes \lambda \;\; \longrightarrow \;\; 2p\lambda \otimes \lambda \;\; under \;\; following \;\; transformations \\
&\lambda \to \lambda' = N\lambda, \;\; p \to p' = \frac{p}{N^2}, \;\; N \neq 0 \; N \in C^\infty(M)
\end{aligned}
\tag{3.19}
$$

KS-λ equation becomes:

$$
\begin{aligned}
\mathcal{L}_\xi \lambda' = \lambda \mathcal{L}_\xi N + N\mathcal{L}_\xi \lambda = \lambda \xi(N) + Nm\lambda& \\
= \frac{\lambda'}{N}\xi N + m\lambda'& \\
= \left(m + \frac{1}{N}\xi N\right)\lambda'& \\
\equiv m'\lambda'&
\end{aligned}
\tag{3.20}
$$

These are tantamount to gauge transformations in the sense that under the following transformations

$$
\lambda \to \lambda' = N\lambda, \quad p \to p' = \frac{p}{N^2}, \quad m \to m' = \left(m + \frac{1}{N}\xi N\right)
\tag{3.21}
$$

the KS equations remain invariant and for this reason p is called the gauge of the metric and m is called the gauge of the null direction λ. One can immediately realize that if $h = 0$, then the KSVF becomes a Killing vector field. Any KSVF which is not

[2] For brevity, from now on we will write "the null direction λ"

a Killing vector field is called a proper KSVF. Now, using $g = \bar{g} + 2V\lambda \otimes \lambda$ and the KS equations, we obtain

$$\begin{aligned}
\mathcal{L}_\xi g &= \mathcal{L}_\xi\big(\bar{g} + 2V\lambda \otimes \lambda\big) \\
&= 2p\lambda \otimes \lambda + 2(\mathcal{L}_\xi V)\lambda \otimes \lambda + 2V\big[(\mathcal{L}\lambda) \otimes \lambda + \lambda \otimes (\mathcal{L}_\xi \lambda)\big] \\
&= 2p\lambda \otimes \lambda + 2(\xi V)\lambda \otimes \lambda + 4mV\lambda \otimes \lambda \\
&= 2\big(p + \xi V + 2mV\big)\lambda \otimes \lambda
\end{aligned} \tag{3.22}$$

Hence, any KSVF of $\bar{g}$ with respect to the null direction λ is a KSVF of g with respect to λ. Clearly, converse statement also holds and thus the two KS-related metrics have the same KSVFs with respect to λ. Incidentally, we observe that the KSVF ξ is a Killing Vector field, locally, of g if $p + \xi V + 2mV = 0$. Recall that, given a metric $\bar{g}$, the KSVFs of $\bar{g}$ are defined with respect to a null direction λ. Next, we consider the set of KSVFs for a metric $\bar{g}$ on M, and a null direction λ:

$$S_{\bar{g},\lambda} = \{\xi \in \mathfrak{X}(M) | \mathcal{L}_\xi \bar{g} = 2p\lambda \otimes \lambda, \quad \mathcal{L}_\xi \lambda = m\lambda \ \ for\ some\, p, m \in C^\infty(M)\}$$

$(S_{\bar{g},\lambda}, +, \cdot)$ is a vector space over the field of reals. Where the binary operation $+$ is the standard addition operations and the scalar multiplication $\cdot$ is the standard multiplication operation. It is clear that $S_{\bar{g},\lambda}$ satisfies the axioms of a vector space; 0 is the identity element: $\xi + 0 = 0 + \xi = \xi$, $-\xi$ is the inverse of ξ : $\xi + (-\xi) = (-\xi) + \xi = 0$, $commutativity: \ \ \xi + \eta = \eta + \xi$, $associativity: \ \ \xi + (\eta + \sigma) = (\xi + \eta) + \sigma$, $distributive\, properties: \ \ (c_1 + c_2)\xi = c_1\xi + c_2\xi \ \ and \ \ c(\xi + \eta) = c\xi + c\eta$ for all c $\in \mathbb{R}$ and $\xi, \eta, \sigma \in S_{\bar{g},\lambda}$. It can also be seen that $S_{\bar{g},\lambda}$ is closed under the addition operation. Given any $\xi, \eta \in \ S_{\bar{g},\lambda}$,

$$\begin{aligned}
\mathcal{L}_\xi \bar{g} &= 2p\lambda \otimes \lambda, \qquad \mathcal{L}_\xi \lambda = m\lambda \\
\mathcal{L}_\eta \bar{g} &= 2q\lambda \otimes \lambda, \qquad \mathcal{L}_\eta \lambda = n\lambda
\end{aligned} \tag{3.23}$$

$\xi + \eta \in \ S_{\bar{g},\lambda}$, since

$$\begin{aligned}
&\mathcal{L}_{\xi+\eta} \bar{g} = \mathcal{L}_\xi \bar{g} + \mathcal{L}_\eta \bar{g} = 2(p + q)\lambda \otimes \lambda \\
&\mathcal{L}_{\xi+\eta} \lambda = 2(m + n)\lambda
\end{aligned} \tag{3.24}$$

However, if we are given any two vector fields $\xi, \eta \ \in \ S_{\bar{g}}$; such that

$$\mathcal{L}_\xi \bar{g} = 2p\lambda \otimes \lambda, \ \ \mathcal{L}_\eta \bar{g} = 2qk \otimes k \tag{3.25}$$

we have $\mathcal{L}_{\xi+\eta}\bar{g} = 2p\lambda \otimes \lambda + 2qk \otimes k$. Thus, if $\bar{g}^{-1}(\lambda, k) \neq 0$ then $\xi + \eta \notin S_{\bar{g}}$. Note that in these calculations we tacitly assumed that λ and k are null one-forms. Now, one can readily observe that

$$S_{\bar{g},\lambda} = S_{\bar{g},\tilde{\lambda}} \quad where \quad \tilde{\lambda} = h\lambda, 0 \neq h \in C^{\infty}(M) \tag{3.26}$$

In fact, we can define an equivalence relation $\sim$ over the set of null one-forms $\Lambda^1_{null}(M)$ by; $\sim$: is equal to a non-zero smooth function multiple of.

It can easily be seen that $\sim$ is

- Reflective, since $\lambda \sim \lambda$
- Symmetric, since $\lambda \sim \omega \Rightarrow \omega \sim \lambda$
- Transitive, since if $\lambda \sim \omega$ and $\omega \sim k$, then $\lambda \sim k$.

for all $\lambda, \omega, k \in \Lambda^1_{null}(M)$. Equivalence class of λ is given by.

$$[\lambda] = \left\{\omega \in \Lambda^1_{null}(M) \mid \omega \sim \lambda\right\} \tag{3.27}$$

Note, in particular, that

$[0] = \{0\}$ (the equivalence class of $\lambda = 0$ has only the $\lambda = 0$ elements)

Hence for the quotient space we have

$\Lambda^1_{null}(M)/_{\sim} \; = \; \{[\lambda] \mid \lambda \in \Lambda^1_{null}(M)\}$

whence we obtain

$$S_{\bar{g}} = \bigcup_{i \in I} S_{\bar{g},\lambda_i} = \left(\bigcup_{i \in I - \{0\}} S_{\bar{g},\lambda_i}\right) \cup S_{\bar{g},0} \tag{3.28}$$

where λ_i is a representative of $[\lambda_i]$ for all $i \neq 0$ and $\lambda_0 = 0$ is a representative of the set of Killing Vector fields of $\bar{g}$. The latter equality in the above equation defines a partition for $S_{\bar{g}}$; that is, it gives a pairwise disjoint, except for the additive identity, union of $S_{\bar{g}}$. An alternative equivalence relation can be defined over the set $\{S_{\bar{g},\lambda} \mid \lambda \in \Lambda^1_{null}(M)\}$, $\sim$: is equal to

which is clearly reflective, symmetric and transitive. Equivalence class of $S_{\bar{g},\lambda}$ is given by

$$[S_{\bar{g},\lambda}] = \{S_{\bar{g},\omega} \mid \omega = h\lambda, for\ some\ h \in C^{\infty}(M),\ h \neq 0\} \tag{3.29}$$

and it gives the same result.

The set of all smooth vector fields on M, $\mathfrak{X}(M)$, has the structure of a vector space and further it becomes a Lie algebra when equipped with the additional binary operation (a multiplication), the Lie bracket [,]. As it is bilinear and anti-commutative and satisfies the Jacobi identity for all $X \in \mathfrak{X}(M)$, these properties hold for all $Y \in (S_{\bar{g},\lambda}, +, \cdot, \mathbb{R})$. Then, the vector space $(S_{\bar{g},\lambda}, +, \cdot, \mathbb{R})$ forms a Lie algebra if it is closed under the Lie bracket [,] (i.e., if [,] is a binary operation); given $\xi, \gamma \in S_{\bar{g},\lambda}$ such that

$$\begin{aligned} \mathcal{L}_\xi \bar{g} &= 2p\lambda \otimes \lambda, \qquad \mathcal{L}_\xi \lambda = m\lambda \\ \mathcal{L}_\gamma \bar{g} &= 2q\lambda \otimes \lambda, \qquad \mathcal{L}_\gamma \lambda = n\lambda \end{aligned} \tag{3.30}$$

we have $[\xi, \gamma] \in (S_{\bar{g},\lambda}, +, \cdot, \mathbb{R})$ if

$$\begin{aligned} &\mathcal{L}_{[\xi,\gamma]} \bar{g} = 2y\lambda \otimes \lambda \\ &\mathcal{L}_{[\xi,\gamma]} \lambda = u\lambda \quad for\ some\ y\ and\ u \in C^\infty(M) \end{aligned} \tag{3.31}$$

The following calculations show that the set of KSVFs is indeed closed under the Lie bracket:

$$\begin{aligned} \mathcal{L}_{[\xi,\gamma]} \bar{g} &= \mathcal{L}_\xi \mathcal{L}_\gamma \bar{g} - \mathcal{L}_\gamma \mathcal{L}_\xi \bar{g} \\ &= \mathcal{L}_\xi \left(2p\lambda \otimes \lambda\right) - \mathcal{L}_\gamma \left(2q\lambda \otimes \lambda\right) \\ &= 2\left(\xi p + 2pm - \gamma q - 2qn\right) \lambda \otimes \lambda \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} \mathcal{L}_{[\xi,\gamma]} \lambda = \mathcal{L}_\xi \mathcal{L}_\gamma \lambda - \mathcal{L}_\gamma \mathcal{L}_\xi \lambda &= \mathcal{L}_\xi (n\lambda) - \mathcal{L}_\gamma (m\lambda) \\ &= (\xi n + nm - \gamma m - mn)\lambda \\ &= (\xi n - \gamma m)\lambda \end{aligned} \tag{3.33}$$

Therefore, $(S_{\bar{g},\lambda}, +, \cdot, [\ ,\ ])$ over $\mathbb{R}$ is a Lie algebra which will denote simply by $\mathfrak{K}_\lambda$. We have

$$\mathfrak{K} = \bigoplus_{i \in I} \mathfrak{K}_{\lambda_i} \ \ where\ \ \mathfrak{K}_{\lambda_0} = \mathfrak{K}_0 = Killing\ \ algebra \tag{3.34}$$

which is not a finite dimensional Lie algebra in general. Note that, we have $\mathcal{L}_\xi \bar{g} = 0$ for Killing vector fields and $\mathcal{L}_\xi \bar{g} = \Psi g$ for conformal Killing vector fields where Ψ is a smooth function. In both cases, the vector field ξ is independent of a null direction λ whereas for KSVFs ξ is defined with respect to λ. For this reason, unlike the cases of Killing algebra and conformal Killing algebra, the Lie algebra of KSVFs can be infinite dimensional. Now, consider a proper non-zero KSVF ξ, parallel to the null direction. $\xi = \Psi \lambda^\sharp$ for smooth function $\Psi \neq 0$. By definition we have

$$\begin{aligned} \mathcal{L}_\xi \bar{g}_{\mu\nu} = \mathcal{L}_{\Psi\lambda^\sharp} \bar{g}_{\mu\nu} &= \Psi \lambda^\alpha \bar{\nabla}_\alpha \bar{g}_{\mu\nu} + \bar{\nabla}_\mu (\Psi \lambda^\alpha) \bar{g}_{\alpha\nu} + \bar{\nabla}_\nu (\Psi \lambda^\alpha) \bar{g}_{\mu\alpha} \\ &= \bar{\nabla}_\mu (\Psi \lambda_\nu) + \bar{\nabla}_\nu (\Psi \lambda_\mu) \end{aligned} \tag{3.35}$$

and

$$\begin{aligned}\mathcal{L}_\xi \lambda_\mu = \mathcal{L}_{\Psi\lambda^\sharp}\lambda_\mu &= \Psi\lambda^\alpha\bar{\nabla}_\alpha\lambda_\mu + \bar{\nabla}_\mu(\Psi\lambda^\alpha)\lambda_\alpha \\ &= \Psi\lambda^\alpha\bar{\nabla}_\alpha\lambda_\mu \\ &= \Psi\mathcal{L}_{\lambda^\sharp}\lambda_\mu\end{aligned} \tag{3.36}$$

From these we get

$$\begin{aligned}2p\lambda_\mu\lambda_\nu &= \bar{\nabla}_\mu(\Psi\lambda_\nu) + \bar{\nabla}_\nu(\Psi\lambda_\mu) \\ &= \Psi(\bar{\nabla}_\mu\lambda_\nu + \bar{\nabla}_\nu\lambda_\mu) + \lambda_\nu\partial_\mu\Psi + \lambda_\mu\partial_\nu\Psi \\ &= \Psi\mathcal{L}_{\lambda^\sharp}\bar{g} + \lambda_\nu\lambda_\nu\partial_\mu\Psi + \lambda_\mu\partial_\nu\Psi\end{aligned} \tag{3.37}$$

and

$$m\lambda_\mu = \Psi\lambda^\alpha\bar{\nabla}_\alpha\lambda_\mu \tag{3.38}$$

where we used the metric compatibility property of $\bar{g}$ and the nullity of λ. Now we can write $\mathcal{L}_{\lambda^\sharp}\bar{g}$ in terms λ, p and Ψ:

$$\begin{aligned}\mathcal{L}_{\lambda^\sharp}\bar{g}_{\mu\nu} &= 2\frac{p}{\Psi}\lambda_\mu\lambda_\nu - \lambda_\nu\frac{1}{\Psi}\partial_\mu\Psi - \lambda_\mu\frac{1}{\Psi}\partial_\nu\Psi \\ &= \Big(\frac{p}{\Psi}\lambda_\mu - \partial_\mu \ln|\Psi|\,\lambda_\nu\Big) + \lambda_\mu\Big(\frac{p}{\Psi}\lambda_\nu - \partial_\nu\ln|\Psi|\Big)\end{aligned} \tag{3.39}$$

Introducing $\omega \in \Lambda^1(M)$, $\omega := \frac{p}{\Psi}\lambda - d\ln|\Psi|$ we obtain

$$\begin{aligned}&\mathcal{L}_{\lambda^\sharp}\bar{g} = \omega\otimes\lambda + \lambda\otimes\omega \;\; and \\ &\mathcal{L}_{\lambda^\sharp}\lambda = \frac{m}{\Psi}\lambda\end{aligned} \tag{3.40}$$

Now, from $\lambda^\mu\bar{\nabla}_\mu\lambda_\nu = \frac{m}{\Psi}\lambda_\nu$ we see that $\lambda^\sharp$ is geodesic vector, albeit not an affinely parametrized one. Further, from

$$\bar{\nabla}_\mu\lambda_\nu + \bar{\nabla}_\nu\lambda_\mu = \left(\frac{p}{\Psi}\lambda_\mu - \partial_\mu\ln|\Psi|\right)\lambda_\nu + \lambda_\mu\left(\frac{p}{\Psi}\lambda_\nu - \partial_\nu\ln|\Psi|\right) \tag{3.41}$$

we obtain, by taking trace of both side, that

$$\begin{aligned}2\bar{\nabla}^\mu\lambda^\nu\bar{\nabla}_{(\mu}\lambda_{\nu)} &= -\lambda_\mu\bar{\nabla}^\mu\lambda^\nu\partial_\nu\ln|\Psi| \\ &= -\frac{m}{\Psi}\lambda^\nu\partial_\nu\ln|\Psi|\end{aligned} \tag{3.42}$$

Conversely, given a null one-form λ and a proper KSVF $\xi \neq 0$ such that

$$\mathcal{L}_{\lambda^\sharp}\bar{g} = \omega\otimes\lambda + \lambda\otimes\omega \quad and \quad \mathcal{L}_\xi\bar{g} = 2p\lambda\otimes\lambda \tag{3.43}$$

where ω is a one-form, $\omega_\mu = \frac{p}{\Psi}\lambda_\mu - \partial_\mu \ln|\Psi|$ with $0 \neq \Psi \in C^\infty(M), p \in C^\infty(M)$, we have

$$\begin{aligned}
\mathcal{L}_{\lambda^\sharp} g_{\mu\nu} &= \left(\frac{p}{\Psi}\lambda_\mu - \partial_\mu \ln|\Psi|\right)\lambda_\nu + \lambda_\mu\left(\frac{p}{\Psi}\lambda_\nu - \partial_\nu \ln|\Psi|\right) \\
\Rightarrow \Psi(\bar{\nabla}_\mu\lambda_\nu + \bar{\nabla}_\nu\lambda_\mu) &= 2p\lambda_\mu\lambda_\nu - \lambda_\nu\partial_\mu\Psi - \lambda_\mu\partial_\nu\Psi \\
\Rightarrow \mathcal{L}_\xi \bar{g}_{\mu\nu} &= \Psi(\bar{\nabla}_\mu\lambda_\nu + \bar{\nabla}_\nu\lambda^\mu) + \lambda_\nu\partial_\mu\Psi + \lambda_\mu\partial_\nu\Psi \qquad (3.44) \\
&= \bar{\nabla}_\mu(\Psi\lambda_\nu) + \bar{\nabla}_\nu(\Psi\lambda_\mu) \\
&= \mathcal{L}_{\Psi\lambda^\sharp}\bar{g}_{\mu\nu}
\end{aligned}$$

Hence, we obtain

$$\mathcal{L}_\xi \bar{g} = \mathcal{L}_{\Psi\lambda^\sharp}\bar{g} \quad \Rightarrow \quad \mathcal{L}_{\xi-\Psi\lambda^\sharp}\bar{g} = 0 \tag{3.45}$$

which holds either if $\xi = \Psi\lambda^\sharp$ or if $\xi - \Psi\lambda^\sharp = \gamma$, γ being a Killing vector field.
The discussions in this chapter can be applied to several specific cases such as 2-dimensional spacetimes, flat spacetime ($\bar{g} = \eta$) etc.

CHAPTER 4

CLASSICAL DOUBLE COPY

The double copy is a proposed correspondence between gauge theories and gravity theories which holds at tree level. This duality is observed by replacing the kinematic factors of the gravity amplitudes by the color factors of the Yang-Mills scattering amplitudes. The result turns out to be that the gravity amplitudes are squares of the scattering amplitudes of Yang-Mills theories and for this reason Yang-Mills theories are called the single copies of the gravity theories. In the case of the double copy for classical theories, the Yang-Mills solutions and the solutions of general relativity are related where the Kerr-Schild ansatz provides a natural framework for examining this idea. For the KS ansatz, there are no dilaton or axion fields since $h_{\mu\nu} = 2V\lambda_\mu\lambda_\nu$ is symmetric and traceless. We will now continue with a brief review of Yang-Mills theory:
Yang-Mills theory is a gauge theory with $SU(N)$ as its symmetry group. The Lagrangian is

$$\mathcal{L}_{YM} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \tag{4.1}$$

where the field strength tensor is given by

$$[D_\mu, D_\nu] = -igF^a_{\mu\nu}t^a \tag{4.2}$$

Here, $D_\mu = \partial_\mu - igA^a_\mu t^a$ is the covariant derivative, g is the coupling constant, t^a are the generators of the Lie algebra associated to the Lie group of gauge symmetries and A^a_μ are gauge vector fields. We obtain

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{4.3}$$

where f^{abc}, defined by $\left[t^a, t^b\right] = if^{abc}t^c$, are the structure constants. As for the equations of motion, we have

$$\frac{\partial \mathcal{L}_{YM}}{\partial A^e_\mu} - \partial_\nu \frac{\partial \mathcal{L}_{YM}}{\partial(\partial_\nu A^e_\mu)} = 0 \tag{4.4}$$

Now since

$$\begin{aligned}
\partial_\nu \left(\frac{\partial \mathcal{L}_{YM}}{\partial(\partial_\nu A^e_\mu)} \right) = \partial_\nu \left(\frac{\partial(-\frac{1}{4}(F^a)^2)}{\partial(\partial_\nu A^e_\mu)} \right) &= -\frac{1}{4} \cdot 2 \cdot \partial_\nu \left(F^{a\rho\sigma} \frac{\partial F^a_{\rho\sigma}}{\partial(\partial_\nu A^e{}_\mu)} \right) \\
&= -\frac{1}{2} \partial_\nu \Big(F^{a\rho\sigma} \big(\delta_\rho{}^\nu \delta_\sigma{}^\mu - \delta_\sigma{}^\nu \delta_\rho{}^\mu \big) \delta^e_a \Big) \\
&= -\partial_\nu F^{e\nu\mu}
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A^e{}_\mu} &= -\frac{1}{4} \cdot 2 \cdot F^{a\rho\sigma} \frac{\partial F^a_{\rho\sigma}}{\partial A^e{}_\mu} \\
&= -\frac{1}{2} F^{a\rho\sigma} g f^{abc} (\delta_e{}^b \delta_\rho{}^\mu A^c_\sigma + A^b_\rho \delta^\mu{}_\sigma \delta^c{}_e) \\
&= -\frac{1}{2} g (f^{aec} F^{a\mu\sigma} A^c{}_\sigma + f^{abe} F^{a\rho\mu} A^b{}_\rho) \\
&= -g f^{aec} F^{a\mu\nu} A^c{}_\nu \\
&= g f^{eca} A^c{}_\nu F^{a\nu\mu}
\end{aligned} \tag{4.6}$$

where we obtained a factor of 2 in the paranthesis since

$$f^{abe} F^{a\rho\mu} A^b{}_\rho = (-f^{aeb})(-F^{a\mu\nu}) A^b{}_\nu$$

The equations of motion are:

$$\partial_\nu F^{a\nu\mu} + g f^{abc} A^b{}_\nu F^{c\nu\mu} = 0 \tag{4.7}$$

Maxwell's theory is an Abelian theory, with $U(1)$ as its symmetry group, and thus $f^{abc} = 0$. In this case, the equations of motion are; with no source term,

$$\partial_\nu F^{\nu\mu} = 0$$

and with source term, $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu J^\mu$,

$$-J^\mu + \partial_\nu F^{\nu\mu} = 0$$

At this point we return to the vacuum field equations of GR with the KS-ansatz with geodesic property. Recall that

$$R^{\mu}_{\ \nu} = \bar{R}^{\mu}_{\ \nu} - 2V\lambda^{\mu}\lambda^{\sigma}\bar{R}_{\sigma\nu} + \bar{\nabla}_{\sigma}\Delta^{\sigma\mu}_{\nu}$$

and

$$R = \bar{R} - 2V\lambda^{\mu}\lambda^{\nu}\bar{R}_{\mu\nu} + 2\bar{\nabla}_{\mu}(\lambda^{\mu}\bar{\nabla}_{\nu}(V\lambda^{\nu}))$$

When the background metric is flat ($\bar{g} = n$), these reduce to

$$\begin{aligned} R^{\mu}_{\ \nu} &= \frac{1}{2}\partial_{\sigma}\Big[- \partial^{\sigma}(V\lambda^{\mu}\lambda_{\nu}) + \partial^{\mu}(V\lambda^{\sigma}\lambda_{\nu}) + \partial_{\nu}(V\lambda^{\sigma}\lambda^{\mu})\Big] \\ R &= 2\partial_{\mu}\big(\lambda^{\mu}\partial_{\nu}(V\lambda^{\nu})\big) \end{aligned} \tag{4.8}$$

since $\bar{R}_{\mu\nu} = \bar{R} = 0$. From the vacuum field equations above, it is possible to extract the equations of motion of an Abelian gauge theory. In order to obtain 4 equations of gauge theory out of 10 equations of gravity, an index of $R^{\mu}_{\ \nu}$ is chosen. Then

$$R^{\mu}_{\ 0} = \frac{1}{2}\partial_{\sigma}\left[-\partial_{\sigma}(V\lambda^{\mu}\lambda_{0}) + \partial^{\mu}(V\lambda^{\sigma}\lambda_{0}) + \partial_{0}(V\lambda^{\sigma}\lambda^{\mu})\right] \tag{4.9}$$

Now, one can observe that if the term $\partial_0(V\lambda^{\sigma}\lambda^{\mu})$ vanishes and λ_0 is taken to be a constant then the paranthesis on the RHS reduces to an expression that is very similar to the field strength. To this end, we assume $\lambda_0 = 1$ and $\partial_0 V = \partial_0\lambda^{\mu} = 0$ i.e. no explicit time dependence. Then

$$R^{\mu}_{\ 0} = \frac{1}{2}\partial_{\sigma}\big(\partial_{\mu}(V\lambda^{\sigma}) - \partial^{\sigma}(V\lambda^{\mu})\big) \tag{4.10}$$

Further, we replace the smooth function V with $\phi/2$ where ϕ is a scalar field. This implies that $\phi\lambda^{\mu}$ can be considered as the gauge vector field A_{μ}. We have

$$\begin{aligned} R^{\mu}_{\ 0} &= \frac{1}{4}\partial_{0}(\partial^{\mu}(\phi\lambda^{\sigma}) - \partial^{\sigma}(\phi\lambda^{\mu})) \\ &= \frac{1}{4}\partial_{\sigma}(\partial^{\mu}A^{\sigma} - \partial^{\sigma}A^{\mu}) \\ &= \frac{1}{4}\partial_{\sigma}F^{\mu\sigma} \end{aligned} \tag{4.11}$$

Hence,

$$\begin{aligned} &\partial_{\sigma}F^{\sigma\mu} = 0 \quad with \quad A^{\mu} = \phi\lambda^{\mu} \\ &where\ \ \partial_0\phi = \partial_0\lambda^{\mu} = 0\ \ and\ \ \lambda_0 = 1 \end{aligned} \tag{4.12}$$

This leads us to consider gravity as a double copy of gauge theory. The remaining field equations are given by

$$\begin{aligned} R^{i}_{\ j} &= \frac{1}{2}\partial_{\sigma}\left[-\partial^{\sigma}(V\lambda^{i}\lambda_{j}) + \partial^{i}(V\lambda^{\sigma}\lambda_{j}) + \partial_{j}(V\lambda^{k}\lambda^{i})\right] \\ &= \frac{1}{2}\partial_{k}\left[-\partial^{k}(V\lambda^{i}\lambda_{j}) + \partial^{i}(V\lambda^{k}\lambda_{j}) + \partial_{j}(V\lambda^{k}\lambda^{i})\right] \end{aligned} \tag{4.13}$$

In particular, $\mu = 0$ equation of $R^{\mu}_{\ 0} = 0$ is

$$\begin{aligned} R^{0}_{\ 0} &= \frac{1}{4}\partial_\sigma \left(\partial^0(\phi\lambda^\sigma) - \partial^\sigma(\phi\lambda^0)\right) \\ &= -\frac{1}{4}\partial_\sigma\partial^\sigma(\phi) \\ &= -\frac{1}{4}\nabla^2\phi \end{aligned} \tag{4.14}$$

Therefore, the scalar field satisfies $\nabla^2\phi = 0$. One may interpret the gauge field $A_\mu = \phi\lambda_\mu$ as the single copy of graviton field $h_{\mu\nu} = \phi\lambda_\mu\lambda_\nu$ and the scalar field ϕ as the zeroth copy. More generally,

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{2}\kappa^2\phi\lambda_\mu\lambda_\nu, \qquad G_{\mu\nu} = \frac{1}{2}\kappa^2 T_{\mu\nu} \tag{4.15}$$

and

$$A^{a}_{\ \mu} = c^a\phi\lambda_\mu, \ \ \Phi^{aa'} = c^a\tilde{c}^{a'}\phi \tag{4.16}$$

where $\kappa^2 = 16\pi G$ and $\Phi^{aa'}$ is the biadjoint scalar field with the following equation of motion,

$$\partial^2\Phi^{aa'} - yf^{abc}f^{a'b'c'}\Phi^{bb'}\Phi^{cc'} = 0 \tag{4.17}$$

and $A^{a}_{\ \mu}$ is a solution to

$$\partial_\nu F^{a\nu\mu} + gf^{abc}A^{b}_{\ \nu}F^{c\nu\mu} = 0$$

Note that the term $yf^{abc}f^{a'b'c'}\Phi^{bb'}\Phi^{cc'}$ vanishes since $\Phi^{bb'}\Phi^{cc'}$ is symmetric in $b' \leftrightarrow c'$ whereas the structure constants f^{abc} and $f^{a'b'c'}$ are antisymmetric. Hence, we simply have

$$\partial^2\Phi^{aa'} = 0 \tag{4.18}$$

which is the $R^{0}_{\ 0}$ equation. Essentially, by replacing one of the null vectors λ_ν from the graviton field $h_{\mu\nu} = \frac{1}{2}\kappa\phi\lambda_\mu\lambda_\nu$ with c^a, we obtain the gauge field $A^{a}_{\ \mu} = c^a\phi\lambda_\mu$ and repeating this we get the biadjoint scalar field $\Phi^{aa'} = c^a\tilde{c}^{a'}\phi$.

As an example, consider the Schwarzschild solution

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu} \tag{4.19}$$

for which we have the following KS-form;

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2GM}{r}\lambda_\mu\lambda_\nu \tag{4.20}$$

where $\lambda_\mu = (-1,1,0,0)$ in spherical coordinates and $\lambda_\mu = (-1, \frac{\vec{x}}{r})$ in Cartesian coordinates. We get

$$\begin{aligned}(8\pi G)\phi = \frac{2GM}{r} \Rightarrow A_\mu = \frac{\kappa}{2}\phi\lambda_\mu &= \frac{\kappa}{2}\frac{M}{4\pi r}\lambda_\mu \\ &= g\frac{c_a t^a}{4\pi r}\lambda_\mu\end{aligned} \tag{4.21}$$

In order to have this correspondence we replace mass with charge and gravitational constant G with the gauge coupling constant g as follows: $M \leftrightarrow c_a t^a$ and $\frac{1}{2}\kappa \leftrightarrow g$. In passing, we see that the gauge vector field A_μ can be gauge transformed such that the transformed gauge field describes a Coulomb charge. There are many examples of this correspondence including the axially symmetric stationary rotations solution, pp-waves etc.

One way to generalize this method is to use multi-KS ansatz

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi_1\lambda_\mu\lambda_\nu + \phi_2 k_\mu k_\nu \tag{4.22}$$

where $\lambda^2 = k^2 = \lambda\cdot k = 0$. However, this has the drawback that the field equations are not linear in the graviton field.[22] One should also note that two gauge equivalent solutions of the gauge theory, with the same source, may lead to different gravity solutions; one with a dilaton field and one without a dilaton field.

CHAPTER 5

CONCLUSIONS

We began with an investigation of the generalized Kerr-Schild spacetimes and discussed various topics related to it such as the algebraic classification of generalized KS metrics, a derivation of the properties of null vector fields of the KS ansatz from the vacuum field equations of general relativity and universality property of Kerr-Schild-Kundt solutions etc. We have provided two corollaries which are useful to find some exact solutions of the field equations and generate new solutions from some given solutions. We finished this chapter by providing the KS forms of some well-known exact solutions of GR. Next, we studied the Kerr-Schild group of coordinate transformations which when applied to a background metric transforms it such that the resulting transformation is the generalized KS ansatz. This discussion was a mathematical one, with no restrictions of the field equations of GR, and included some parts of the study of isometries as a special case. By choosing the background metric to be a conformally flat spacetime, one can extend this idea to discuss the conformal Killing vector fields. The existence of a null direction changes the discussion drammatically from that of isometries and in fact we find out that even the set of KSVFs with respect to some certain null directions is not finite dimensional in general. In the last chapter, we examined the classical double copy idea by showing that the Maxwell's equations can be obtained from the field equations of GR once we assume that both the scalar field and the null vector are not explicitly time-dependent and time component of the null vector is a constant. Next, by using the identifications of the BCJ duality, we applied this procedure for the case of the Schwarzschild solution and found that the gauge theory it corresponds to is gauge equivalent to that of a Coulomb charge.

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