SOLUTION OF THE ALMGREN-CHRISS MODEL WITH QUADRATIC MARKET VOLUME VIA SPECIAL FUNCTIONS

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ABSTRACT

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One of the current topics of research in mathematical finance is the scheduling of buy or sell orders to liquidate a position. A well-known framework for this problem is the one proposed by Almgren and Chriss that poses the problem as the maximization of the expected utility of the final terminal wealth. In the simplest formulation of the problem the market trading volume is taken as a constant. Under this and other assumptions an optimal trading curve can be computed in terms of the $\sinh$ function. This study aims to consider the same model under the assumptions that the market trading volume is an affine function of time, i.e, $V_t = at + b$ and a quadratic function of time i.e, $V_t = at^2 + bt + c$. A solution of the differential equations arising from the Almgren Chriss model under these assumptions is given in terms of the Modified Bessel function (for the affine volume curve) and the confluent hypergeometric function (for the quadratic volume curve). We also provide numerical examples on how the optimal trading curve varies with the model parameters a, b and c.

Keywords: Optimal Liquidation, Execution Strategy, Optimal Trading Volume

ÖZ

İKİNCİ MERTEBEDEN MARKET HACİMLİ ALMGREN-CHRİSS MODELİNİN ÖZEL FONKSİYONLAR YARDIMIYLA ÇÖZÜMÜ

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Günümüz finansal matematiğindeki güncel araştırma konulardan biri de pozisyonu tasfiye etmek amacıyla alım ya da satım emirleri hakkında planlamayı yapmaktır. Bu alandaki bilinen çalışmalardan biri Almgren ve Chriss tarafından önerilen son durumda beklenen faydanın maksimizasyonu çerçevesidir. Problemin en basit formülasyonunda market işlem hacmi sabit bir sayı olarak alınmıştır. Bu ve diğer varsayımlar altında optimal işlem eğrisi $\sinh$ fonksiyonu cinsinden hesaplanmıştır. Bu çalışmada aynı modelin market işlem hacmi zamana bağlı lineer fonsiyon, yani $V_t = at + b$, ve market işlem hacmi ikinci dereceden zamana bağlı bir fonksiyon, yani $V_t = at^2 + bt + c$, olması durumu irdelenmiştir. Bu varsayımlar altında Almgren ve Chriss modelinden doğan diferansiyel denklemler modifiye Bessel fonksiyonu (lineer hacim eğrisi için) ve bileşik hipergeometrik fonksiyon (ikinci dereceden market eğrisi için) cinsinden ifade edilmiştir. Market işlem eğrisinin $a,\ b$ ve c parametrelerine göre nasıl değiştiğine dair nümerik örnekler sunulmuştur.

Anahtar Kelimeler: Optimal Likidite, Çıkış Stratejisi, Optimal İşlem Hacmi

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q_t	Process for the number of shares in a portfolio at time t.
v_t	The trading velocity at time t.
S_t	Mid-price of the stock at time t.
X_t	The amount of cash on the trader's account at time t.
V_t	Market volume traded by others at time t.
W_t	Brownian motion.
σ	The arithmetic volatility of the stock which is positive.
k	The magnitude of the permanent market impact which is non negative.
L	The execution cost function.
H	The Legendre-Fenchel transformation where $H(p) = \sup_\rho \rho p - L(\rho)$
γ	The absolute risk aversion coefficient of trader which is positive.
$A_i(x)$	The Airy function of the first kind.
$B_i(x)$	The Airy function of the second kind.
$I_\nu(x)$	The modified Bessel function of the first kind.
$H_n(z)$	The Hermite Polynomial of order n.
$\Phi(\alpha, \beta, z)$	The confluent hypergeometric function.

CHAPTER 1

INTRODUCTION

A basic assumption in classical mathematical finance is that the market price of a traded asset arises from the collective actions of many market participants and that the actions of any single participant cannot influence the price by itself. Therefore, all classical mathematical finance models from derivatives pricing to optimal investment assume that the market participant can sell and buy any number of assets instantly at the market price. This assumption becomes nonrealistic for trades that are large compared to the market trading volume of the underlying asset. In the presence of a large sell or buy order, market participants can quickly react to this sudden increase in market volume and change their bid and ask prices. An investor with a large position to liquidate needs to take into consideration the impact the liquidation will make to the price. Starting with the seminal work of Bertsimas and Lo [16] a considerable amount of research has been conducted in optimal liquidation, see, e.g., [11, 2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 17, 18].

One of the main models in market liquidation is the one initiated by Almgren and Chriss in [3], which extends the model of [16]. Our main focus in this thesis will be on this framework. We adopt our exposition of this framework form [11, Chapter 3]. This model assumes that an initial position q_0 is to be closed in a fixed time interval $[0, T]$ using a strategy q_t, $t \in [0, T]$. The mathematical problem is to choose q so that the expected utility of the final terminal wealth is maximized. The mid-price process is assumed to be a simple Brownian motion plus terms associated with short and long term impacts of q on the price. Under the assumption of quadratic execution costs and deterministic market trading volume, the problem reduces to the solution of a linear second order ordinary differential equation. In the case when the market trading volume is constant, the equation has in fact constant coefficients and admits a solution in terms of the $\sinh$ function. For the precise statement of the model and the ODE we refer the reader to Chapter 2 of this thesis.

In real life, the trading volume of any given asset obviously changes with time. The goal of this thesis is to derive solutions to the above mentioned ODE that describes the optimal trading curve within the Almgren-Chris framework under the assumption that the volume process is either of the form

$$V_t = at + b \qquad t \in [0, T]$$

(the affine case), or of the form

$$V_t = at^2 + bt + c \quad t \in [0, T]$$

(the quadratic case). Note that these simple curves already allow for a range of different types of volume behaviour, for example: a trading curve that is increasing or decreasing in time, or concave/convex trading curves. We further discuss these possibilities in Chapter 5. Given that the resulting ODE are second order linear equations with polynomial coefficients, one expects that one can derive explicit solutions to these equations in terms of classical functions of analysis. In Chapter 3 such a derivation is given for the affine volume case. For the solutions we follow [19]. As explained in [19], an affine change of variables reduce the ODE (3.1) to equation to Airy's Equation whose solution is known in terms of the modified Bessel functions. Airy's equation and its solution in terms of Bessel functions is discussed in Section 3.3. In Chapter 4, the same ODE but this time with quadratic market volume is solved in terms of special functions. Two transformations given in [19] convert the ODE to the degenerate hypergeometric equation, whose general solution can be expressed in terms of the confluent hypergeometric function. The work [15] provide a solution to the (3.1) for affine V_t using Taylor expansions. In Chapter 3 we discuss this work as well.

Chapter 5 begins with several numerical examples verifying that the formulas given in earlier chapters are correct by comparing them with the $\sinh$ function when $a = 0$ for the affine case and $a = b = 0$ for the quadratic case. The impact of the a variable is already discussed in [15]. Therefore, in our study of the impact of the shape of the volume curve to the optimal trading curve we only consider the quadratic case. The new variable here is the coefficient a of the quadratic term; in four numerical studies we vary this parameter to see how it changes the optimal trading curve. We see that within the range of parameters we look at the trading curve remains mostly convex. The influence of the a parameter is larger for small c and negligable for large c where c is the constant in the volume curve. Further comments on this study is given in Chapter 5. Finally, the conclusion (Chapter 6) provides a summary of our findings and comments on future research.

CHAPTER 2

ALMGREN AND CHRISS MODEL

The material in this chapter is based on *The Financial Mathematics of Market Liquidty: From Optimal Execution to Market Making* by O. Guéant [11] it provides a summary of the Almgren-Chriss framework. After understanding their model, in Chapter 3 the trading volume $V_t = at + b$ and in Chapter 4 the trading volume $V_t = at^2 + bt + c$ will be studied within this framework.

Consider a single-stock portfolio with q_0 shares initially. The trader's position at time $t \in [0, T]$ is modelled by

$$dq_t = v_t dt. \tag{2.1}$$

where v_t is progressively measurable control process (we can call as trading velocity) satisfying the unwinding constraint $\int_0^\infty v_t dt = -q_0$ and additional technical condition $\int_0^T |v_t|$ is in the set of almost surely (a.s) bounded random variables. The trading velocity of the trader v_t has a lasting effect on the price of the security; this is formalized as

$$dS_t = \sigma dW_t + k v_t dt, \tag{2.2}$$

where S_t is the mid-price of the stock, σ is the arithmetic volatility of the stock and $k > 0$ is a parameter of the permanent market impact. In fact, the actual trading price includes a transaction cost that is dependent on the not only traded volume, but also the market volume, which is denoted by V_t; V_t is a deterministic, continuous, positive and bounded process. We observe the cash position of the trader at particular time t as

$$dX_t = -v_t S_t d_t - V_t L\left(\frac{v_t}{V_t}\right) d_t \tag{2.3}$$

where L is the instantaneous execution cost function. The assumptions on the instantaneous cost function $L : \mathbb{R} \to \mathbb{R}$ are as follows:

- H_1: No fixed cost, i.e., $L(0) = 0$,
- H_2: L is strictly convex, increasing on $\mathbb{R}_+$ and decreasing on $\mathbb{R}_-$,
- H_3: L is asymptotically super linear, i.e., $\lim_{|\rho|\to+\infty} \dfrac{L(\rho)}{|\rho|} = +\infty$

Here, H_1 means that no execution costs to be charged when there is no transaction, H_2 implies there is always a cost to trade, and H_3 is a technical assumption. In the Almgren-Chiss model $L(\rho)$ correspond to a quadratic function, i.e., $L(\rho) = \eta\rho^2$.

2.1 Optimal Strategy

In order to find optimal strategy to liquidate the portfolio, the main goal is to find optimal (q_t) (or equivalently its derivative v_t defined in (2.1)). For this purpose, Bertsimas and Lo in [16] worked on maximizing $E[X_T]$.In [3], [4] Almgren and Chriss add varience to this frame and they try to maximize

$$E[X_T] - \frac{\gamma}{2} V[X_T],$$

where γ is a positive constant. Almgren and Chriss framework as presented in [Chapter 3, [11]] focuses on a utility function of the form

$$\mathbb{E}[-e^{-\gamma X_T}]$$

where γ is called the absolute risk aversion coefficient of the trader which is positive. In order to solve this problem, we start with the assumption that the class of admissible controls v_t to be deterministic.

Recall that (2.1), (2.2), (2.3) are defined as below

$$\begin{aligned} dq_t &= v_t dt \\ dS_t &= \sigma dW_t + k v_t dt \\ dX_t &= -v_t S_t - V_t L\left(\frac{v_t}{V_t}\right) dt \end{aligned}$$

By using (2.1), (2.2), (2.3) and integration by parts (comprehensively explained in [Chapter 3, [11]]), the final value of the cash X_T as a function of q as follows:

$$X_T = X_0 + q_0 S_0 - \frac{k}{2}{q_0}^2 + \sigma \int_0^T q_t dW_t - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt. \tag{2.4}$$

For deterministic q, the final value of the cash process X_T is Normally distributed with the following variance and mean:

$$V[X_T] = \sigma^2 \int_0^T {q_t}^2 dt. \tag{2.5}$$

$$E[X_T] = \underbrace{X_0 + q_0 S_0}_{\text{MtM value}} - \underbrace{\frac{k}{2}{q_0}^2}_{\text{perm. m. i.}} - \underbrace{\int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt}_{\text{execution cost}} \tag{2.6}$$

We can mention the first term as the mean of X_T is Mark-to Market value of the portfolio i.e. the cash value of the portfolio if it were possible to liquidate the shares instantly at the real time market price. The second one is about costs from permanent ingredient of the market impact. The third is term is v_t , which is also called the execution costs.

The last term of equation (2.4) is minimal when v_t is proportional to V_t since L is assumed to be convex. This is the strategy proposed by Bertsimas and Lo in [16]. However, this strategy is not optimal in Almgren-Chriss framework since they take the varience into account. The variance of X_T is an increasing function of the volatility parameter σ, and it depends on the strategy throught the term $\int_0^T {q_t}^2 dt$. To minimize this variance, the trader must quickly liquidate. The trade-off between execution costs and price risk is here a trade-off between minimizing the last term of (2.4), and minimizing the variance of X_T

After using equation (2.5), equation (2.6) and the Laplace transform of a Gaussian variable, objective function can be obtained as

$$\begin{aligned} E[-\exp(-\gamma X_T)] &= -\exp\left(-\gamma E[X_T] + \frac{1}{2}\gamma^2 V[X_T]\right) \\ &= -\exp\left(-\gamma\left(X_0 + q_0 S_0 - \frac{k}{2}q_0^2\right)\right) \\ &\quad \times \exp\left(\gamma\left(\int_0^T V_0 L\left(\frac{v_t}{V_t}\right) dt + \frac{\gamma}{2}\sigma^2 \int_0^T q_t^2 dt\right)\right). \end{aligned}$$

Eventually, the problem reduces to finding a control process (q_t) (called a Bolza problem) which minimizes the following

$$J(q) = \int_0^T \left(V_t L\left(\frac{q'(t)}{V_t}\right) + \frac{1}{2}\gamma\sigma^2 q(t)^2 \right) dt, \tag{2.7}$$

where, q is differentiable in t and satisfies $q(0) = q_0$ and $q(T) = 0$. Theorem 3.1 in [11] guranteed that such an minimizer exists and unique. Before we proceed let us recall the statement of theorem without giving any proof. (for further details of proof we refer the reader to [Chapter 3] in [11]).

Theorem: There exists a unique minimizer q^* of the function J over the set $C = \{q \in W^{1,1}(0,T),\ q(0) = q_0, (T) = 0\}$. Furthermore, q^* is a monotone function:

i if $q_0 \geq 0$, q^* is a nonincreasing function of time and,

ii if $q_0 \leq 0$, q^* is a nondecreasing function of time.

In order to find minimizer q^* of equation (2.7), one can use either a Euler-Lagrange characterization or a Hamiltonian characterization. Since there is no assumption on regularity on the function L, the Euler-Lagrange equation is of the following form

$$\begin{cases} p'(t) &= \gamma\sigma^2 q^*(t), \\ p(t) &\in \partial^- L\left(\frac{{q^*}'(t)}{V_t}\right), \\ q^*(0) &= q_0, \\ q^*(T) &= 0, \end{cases}$$

where $\partial^- L$ represents the sub-differential of L. If L is differentiable, the subdiffirental of L is simply its derivative, therefore, the Euler-Lagrange equation for differentiable L becomes

$$\begin{cases} p'(t) &= \gamma\sigma^2 q^*(t), \\ p(t) &= L'\left(\frac{q^{*\prime}(t)}{V_t}\right), \\ q^*(0) &= q_0, \\ q^*(T) &= 0. \end{cases}$$

The convexity of L and convex duality imply that the optimal solution can also be represented as the solution of the following Hamiltonian ODE:

$$\begin{cases} p'(t) &= \gamma\sigma^2 q^*(t), \\ q^{*\prime}(t) &= V_t H'(p(t)), \\ q^*(0) &= q_0, \\ q^*(T) &= 0, \end{cases} \tag{2.8}$$

where H is the Legendre-Fenchel transform of the function L defined by

$$H(p) = \sup_{\rho} \{\rho p - L(\rho)\}. \tag{2.9}$$

Since V_t is assumed to be continuous, from this characterization one can conclude that q^* is a function of class C^1. Consequently, p is of class C^2 with respect to the same hypothesis. However, q^* may not be of class C^2. For instance, consider $L(\rho) = \eta|\rho|^{1+\phi} + \psi|\rho|$ the H function is computed to be

$$H(p) = \begin{cases} 0, & \text{if } |p| \le \psi, \\ \phi\eta(\frac{|p|-\psi}{\eta(1+\phi)})^{1+\frac{1}{\phi}}, & \text{otherwise.} \end{cases} \tag{2.10}$$

clearly H is C^1 but not C^2 in general, and $q^{*'}$ need not to be differentiable. We will refer this result in Chapter 3.
This characterization lead to another important characterization that is the dual variable p is solution of

$$p''(t) = \gamma\sigma^2 V_t H'(p(t)) \tag{2.11}$$

with the boundary conditions $p'(0) = \gamma\sigma^2 q_0$ and $p'(T) = 0$. Main approach of this thesis is to find analytic solutions of (2.11) with linear (Chapter 3) and quadratic (Chapter 4) market volume V_t.

2.2 The case of quadratic execution cost

Almgren and Chriss initially studied a model where the additional cost per share due to limited liquidity was a linear function of the number of shares. This corresponds to g linear, or equivalently L quadratic function [11]. For this framework, $L(\phi) = \rho\phi^2$ which is quadratic function, by equation (2.9) the associated function H is given by $H(p) = \dfrac{p^2}{4\rho}$.

Therefore, the ODE for optimal execution strategy reduces to

$$\begin{cases} p'(t) & = \gamma\sigma^2 q^*(t), \\ {q^*}'(t) & = \frac{V_t}{2\rho} p(t), \\ q^*(0) & = q_0, \\ q^*(T) & = 0. \end{cases} \tag{2.12}$$

Thus, q^* is the unique solution of the following ODE:

$$q^{*''}(t) = \frac{\gamma\sigma^2 V_t}{2\rho} q^*(t), \tag{2.13}$$

with the boundary conditions are $q^*(0) = q_0$ and $q^*(T) = 0$.

In order to get Eq. (2.13), trading volume V_t assumed to be constant by taking $V_t = V$, we can get the classical hyperbolic sine formula of Almgren and Chriss

$$q^*(t) = q_0 \frac{\sinh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\rho}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\rho}} T\right)}. \tag{2.14}$$

With Eq. (2.14) Almgren and Chriss found their optimal solution for the liquidation trade-off. For comments on this solution and numerical examples we refer the reader to [11].

CHAPTER 3

ANALYTIC SOLUTION OF ALMGREN CHRIS MODEL WITH AFFINE MARKET VOLUME

In this chapter, we will study the analytic solution of Almgren Chriss model with linear V_t. From Chapter 2, Almgren Chriss model for quadratic execution cost end up with the following equation:

$$\begin{cases} p'(t) &= \gamma\sigma^2 q^*(t), \\ q^{*\prime}(t) &= \frac{V_t}{2\rho} p(t), \\ q^*(0) &= q_0, \\ q^*(T) &= 0. \end{cases} \tag{3.1}$$

Before we proceed to the solution of Almgren Chriss model with affine market volume, we summarize the results of [15] on this model.

3.1 Almgren Chriss Model with Linear V_t

In this section, we give a summary of the results in [15] on Almgren Chris Model with affine V_t, i.e., V_t of the form

$$V_t = at + b.$$

Without loss of generality, all constants in the equation (3.1) assumed to be 1; setting $T = 1$ and $q_0 = 1$ this reduces the equation to:

$$\begin{cases} p'(t) &= q^*(t), \\ q^{*\prime}(t) &= V_t p(t), \\ q^*(0) &= 1, \\ q^*(1) &= 0. \end{cases}$$

In this study, this system of equations is solved by using following Taylor series:

$$p = \sum_{j=0}^{\infty} \alpha_j t^j$$

and

$$q = \sum_{j=0}^{\infty} \beta_j t^j.$$

Substituting these infinite series into equation, following recurrence relation can be found:

$$\beta_{j+1} = \frac{b\alpha_j + \alpha_{j-1}a}{j+1}.$$

where a, b are constants in market volume V_t. In order to solve the above relation numerically a binary search is used. Main result in this study is the parameter a carries a strong impact on the trading curve when it is both negative and positive when b is small compared to initial position, namely q_0. Trading curve deflects from the $\sinh$ trading curve and even becomes concave; this implies a slow liquidation process and delay for a future time in which the market trading volume is high. However, we see a change in this behaviour when b increases. When the market is volume is decreasing,i.e., a is negative, resulting strategy continues to deviate from the original curve of Almgreen Chriss framework. But, since initial volume is already high, the trading strategy does not change very much with the speed (the $a > 0$ parameter) at which the market trading volume is increasing.

3.2 Analytical solution of the ODE with affine volume

Proceeding to our study, firstly note that the equation (3.1) cannot be reduced to a second order ODE with respect to q^* since q^* need not to be of class C^2 as mentioned in Chapter 2. However, it can be reduced to a second order ODE with respect to p. Consequently we have,

$$p''(t) = \frac{\gamma\sigma^2}{2\rho}\, V_t\, p(t) \tag{3.2}$$

with terminal conditions $p'(T) = 0$ and $p'(0) = \gamma\sigma^2 q_0$. Let us take the market volume as $V_t = at + b$ and for simplicity constants appearing in the equation as 1. Furthermore, let us assume that trader's position $t = 0$ is 1 and terminal time $T = 1$. Then (3.2) will be of the form:

$$p''(t) = (at + b)\, p(t) \tag{3.3}$$

with terminal conditions $p'(1) = 0$ and $p'(0) = 1$. Before we proceed to the solution of (5.1), first let us recall one of the famous special function so called *Airy Function* and its application in the solutions of second order ODE.

3.3 Airy's Equation

In this section main reference is book of J.R. Brannan [8]. The second order linear differential equations when coefficients are functions of independent variable ξ can be generalized as follows,

$$P(\xi)\frac{d^2y}{d\xi^2} + Q(\xi)\frac{dy}{d\xi} + R(\xi)y = 0.$$

If we take $P(\xi) = 1$, $Q(\xi) = 0$ and $R(\xi) = -\xi$, we will have

$$y'' - \xi y = 0 \tag{3.4}$$

where $-\infty < \xi < \infty$. (3.4) is called *Airy's Equation.*[1] Since $P(\xi) = 1$, $Q(\xi) = 0$ and $R(\xi) = -\xi$, every point is an ordinary point. We assume that

$$y = \sum_{n=0}^{\infty} a_n \xi^n$$

and the series converges in some interval $|\xi| < \rho$. We can write series for y'' as follows

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\xi^n$$

[1] Sir George Biddell Airy (1801–1892), an English astronomer and mathematician, studied the equation named for him in an 1838 paper on optics.

then substitude the series for y and y'' into the (3.4) we have

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\xi^n = \xi\sum_{n=0}^{\infty}a_n\xi^n = \sum_{n=0}^{\infty}a_n\xi^{n+1}.$$

We shift the index of summation in the series on the right side, we have

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}\xi^n = \sum_{n=1}^{\infty}a_{n-1}\xi^n$$

the coefficients of like powers of ξ must be equal; hence $a_2 = 0$ and we obtain the recurrence relation

$$(n+2)(n+1)a_{n+2} = a_{n-1}$$

for $n = 1, 2, 3.....$ Thus, coefficients are determined in steps of three, that is,

- a_0 determines a_3, which in turn determines a_6 ...
- a_1 determines a_4, which in turn determines a_7 ...

Recall that $a_2 = 0$, thus $a_2 = a_5 = a_8 = a_{11} = ... = 0$. For sequence $a_0, a_3, a_6, ...$ we set $n = 1, 4, 7, ..$

$$a_3 = \frac{a_0}{2\cdot 3},\ \ a_6 = \frac{a_3}{5\cdot 6} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6},\ \ a_9 = \frac{a_6}{8\cdot 9} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}$$

Hence, one can generalized this results as follows

$$a_{3n} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6...\cdot(3n-1)\cdot(3n)}$$

for $n \geq 4$. A similar argument is valid for $a_1, a_4, a_7, ...$ hence,

$$a_{3n+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7...\cdot(3n)\cdot(3n+1)}$$

for $n \geq 4$. Thus general solution for Airy's Equation is of the form

$$y = a_0\underbrace{[1 + \frac{\xi^3}{2\cdot 3} + \frac{\xi^6}{2\cdot 3\cdot 5\cdot 6} + ... + \frac{\xi^{3n}}{2\cdot 3\cdot 5\cdot 6.....(3n-1)(3n)} + ...]}_{y_1(\xi)}$$
$$+ a_1\underbrace{[\xi + \frac{\xi^4}{3\cdot 4} + \frac{\xi^7}{3\cdot 4\cdot 6\cdot 7} + ... + \frac{\xi^{3n+1}}{3\cdot 4\cdot 6\cdot 7.....(3n)(3n+1)} + ...]}_{y_2(\xi)}.$$

One can show that by using ratio test both series converge for all ξ, thus let $y_1(\xi)$ and $y_2(\xi)$ be functions shown above. Then, setting $a_0 = 1$, $a_1 = 0$ and then $a_0 = 0$,

$a_1 = 1$ gives us both $y_1(\xi)$ and $y_2(\xi)$ are individual solutions of the equation (3.4). Note that y_1 and y_2 satisfies the initial conditions, namely $y_1(0) = 1$, $y_1'(0) = 0$ and $y_2(0) = 1$, $y_2'(0) = 1$. Since $W[y_1, y_2] = 1$, y_1 and y_2 are linearly independent. Hence, general solution of equation (3.4) is of the form

$$y(\xi) = a_0\, y_1(\xi) + a_1\, y_2(\xi). \tag{3.5}$$

According to [20], G.B. Airy in [1] propose a solution for a physical phenomena by solving (3.4). In 1928, Jeffreys introduce the solution of (3.4) in the following form

$$A_i(\xi) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + \xi t\right) dt. \tag{3.6}$$

$B_i(\xi)$ was also defined as (provided that $A_i(\xi)$ and $B_i(\xi)$ are linearly independent)

$$B_i(\xi) = \frac{1}{\pi}\int_0^\infty \left(e^{-\frac{1}{3}t^3+\xi t} + \sin\left(\frac{1}{3}t^3 + \xi t\right)\right) dt. \tag{3.7}$$

For decades mathematicians extensively studied these two special functions. Furthermore, it is known that $y_1(\xi)$ and $y_2(\xi)$ from (3.5) appear in the ascending series of $A_i(\xi)$ and $B_i(\xi)$ as follows

$$A_i(\xi) = c_1\, y_1(\xi) - c_2\, y_2(\xi) \tag{3.8}$$

$$B_i(\xi) = \sqrt{3}(c_1\, y_1(\xi) + c_2\, y_2(\xi)) \tag{3.9}$$

where $c_1 = (3^{\frac{2}{3}}\Gamma(\frac{2}{3}))^{-1}$ and $c_2 = (3^{\frac{1}{3}}\Gamma(\frac{1}{3}))^{-1}$. Since $W[A_i(\xi), B_i(\xi)] = \pi^{-1}$, they are linearly independent. Also, they are linear combinations of $y_1(\xi)$ and $y_2(\xi)$. Therefore, since equation (3.4) is linear, $A_i(\xi)$ and $B_i(\xi)$ are also fundamental set of solutions to Airy's Equation as required.

3.4 Analytic Solution of Almgren Chriss Model with Linear V_t.

The main target of this chapter is to solve

$$p''(t) = (at + b)\, p(t)$$

with terminal conditions $p'(1) = 0$ and $p'(0) = 1$. Following [19, page 213], we use substitution $\xi = a^{-\frac{2}{3}}(at + b)$ then by using chain rule we have

$$\begin{aligned} p'(t) = \frac{dp}{dt} &= \frac{dp}{d\xi}\,\frac{d\xi}{dt} \\ &= \frac{dp}{d\xi}\, a^{\frac{1}{3}} \end{aligned}$$

$$
\begin{aligned}
p''(t) = \frac{dp}{dt}\left[\frac{dp}{d\xi}\, a^{\frac{1}{3}}\right] &= \frac{d^2p}{d\xi^2}\, a^{\frac{1}{3}}\, \frac{d\xi}{dt} \\
&= \frac{d^2p}{d\xi^2}\, a^{\frac{2}{3}}
\end{aligned}
$$

For simplicity let us denote p' as a derivative of p with respect to t and $\dot{p}$ as a derivative of p with respect to ξ. Hence,

$$p''(t) = a^{\frac{2}{3}}\, \ddot{p}(\xi).$$

Substitute into equation (5.1), we have

$$
\begin{aligned}
a^{\frac{2}{3}}\, \ddot{p}(\xi) - \xi\, a^{\frac{2}{3}} p(\xi) = 0 \\
\ddot{p}(\xi) - \xi\, p(\xi) = 0 \qquad (3.10)
\end{aligned}
$$

with terminal conditions $\dot{p}(a^{-\frac{2}{3}}b) = 1$ and $\dot{p}(a^{-\frac{2}{3}}(a+b)) = 0$. Equation (3.10) become well known Airy's Equation. By using results in the previous section, we can write solution of (3.10) as

$$p(\xi) = C_1\, A_i(\xi) + C_2\, B_i(\xi) \qquad (3.11)$$

where $A_i(\xi)$ and $B_i(\xi)$ are the Airy's function of first and second kind respectively with terminal conditions $\dot{p}(a^{-\frac{2}{3}}b) = 1$ and $\dot{p}(a^{-\frac{2}{3}}(a+b)) = 0$.

According to [20] first derivative of Airys function of first and second kind can be represented by using *Modified Bessel Functions*:

$$
\begin{aligned}
\dot{A}_i(\xi) &= -\frac{\xi}{3}\left[I_{-2/3}(z) - I_{2/3}(z)\right] \\
\dot{B}_i(\xi) &= \frac{\xi}{\sqrt{3}}\left[I_{-2/3}(z) + I_{2/3}(z)\right]
\end{aligned}
$$

where $z = \dfrac{2}{3}\xi^{\frac{3}{2}}$ and $I_\nu(x)$ is *Modified Bessel Functions* defined as

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\nu}}{k!\Gamma(\nu+k+1)}$$

where $\Gamma(x)$ is the gamma function. Hence, let us take first derivative of both sides with respect to ξ in equation (3.11)

$$\dot{p}(\xi) = C_1 \frac{\xi}{3}\left[I_{-2/3}(z) - I_{2/3}(z)\right] + C_2 \frac{\xi}{\sqrt{3}}\left[I_{-2/3}(z) + I_{2/3}(z)\right]$$

where $z = \frac{2}{3}\xi^{\frac{3}{2}}$. Recall that terminal conditions are $\dot{p}(a^{-\frac{2}{3}}b) = 1$ and $\dot{p}(a^{-\frac{2}{3}}(a+b)) = 0$. Let us denote $k_1 = \frac{2}{3}(a^{-\frac{2}{3}}(a+b))^{\frac{3}{2}}$ and $k_2 = \frac{2}{3}(a^{-\frac{2}{3}}b)^{\frac{3}{2}}$. Thus,

$$0 = C_1 \frac{a^{-\frac{2}{3}}(a+b)}{3}\left[I_{-2/3}(k_1) - I_{2/3}(k_1)\right] + C_2 \frac{a^{-\frac{2}{3}}(a+b)}{\sqrt{3}}\left[I_{-2/3}(k_1) + I_{2/3}(k_1)\right]$$
$$C_1 = -C_2 \cdot A \tag{3.12}$$

where $A = \sqrt{3}\frac{I_{-2/3}(k_1) + I_{2/3}(k_1)}{I_{-2/3}(k_1) - I_{2/3}(k_1)}$. Note that A is a constant parameter depends on choice of a and b. Hence, using this result to other terminal condition we have following

$$1 = -C_2 \cdot A \frac{a^{-\frac{2}{3}}b}{3}\left[I_{-2/3}(k_2) - I_{2/3}(k_2)\right] + C_2 \frac{a^{-\frac{2}{3}}b}{\sqrt{3}}\left[I_{-2/3}(k_2) + I_{2/3}(k_2)\right]$$
$$C_2 = \frac{1}{\frac{a^{-\frac{2}{3}}b}{\sqrt{3}}\left[I_{-2/3}(k_2) + I_{2/3}(k_2)\right] - A\frac{a^{-\frac{2}{3}}b}{3}\left[I_{-2/3}(k_2) - I_{2/3}(k_2)\right]} \tag{3.13}$$

Remember that from (3.1) and assuming that constants in that equation as 1, we have

$$p'(t) = q^*(t).$$

Therefore, to find optimal trading curve $q^*(t)$ is

$$q^*(t) = C_1 \frac{a^{-\frac{2}{3}}(at+b)}{3}\left[I_{-2/3}(z) - I_{2/3}(z)\right] + C_2 \frac{a^{-\frac{2}{3}}(at+b)}{\sqrt{3}}\left[I_{-2/3}(z) + I_{2/3}(z)\right] \tag{3.14}$$

where $z = \frac{2}{3}(a^{-\frac{2}{3}}(at+b))^{\frac{3}{2}}$.

CHAPTER 4

ANALYTIC SOLUTION OF ALMGREN CHRIS MODEL WITH QUADRATIC MARKET VOLUME

In this chapter, we study the analytic solution of Almgren Chriss model with quadratic V_t. Recall from Chapter 2 that, Almgren Chriss model for quadratic execution cost end up with the following equations:

$$\begin{cases} p'(t) &= \gamma\sigma^2 q^*(t), \\ q^{*\prime}(t) &= \frac{V_t}{2\rho} p(t), \\ q^*(0) &= q_0, \\ q^*(T) &= 0. \end{cases}$$

Furthermore, we will solve this system with respect to p and again without loss of generality we will assume constants in this system as 1. Moreover, let us assume that trader's position $t = 0$ is 1 and terminal time $T = 1$. We are focusing on the case $V_t = at^2 + bt + c$. Then the system will be of the form:

$$p''(t) = (at^2 + bt + c)\, p(t) \tag{4.1}$$

with terminal conditions $p'(1) = 0$ and $p'(0) = 1$ provided that $a \neq 0$. Below we present a solution to this equation given in [19]. See Section 4.3 at the end of this chapter for comments on the presented solution.

Following [19, page 214], we begin by an affine change variables $\xi = t + \dfrac{b}{2a}$ to get rid of the linear term in V_t:

$$\begin{aligned} p'(t) = \frac{dp}{dt} &= \frac{dp}{d\xi}\,\frac{d\xi}{dt} \\ &= \frac{dp}{d\xi} \end{aligned}$$

$$
\begin{aligned}
p''(t) = \frac{dp}{dt}\left[\frac{dp}{d\xi}\right] &= \frac{d^2p}{d\xi^2}\,\frac{d\xi}{dt} \\
&= \frac{d^2p}{d\xi^2}
\end{aligned}
$$

Let us denote by p' the derivative of p with respect to t and by $\dot{p}$ the derivative of p with respect to ξ. Hence, we have

$$p''(t) = \ddot{p}(\xi), \;\; p'(t) = \dot{p}(\xi)$$

Let us substitute this into equation (4.1) to obtain

$$\ddot{p}(\xi) = \left(a\left(\xi^2 - \frac{\xi b}{a} + \frac{b^2}{4a^2}\right) + b\left(\xi - \frac{b}{2a}\right) + c\right)\, p(\xi)$$

After rearranging the terms, we have

$$\ddot{p}(\xi) - (a\xi^2 + c - \frac{b^2}{4a})p(\xi) = 0 \tag{4.2}$$

with boundary conditions $\dot{p}\left(\frac{b}{2a}\right) = 1$ and $\dot{p}\left(1 + \frac{b}{2a}\right) = 0$. Equation (4.2) is referred to as a *Weber Equation*. It is noted in [19] that there is a general solution of this equation for $a > 0$ and in the special case that $a = k^2 > 0$ and $\left(c - \frac{b^2}{4a}\right) = (2n+1)k$ where $n = 1, 2, 3...$ there is even a simpler solution of equation (4.2). We first review this special case:

4.1 Special case

If $a = k^2 > 0$ and $\left(c - \frac{b^2}{4a}\right) = (2n+1)k$ where $n = 1, 2, 3...$ there is a solution of the form

$$p(\xi) = e^{\left(-\frac{1}{2}k\xi^2\right)} H_n(\sqrt{k}\,\xi) \tag{4.3}$$

for $k > 0$ where $H_n(z) = (-1)^n exp(z^2)\frac{d^n}{dz^n}(exp(-z^2))$ is *Hermite Polynomial* of order n.

4.2 General case

We now treat the solution of (4.2) in general. Following [19] once again we transform (4.2), by the transformations $z = \xi^2\sqrt{a}$ and $u = e^{z/2}p(\xi)$ to the so called degenerate hypergeometric ODE. We begin by providing the details of this transformation and the resulting ODE.

4.2.1 Reduction to hypergeometric ODE

Using chain rule we will have:

$$\dot{p}(\xi) = \frac{dp}{dz}\frac{dz}{d\xi} = \frac{dp}{dz}\, 2\xi\sqrt{a}$$

Note that u_z will represent the first derivative of u with respect to z, then

$$\begin{aligned}
u_z &= \frac{1}{2}e^{\frac{z}{2}}p(\xi) + e^{\frac{z}{2}}\frac{\dot{p}(\xi)}{2\xi\sqrt{a}} \\
u_z &= \frac{1}{2}u + e^{\frac{z}{2}}\frac{\dot{p}(\xi)}{2\xi\sqrt{a}} \\
u_z - \frac{1}{2}u &= e^{\frac{z}{2}}\frac{\dot{p}(\xi)}{2\xi\sqrt{a}} \qquad (4.4)
\end{aligned}$$

Then, let us calculate second derivative of u with respect to z

$$u_{zz} = \frac{1}{2}u_z + \frac{1}{2}e^{\frac{z}{2}}\frac{\dot{p}(\xi)}{2\xi\sqrt{a}} + e^{\frac{z}{2}}\frac{d}{dz}\left[\frac{\dot{p}(\xi)}{2\xi\sqrt{a}}\right] \qquad (4.5)$$

Let $g(\xi) = \dfrac{\dot{p}(\xi)}{2\xi\sqrt{a}}$, then again by the chain rule

$$\begin{aligned}
\frac{d(g(\xi))}{d\xi} &= \frac{d(g(\xi))}{dz}\frac{dz}{d\xi} \\
\frac{\ddot{p}(\xi)2\xi\sqrt{a} - \dot{p}(\xi)2\sqrt{a}}{4\xi^2 a} &= \frac{d(g(\xi))}{dz}2\xi\sqrt{a} \\
\frac{d(g(\xi))}{dz} &= \frac{\ddot{p}(\xi)\xi - \dot{p}(\xi)}{4\xi^3 a} \qquad (4.6)
\end{aligned}$$

Substitute result in (4.6) into (4.5), we have

$$u_{zz} = \frac{1}{2}u_z + \frac{1}{2}e^{\frac{z}{2}}\frac{\dot{p}(\xi)}{2\xi\sqrt{a}} + \frac{e^{\frac{z}{2}}\ddot{p}(\xi)}{4\xi^2 a} - \frac{e^{\frac{z}{2}}\dot{p}(\xi)}{4\xi^3 a}$$

Using (4.4) and rearranging terms we have

$$u_{zz} = u_z\left(1 - \frac{1}{2\xi^2\sqrt{a}}\right) + u\left(\frac{1}{4\xi^2\sqrt{a}} - \frac{1}{4}\right) + \frac{e^{\frac{z}{2}}\ddot{p}(\xi)}{4\xi^2 a} \qquad (4.7)$$

Multiply equation (4.7) with z and substitute $\ddot{p}(\xi)$ from equation (4.2), we finally have

$$zu_{zz} = u_z\left(z - \frac{1}{2}\right) + u\left(\frac{1}{4} + \frac{c - \frac{b^2}{4a}}{4\sqrt{a}}\right) \qquad (4.8)$$

with new terminal conditions

$$u_z(\sqrt{a}(1+\frac{b}{2a})^2) = \frac{1}{2}u(\sqrt{a}(1+\frac{b}{2a})^2) \tag{4.9}$$

$$u_z(\sqrt{a}(\frac{b}{2a})^2) = \frac{1}{2}u(\sqrt{a}(\frac{b}{2a})^2) + \frac{e^{\frac{\sqrt{a}}{2}(\frac{b}{2a})^2}}{2\sqrt{a}(\frac{b}{2a})}$$

The final result equation (4.8) is called *the degenerate hypergeometric equation* which has the general form $xy''_{xx} + (\beta - x)y'_x - \alpha y = 0$. For (4.8)

$$zu_{zz} + u_z\left(\underbrace{\frac{1}{2}}_{\beta} - z\right) - u\left(\underbrace{\frac{1}{4} + \frac{c - \frac{b^2}{4a}}{4\sqrt{a}}}_{\alpha}\right) = 0. \tag{4.10}$$

This is a linear second order ODE, therefore the general solution to it will be spanned by two linearly independent solutions. We next present the general solution to this equation (a linear combination of two linearly independent solutions) given in [19]:

4.2.2 General solution of (4.10)

According to [19] if β is not an integer, as in (4.10), a general solution of the degenerate hypergeometric equation can be written in the form

$$u = C_1\Phi(\alpha, \beta; z) + C_2\sqrt{z}\Phi(\alpha - \beta + 1, 2 - \beta; z) \tag{4.11}$$

where Φ is the confluent hypergeometric function (also known as a *Kummer series*) defined as follows:

$$\Phi(\alpha, \beta; z) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k z^k}{(\beta)_k k!}$$

where $(\alpha)_k = \alpha(\alpha+1)....(\alpha+k-1)$ and $(\alpha)_0 = 1$. In our calculations below we will need derivatives of Φ, this is given by:

$$\frac{d^n}{dz^n}\Phi(\alpha, \beta; z) = \frac{(\alpha)_n}{(\beta)_n}\Phi(\alpha+n, \beta+n; z)$$

Applying this rule to the equation (4.11), we have

$$u_z = C_1\frac{\alpha}{\beta}\Phi(\alpha+1, \beta+1; z) + C_2\frac{1}{2\sqrt{z}}\Phi(\alpha-\beta+1, 2-\beta; z) + C_2\sqrt{z}\frac{\alpha-\beta+1}{2-\beta}\Phi(\alpha-\beta+2, 3-\beta; z) \tag{4.12}$$

4.2.3 Computation of C_1 and C_2

We are seeking a solution of the form (4.11) to the equation (4.8) and the boundary conditions (4.9). The only unknowns here are the constants C_1 and C_2, which we next determine using the boundary conditions. Substituting (4.11) into (4.9) gives us a pair of linear equations that C_1 and C_2 satisfy. Solving these equations for C_1 and C_2; we obtain explicit formulas for these constants in terms of model parameters. The details are as follows.

For simplicity let us denote $z_1 = \sqrt{a}(1 + \frac{b}{2a})^2$ and $z_2 = \sqrt{a}(\frac{b}{2a})^2$. Hence,

$$u_z(z_1) = \frac{1}{2}(C_1\Phi(\alpha, \beta; z_1) + C_2\sqrt{z_1}\Phi(\alpha - \beta + 1, 2 - \beta; z_1)) \tag{4.13}$$

$$u_z(z_2) = \frac{1}{2}(C_1\Phi(\alpha, \beta; z_2) + C_2\sqrt{z_2}\Phi(\alpha - \beta + 1, 2 - \beta; z_2)) + \frac{e^{\frac{\sqrt{a}}{2}(\frac{b}{2a})^2}}{2\sqrt{a}(\frac{b}{2a})} \tag{4.14}$$

After rearranging terms in (4.14) by using (4.13) we have,

$$C_1 = -X \cdot C_2 \tag{4.15}$$

where

$$X = \frac{\frac{1 - z_1}{2\sqrt{z_1}}\Phi(\alpha - \beta + 1, 2 - \beta; z_1) + \sqrt{z_1}\frac{\alpha - \beta + 1}{2 - \alpha}\Phi(\alpha - \beta + 2, 3 - \beta; z_1)}{\frac{\alpha}{\beta}\Phi(\alpha + 1, \beta + 1; z_1) - \frac{1}{2}\Phi(\alpha, \beta; z_1)}$$

Note that, X is a constant variable depends on the choice of a, b and c. Substituting (4.16) into (4.17).

$$C_2 \cdot Y = \frac{e^{\frac{\sqrt{a}}{2}(\frac{b}{2a})^2}}{2\sqrt{a}(\frac{b}{2a})} \tag{4.16}$$

where

$$\begin{aligned} Y = \frac{1 - z_2}{2\sqrt{z_2}}\Phi(\alpha - \beta + 1, 2 - \beta; z_2) + \sqrt{z_2}\frac{\alpha - \beta + 1}{2 - \alpha}\Phi(\alpha - \beta + 2, 3 - \beta; z_2) \\ - X\frac{\alpha}{\beta}\Phi(\alpha + 1, \beta + 1; z_2) + \frac{1}{2}X\Phi(\alpha, \beta; z_2) \end{aligned}$$

These give explicit formulas for C_1 and C_2 in terms of model parameters.

4.2.4 Formula for the optimal trading curve

Recall that in order to find (4.11), according to Polyanin and Zaitsev's book we have used $u = e^{z/2}p(\xi)$ and $z = \xi^2\sqrt{a}$. Hence by back substitution,

$$p(\xi) = \frac{1}{e^{\xi^2\sqrt{a}}}\left(C_1\Phi(\alpha, \beta; \xi^2\sqrt{a}) + C_2\xi\sqrt[4]{a}\Phi(\alpha - \beta + 1, 2 - \beta; \xi^2\sqrt{a})\right)$$

Remember that from (3.1) and assuming that constants in that equation as 1, we have

$$p'(t) = q^*(t).$$

Therefore, to find optimal trading curve $q^*(t)$, it suffices to calculate the derivative of $p(\xi)$:

$$\begin{aligned} q^*(t) = \dot{p}(\xi) = &\frac{-2\xi\sqrt{a}}{e^{\xi^2\sqrt{a}}}\left(C_1\Phi(\alpha, \beta; \xi^2\sqrt{a}) + C_2\xi\sqrt[4]{a}\Phi(\alpha - \beta + 1, 2 - \beta; \xi^2\sqrt{a})\right) \\ &+ \frac{1}{e^{\xi^2\sqrt{a}}}(C_1\frac{\alpha}{\beta}2\xi\sqrt{a}\Phi(\alpha + 1, \beta + 1; \xi^2\sqrt{a}) \\ &+ C_2\sqrt[4]{a}\Phi(\alpha - \beta + 1, 2 - \beta; \xi^2\sqrt{a}) \\ &+ C_2\frac{\alpha - \beta + 1}{2 - \beta}2\xi^2 a^{\frac{3}{4}}\Phi(\alpha - \beta + 2, 3 - \beta; \xi^2\sqrt{a})) \end{aligned} \tag{4.17}$$

where C_1 and C_2 are given by formulas in the previous subsection and $\xi = t + \dfrac{b}{2a}$.

4.3 Comments on the solution

Some care must be taken in using the formula (4.18) as a solution to the ODE (4.1) and the boundary conditions $p'(1) = 0$ and $p'(0) = 1$. Recall that one of the steps in the derivation of (4.18) is the change of variables $z = \sqrt{a}\xi^2$. Note that this change of variables is bijective only when ξ doesn't change sign (i.e., when ξ is always positive or negative). Therefore, the formula (4.18) can be used directly only when this condition is satisfied. Recall that $\xi = t + b/2a$ and $t \in (0, 1)$. It follows that $\xi \mapsto \xi^2$ is bijective if and only if $b/2a > 0$ or $b/2a + 1 < 0$. It follows from these that for the formula (4.18) to be applicable directly we have to exlude the case $-1 \leq b/2a \leq 0$. In the numerical analysis of Chapter 5 the parameter values are always chosen so that $-1 \leq b/2a \leq 0$ doesn't hold.

Another issue that may raise problems in numerical calculations is the following. Note that the solution (4.18) and the formulas for the constants C_1 and C_2 involve the term $e^{\xi^2\sqrt{a}}$. This term can be very large even for moderate values of $\xi = b/2a + 1$. For example, for $b = 20$ and $a = 1$ we have $e^{\xi^2\sqrt{a}} = e^{121}$. At least in a direct implementation of (4.18) such large terms destabilize the calculation of (4.18). Therefore, in our numerical study in Chapter 5 parameters a and b are kept in the ranges where this term doesn't cause numerical instabilities.

CHAPTER 5

NUMERICAL STUDY

In this chapter, we provide numerical examples showing that the solutions given in the last two chapters reduce to the $\sinh$ curve when the volume curve is taken to be a constant. In the affine trading volume model the new parameter (as compared to the constant volatility model) is the a parameter; $a > 0$ corresponds to an increasing trading volume and $a < 0$ corresponds to a decreasing trading volume. The dependence of the optimal trading curve on a was studied in [15]; so for comments on the affine case we refer the reader to that work. The second goal of this section is a similar sensitivity analysis for the quadratic case.

As noted in [15], there is no loss of generality in assuming $T = 1$, $q_0 = 1$ and $\frac{\gamma\sigma^2}{2\rho} = 1$; every other cases can be reduced to this case by scaling the p and q processes and time.

5.1 Almgren Chriss Model with Affine V_t

Recall that in Chapter 3, we presented a solution of the differential equation

$$p''(t) = (at + b)\, p(t).$$

using methods in [19] and (3.1) and obtained the following optimal trading curve for affine market volume:

$$q^*(t) = C_1 \frac{a^{-\frac{2}{3}}(at+b)}{3} \left[I_{-2/3}(z) - I_{2/3}(z)\right] + C_2 \frac{a^{-\frac{2}{3}}(at+b)}{\sqrt{3}} \left[I_{-2/3}(z) + I_{2/3}(z)\right] \tag{5.1}$$

where C_1 and C_2 have an explicit formula in terms of a and b as given in Chapter 3.

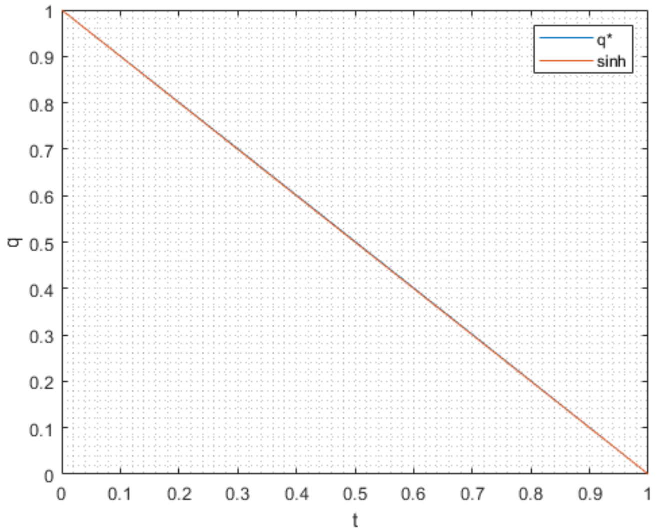


Figure 5.1: Graph of q with respect to t where a is near 0

For the affine volume case we provide a numerical example that shows that the above formula reduces to $\sinh$ for $a = 0$ (i.e., when the market volume process is constant. Because the change of variable $z = at + b$ is degenerate for $a = 0$, to compute (5.1) for $a = 0$ one in fact has to compute this case as a limit as $a \to 0$. Instead of computing the limit we take a to be very close to 0, e.g., $a = 10^{-5}$ volume V_t is reduces to a constant trading curve is famous $\sinh$ formula (see (2.14)). Figure 5.1 gives the graph of q^* and $\sinh$ for this case by using MATLAB. We see that the graphs overlap, as expected.

We do not provide a sensitivity analysis of q^* as a function of a as this has already been done in [15]. For verification purposes we did repeat several of the calculations in that work and observed that calculations made with (5.1) agree with those presented in that work. Since the results are identical we don't reproduce them here.

5.2 Almgren Chriss Model with Quadratic V_t.

In this section, we will give some numerical examples for the case market trading volume is a quadratic function of time, i.e. $V_t = at^2 + bt + c$. In Chapter 4, we obtained the solution of the ODE

$$p''(t) = (at^2 + bt + c)\, p(t)$$

with terminal conditions

$$p'(0) = 1, p'(1) = 0,$$

where p is the integral of the optimal trading curve q in the Almgren Chris model (under the assumption that the trading volume is quadratic $V_t = (at^2 + bt + c)$ in terms of the confluent hypergeometric function Φ:

$$\begin{aligned} q^*(t) = &\frac{-2\xi\sqrt{a}}{e^{\xi^2\sqrt{a}}}\left(C_1\Phi(\alpha,\beta;\xi^2\sqrt{a}) + C_2\xi\sqrt[4]{a}\Phi(\alpha-\beta+1,2-\beta;\xi^2\sqrt{a})\right) \\ &+ \frac{1}{e^{\xi^2\sqrt{a}}}\Bigg(C_1\frac{\alpha}{\beta}2\xi\sqrt{a}\Phi(\alpha+1,\beta+1;\xi^2\sqrt{a}) \\ &+ C_2\sqrt[4]{a}\Phi(\alpha-\beta+1,2-\beta;\xi^2\sqrt{a}) \\ &+ C_2\frac{\alpha-\beta+1}{2-\beta}2\xi^2 a^{\frac{3}{4}}\Phi(\alpha-\beta+2,3-\beta;\xi^2\sqrt{a})\Bigg), \end{aligned} \tag{5.2}$$

where $\xi = t + \frac{b}{2a}$ and C_1 and C_2 are constants depending on a, b, c whose explicit formulas are given in the previous chapter. In all of the numerical examples in this section we will stay within the part of the parameter space described in Section 4.3.

We begin by checking whether (5.2) reduces to the $\sinh$ function (see (2.14)) when $a = b = 0$, i.e., when V_t reduces to a constant once again, because ξ is undefined for $a = 0$ one cannot directly set $a = b = 0$ in (5.2) and instead a limiting process needs to be used. Instead of taking limits, we set $a = b$ to be a small number, e.g., 10^{-12} and compute (5.2) with these. By using MATLAB, the resulting graph of (5.2) with $\sinh$ of (2.14) is given in Figure 5.2, as expected the solutions overlap.

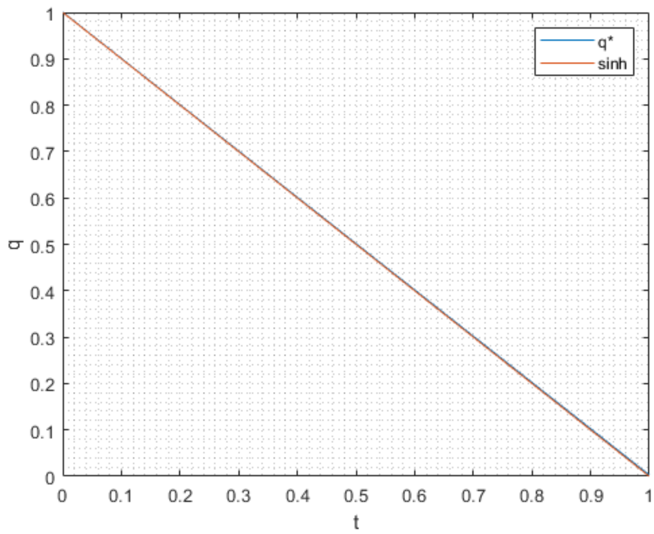


Figure 5.2: Graph of q with respect to t where a and b are near 0

In the rest of this section we will present several numerical experiments showing the impact of the a parameter on the optimal trading curve. Recall that $q^*(0) = 1$. We present four different scenarios: c large, V_t increasing, c large, V_t decreasing, c small, V_t increasing and c small, V_t decreasing. Recall that c is the constant component of the trading volume; therefore it determines the overall basic market volume. A large c corresponds to a market that has overall a bigger trading volume in comparison to the initial position of the trader. A small c corresponds to a market whose overall volume is comparable to the position of the trader. An increasing V_t is represented by $b > 0$ and a decreasing V_t is represented by $b < 0$. We will use the a parameter to concavely or convexly perturb these scenarios. The base scenario is given by the affine trading volume $V_t^A = bt + c$. To perturb this with a we add $a(t^2 - t)$ to V_t^A:

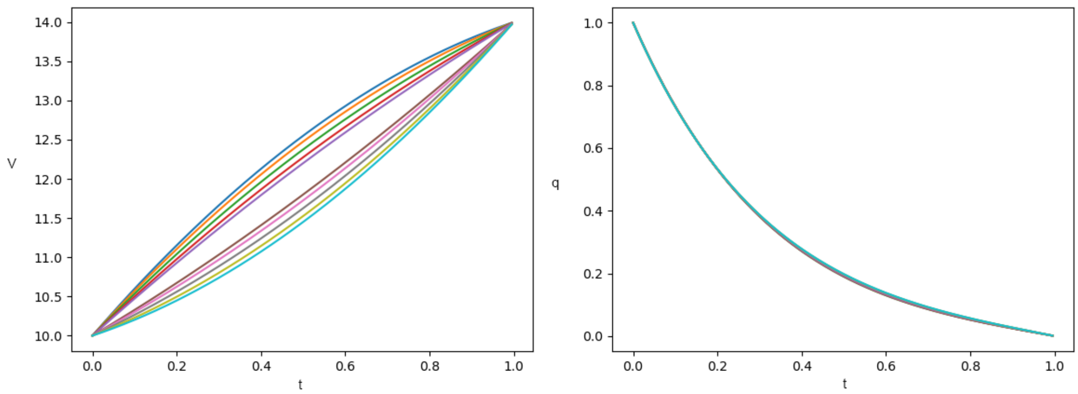
$$V_t = V_t^{(A)} + a(t^2 - t) = at^2 + (b - a)t + c. \tag{5.3}$$

Note that $t^2 - t = 0$ for $t = 0$ and $t = 1$, therefore, adding it to V_t^A does not change it at $t = 1$ and $t = 0$ and this is how (5.3) corresponds to a perturbation of V_t^A: $a < 0$ corresponds to a concave perturbation whereas $a > 0$ corresponds to a convex perturbation (see the volume curves in the figures below). In all of the numerical examples below the market volume is computed by using the formula (5.3) and Python.

Let us start by taking $c = 10$, i.e., the constant market volume is 10 times the position to be liquidated and $b = 4$; i.e., we are assuming that the market trading volume increases in time. With these parameters we allow a to vary in the interval $[-2.2, 2.2]$. The resulting family of trading volumes is given on the left side of Figure 5.3. The corresponding optimal trading curves, computed using (5.2) is shown on the right side of the same figure. We note that for c large, the convex/concave perturbations to the market volume have little impact on the optimal trading curve.

Next we repeat the same computation, this time taking $b = -4$. This corresponds to the same setup as in the previous example except that the market volume is now decreasing in time. Allowing the a parameter to vary in the interval $[-2.2, 2.2]$ we get the graphs shown in Figure 5.4. Qualitatively the results are similar to those for $b > 0$ discussed above.

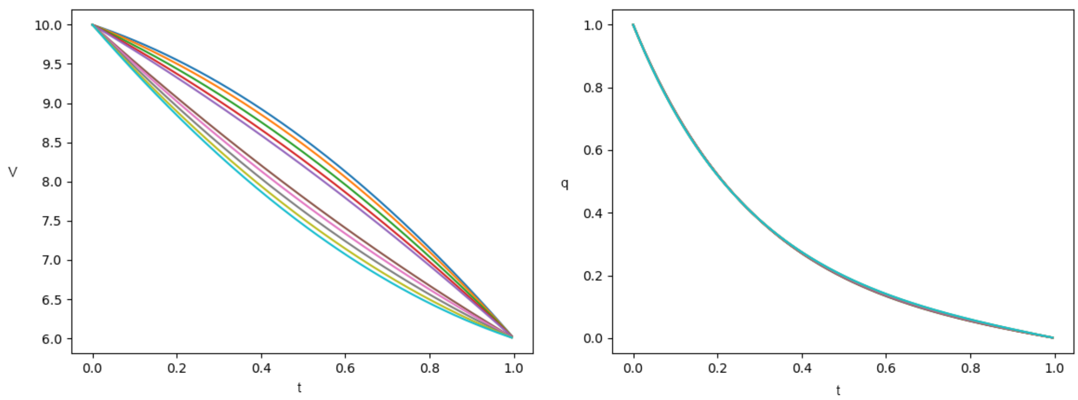
We next consider the case $c = 1$. This corresponds to a market where the constant



(a) Graph of V_t with respect to t

(b) Graph of q^* with respect to t

Figure 5.3: Graph of market volume and trading strategy with respect to t where $a \in [-2.2, 2.2] - \{0\}, b = 4 - a$ and $c = 10$

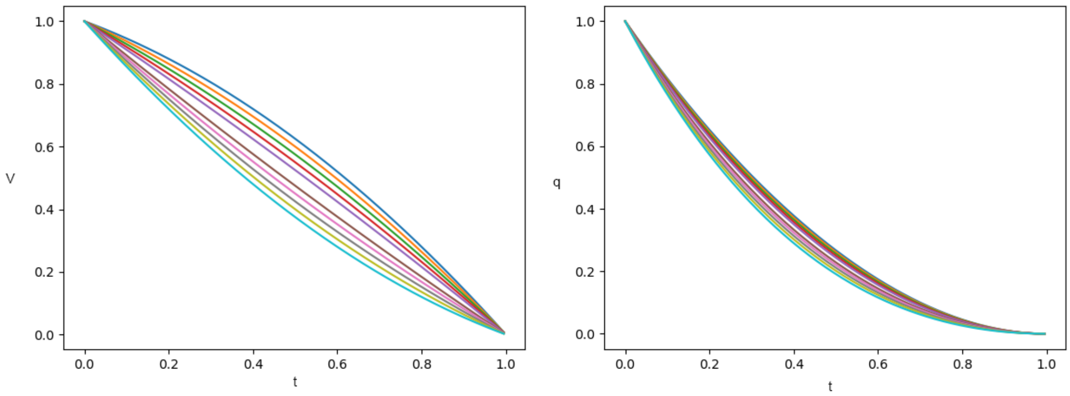


(a) Graph of V_t with respect to t

(b) Graph of q^* with respect to t

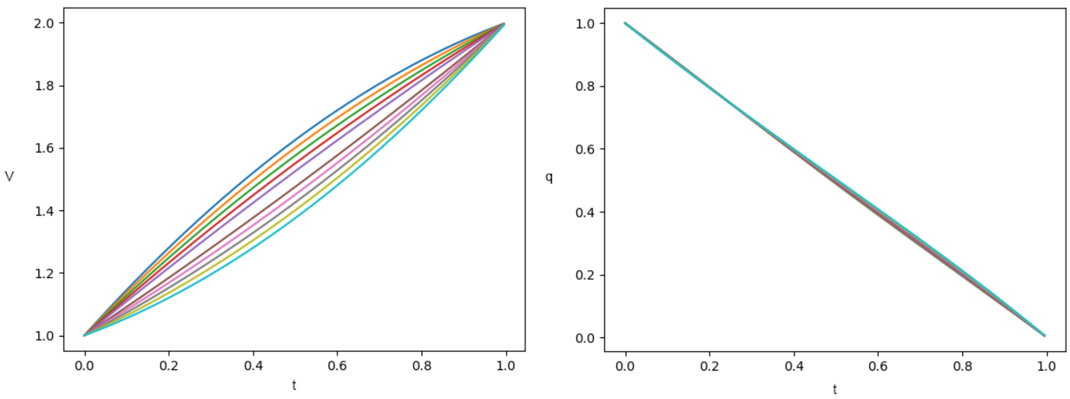
Figure 5.4: Graph of market volume and trading strategy with respect to t where $a \in [-2.2, 2.2] - \{0\}, b = -a - 4$ and $c = 10$

component market volume equals the initial position to be liquidated, i.e., the position to be liquidated is very large. We again consider two cases $b = 1$ and $b = -1$ which represent increasing and decreasing market volumes. Allowing a now to vary in the interval $[-0.5, 0.5]$ get the market volumes and optimal trading curves shown in Figures 5.5 and 5.6. As opposed to the case $c = 10$, we see that now the a parameter has a significant impact on the optimal trading curve for both b negative and positive. In both cases the optimal trading curve remain mostly convex, although for $b < 0$ we observe that q^* is very slightly concave for $|a|$ large.



(a) Graph of V_t over time (b) Graph of q^* over time

Figure 5.5: Graph of trading strategy with respect to t where $a \in [-0.5, 0.5] - \{0\}$, $c = 1$ and $b = -a - 1$



(a) Graph of V_t over time (b) Graph of q^* over time

Figure 5.6: Graph of trading strategy with respect to t where $a \in [-0.5, 0.5] - \{0\}$, $c = 1$ and $b = 1 - a$

CHAPTER 6

CONCLUSION

Scheduling of buy or sell orders to liquidate a position has been one of the most popular topics in mathematical finance for decades. Almgren and Chriss modelled the solution to the problem by the maximization of the expected utility of the final terminal wealth. In their framework the market trading volume is taken as a constant and by using other assumptions, an optimal trading curve can be computed in terms of the $\sinh$ function.

In this thesis, we give explicit formulas for the optimal liquidation strategy q^* of the Almgren and Chriss model for market volumes that are affine or quadratic functions of time in terms of the special functions of mathematical analysis using the transformations and formulas given in [19]. A parameter sensitivy analysis for the affine case was already available in [15]. For this reason, in the rest of the thesis we focuse on the sensitivity of the optimal trading curve to the a parameter (the novel feature of the quadratic volume model). We find out that when the overall market volume is large compared to the initial position to be liquidated the a parameter has little impact on the optimal trading strategy whereas when the overall market volume is close to the initial position to be liquidated, the trading curve changes in a nontrivial way with a. In this case, the formulas given in the present work can be used to quickly and easily compute the liquidation strategy.

Recall that we work under several constraints on the parameters a, b and c which are discussed in Section 4.3. Immediate future work can try to develop formulas that continue to work outside of these constraints. Future work can also consider the development of formulas for portfolios consisting of positions on multiple assets.

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