Identifying Stability Properties of a Hybrid Spring–Mass–Damper via Piecewise LTI Approximation and Harmonic Transfer Functions

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1 Introduction

Modeling of legged locomotion as a nonlinear rhythmic dynamical system that operates near a (stable) limit cycle is a common practice in robotics and biology. Under this modeling framework a wide variety of methods have been developed for the identification of the limit cycle as well as the dynamics near, but off of the limit cycle. For example, it has been showed that simple low dimensional spring–mass models can accurately capture the steady-state characters of locomotor systems. However, unlike their success in characterization of the limit cycle, they fail at predicting the stability properties of such systems, e.g. how the dynamics recover from perturbations [1]. An alternative to such "first-principles" (or "grey-box") models—which have inherent structural limitations—is a set of "block-box" data-driven system identification approaches that may better capture the dynamics around the limit cycle [1, 3].

However, existing black-box methods for periodic systems have three basic limitations. First, most existing methods presume the ability to measure all state variables. Second, they significantly down samples the raw data to construct multiple cross sections within a cycle. Finally, these methods are typically "input free" and the identification is performed only based on the output measurements. In the context of system identification of non-rhythmic systems we know that "input–output" approaches can be powerful since they typically have higher signal to noise ratio and, critically, they address the problem of hidden states and/or delays in dynamics.

In this study our goal is to develop a method for the identification of stability properties of locomotor dynamics using an input–output approach which does not require downsample the raw data and does not presume full state measurement. We have previously shown that an input–output linear time-periodic (LTP) system structure can be used to represent rhythmic locomotor behaviors around their limit cycles. This approach considers legged locomotion models as hybrid dynamical systems with state-dependent transitions between system phases being approximated as *time*-dependent transitions [2]. Here we show that our the data-driven system identification technique can be utilized to characterize stability properties of limit cycles for clock-driven legged locomotion models. To this end, we use the concept of harmonic transfer functions (HTF), first to obtain a non-parametric system model based on input–output data and then to identify an associated, explicitly parameterized model to estimate the eigenvalues of a suitably defined Poincaré map. We present simulation studies for a simple hybrid spring–mass–damper across a range of system parameters to illustrate the accuracy of this data-driven method.

2 Modeling Legged Locomotion as an LTP System

Locomotion systems are generally nonlinear, hybrid dynamical systems with stable periodic orbits. We begin by linearizing these systems around their limit cycles to obtain an LTP representation. In doing so, our first assumption is that phase transitions that are normally state-dependent can be approximated as being only time-dependent around the limit cycle. We then introduce another approximation to reduce system dynamics to a finite dimensional piecewise LTI representation, admitting a practical parametric identification framework while preserving its LTP nature. The final LTP equations of motion hence take the form

$$\dot{x}(t) = \begin{cases} A_0 x(t) + B_0 u(t), & \text{if } \operatorname{mod}(t,T) \in [0,\hat{t}) \\ A_1 x(t) + B_1 u(t), & \text{if } \operatorname{mod}(t,T) \in [\hat{t},T) \end{cases}$$

with a similar structure for output dynamics. This formulation constitutes our framework for analyzing and identifying clock-driven legged locomotion models in this study.

3 The Hybrid Spring–Mass–Damper Model

Figure 1(A) illustrates a simple, vertically constrained spring–mass–damper system, consisting of a point mass attached to a spring–damper mechanism connected in parallel with a linear actuator. The hybrid nature of the model is introduced through the leg damping, which is turned off and on during the compression and decompression phases, respectively. The force transducer is used both as an active energy input $f_0(t)$ to maintain a stable limit cycle and as an exogenous input $u(t)$ to support data-driven system identification. We use this model as an illustrative example for our identification method.

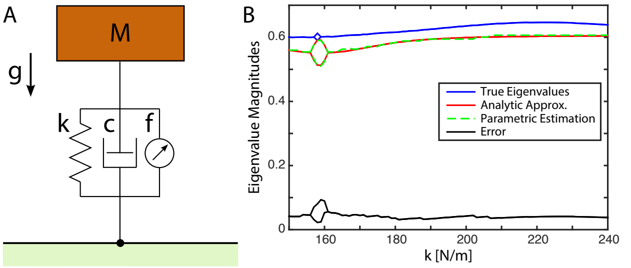

Figure 1: (A) Hybrid, vertical spring–mass–damper system. (B) Eigenvalues of the linearized return map for the dynamics around the limit cycle, computed using three different methods as a function of the spring stiffness.

The equations of motion for this model are given by

$$m\ddot{x} = \begin{cases} -mg - c\dot{x} - k(x - x_0) + f(t), & \text{if } \dot{x} > 0 \\ -mg - k(x - x_0) + f(t), & \text{otherwise,} \end{cases} \tag{1}$$

Simulations in this study use $g = 9.81\text{m/s}^2$, $k = [150 - 240]\text{N/m}$, $c = 2\text{Ns/m}$, $m = 1\text{kg}$ and $x_0 = 0.2\text{m}$ with the force transducer input constructed as $f(t) = f_0(t) + u(t)$. Following the modeling principles described in Section 2, we obtain the LTP system matrices as

$$A_0 = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}.$$
$$B_0 = B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_0 = C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \tag{2}$$

Using the framework described in [4] on these LTP dynamics yields analytic solutions to the harmonic transfer functions.

4 Method for Estimating the Linearized Return Map

Let Σ_0 and Σ_1 be two Poincaré sections associated with the two distinct hybrid transitions for our model. This results in a nonlinear map $P_{0\to1} : \Sigma_0 \to \Sigma_1$ defined by continuous forward trajectories of the system from Σ_0 until their intersection with Σ_1. Likewise, $P_{1\to0} : \Sigma_1 \to \Sigma_0$, resulting in an overall, single-stride return map on Σ_0 constructed as $P_{0\to0} := P_{1\to0} \circ P_{0\to1}$.

Approximating the hybrid equations of motion around a limit cycle as a Piecewise LTI system, linearized versions of $P_{0\to1}$ and $P_{1\to0}$ correspond to transition states between the two linear systems $\dot{x}(t) = A_0 x(t),\ x(t_0) = x_0$, at time $t = t_0 + \hat{t}$ and $\dot{x}(t) = A_1 x(t),\ x(t_1) = x_1$, at time $t = t_1 + (T - \hat{t})$, respectively. The overall linearized return map on Σ_0 is then computed as

$$DP_{0\to0} = e^{A_1(T-\hat{t})} e^{A_0\hat{t}}. \tag{3}$$

In this study, we both analytically derive and parametrically estimate A_0 and A_1 matrices for the hybrid model of Section 3. However, one should note that even the analytic version of (3) is an approximation to the "true" linearized return map, since we approximate hybrid transitions as being dependent on time rather than state in close proximity to the limit cycle. For this reason, we use a numerically computed Jacobian to the return map for the hybrid model as a ground truth against which we compare the eigenvalues estimated with both the parametric LTP model as well as our analytic approximations.

5 Results

For the analytic approximation, we explicitly derive the system matrices in (2) using the simulated system parameters. For the data-driven identification step, we first estimate the non-parametric HTFs using the input–output method detailed in [2]. Subsequently, we perform a parametric fitting to estimate the values for k and c that minimize the error between non-parametric HTFs and analytically derived HTFs of the explicitly parameterized LTP system structure in (2) (See [2] for details). Having a piecewise LTI representation with explicit estimates of system parameters, we obtain the eigenvalues of the LTP system as described in Section 4 for both analytic and the identified model.

We repeated the above steps for different values of spring constants as illustrated in Fig. 1(B). Note that our model is a two-dimensional system due to its clock-driven structure and we observe two complex conjugate eigenvalues except around $k = 160$, where we obtain two distinct real eigenvalues. Our results show that the LTP framework yields accurate estimates for the true eigenvalues of the system. We believe that the small error (see Fig. 1(B)) originates primarily from our approximating of state-dependent hybrid transitions as time dependent away from steady state.

In future work, we plan to extend this study to more realistic legged locomotion models that are not clock-driven and incorporate separate flight and stance phases with different, physically relevant mechanisms for regulating system energy. We also hope to extend the parametric identification work to more general models without having a priori knowledge of system order.

References

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