DECENTRALIZED INVENTORY PLANNING WITH TRANSSHIPMENTS IN A MULTI-PROJECT ENVIRONMENT

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ABSTRACT

DECENTRALIZED INVENTORY PLANNING WITH TRANSSHIPMENTS IN A MULTI-PROJECT ENVIRONMENT

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Many companies in project-based industries manage their inventories in a decentralized manner on a project-by-project basis. Even if one project has excess supply of a material, the other may have excess demand of it, which increase inventory costs of the company. These costs can be decreased if material transshipments between projects are allowed. In this study, motivated by the defense industry applications in Turkey, we investigate ordering decisions of two different agents, who make inventory decisions for two different projects, in a two-period problem. We study a transshipment mechanism where a project transfers its excess inventory at a unit transfer price to a project with excess demand. We prove the existence and uniqueness of the corresponding equilibrium between agents and characterize it. We also study two benchmark settings, decentralized setting without transfers and centralized setting, to discuss the benefits of allowing transfers. Numerical analyses show that agents achieve significantly more profits with transshipments compared to decentralized setting without transfers. We also show that the amount of benefits introduced by transshipments depend on the right selection of transfer prices.

Keywords: inventory management, game theory, project-based companies

ÖZ

TRANSFERLERE İZİN VERİLEN ÇOKLU PROJE ORTAMINDA MERKEZİ OLMAYAN ENVANTER PLANLAMA

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Proje bazlı çalışan birçok firma için envanter yönetimi, proje bazında merkezi olmayan bir şekilde yapılmaktadır. Bir proje bir malzemenin fazla arzına sahip olsa bile, diğerinin fazla talebi olabilir ve bu da şirketin envanter maliyetlerini artırır. Projeler arasında malzeme transferlerine izin verilirse bu maliyetler azaltılabilir. Türkiye merkezli savunma sanayi uygulamalarından hareketle yürütülen bu çalışmada, iki periyotlu bir problemde iki farklı projenin karar vericilerinin aynı malzemeyi tedarik etme politikaları incelenmiştir. Bir projenin fazla arzını, belirli bir transfer fiyatı karşılığında, fazla talebi olan bir projeye transfer ettiği bir transfer mekanizmasını inceliyoruz. Ayrıca, transferlerin getirdiği faydanın değerini tartışmak için iki karşılaştırma noktası, iki projenin de birbirinden bağımsız yönetildiği ve ikisinin de merkezi karar verici tarafından yönetildiği problemler üzerinde çalışıyoruz. Sayısal analiz, transferlerin varlığında ulaşılan çözümün, merkezi olmayan yönetimle gelen çözümden önemli ölçüde daha fazla kar elde getirdiğini göstermektedir. Ayrıca, transferlerin sağladığı avantajların miktarının, transfer fiyatlarının doğru seçimine bağlı olduğu da

sayısal analizlerle gösterilmiştir.

Anahtar Kelimeler: envanter yönetimi, oyun teorisi, proje bazlı şirketler

To my family

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CHAPTER 1

INTRODUCTION

Inventory management has its own challenges in defense and aerospace industries, where various types of complex-structured final products are produced. Although their production quantities are certain as they usually work with their customers on contracts, there are several reasons creating uncertainty in their production quantities and schedules.

To begin with, complex structure of the final product causes variance in compliance to the specifications for the intermediate and final products, resulting in high wastage rates and randomness in total production quantity. Continuous changes in the bill of materials (BOM) and/or production process of the intermediate and final products in order to keep up with developing technology bring additional uncertainty to total production quantities. Moreover, although defense companies mostly work on contract basis, non-contract development activities are also carried out intensively in these companies. Both BOM and production processes become clear over time in these activities, which results as another source of randomness in total production quantity. Uncertainties in total production quantities and BOM are reflected as uncertainties in raw material needs.

For those companies whose customers are states and armies, production schedules tend to change due to the uncertainties arising from cross-country diplomacy. For example, a COTS (commercial of the shelf) fastener purchased from an abroad country may not be supplied due to political reasons, even though the materials are available at the customer. Their production schedules are also affected by the constant changes in the production processes to keep up with developing technology and the instantly changing needs of their customers. Uncertainty in the production schedule affects material requirement and workforce plans.

Project based management is another source of uncertainty for these companies. Due to customer-based deliveries, production and inventory management is carried out on a project basis. The decisions on production and inventory management are made by separate decision makers for each project. One of the motivations for this type of management is that the company has different budget constraints on all projects. For example, the budget of a newly started design project and the budget of a product that has started mass production within the scope of a customer contract are required to be followed separately. Moreover, with different penalty costs of not delivering the products to customers on time, each project has different priority levels for the company, which requires allocation of raw materials and resources among projects. As these penalty costs are much higher in defense sector compared to other sectors, decision makers of each project tend to carry more inventory separately. On the other hand, because of the uncertainties above, there are also cases that while a project has excess supply of a material, another project of the same company may have to suspend production due to the lack of this material. As a result, project-based management creates inefficiencies in inventory management.

In this thesis, we study the quantification of the inefficiencies generated due to the simultaneous existence of excess supply and excess demand in project-based companies. If a centralized decision maker could make inventory and production decisions for all projects, these inefficiencies could be eliminated. On the other hand, due to the motivations mentioned above, this solution is not applicable for the project-based companies. Therefore, we investigate the effects of allowing transshipments among parties on inventory costs.

Our problem involves ordering behavior of two separate decision makers, agents, to meet their material demand over two-period horizon, where both agents are required to determine their order quantities at the beginning of the planning horizon. Demand of both agents has randomness in quantity. Both agents have perfect information on all parameters. The agents are asymmetric in the sense that the period of demand realization is certain for agent 2 whereas it has a probabilistic nature for agent 1. We investigate the benefits of allowing transshipments among parties. When one of the agents has excess demand whereas the other has excess supply, transshipments of materials between agents, which will be referred as transfers in the remainder of the thesis, are allowed at a unit transfer price. Given transfer opportunity, decisions of both agents affect each other. Hence, with both parties aiming to maximize their profits, we consider this problem using game theoretical concepts and show that there exists a unique non-zero equilibrium for this problem.

In order to quantify these inefficiencies generated due to the project-based management, we investigate decentralized setting, where each agent makes procurement decisions independently, and centralized setting, where a single decision maker makes procurement decisions for the same problem environment. We observe that as the level of coordination among the decision makers increases, total profits increase. In other words, total profits achieved when transfers are allowed are higher than that in decentralized setting whereas they are lower than that in centralized setting. We conduct numerical analysis in order to investigate how the equilibrium is affected by the problem parameters. Numerical analysis also shows that the benefits of transfers depend on the transfer prices. We observe that selection of the best transfer price per unit gains importance as the procurement cost per unit increases. We also show that when the period in which demand will be realized is probabilistic, there may not be a transfer price that will enable the centralized optimal solution.

In the remainder of the thesis, we continue with the review of the literature in Chapter 2. We introduce a detailed problem definition and investigate ordering behaviors of benchmark settings in Chapter 3. Chapter 4 involves a detailed analysis of ordering decisions when transfers are allowed. We provide numerical studies in Chapter 5 and conclude with our findings and future research suggestions in Chapter 6.

CHAPTER 2

LITERATURE REVIEW

There are many studies in literature on the use of game theory concepts in supply chain management. Cachon and Netessine [4] provide a review of the existing literature based on a classification of game theoretical concepts. Leng and Parlar [8] also present a review based on a classification of supply chain management topics.

According to Cachon and Netessine [4], use of cooperative game theory to investigate supply chain topics is becoming more popular with the increasing attention paid to bargaining and negotiations. Özen et al. [11] provide a review of studies specifically on inventory cooperation using newsboy models where the solution is established using the concept of the core. The concept of a non-empty core is crucial for cooperative games since it ensures the existence of allocations for which no sub coalitions can be formed. Hartman et al. [6] made an important contribution to the literature with the study on inventory centralization of n retailers selling an identical good over a single-period horizon and provide conditions for a non-empty core. Müller et al. [12] and Slikker et al. [18] independently show that in the problem setting defined by Hartman et al. [6], if players are allowed to make transshipments after demand realization, the core is always non-empty.

As an extension to these studies, Özen et al. [15] consider a supply chain of n retailers and m warehouses. They consider a two-stage problem. At the first stage, retailers determine their order quantities, and these quantities become available at warehouses. The second stage starts with the realization of demand for all retailers. Retailers determine the quantity to be transferred from the warehouses at this stage. Özen et al. [15] investigate the joint profits of cooperative game and show that this game has a non-empty core. As a further extension, Özen et al. [16] introduce a general framework of newsvendor problems with warehouses. Using a similar two-stage stochastic programming approach, they prove that every general newsvendor game with warehouses has a non-empty core. Wang and Parlar [19] investigate the impact of coordination among three players. They use the concept of core for cooperation model and present the conditions for non-empty core. Moreover, they prove the existence of Nash equilibrium under cooperation.

In their review, Leng and Parlar [8] state that non-cooperative game models are more common in literature compared to cooperative ones. They highlight the common use of Nash and Stackelberg equilibria as solution models in non-cooperative games. Lippman and McCardle [9] illustrate the existence of equilibrium in a single-period inventory problem of two-players and n-players under substitution. They consider total industry demand as a random variable and assume that the allocation of demand to the players is based on a pre-determined splitting rule. They show that total inventory level of the players always increases in the case of perfect substitution, and investigate the relation between the inventory levels at equilibrium and the splitting rule. Parlar [17] investigates a single-period inventory problem of two substitutable products. Different than Lippman and McCardle, he considers that the products have independent random demands. He shows the existence of a unique Nash equilibrium in this non-cooperative game and provides upper and lower bounds for the solution. Parlar [17] also investigates the problem where the players act irrationally to damage the other one, called as the maximin problem, and shows that it's solution is equivalent to that of a classical newsvendor problem.

As an extension to Parlar [17], a single-period game of three competing players with independent random demands is investigated by Wang and Parlar [19]. They prove the existence of Nash equilibrium when players make independent ordering decisions. Similar to Lippman and McCardle [9], in this case they also observe that the optimal order quantity of players is larger than that under the newsboy model since the players have motivation of sales through substitution. The work of Parlar [17] is also extended to n products by Netessine and Rudi [13]. They study a problem setting under consumer-driven substitution where unsatisfied demand for one product can be satisfied by another product in deterministic proportions. They also investigate centralized inventory management and the effect of demand correlation on profits for this problem setting. Extending the work in [17], Avşar and Baykal-Gürsoy [2] focus on a substitutable product inventory problem of two players over an infinite horizon with lost sales, where each player aims to minimize discounted costs over the planning horizon. They prove the existence and uniqueness Nash equilibrium within the class of stationary base stock strategies. Similar to Parlar [17], Kapur et al. [7] investigate the existence of Nash equilibrium in a two-retailer, single-period game, where transshipments among retailers are allowed at a transshipment price. The main difference is that Parlar [17] studies the case where customer demand is transferred between retailers whereas Kapur et al. [7] study the case where available units are transferred between retailers. Kapur et al. [7] also argue that the joint profits depend on the transshipment price, and prove the existence of a set of transshipment prices which achieves the joint optimal solution.

There are also studies on non-cooperative game models in multi-echelon supply chain settings. Cachon [3] investigates a single-item problem of a supplier and a retailer over infinite planning horizon where each party uses a base stock policy. Given stochastic interarrival times of customers to retailers, he uses continuous Markov chain concepts and prove the existence of equilibrium for the optimal base stock levels in this supply chain. Cachon [3] also shows that the joint profits obtained with Nash equilibrium is always less than the optimal profits and investigates possible contracts that may result in supply chain optimal base stock levels. Güllü et al. [5] consider a decentralized supply chain of one supplier and two retailers. Each retailer faces random demand and follows a periodic review system. In their study, at the end of supplier lead time, the retailers have the opportunity to revisit their order quantity, which are determined at the beginning of supplier lead time, without changing total quantity ordered from supplier. The costs of these transfers are assumed to be negligible. The opportunity of transfer would be beneficial for both parties as demand during supplier lead time is random. They show the existence of a unique Nash equilibrium for optimal-order up-to levels of retailers.

Mahajan and Ryzin [10] examine a non-cooperative game with a two-echelon supply chain of a monopolist retailer and n horizontally competing suppliers. They show that a mixed coordination of Retailer Managed Invetory (RMI) and Vendor Managed Inventory (VMI) can improve total system profits. In their study, Netessine and

Shumsky [14] establish the existence of a unique Nash equilibrium for the seat inventory control problem in airline sector under both horizontal and vertical competition, and utilize revenue sharing contracts as a mean of coordination. Anupindi et al. [1] investigate the case of n retailers and m warehouses over single-period horizon using a mix of competitive and cooperative framework. In their study, n retailers decide on the quantity that will arrive to the retailer and be stocked at the warehouse on the behalf of the retailer. After demand realization, retailers decide on the allocation of local residual stocks and pooled warehouse stocks to satisfy excess demand of all parties. Anupindi et al. [1] construct a mathematical model and establish the conditions for the existence of a Nash equilibrium.

Our study can be considered as an extension to Parlar [17] and Kapur et al. [7]. We study a non-cooperative game of two agents having random demands over a two-period horizon, where both agents aim to maximize their expected profits. We assume that both agents determine their order quantities for both periods at the beginning of the period. When one of the agents has excess supply whereas the other has excess demand, transfers of materials between the parties are allowed at a transfer price of t in the second period. These transfers construct the base for the use of non-cooperative game theory. Different than the existing literature, there is a difference between the levels of uncertainty faced by the agents in our study: While agent 2 knows that the demand will be realized at the second period, agent 1 knows the probability that demand is realized at the second period. The existence of holding and penalty costs charged at the end of the first period also makes our study different from the existing literature.

CHAPTER 3

PROBLEM DEFINITION AND BENCHMARK SETTING

We investigate a two-period inventory problem of two decision makers, called as agent 1 and agent 2. In our study, we represent two different types of projects in the defense industry industries with two different agents. Although the production time is certain in contract-based mass production projects, the amount of material to be transferred to the production line is uncertain. On the other hand, in design projects which are carried out for development purposes, both the production time and the amount of material to be transferred to production have uncertainty. With this motivation, we are working on a stylized profit maximization problem of 2-agents. In the problem, agent 1 represents design projects and agent 2 represents mass production projects.

In order to meet their random demand, both agents are required to determine their order quantities at the beginning of the planning horizon before any demand uncertainty resolves. While demand faced by agent 2 is ensured to be realized in the second period, demand faced by agent 1 has a probability of occurrence at each period. Hence, agent 2 places an order only for the second period whereas agent 1 can place an order for both periods. Both agents have information on all demand parameters and models. Our aim is to analyze the benefit of allowing transfers between two agents in the second period if one agent has excess demand and the other has excess supply.

We provide the parameters and decision variables in the following table. Below, we describe the structure of costs and revenues (see Table 3.1).

- Quantity remaining on hand at the end of the first period is charged by a holding cost per unit, h.

Table 3.1: Parameters and Decision Variables

X	Demand of agent 1
Y	Demand of agent 2
Z	Sum of demands X and Y
$f(x)$	Probability density function of X
$g(y)$	Probability density function of Y
$v(z)$	Probability density function of Z
k	Probability that X is realized in the first period
r	Sales price per unit
c	Purchase cost per unit
h	Holding cost per unit per period
w	Backorder cost per unit for the first period
l	Lost sales cost per unit for the second period
s	Salvage value per unit
t	Transfer price per unit
q_{1i}^{d}	Quantity ordered by agent 1 to arrive in period i in the decentralized setting
Q_1^d	Total quantity ordered by agent 1 in the decentralized setting
q_2^{d}	Quantity ordered by agent 2 to arrive in period 2 in the decentralized setting
q_i^{c}	Quantity ordered to arrive in period i in the centralized setting
Q^c	Total quantity ordered in the centralized setting
q_{1i}	Quantity ordered by agent 1 to arrive in period i in the 2-agent setting
q_2	Quantity ordered by agent 2 to arrive in period 2 in the 2-agent setting
Q_1	Total quantity ordered by agent 1 in the 2-agent setting

- If demand realizes in the first period and quantity on hand is not enough to fulfill it, a backorder cost is charged per unit, w, for the amount of underage. Excess demand at the end of the first period will be met in the second period if possible.

- Quantity remaining on hand at the end of the second period is salvaged at a salvage value per unit, s.

- Demand not fulfilled at the end of the second period is charged by a lost sales cost per unit, l.

- If transfers are allowed, excess inventory at the end of the second period will be transferred between players at a transfer price per unit, t.
- All costs and revenue parameters are exogenous.

In this chapter, we introduce decentralized and centralized settings as building blocks. In the decentralized setting, we analyze the problem of agent 1, who faces demand X (see Table (3.1) for notation). We also show that the problem of agent 2, who faces demand Y, is a special case of the problem of agent 1 in the form of a classical newsboy problem. In the centralized setting, we consider a central agent facing both X and Y.

We assume $r + l > c$, $t > s$ and $c > s$ without loss of generality as the system would not operate otherwise.

3.1 Decentralized Setting

In this problem setting, two agents place orders for the coming two periods at the beginning of the first period without interaction. We start with introducing the sequence of events for agent 1:

1. The agent determines order quantities for period 1 and period 2 at the beginning of the first period.
2. Order placed for the first period, q_{11}^d, arrives at the beginning of the first period.
3. If demand realizes in the first period, then quantity available on hand is used to fulfill the demand. If quantity on hand is not enough to fulfill demand in the first period, excess demand will be met in the second period if possible.

   Quantity remaining on hand at the end of the first period will be carried over to the second period.
4. End of period 1 costs are incurred.

5. Order placed for the second period, q_{12}^d, arrives at the beginning of the second period.

6. If demand realizes in the first period and excess demand is carried to the second period, then quantity available on hand is used to fulfill this demand. If it realizes in the second period, then total quantity ordered is used to fulfill the demand.

   Quantity remaining on hand is salvaged, and excess demand becomes lost sales at the end of second period.

7. End of period 2 costs and revenues are computed.

Note that events 2, 3, 4 and 5 do not take place for agent 2 since she faces demand Y which occurs in the second period.

We proceed with the profit function of agent 1. Let x represent the demand realization. If it occurs in the first period, the profit function of agent 1, denoted as π', is given as follows:

$$\begin{aligned}\pi' &= r \cdot min\{x, Q_1^d\} + s\left[Q_1^d - min\{x, Q_1^d\}\right] - l\left[x - min\{x, Q_1^d\}\right] - cQ_1^d \\ &- h\Big[q_{11}^d - min\{x, q_{11}^d\}\Big] - w\Big[x - min\{x, q_{11}^d\}\Big]\end{aligned} \tag{3.1}$$

where $Q_1^d = q_{11}^d + q_{12}^d$.

In (3.1), the first two terms are revenues from sales and salvaging. The remaining terms are lost sales, procurement, holding and backorder costs.

If demand occurs in the second period, the profit function of agent 1, denoted as π'', is written as follows:

$$\begin{aligned}\pi'' &= r \cdot min\{x, Q_1^d\} + s\left[Q_1^d - min\{x, Q_1^d\}\right] - l\left[x - min\{x, Q_1^d\}\right] \\ &- cQ_1^d - hq_{11}^d\end{aligned} \tag{3.2}$$

Comparing π' and π'', we observe that both expressions have the same revenues of sales and salvage, and the same costs of lost sales and procurement. The difference between them is due to the holding and backorder costs incurred in the first period. Hence sales, salvage revenues, lost sales costs and procurement cost do not depend on which period demand X is realized whereas holding and backorder costs depend on this realization. As a result, expected profit function of agent 1, denoted as $E[\pi_1^d]$, can be constructed as follows:

$$\begin{aligned}
E[\pi_1^d] &= kE[\pi'] + (1-k)E[\pi''] \\
&= r\left[\int_0^{Q_1^d} xf(x)dx + \int_{Q_1^d}^{\infty} Q_1^d f(x)dx\right] + s\int_0^{Q_1^d}(Q_1^d - x)f(x)dx \\
&- l\int_{Q_1^d}^{\infty}(x - Q_1^d)f(x)dx - c(Q_1^d) - (1-k)hq_{11}^d \\
&- k\left[h\left(\int_0^{q_{11}^d}(q_{11}^d - x)f(x)dx\right) + w\left(\int_{q_{11}^d}^{\infty}(x - q_{11}^d)f(x)dx\right)\right]
\end{aligned} \tag{3.3}$$

We will find the optimal order quantities given this expected profit function. We first investigate concavity of $E[\pi_1^d]$.

Lemma 3.1.1. *$E[\pi_1^d]$ is jointly concave in q_{11}^d and q_{12}^d.*

Proof. All the proof are in the Appendices. ■

Proposition 3.1.1. *Optimal order quantities for the expected profit function (3.3) are determined according to the following framework.*

$$({q_{11}^d}^*, {q_{12}^d}^*) = \begin{cases} (0, F^{-1}(\rho_1)) & \text{if } \rho_2 \leq 0 \\ (F^{-1}(\rho_2), F^{-1}(\rho_1) - F^{-1}(\rho_2)) & \text{if } \rho_1 \geq \rho_2 > 0 \\ (F^{-1}(\rho_3), 0) & \text{if } \rho_2 > \rho_1 > 0 \end{cases}$$

where $\rho_1 = \left(\frac{r+l-c}{r+l-s}\right)$, $\rho_2 = \left(\frac{kw-h(1-k)}{k(h+w)}\right)$ *and* $\rho_3 = \left(\frac{r+l-c+kw-h(1-k)}{r+l-s+k(h+w)}\right)$.

An initial observation from Proposition 3.1.1 is about the ratios which affect the optimal order decision of agent 1. We showed that the holding and backorder costs depend on the period X is realized whereas sales revenues, procurement costs, salvage revenues and lost sales costs are independent of that. Using sales revenues, procurement costs, salvage revenues and lost sales costs, we can interpret underage costs as $c_u = r + l - c$ and overage costs as $c_o = c - s$, yielding the critical fractile of $\rho_1 = \frac{r+l-c}{r+l-s}$.

Moreover, given Q_1^d, increasing q_{11}^d by one unit would increase the expected overage costs from the first period by $h\left(kF(x) + (1-k)\right) = hkF(x) + h(1-k)$ and decrease the expected underage costs from the first period by $w\left(k\left(1 - F(x)\right)\right) = kw -$

$kwF(x)$. As the expected overage and underage costs of an additional unit would be equal for the optimal order quantity, we obtain that $hkF(x)+h(1-k) = kw-kwF(x)$ yielding the ratio of $F(x) = \frac{kw-h(1-k)}{k(h+w)} = \rho_2$. We also observe that whether to place an order for the first period depends on the value of ρ_2: An order is placed for the first period if $\rho_2 > 0$, i.e., $kw > h(1-k)$.

Note that kw can be interpreted as the cost of understocking in period 1 whereas $h(1-k)$ can be considered as the cost of overstocking. Hence, we can argue that an order is placed for the first period if cost of understocking is larger than cost of overstocking. Obviously, a higher value of backorder cost and a higher probability that X occurs in the first period favor an order for the first period whereas a higher holding cost creates an incentive not to order.

If it is not profitable to order for the first period, i.e., $kw < h(1-k)$, holding and backorder costs do not affect total quantity ordered. That is, it is optimal to order only for the second period as holding and backorder costs are irrelevant. In such a case, the optimal solution is characterized by the critical fractile solution of a newsvendor facing demand X.

When it gets profitable to order for the first period, i.e., as ρ_2 increases beyond 0, holding and backorder costs affect quantity ordered by agent 1. If $h + w$ is high and procurement cost, c, is close to salvage value, s, under this condition, i.e., $\rho_1 \geq \rho_2$, it is interesting to note that the total order quantity, Q_1^d, is determined by ρ_1, which is independent of holding and backorder costs. However, backorder and holding costs are not irrelevant since they determine the allocation of total order quantity among two periods. On the other hand, if $h + w$ becomes lower and/or procurement cost, c, becomes higher relative to salvage value, s, i.e., $\rho_2 \geq \rho_1$, the optimal solution suggests to place a single order for the first period.

As a result, if an order is placed for the second period, i.e., ${q_{12}^d}^* > 0$, backorder and holding costs do not have an impact on total quantity ordered, which is equal to $F^{-1}(\rho_1)$.

Finally, we observe that agent 2's problem is a specific version of agent 1's problem. When $k = 0$ and $X = Y$, the optimal order quantity of agent 1 is characterized by the critical fractile solution of a newsvendor. Moreover, when $k = 0$ and/or $w = 0$, agent 1 does not order for the first period to avoid overage costs. This problem also can be considered as a newsvendor problem. Note that given $X = Y$, when $k = 0$ and/or $w = 0$, the problem of two agents become exactly the same, and we observe that the expected profits of both agents are equal in the decentralized setting.

3.2 Centralized Setting

In this section, we consider the problem of a central decision maker who faces demands X and Y and gives orders at the beginning of the planning horizon to arrive at the beginning of both periods. The sequence of events for this case is as follows:

- The centralized agent places orders for the two periods.
- Order placed for period 1, q_1^c, arrives at the beginning of the first period.
- If X is realized in the first period, then quantity available on hand is used to fulfill the demand.
- End of period 1 costs and revenues are incurred.
- Order placed for period 2, q_2^c, arrives at the beginning of the second period.
- Y realizes. If X is realized in the first period and excess demand is carried to the second period, then quantity available on hand is used to fulfill demand Y and excess demand of X. If no excess demand of X remains from period 1, quantity available is used to fulfill demand Y. If X is realized in the second period, then total quantity ordered is used to fulfill both demands. Quantity remaining on hand at the end of the second period is salvaged, and quantity not fulfilled is charged by lost sales cost.
- End of period 2 costs and revenues are incurred.

The decision variables and parameters presented in Table (3.1) are used to express the profit function of the centralized agent. Recall that $Q^c = q_1^c + q_2^c$.

If X is realized in the first period, the profit function, $\pi^{c\prime}$ is as follows:

$$\begin{aligned}\pi^{c\prime} =& r \cdot min\Big\{x+y, Q^c\Big\} - l\Big[x+y-min\Big\{x+y, Q^c\Big\}\Big] - cQ^c \\ &+ s\Big[Q^c - min\Big\{x+y, Q^c\Big\}\Big] - h\Big[q_1^c - min\{x, q_1^c\}\Big] - w\Big[x - min\{x, q_1^c\}\Big]\end{aligned} \tag{3.4}$$

In (3.4), the first two terms are revenues from sales and salvaging. The remaining terms are lost sales, procurement, holding and backorder costs.

The profit function when X occurs in the second period, denoted as $\pi^{c\prime\prime}$, is written as follows:

$$\begin{aligned}\pi^{c\prime\prime} =& r \cdot min\Big\{x+y, Q^c\Big\} - l\Big[x+y-min\Big\{x+y, Q^c\Big\}\Big] - cQ^c \\ &+ s\Big[Q^c - min\Big\{x+y, Q^c\Big\}\Big] - hq_1^c\end{aligned} \tag{3.5}$$

Comparing $\pi^{c\prime}$ and $\pi^{c\prime\prime}$, sales revenues, salvage revenues, lost sales costs and procurement costs are the same in both expressions since these are independent of the period X is realized as in the decentralized case. The costs incurred at the end of period 1 differ between the two expressions since the period X is realized affects end of period 1 costs.

Using the realizations $\pi^{c\prime}$ and $\pi^{c\prime\prime}$, and the notation of convolution demand Z, the expected profit function is written as follows:

$$\begin{aligned}E[\pi^c] &= kE[\pi^{c\prime}] + (1-k)E[\pi^{c\prime\prime}] \\ &= r\left[\int_0^{Q^c} zv(z)dz + \int_{Q^c}^{\infty} Q^c v(z)dz\right] + s\int_0^{Q^c} (Q^c - z)v(z)dz \\ &- l\int_{Q^c}^{\infty} (Q^c - z)v(z)dz - cQ^c + (1-k)hq_1^c \\ &- k\left[h\int_0^{q_1^c} (q_1^c - x)f(x)dx + w\int_{q_1^c}^{\infty} (x - q_1^c)f(x)dx\right]\end{aligned} \tag{3.6}$$

Lemma 3.2.1. *$E[\pi^c]$ is jointly concave in q_1^c and q_2^c.*

Definition 3.2.1. *Let $q_1^{c\prime}$ denote the order quantity for period 1 which satisfy*

$$0 = -(r+l-s)V(Q_c) + r + l - c - (k(h+w))F(q_1^c) + kw - h(1-k) \tag{3.7}$$

for $q_2^c = 0$.

Proposition 3.2.1. *The optimal order quantities of a central player is given by:*

$$(q_1^{c*}, q_2^{c*}) = \begin{cases} (0, V^{-1}(\rho_1)) & \textit{if } \rho_2 \leq 0 \\ (F^{-1}(\rho_2), V^{-1}(\rho_1) - F^{-1}(\rho_2)) & \textit{if } \rho_1 \geq \rho_2 > 0 \\ (q_1^{c\prime}, 0) & \textit{if } \rho_2 > \rho_1 > 0 \end{cases}$$

Recall that $\rho_1 = \left(\frac{r+l-c}{r+l-s}\right)$ and $\rho_2 = \left(\frac{kw-h(1-k)}{k(h+w)}\right)$.

Considering Proposition 3.1.1 and Proposition 3.2.1, it should be remarked that the expressions defining the structure of the optimal solution are similar for the decentralized and centralized settings. We observe that cost of not fulfilling demand in the first period is the motivation of decision maker to order for the first period. This motivation defines the same critical fractile, ρ_2, in both settings.

On the other hand, there are differences in optimal order quantities between the two settings. Table (3.2) below provides a summary of these quantities.

$kw < h(1-k)$	$kw \geq h(1-k)$	
$(\rho_1 > 0 > \rho_2)$	$\rho_1 \geq \rho_2$	$\rho_1 < \rho_2$
$Q_1^{d*} = F^{-1}(\rho_1)$ $Q^{c*} = V^{-1}(\rho_1)$ $Q_1^{d*} \leq Q^{c*}$	$Q_1^{d*} = F^{-1}(\rho_1)$ $Q^{c*} = V^{-1}(\rho_1)$ $Q_1^{d*} \leq Q^{c*}$	$Q_1^{d*} = F^{-1}(\rho_3)$ $Q^{c*} = q_1^{c\prime}$ $Q_1^{d*} \leq Q^{c*}$
$0 = {q_{11}^d}^* = q_1^{c*}$	$0 \leq {q_{11}^d}^* = q_1^{c*}$	$0 < {q_{11}^d}^* \leq q_1^{c*}$
$0 < {q_{12}^d}^* \leq q_2^{c*}$	$0 < {q_{12}^d}^* \leq q_2^{c*}$	$0 = {q_{12}^d}^* = q_2^{c*}$

Table 3.2: Comparison of the optimal ordering decisions in the decentralized and centralized settings

Recall that agent 1 in the decentralized setting faces X, whereas the central agent faces $X + Y$. Thus, the central agent orders at least as many as agent 1 in total.

If an order is placed for the second period, i.e., $\rho_1 \geq \rho_2$, the first period order quantities are the same for both settings because the motivation behind placing an order is the expected cost of not fulfilling demand in the first period, which remains the same for both settings. Due to the reasoning above, the second period order quantity of the central agent is at least as much as that of agent 1 in the decentralized setting. In addition, if $\rho_1 \geq \rho_2 > 0$ the first period order quantity is equal to $F^{-1}(\rho_2)$ in both settings. In fact, we show in the proofs of Propositions 3.1.1 and 3.2.1 that the maximum quantity that can be attained for the first period is also equal to $F^{-1}(\rho_2)$.

On the other hand, if an order is placed only for the first period, i.e., $\rho_2 > \rho_1$, the central agent orders at least as many as agent 1 for the first period.

CHAPTER 4

DECENTRALIZED AGENTS IN THE PRESENCE OF TRANSFER OPPORTUNITY

In this chapter, we consider two rational agents acting independently to maximize their individual profits. In the beginning of the planning horizon, agent 1 faces with demand X and is expected to place an order for both periods denoted as q_{11} and q_{12} while agent 2 faces with demand Y and places an order to arrive at period 2, denoted as q_2. The sequence of events for the two-agent setting is summarized below:

- Agents make their ordering decisions independently at the beginning of period 1.
- Order placed by agent 1 for the first period, q_{11}, arrive at the beginning of the first period.
- If demand for agent 1, X, realizes in the first period, agent 1 uses quantity available on hand to fulfill the demand.
- End of period 1 costs and revenues are incurred for agent 1.
- Orders placed for the second period by both agents, q_{12} and q_2, arrive at the beginning of the second period.
- If X is realized in the first period and excess demand is carried to the second period, agent 1 uses its order arrived in the second period to fulfill this excess demand. If X is realized in the second period, then agent 1 uses total quantity ordered to fulfill the demand.
- At the same time, demand for agent 2, Y, realizes. Agent 2 uses quantity on hand to meet the demand.

- Transfers between agents are possible if one of them has excess demand while the other has excess supply. Transfer quantity is determined as the minimum of excess demand and excess supply. The agent who has excess demand pays a transfer price of t per unit to the agent with excess supply and satisfy its demand if possible.

- Quantity remaining on hand at the end of the second period is salvaged at s, and demand not fulfilled is charged by lost sales cost for both agents.

- End of period 2 costs and revenues are accounted.

Transfer payments and transfer earnings create the difference of this setting from the decentralized and centralized settings. The amount of transfers between agents affects the sales, excess supply and excess demand of both agents.

Let x be a realization for X and y be a realization for Y. The following four events may take place at the end of the second period:

1. If $Q_1 \leq x$ and $q_2 \leq y$, no transfers will take place and both agents sell the units available individually. Recall $Q_1 = q_{11} + q_{12}$ (see Table (3.1) for notation).

2. If $Q_1 \leq x$ and $q_2 > y$, transfers from agent 2 to agent 1 will take place. The transfer quantity will be equal to either excess demand of agent 1 or excess supply of agent 2.

3. If $Q_1 > x$ and $q_2 \geq y$, no transfers will take place and both agents will sell the units available individually.

4. If $Q_1 > x$ and $q_2 < y$, transfers from agent 1 to agent 2 will take place. The transfer quantity will be equal to either excess demand of agent 2 or excess supply of agent 1.

Figure (4.1) provides an illustration for possible transfer quantities between agents. Moreover, the amount of sales, transfers, excess demand and excess supply to be realized by the end of the second period under these four scenarios can be seen in Table (4.1) separately for both agents. In this table, T_{ij} denotes the quantity transferred from

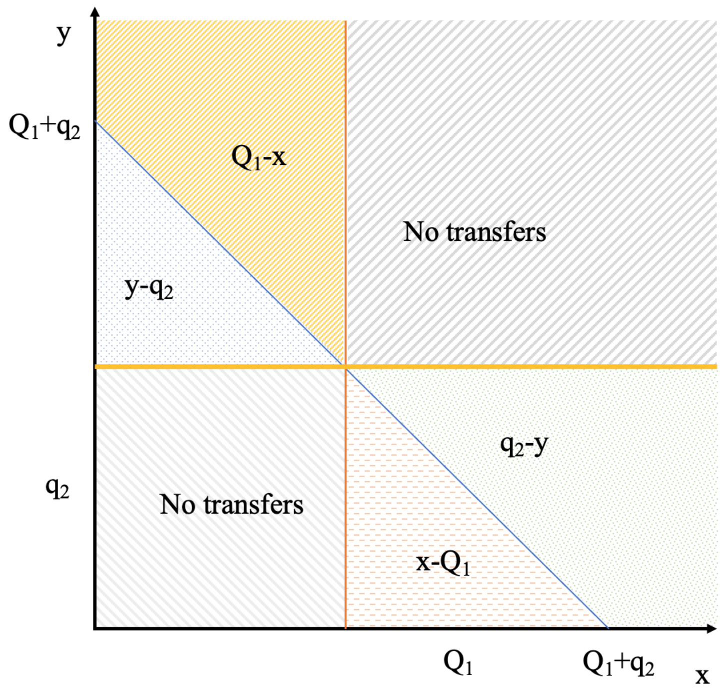


Figure 4.1: Transfer quantities between agents

	$Q_1 \geq x$		$Q_1 < x$	
	$q_2 \geq y$	$q_2 < y$	$q_2 \geq y$	$q_2 < y$
T_{12}	0	$min\{y - q_2, Q_1 - x\}$	0	0
T_{21}	0	0	$min\{q_2 - y, x - Q_1\}$	0
S_1	Q_1	x	$Q_1 + T_{21}$	Q_1
S_2	y	$q_2 + T_{12}$	y	q_2
O_1	$Q_1 - x$	$Q_1 - x - T_{12}$	0	0
O_2	$q_2 - y$	0	$q_2 - y - T_{21}$	0
U_1	0	0	$x - Q_1 - T_{21}$	$x - Q_1$
U_2	0	$y - q_2 + T_{12}$	0	$y - q_2$

Table 4.1: Transfer, sales, overstock and understock quantities of both agents for possible demand realizations

agent i to agent j whereas S_i, O_i and U_i denote the sales, overstock and understock quantities of agent i, respectively.

We continue with analyzing the expected profit function of agent 1. Using the quantities provided in Table (4.1), when X is realized in the first period, the profit function of agent 1, π_1', can be written as follows:

$$\begin{aligned}\pi_1' =& r \cdot min\left\{x, Q_1 + max\{0, q_2 - y\}\right\} + s \cdot max\left\{0, (Q_1 - x - max\{0, y - q_2\})\right\}\\ &- l \cdot max\left\{0, (x - Q_1 - max\{0, q_2 - y\})\right\} - cQ_1\\ &- t \cdot max\left\{0, min\left\{q_2 - y, x - Q_1\right\}\right\} + t \cdot max\left\{0, min\left\{y - q_2, Q_1 - x\right\}\right\}\\ &- h\Big[q_{11} - min\{x, q_{11}\}\Big] - w\Big[x - min\{X, q_{11}\}\Big]\end{aligned} \tag{4.1}$$

Given that X is realized in the second period, the profit function of agent 1, π_1'', is as follows:

$$\begin{aligned}\pi_1'' =& r \cdot min\left\{x, Q_1 + max\{0, q_2 - y\}\right\} + s \cdot max\left\{0, (Q_1 - x - max\{0, y - q_2\})\right\}\\ &- l \cdot max\left\{0, (x - Q_1 - max\{0, q_2 - y\})\right\} - cQ_1\\ &- t \cdot max\left\{0, min\left\{q_2 - y, x - Q_1\right\}\right\} + t \cdot max\left\{0, min\left\{y - q_2, Q_1 - x\right\}\right\}\\ &- hq_{11}\end{aligned} \tag{4.2}$$

We know from the decentralized and centralized settings that the sales revenues, procurement costs, salvage revenues and lost sales costs are independent of the period X is realized, whereas backorder and holding costs depend on the period of realization. Comparing (4.1) and (4.2), we observe that the period at which X occurs does not affect these revenues and costs in this setting either. In addition, transfer quantities are also independent of the period X is realized.

Given π_1' and π_1'', the expected profit function of agent 1, $E[\pi_1]$, can be derived as follows:

$$E[\pi_1] = kE[\pi_1'] + (1-k)E[\pi_1'']$$

$$
\begin{aligned}
E[\pi_1] = r \Bigg[& \int_0^{Q_1} x f(x) dx + \int_0^{q_2} \int_{Q_1+q_2-y}^{\infty} (Q_1 + q_2 - y) f(x) g(y) dx dy \\
& + \int_{Q_1}^{Q_1+q_2} \int_0^{Q_1+q_2-x} x f(x) g(y) dy dx + \int_{Q_1}^{\infty} \int_{q_2}^{\infty} Q_1 f(x) g(y) dy dx \Bigg] \\
& + s \Bigg[\int_0^{Q_1} \int_0^{q_2} (Q_1 - x) f(x) g(y) dy dx \\
& + \int_0^{Q_1} \int_{q_2}^{Q_1+q_2-x} (Q_1 + q_2 - x - y) f(x) g(y) dy dx \Bigg] \\
& - l \Bigg[\int_{Q_1}^{\infty} \int_{q_2}^{\infty} (x - Q_1) f(x) g(y) dy dx \\
& + \int_0^{q_2} \int_{Q_1+q_2-y}^{\infty} (x + y - Q_1 - q_2) f(x) g(y) dx dy \Bigg] \\
& + t \Bigg[\int_0^{Q_1} \int_{q_2}^{Q_1+q_2-x} (y - q_2) f(x) g(y) dy dx \\
& + \int_0^{Q_1} \int_{Q_1+q_2-x}^{\infty} (Q_1 - x) f(x) g(y) dy dx \Bigg] \\
& - t \Bigg[\int_{Q_1}^{Q_1+q_2} \int_{Q_1+q_2-x}^{q_2} (q_2 - y) f(x) g(y) dy dx \\
& + \int_0^{q_2} \int_{Q_1+q_2-y}^{\infty} (q_2 - y) f(x) g(y) dx dy \Bigg] - cQ_1 \\
& - k \Bigg[h \int_0^{q_{11}} (q_{11} - x) f(x) dx + w \int_{q_{11}}^{\infty} (x - q_{11}) f(x) dx \Bigg] - (1-k) h q_{11}
\end{aligned}
\tag{4.3}
$$

We continue with analyzing the expected profit function of agent 2.

Using Table (4.1), the profit function of agent 2 for a given realization of X and Y, denoted as π_2, can be constructed as follows:

$$
\begin{aligned}
\pi_2 = & r \cdot min\,\{y, q_2 + max\{0, Q_1 - x\}\} + s \cdot max\,\{0, (q_2 - y - max\{0, x - Q_1\})\} \\
& - l \cdot max\,\{0, (y - q_2 - max\{0, Q_1 - x\})\} - cq_2 \\
& - t \cdot max\,\{0, min\,\{y - q_2, Q_1 - x\}\} + t \cdot max\,\{0, min\,\{q_2 - y, x - Q_1\}\}
\end{aligned}
$$

Recall from the discussions about agent 1 that only holding and backorder costs depend on the period X is realized. As a result, we observe that the profit function of agent 2 is independent of the probability that X is realized in period 1, k.

Given π_2, we derive the expected profit function of agent 2, $E[\pi_2]$, as follows:

$$\begin{aligned}
E[\pi_2] =& r\Bigg[\int_0^{q_2} yg(y)dy + \int_0^{Q_1}\int_{q_2}^{Q_1+q_2-x} yf(x)g(y)dydx \\
&+ \int_0^{Q_1}\int_{Q_1+q_2-x}^{\infty} (Q_1+q_2-x)f(x)g(y)dydx + \int_{Q_1}^{\infty}\int_{q_2}^{\infty} q_2 f(x)g(y)dydx\Bigg] \\
&+ s\Bigg[\int_0^{Q_1}\int_0^{q_2} (q_2-y)f(x)g(y)dydx \\
&+ \int_{Q_1}^{Q_1+q_2}\int_0^{Q_1+q_2-x} (Q_1+q_2-x-y)f(x)g(y)dydx\Bigg] \\
&- l\Bigg[\int_0^{Q_1}\int_{Q_1+q_2-x}^{\infty} (x+y-Q_1-q_2)f(x)g(y)dydx \\
&+ \int_{Q_1}^{\infty}\int_{q_2}^{\infty} (y-q_2)f(x)g(y)dydx\Bigg] \\
&+ t\Bigg[\int_0^{q_2}\int_{Q_1+q_2-y}^{\infty} (q_2-y)f(x)g(y)dxdy \\
&+ \int_{Q_1}^{Q_1+q_2}\int_0^{Q_1+q_2-x} (x-Q_1)f(x)g(y)dydx\Bigg] \\
&- t\Bigg[\int_0^{Q_1}\int_{q_2}^{Q_1+q_2-x} (y-q_2)f(x)g(y)dydx \\
&+ \int_0^{Q_1}\int_{Q_1+q_2-x}^{\infty} (Q_1-x)f(x)g(y)dydx\Bigg] - cq_2
\end{aligned} \tag{4.4}$$

Since the agents determine their order quantities simultaneously at the beginning of the planning horizon, this inventory problem with transfers can be considered as a simultaneous move non-zero sum game. We proceed with analyzing the best response functions (BRF) of both parties to find the outcome of the corresponding game.

4.1 Best Response Function of Agent 1

We investigate the BRF of agent 1 in this section. For this purpose, we first argue that the expected profit function of agent 1 is concave. We introduce definitions of functions which characterize the BRF. We establish the BRF and provide our related findings.

Lemma 4.1.1. *$E[\pi_1]$ is jointly concave in q_{11} and q_{12}.*

We make the following definitions and observations in order to establish the optimal solution to the first agent's problem.

Definition 4.1.1. *Let $q_{11}^F(q_2)$ denote q_{11} value that satisfies $\frac{\partial E[(\pi_1)]}{\partial q_{11}} = 0$ for a given q_2 when $q_{12} = 0$ (see (B.1) in Appendix).*

Lemma 4.1.2. *$q_{11}^F(q_2)$ is continuous and monotonically decreasing in q_2.*

Definition 4.1.2. *Let $q_{12}^F(q_2)$ denote q_{12} value that satisfies $\frac{\partial E[(\pi_1)]}{\partial q_{12}} = 0$ for a given q_2 when $q_{11} = 0$ (see (B.2) in Appendix).*

Lemma 4.1.3. *$q_{12}^F(q_2)$ is continuous and monotonically decreasing in q_2.*

Definition 4.1.3. *Let $q_{12}^T(q_2)$ denote q_{12} value that satisfies $\frac{\partial E[(\pi_1)]}{\partial q_{12}} = 0$ for a given q_2 when $q_{11} = F^{-1}(\rho_2)$ (see (B.2) in Appendix).*

Lemma 4.1.4. *$q_{12}^T(q_2)$ is continuous and monotonically decreasing in q_2.*

In decentralized setting with transfer opportunity, we introduce $\rho_1^n = \frac{r+l-c}{r+l-t}$ and $\rho_3^n = \frac{r+l-c+hk-h+kw}{r+l-t}$. Recall that ρ_1 and ρ_3 in the decentralized and centralized settings depend on the salvage value, whereas ρ_1^n and ρ_3^n depend on the transfer price, t. Since agents transfer goods before salvaging if possible, here, $c-t$ can be interpreted as the overage cost while underage costs remain the same, resulting in these changes in the ratios. Recall that $\rho_2 = \frac{kw-h(1-k)}{k(h+w)}$.

Lemma 4.1.5. *Given $\rho_2 > 0$, $q_{11} = F^{-1}(\rho_2)$ and $q_{12} = 0$, there exists a unique q_2, denoted as q_2', which satisfies $\frac{\partial E[(\pi_1)]}{\partial q_{11}} = 0$ and $\frac{\partial E[(\pi_1)]}{\partial q_{12}} = 0$ iff the condition $\frac{t-c}{t-s} \leq \rho_2 \leq \rho_1^n$ holds.*

Definition 4.1.4. *Let q_2^T denote the order quantity of agent 2, q_2, defined as*

$$q_2^T = \begin{cases} -\infty & \text{if } \rho_2 > \rho_1^n \text{ and } \rho_2 > \frac{t-c}{t-s} \\ q_2' & \text{if } \frac{t-c}{t-s} \leq \rho_2 \leq \rho_1^n \\ \infty & \text{if } \rho_2 < \rho_1^n \text{ and } \rho_2 < \frac{t-c}{t-s} \end{cases}$$

Proposition 4.1.1. *The best response function of agent 1 is characterized as follows:*

i. If $\rho_2 < 0$

a. If $t > c$

$$(q_{11}^*, q_{12}^*) = \big(0, q_{12}^F(q_2)\big)$$

b. If $t < c$

$$(q_{11}^*, q_{12}^*) = \begin{cases} (0,0) & \text{if } q_2 \geq V^{-1}(\rho_1^n) \\ \big(0, q_{12}^F(q_2)\big) & \text{if } q_2 < V^{-1}(\rho_1^n) \end{cases}$$

ii. If $\rho_2 > 0$

c. If $kw + t > h(1-k) + c$

$$(q_{11}^*, q_{12}^*) = \begin{cases} \big(q_{11}^F(q_2), 0\big) & \text{if } q_2 \geq q_2^T \\ \big(F^{-1}(\rho_2), q_{12}^T(q_2)\big) & \text{if } q_2 < q_2^T \end{cases}$$

d. If $kw + t < h(1-k) + c$

$$(q_{11}^*, q_{12}^*) = \begin{cases} (0,0) & \text{if } q_2 > V^{-1}(\rho_3^n) \\ \big(q_{11}^F(q_2), 0\big) & \text{if } V^{-1}(\rho_3^n) \geq q_2 \geq q_2^T \\ \big(F^{-1}(\rho_2), q_{12}^T(q_2)\big) & \text{if } q_2 < q_2^T \end{cases}$$

Investigating the BRF for agent 1 in Proposition 4.1.1, we observe that it has similarities with the optimal ordering decisions in the decentralized and centralized settings. Below, we elaborate on these similarities.

As in the centralized setting and the decentralized setting with no transfers, agent 1 does not order when holding cost is relatively larger than the backorder costs for the first period, regardless of the second agent's order. However, the opposite is no longer true. That is, when $kw > h(1-k)$, agent 1 might still not order for the first period depending on q_2 and t.

We should emphasize that agent 1 of this setting considers possible benefits of transfers: For agent 1, $kw + t$ can be interpreted as the cost of fulfilling demand from agent 2 in the second period whereas $h(1-k) + c$ can be interpreted as the cost of fulfilling by ordering in first period. If $kw + t < h(1-k) + c$ and if agent 2 orders beyond the threshold of $V^{-1}(\rho_3^n)$, agent 1 is better off with not placing any orders despite the fact that $kw > h(1-k)$. We can explain the reasoning as follows: Although it is more costly not to order for the first period without the second agent, i.e., $kw > h(1-k)$, it may be less costly not to order given the supply opportunity from agent 2. Agent 1 can use this opportunity if agent 2 carries enough inventory for both agents, i.e $q_2 \geq V^{-1}(\rho_3^n)$. On the other hand, she definitely places an order for period 1 if any of the conditions $kw + t < h(1-k) + c$ and $q_2 > V^{-1}(\rho_3)$ does not hold. That is, given $kw > h(1-k)$, if it is not beneficial to backorder and satisfy demand with transfers, i.e., $kw + t > h(1-k) + c$, or agent 2 does not have enough units for both agents, $q_2 < V^{-1}(\rho_3^n)$, agent 1 certainly places an order for the first period.

Similar to the previous settings, it was presented in the proof of Proposition 4.1.1 that the maximum quantity that can be ordered by agent 1 for period 1 is also equal to $F^{-1}(\rho_2)$. We also observe that if agent 1 places a non-zero order for both periods, the order quantity of period 1 depends on ρ_2 as in previous settings, i.e., $q_{11}^* = F^{-1}(\rho_2)$.

It was further shown in the decentralized and centralized settings that total order quantity and allocation decisions are shaped by the relation between ρ_1 and ρ_2. However, we show that there is a difference in two-agent setting with transfers: The order quantity of agent 2 has an impact on agent 1's decision in two-agent settings. To explain the reasoning, we recall that for the previous settings, if $\rho_1 \geq \rho_2 > 0$, it is optimal to order for both periods; and total and the first period order quantities are determined according to ρ_1 and ρ_2, respectively. If $\rho_2 > \rho_1$, it is optimal to order for the first period only and the order quantity is determined according to the critical fractile of ρ_2. On the other hand, these relations cannot be used to describe the optimal behavior of agent 1 when two agents interact. Given $kw > h(1-k)$, the optimal order decision of agent 1 depends on the order quantity of agent 2: If agent 2 does not order enough for both agents, i.e., $q_2 < q_2^T$, agent 1 will place an order for both periods.

4.2 Best Response Function of Agent 2

In this section, we proceed to the BRF of agent 2. After arguing that its expected profit function is concave, we make definitions of functions that characterize the BRF of agent 2. We complete the section with the BRF and our related findings.

Lemma 4.2.1. *$E[\pi_2]$ is concave in q_2.*

Lemma 4.2.2. *Let $q_2^F(Q_1)$ denote q_2 value that satisfies $\frac{\partial E[(\pi_2)]}{\partial q_2} = 0$ for a given Q_1. There exists $q_2^F(Q_1) \geq 0$ for $\forall Q_1 \in (0, \infty)$ (see (B.21) in Appendix).*

Lemma 4.2.3. *$q_2^F(Q_1)$ is continuous and monotonically decreasing in Q_1.*

Proposition 4.2.1. *The best response of agent 2 is characterized as follows:*

a. If $t > c$

$$q_2^* = q_2^F(Q_1)$$

b. If $t < c$

$$q_2^* = \begin{cases} 0 & \text{if } Q_1 \geq V^{-1}(\rho_1^n) \\ q_2^F(Q_1) & \text{if } Q_1 < V^{-1}(\rho_1^n) \end{cases}$$

While determining its order quantity, agent 2 also considers possible benefits of transfers: For agent 2, t can be interpreted as the cost of fulfilling demand with units transferred from agent 1 in the second period, whereas c can be interpreted as cost of fulfilling by ordering from the supplier. If $t < c$, agent 2 may be better off with not placing any orders and supplying all required units from agent 1. Agent 2 can use this opportunity if agent 1 has enough inventory for both agents, i.e., $V^{-1}(\rho_1)$. However, the agent definitely places an order if it is not beneficial to satisfy demand with transfers, i.e., $t > c$, or agent 1 does not have enough units for both agents, $Q_1 < V^{-1}(\rho_1)$.

Another observation is that the allocation decision of agent 1 does not affect the ordering decision of agent 2. Since transfers take place in the second period and excess demand or supply is carried to the second period, quantity allocated by agent 1 for the first period does not have an impact on order quantity of agent 2.

4.3 Nash Equilibrium

With the interacting choices of two agents, we can investigate their optimal ordering decision using the concept of a two-person simultaneous move non-zero sum game.

We will investigate the existence and uniqueness of Nash equilibrium in the presence of transfers. As the best response of agent 2 depends on $q_{11} + q_{12}$ rather than their individual values, the outcome of the game can be characterized on the (q_2, Q_1) plane.

Continuity of and boundedness of best response functions is necessary and sufficient for the existence of Nash equilibrium. We use the optimal solution structures in Proposition 4.1.1 and (4.2.1) to prove continuity since it is not possible to derive the BRFs explicitly.

Lemma 4.3.1. *The best response functions are continuous and bounded on the (q_2, Q_1) plane.*

Theorem 4.3.1. *The corresponding simultaneous move non-zero sum game has a Nash equilibrium.*

Given its existence, we continue the analysis on the uniqueness of Nash equilibrium. If the best response function of one agent is always decreasing faster than that of the other, then these two functions will intersect at a unique point. Hence, we continue by analyzing the slopes of the functions that characterize the BRFs.

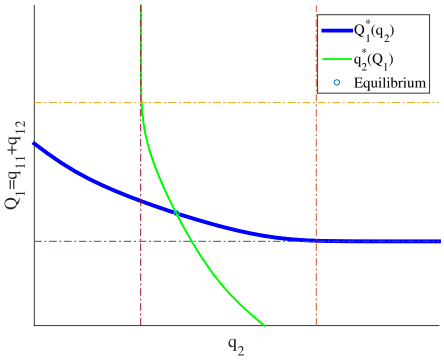
Lemma 4.3.2. *The slope of q_2^F is always strictly smaller than the slope of q_{11}^F, q_{12}^F and q_{12}^T.*

Theorem 4.3.2. *The corresponding game has a unique Nash equilibrium.*

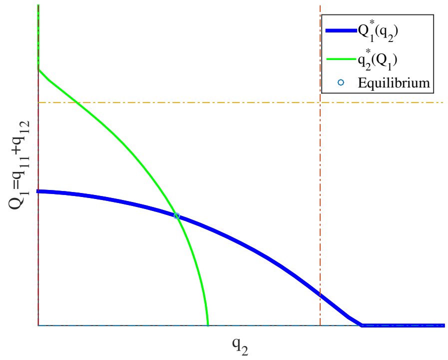
Corollary 4.3.2.1. *$Q_1^* > 0$ and $q_2^* > 0$ at equilibrium.*

As a result, we conclude that there exists a unique non-zero equilibrium in the decentralized setting with transfers. Figure (4.2) illustrates the bounds on the best response functions and the unique non-zero equilibrium for all possible parameter alignments.

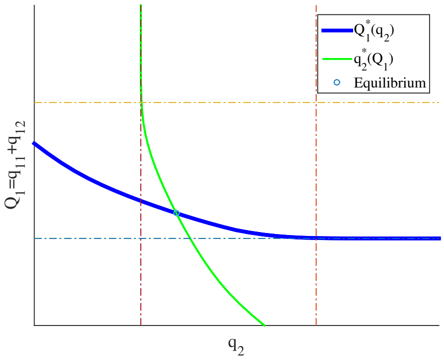
Gathering the results in Propositions 4.1.1 and 4.2.1 with the observations in Figure (4.2), we can obtain important information about the optimal order quantities at the equilibrium. Recall from these propositions that depending on the relation between t and c, optimal order quantity of the agents can be equal to 0. From the best response function of agents in Figure (4.2), we can see that order quantities at equilibrium are



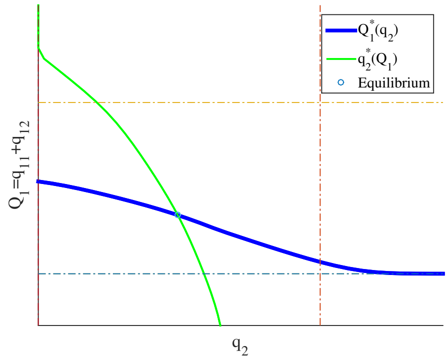
(a) $\rho_2 < 0$ and $t > c$



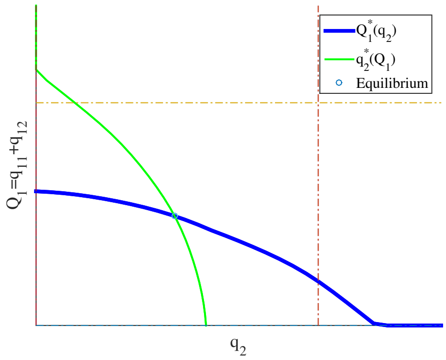
(b) $\rho_2 < 0$ and $t < c$



(c) $\rho_2 > 0$ and $t > c$



(d) $\rho_2 > 0, t < c$ and $kw + t > h(1-k) + c$



(e) $\rho_2 > 0, t < c$ and $kw + t < h(1-k) + c$

Figure 4.2: Illustration of best response functions for all possible parameter settings

non-zero, as stated in Corollary 4.3.2.1. It is noteworthy that both agents place an order even if the transfer price between agents is lower than the price of the product. This is due to the fact that no agent will order as much as the central decision maker.

As a result, regardless of the relationship between t and c, the optimal order quantity of agent 2 at equilibrium is characterized as $q_2^* = q_2^F(Q_1)$ and that of agent 1 can be given as:

i. If $\rho_2 \leq 0$

$$(q_{11}^*, q_{12}^*) = \big(0, q_{12}^F(q_2)\big)$$

ii. If $\rho_2 \geq 0$

$$(q_{11}^*, q_{12}^*) = \begin{cases} \big(q_{11}^F(q_2), 0\big) & \text{if } q_2 \geq q_2^T \\ \big(F^{-1}(\rho_2), q_{12}^T(q_2)\big) & \text{if } q_2 < q_2^T \end{cases}$$

Observe that if $\rho_2 \leq 0$, agent 1 is ensured to place an order for the second period. The agent does not place any orders for the first period since the expected holding costs are higher. Also, observe that if $\rho_2 \geq 0$, agent 1 is ensured to place an order for the first period to avoid expected backorder costs. Whether to place an order for the second period depends on the optimal order quantity of agent 2 at this condition. If agent 2 places a sufficiently large order, agent 1 is better off with placing an order only for the first period.

CHAPTER 5

COMPUTATIONAL ANALYSIS

In this section, we perform a thorough computational analysis using Matlab in order to evaluate the effects of introducing transfer opportunity in a decentralized environment. We will investigate whether the profits obtained at Nash equilibrium achieve the joint profits obtained in the centralized setting. In addition, we will observe the effects of problem parameters on the solutions and improvement gained through transfers.

For this purpose, we introduce a base parameter setting (see Table (5.1)) and compute optimal order quantities and corresponding expected profits for three settings (see Table (5.2)). We assume that demand for both agents are assumed to be normally distributed. Recall that $\rho_1 = \frac{r+l-c}{r+l-s}$, $\rho_1^n = \frac{r+l-c}{r+l-t}$, $\rho_2 = \frac{kw-h(1-k)}{hk+kw}$, $\rho_3 = \frac{r+l-c+kw-h(1-k)}{r+l-s+hk+kw}$ and $\rho_3^n = \frac{r+l-c+kw-h(1-k)}{r+l-t}$.

Table 5.1: Parameters and Ratios in Computational Analysis

r	50	c	20	h	5	w	10
l	0	s	0	k	0.5	t	25
μ_x	100	σ_x	25	μ_y	100	σ_y	25
ρ_1	0.60	$F^{-1}(\rho_1)$	106.34	$V^{-1}(\rho_1)$	208.96	ρ_1^n	1.20
$V^{-1}(\rho_1^n)$	∞	ρ_2	0.33	$F^{-1}(\rho_2)$	89.23	ρ_3	0.57
$F^{-1}(\rho_3)$	104.11	ρ_3^n	1.30	$V^{-1}(\rho_3^n)$	∞		

In Table 5.2, q_{11} and q_{12} represents the order quantities of the agents facing demand X and q_2 represents the order quantity of the agent facing demand Y. In addition, we use D for decentralized setting, C for centralized setting and N for decentralized setting with transfer opportunity, which we will interchangeably refer as 2-agents setting in the remainder of the chapter.

Table 5.2: Optimal order quantities and expected profits

Setting	Agent 1			Agent 2		Total Profits
	q_{11}	q_{12}	$E[\pi_1]$	q_2	$E[\pi_2]$	$E[\pi_1] + E[\pi_2]$
D	89.23	17.11	2198.91	106.34	2517.08	4715.99
C	89.23	119.72	4998.87	N/A	N/A	4998.87
N	89.23	16.02	2339.95	105.25	2658.12	4998.07

In our numerical studies, we often make similar comments for the agents who face demand X, which are the central agent and agent 1 in decentralized settings, i.e., the decentralized and 2-agents settings. Therefore, in the remainder of this chapter, unless specifically stated otherwise, the definition of agent 1 includes both agent 1 in these decentralized settings and the central agent.

We start with analyzing the solutions of the base case. In our base case, demands X and Y have the same distribution. Therefore, if demand X is ensured to realize in the second period like demand Y, i.e., if $k = 0$, both agents would have the same optimal decision in the decentralized and 2-agents settings. Thus, an initial observation is that the differences in agents' ordering decisions are due to the positive value of the probability that X is realized in the first period, i.e., $k > 0$. We also observe that $E[\pi_1] < E[\pi_2]$ in row D and N of Table (5.2) as a result of this additional uncertainty.

We should highlight that the total order quantity of both agents is non-zero in 2-agents setting, as shown in Corollary 4.3.2.1. Moreover, observing that agent 1 places an order for both periods, we conclude that order quantity of agent 2 is not sufficiently enough, i.e., $q_2 < q_2^T$.

In fact, we observe that agent 1 places an order for both periods in all settings. When agent 1 orders for both periods, the total order quantity will not be affected by the holding and backorder costs incurred at the end of the first period. Therefore, total order quantities of agent 1 and agent 2 are equal in the decentralized and 2-agents settings.

Although holding and backorder costs do not have an impact on the total quantity ordered, meaning that $\rho_2 < \rho_1$, a positive value of ρ_2 implies that it is in the best interest of agent 1 to place an order for period 1 in all settings. These holding and backorder costs are taken into account for setting the order quantity for the first period, hence we observe $q_{11} = F^{-1}(\rho_2)$ in all rows of Table (5.2). On the other hand, we also observe that the central agent places a larger order for the second period compared to agent 1 in the decentralized and 2-agents settings in order to meet demand Y.

A final remark is that under transfers, total expected profit is always greater than that of decentralized solution. In fact, the expected profits of both players in the 2-agents setting are larger than those without transfer opportunity since transfers make it possible for agents to share their excess demand and excess supply if possible.

The following figure demonstrates the best response functions of agents and the unique equilibrium achieved in decentralized setting with transfers.

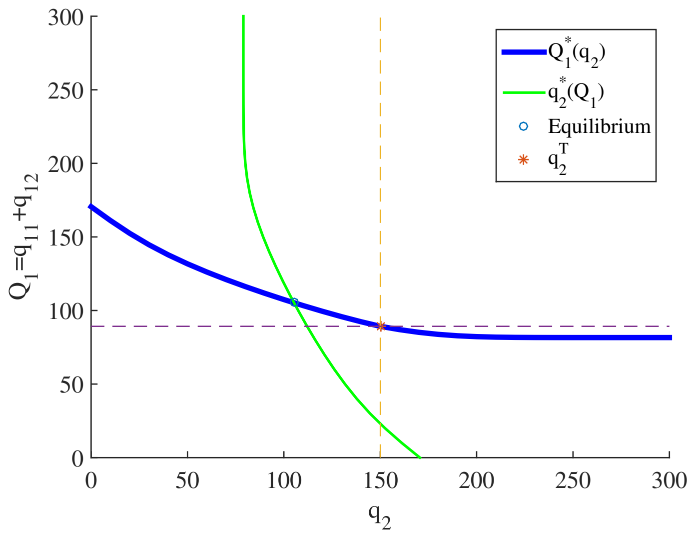


Figure 5.1: Best response functions in the base case

In Figure (5.1), we can observe that q_2^T is finite under the base setting. Using Proposition 4.1.1, we can show that the part of the blue line, i.e., $Q_1^*(q_2)$, which lies at the right of $q_2 = q_2^T$ corresponds to the function $q_{11}^F(q_2)$, whereas that lies at the left of $q_2 = q_2^T$ corresponds to the function $q_{12}^T(q_2)$. Observe that at the equilibrium point,

$q_2^* < q_2^T$. Thus, agent 1 places an order for both periods.

Recall that when $\rho_2 > 0$, agent 1 definitely places an order for the first period. Also recall that order quantity for the first period is bounded by $F^{-1}(\rho_2)$ (see Proof of Proposition 4.1.1 in Appendix). We observe that total order quantity of agent 1 is higher than $F^{-1}(\rho_2)$, which is in line with the fact that agent 1 places an order for both periods.

In order to investigate the improvement achieved with transfers, we introduce a performance indicator, P, defined as

$$P = \frac{E[\pi_1] + E[\pi_2] - \big(E[\pi_1^d] + E[\pi_2^d]\big)}{E[\pi^c] - \big(E[\pi_1^d] + E[\pi_2^d]\big)}$$

P measures the percentage of gain achieved by transfer opportunity over the decentralized setting compared to that by the centralized solution. A high P-value indicates that total profit obtained at Nash equilibrium is close to the profits obtained in the centralized solution, i.e., the joint optimal solution.

In numerical analysis, we encountered scenarios where the P value is relatively low. In these scenarios, it is observed that although the profit obtained at Nash equilibrium is not close to the joint optimal profit, it is significantly higher than that in the decentralized solution. Therefore, we defined another performance indicator, I, which measures the percentage improvement achieved by transfers in terms of decentralized solution.

$$I = \frac{E[\pi_1] + E[\pi_2] - \big(E[\pi_1^d] + E[\pi_2^d]\big)}{\big|E[\pi_1^d] + E[\pi_2^d]\big|}$$

We will now continue with analyzing changes in the optimal order decisions with changes in parameters. For this purpose, we take the base parameter set and change one parameter at a time.

5.1 Effect of uncertainty about when X occurs, k

Table (5.3) shows the results for $\forall k \in \{0, 0.1, ..., 1\}$.

Note that order quantity of agent 2 in decentralized setting is constant for $\forall k$ in Table 5.3 since expected profits of agent 2 in decentralized setting does not depend demand

Table 5.3: Order decisions and optimal profits in changing k

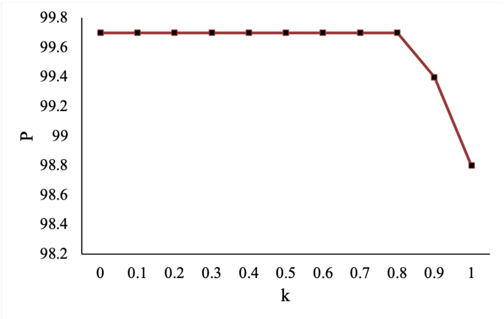
k	$E[\pi_1^d]$	$E[\pi_2^d]$	$E[\pi_1]$	$E[\pi_2]$	$E[\pi^c]$	q_{11}^d	q_{12}^d	q_2^d	q_{11}	q_{12}	q_2	q_1^c	q_2^c	P	I
0	2517.1	2517.1	2658.1	2658.1	5317	0	106.3	106.3	0	105.3	105.3	0	209	99.7	5.6
0.1	2417.1	2517.1	2558.1	2658.1	5217	0	106.3	106.3	0	105.3	105.3	0	209	99.7	5.7
0.2	2317.1	2517.1	2458.1	2658.1	5117	0	106.3	106.3	0	105.3	105.3	0	209	99.7	5.8
0.3	2217.1	2517.1	2358.1	2658.1	5017	0	106.3	106.3	0	105.3	105.3	0	209	99.7	6
0.4	2179.6	2517.1	2320.6	2658.1	4979.6	75.8	30.5	106.3	75.8	29.4	105.3	75.8	133.2	99.7	6
0.5	2198.9	2517.1	2340	2658.1	4998.9	89.2	17.1	106.3	89.2	16	105.3	89.2	119.8	99.7	6
0.6	2228.2	2517.1	2369.2	2658.1	5028.2	96.5	9.8	106.3	96.5	8.7	105.3	96.5	112.5	99.7	5.9
0.7	2262.6	2517.1	2403.6	2658.1	5062.5	101.5	4.8	106.3	101.5	3.8	105.3	101.5	107.5	99.7	5.9
0.8	2300	2517.1	2441.1	2658.1	5100	105.3	1	106.3	105.3	0	105.3	105.3	103.7	99.7	5.9
0.9	2339.3	2517.1	2480.3	2657.5	5139.6	106.7	0	106.3	106.1	0	104.9	108.3	100.7	99.4	5.8
1	2379.1	2517.1	2520.1	2657.1	5180.7	107.3	0	106.3	106.9	0	104.6	110.8	98.2	98.8	5.7

of agent 1.

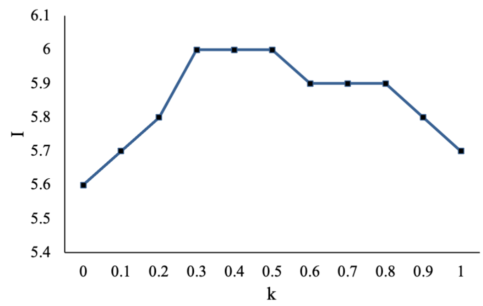
Expected backorder costs of not satisfying demand in the first period increases as k increases. In decentralized setting, when probability that X occurs in the first period is low, i.e., $k < 0.33$, agent 1 does not place an order for period 1. When it is moderate, it places an order for both periods and aggregate order quantity is constant. If this probability is sufficiently large, i.e., $k > 0.83$, then the cost of not satisfying demand in the first period becomes higher than that of not satisfying demand at all and agent 1 places an order only for the first period. Although there is a similar behavior of agent 1 in the 2-agents setting, we should highlight that this behavior depends on both expected backorder cost of not satisfying demand in the first period and the order quantity of agent 2. It should also be underlined that the profit figures with transfers improve considerably in 2-agents setting compared to the decentralized solution. Another interesting finding is that both agents decrease their total order quantities with the motivation of transfers from the other agent for $\forall k$ under the base case.

We observe in Table 5.3 that total order quantity of the central agent is constant for $\forall k$ and it always places a non-zero order for period 2. The reason is that the central agent is required to fulfill the demand $X + Y$. Therefore, even if it is certain that X will occur in the first period, i.e., $k = 1$, the central agent can minimize the expected backorder costs of not satisfying demand in the first period by ordering a portion of its total order amount in the first period. As k becomes moderate, this probability decreases, hence we observe that the order quantity of the central agent for the first period decreases. When it backorder costs are sufficiently low, i.e., $k < 0.33$, the central agent also does not place an order for period 1 as agent 1 in decentralized settings. It is significant that the central agent's total order quantity is less than that in the 2-agents setting. Moreover, the gap between total quantities increase for $k \geq 0.8$. For $k \geq 0.8$, agent 1 places an order only for the first period. This leads to a slight decrease in the level of coordination among parties in the 2-agents setting, hence total order quantity at equilibrium diverges from that in joint optimal solution.

The following figures illustrate how the performance measures change in k. With $P > 0$ and $I > 0$ for $\forall k$, we can argue that the solution at Nash equilibrium always performs better than that of the decentralized setting. Also observe that P decreases for $k \geq 0.8$ in Table (5.3) due to the slight decrease in the level of coordination among parties mentioned above.



(a) P vs k



(b) I vs k

Figure 5.2: Changes in P and I with k

The following figure illustrates how total profits change in k in all settings.

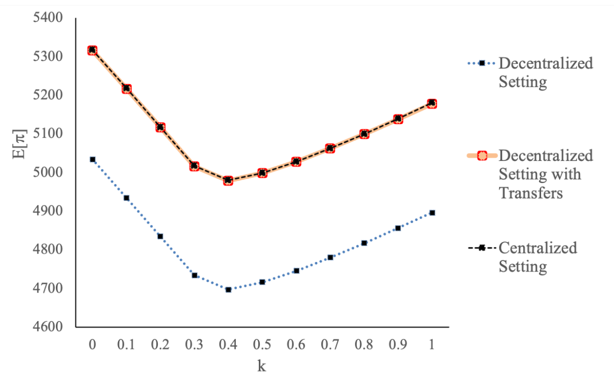


Figure 5.3: Changes in expected profits with k

Observe in Figure (5.3) that profits have an inverse unimodal shape with respect to k. As k gets closer to 0 or 1, the uncertainty on the period which X is realized decreases, explaining the shape of these profit functions. Also observe in Figure (5.2b) that I is unimodal in k. We know that parties suffer less from the increased level of uncertainty as the level of coordination increase. Thus, solutions at Nash equilibrium decrease less compared to decentralized solution, explaining the shape of I.

5.2 Effect of backorder cost, w

Table (5.4) presents the optimal order quantities and profits for $\forall w \in \{0, 2, ..., 30\}$.

Firstly, note that the optimal decision of agent 2 in the 2-agents setting remains the same for $\forall w$ since its expected profit function does not depend on this parameter. On the other hand, as w increases, expected costs of not placing an order for the first period increases for the agent 1 in all settings. Thus, the motivation to order for the first period increases.

When backorder costs are low, i.e., $w \leq 5$, agent 1 does not place an order for the first period. As w increases, expected penalty costs from period 1 become unavoidable and force agent 1 to place an order for the first period. If backorder costs are moderate, i.e., $w < 20$, agent 1 in decentralized setting place an order for both periods. We observe a similar behavior in the setting with transfers. Besides that the backorder costs are moderate, we also conclude that order quantity of agent 2 is low enough for agent 1 to place an order for both periods in the 2-agents setting.

Another interesting result is that total order quantity of agent 1 is constant if the backorder costs are not high, i.e., $w < 20$. Total order quantity of the central agent is also constant for $20 \leq w \leq 30$ since it can manage the increase in backorder costs by allocating its total order quantity over two periods. We also observe that when agent 1 in all settings place an order for both periods, the order quantity for this period is equal and depends on the critical fractile of holding and backorder costs, i.e., ρ_2.

When backorder costs are considerably high, i.e., $w \geq 20$, agent 1 in the decentralized setting places a single order for the first period. We observe the same ordering behavior in agent 1 under transfers. Hence, we can conclude that order quantity of agent 2 is sufficiently high for agent 1 to place a single order for period 1. We can also conclude that when backorder costs are high, agent 1 becomes motivated to fulfill all its demand in the first period in the decentralized and 2-agents settings, resulting in an increase in its total order quantity without placing an order for the second period. This situation deteriorates the level of coordination provided by transfers among parties compared to that by the centralized solution.

Table 5.4: Order decisions and optimal profits in changing w

w	$E[\pi_1^d]$	$E[\pi_2^d]$	$E[\pi_1]$	$E[\pi_2]$	$E[\pi^c]$	q_{11}^d	q_{12}^d	q_2^d	q_{11}	q_{12}	q_2	q_1^c	q_2^c	P	I
0	2517.1	2517.1	2658.1	2658.1	5317	0	106.3	106.3	0	105.3	105.3	0	209	99.7	5.6
2	2417.1	2517.1	2558.1	2658.1	5217	0	106.3	106.3	0	105.3	105.3	0	209	99.7	5.7
4	2317.1	2517.1	2458.1	2658.1	5117	0	106.3	106.3	0	105.3	105.3	0	209	99.7	5.8
6	2244.6	2517.1	2385.6	2658.1	5044.5	66.6	39.7	106.3	66.6	38.6	105.3	66.6	142.3	99.7	6
8	2217.7	2517.1	2358.7	2658.1	5017.6	81.6	24.7	106.3	81.6	23.7	105.3	81.6	127.4	99.7	6
10	2198.9	2517.1	2340	2658.1	4998.9	89.2	17.1	106.3	89.2	16	105.3	89.2	119.7	99.7	6
12	2184.4	2517.1	2325.4	2658.1	4984.4	94.4	11.9	106.3	94.4	10.8	105.3	94.4	114.5	99.7	6
14	2172.5	2517.1	2313.6	2658.1	4972.5	98.4	8	106.3	98.4	6.9	105.3	98.4	110.6	99.7	6
16	2162.6	2517.1	2303.6	2658.1	4962.5	101.5	4.8	106.3	101.5	3.8	105.3	101.5	107.5	99.7	6
18	2153.9	2517.1	2295	2658.1	4953.9	104.1	2.2	106.3	104.1	1.1	105.3	104.1	104.8	99.7	6
20	2146.4	2517.1	2287.4	2657.9	4946.3	106.3	0	106.3	105.5	0	105.1	106.3	102.6	99.6	6
22	2139.3	2517.1	2280.3	2657.5	4939.6	106.7	0	106.3	106.1	0	104.9	108.3	100.7	99.4	6
24	2132.4	2517.1	2273.4	2657.2	4933.5	107.1	0	106.3	106.6	0	104.7	110	99	99	6.1
26	2125.7	2517.1	2266.8	2657	4928	107.5	0	106.3	107.1	0	104.5	111.5	97.4	98.5	6.1
28	2119.1	2517.1	2260.4	2656.8	4923	107.9	0	106.3	107.6	0	104.3	112.9	96.1	98	6.1
30	2112.7	2517.1	2254.1	2656.7	4918.3	108.3	0	106.3	108.1	0	104.1	114.2	94.8	97.4	6.1

Another observation is that $P > 0$ and $I > 0$ for $\forall w$, concluding that transfer option leads to a higher profit. In addition, also observe that P is constant for $w \leq 18$. In scenarios where $w \leq 18$, we observe that the total order quantity is constant for all players. We also observe that the order quantity in the first period is equal for agent 1 in all settings. For instance, when $w = 14$, agent 1 order $F^{-1}(\rho_2) = 98.4$ for the first period. When $w = 16$, its order quantity for period 1 increase to $F^{-1}(\rho_2) = 101.5$. Since the first period order quantities remain equal while total order quantity is constant, a change in backorder cost will affect the total profits by the same amount in all settings. In other words, the differences in total profits will remain equal. Thus, as a ratio of these differences, we observe P is constant for $w \leq 18$. On the other hand, although the improvement with transfers compared to decentralized setting is constant for $w \leq 18$, total profits in the decentralized settings decrease. That is why we observe an increase in I for $w \leq 18$.

The decrease in P for $w \geq 20$ in Table (5.3) can be explained by the deterioration in the level of coordination as mentioned above.

We also find that I increases in w for $w \geq 20$. The reason is that although total profits at Nash equilibrium decrease since the level of coordination introduced by transfers decreases, total profits in the decentralized setting decrease more.

5.3 Effect of holding costs, h

Table (5.5) presents the optimal solution for $h \in \{0, 1, ..., 10\}$.

Increase in h creates the opposite effects of that in w and k: As holding costs increase, agent 1 in all settings will decrease the order quantity for period 1 as expected costs of overstocking for the first period increase.

For $h < 10$, i.e., when $\rho_2 > 0$, expected backorder costs are higher than expected holding costs from period 1, which makes it profitable to for agent 1 to place an order for the first period. As h increases further, holding costs dominate backorder costs from period 1 and force to decrease the order quantity for the first period in all settings. When $h \geq 10$ and $\rho_2 \leq 0$, we observe that agent 1 does not place an order

Table 5.5: Order decisions and optimal profits in changing h

h	$E[\pi_1^d]$	$E[\pi_2^d]$	$E[\pi_1]$	$E[\pi_2]$	$E[\pi^c]$	q_{11}^d	q_{12}^d	q_2^d	q_{11}	q_{12}	q_2	q_1^c	q_2^c	P	I
0	2483.8	2517.1	2625.7	2656.6	5317	108.7	0	106.3	108.6	0	103.9	209	0	89	5.6
1	2422.4	2517.1	2563.5	2656.9	5230.7	107.7	0	106.3	107.3	0	104.4	122.7	86.2	96.5	5.7
2	2361.8	2517.1	2502.7	2657.5	5162.5	106.8	0	106.3	106	0	104.9	110.8	98.2	99.2	5.8
3	2302.6	2517.1	2443.6	2658.1	5102.5	102.4	3.9	106.3	102.4	2.8	105.3	102.4	106.5	99.7	5.9
4	2248.4	2517.1	2389.4	2658.1	5048.3	95.5	10.8	106.3	95.5	9.8	105.3	95.5	113.5	99.7	5.9
5	2198.9	2517.1	2340	2658.1	4998.9	89.2	17.1	106.3	89.2	16	105.3	89.2	119.7	99.7	6
6	2153.5	2517.1	2294.6	2658.1	4953.5	83.1	23.2	106.3	83.1	22.1	105.3	83.1	125.8	99.7	6
7	2112	2517.1	2253.1	2658.1	4912	76.8	29.6	106.3	76.8	28.5	105.3	76.8	132.2	99.7	6.1
8	2074.5	2517.1	2215.5	2658.1	4874.4	69.5	36.8	106.3	69.5	35.8	105.3	69.5	139.5	99.7	6.1
9	2041.6	2517.1	2182.6	2658.1	4841.5	59.5	46.8	106.3	59.5	45.7	105.3	59.5	149.4	99.7	6.2
10	2017.1	2517.1	2158.1	2658.1	4817	0	106.3	106.3	0	105.3	105.3	0	209	99.7	6.2
11	2017.1	2517.1	2158.1	2658.1	4817	0	106.3	106.3	0	105.3	105.3	0	209	99.7	6.2
12	2017.1	2517.1	2158.1	2658.1	4817	0	106.3	106.3	0	105.3	105.3	0	209	99.7	6.2
13	2017.1	2517.1	2158.1	2658.1	4817	0	106.3	106.3	0	105.3	105.3	0	209	99.7	6.2
14	2017.1	2517.1	2158.1	2658.1	4817	0	106.3	106.3	0	105.3	105.3	0	209	99.7	6.2
15	2017.1	2517.1	2158.1	2658.1	4817	0	106.3	106.3	0	105.3	105.3	0	209	99.7	6.2

for the first period, whereas its total order quantity is constant.

Also observe that the 2-agents setting is beneficial for both parties with $P > 0$ and $I > 0$ in Table (5.5) for $\forall h$. For $h \leq 2$, agent 1 is motivated to satisfy its demand in the first period. Since transfer opportunity exists only for the second period, it tries to meet its demand without using this opportunity. Hence, we can conclude that the level of coordination among parties is relatively low when holding costs are low. Thus, as h increases up to 3, we observe an increase in P and I.

In scenarios where $h \geq 3$, we observe that the total order quantity is constant for all players. Additionally, in these scenarios, we observe that the order quantity for the first period is the same for agent 1 in all settings. For example, when $h = 4$, agent 1 orders $F^{-1}(\rho_2) = 95.5$ for the first period. When $h = 5$, this order quantity falls to $F^{-1}(\rho_2) = 89.2$. Since their first period order quantities remain equal while their total order quantity is constant, a change in holding cost will affect total profits by an equal amount in all settings. The differences in total profits will be the equal. Thus, as a ratio of these differences, we observe P is constant for $h \geq 3$. On the other hand, although the improvement with transfers compared to decentralized setting is constant for $h \geq 3$, total profits in decentralized settings decrease. Therefore, I increases in h for $h \geq 3$.

5.4 Effect of procurement costs, c

Table (5.6) presents the optimal solution for $c \in \{10, 12, ..., 40\}$.

As procurement costs increase, profits and total order quantities in all settings tend to decrease in (Table (5.6)).

When procurement cost is not high, i.e., $c \leq 33.3$, agent 1 in decentralized setting places an order for each period since expected cost of not meeting demand in the first period is less than expected cost of not meeting demand at all, i.e., $\rho_1 > \rho_2$. Order quantity of agent 1 for the first period remains constant and equal to $F^{-1}(\rho_2)$ for $c \leq 33.3$ as it is proposed in Proposition 3.1.1. When procurement costs are sufficiently high, expected backorder costs become relatively higher than expected

Table 5.6: Order decisions and optimal profits in changing c

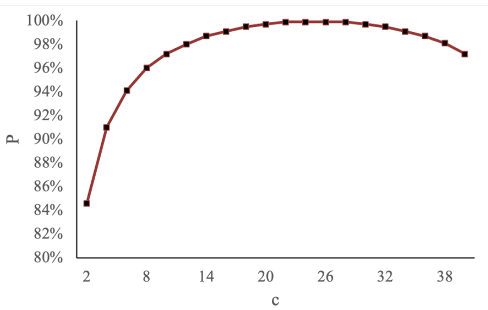
c	$E[\pi_1^d]$	$E[\pi_2^d]$	$E[\pi_1]$	$E[\pi_2]$	$E[\pi^c]$	q_{11}^d	q_{12}^d	q_2^d	q_{11}	q_{12}	q_2	q_1^c	q_2^c	P	I
10	3331.9	3650.1	3431.5	3749.7	7186.9	89.2	31.8	121	89.2	28.4	117.6	89.2	140.5	97.2	2.9
12	3093.2	3411.4	3204.8	3523	6732.3	89.2	28.4	117.7	89.2	25.5	114.7	89.2	135.7	98	3.4
14	2861.1	3179.2	2982.7	3300.8	6286.8	89.2	25.3	114.6	89.2	22.9	112.1	89.2	131.4	98.7	4
16	2634.8	2953	2764.6	3082.8	5849.7	89.2	22.5	111.7	89.2	20.5	109.7	89.2	127.3	99.1	4.6
18	2414.2	2732.4	2550.4	2868.6	5420.5	89.2	19.7	109	89.2	18.2	107.4	89.2	123.4	99.5	5.3
20	2198.9	2517.1	2340	2658.1	4998.9	89.2	17.1	106.3	89.2	16	105.3	89.2	119.7	99.7	6
22	1988.8	2307	2133	2451.2	4584.6	89.2	14.5	103.8	89.2	13.9	103.1	89.2	116.1	99.9	6.7
24	1783.8	2102	1929.6	2247.7	4177.5	89.2	12	101.3	89.2	11.8	101	89.2	112.5	99.9	7.5
26	1583.8	1902	1729.6	2047.7	3777.5	89.2	9.5	98.7	89.2	9.7	99	89.2	109	99.9	8.4
28	1388.8	1707	1533	1851.2	3384.6	89.2	7	96.2	89.2	7.6	96.9	89.2	105.4	99.9	9.3
30	1198.9	1517.1	1339.9	1658.1	2998.9	89.2	4.4	93.7	89.2	5.5	94.8	89.2	101.8	99.7	10.4
32	1014.2	1332.4	1150.4	1468.6	2620.5	89.2	1.8	91	89.2	3.3	92.6	89.2	98.1	99.5	11.6
34	834.8	1153	964.6	1282.8	2249.7	88.4	0	88.3	89.2	1.1	90.3	89.2	94.2	99.1	13.1
36	660.4	979.2	782.4	1101.2	1886.8	85.9	0	85.4	88.1	0	87.8	89.2	90.2	98.7	14.9
38	491.1	811.4	603.6	924.4	1532.3	83.3	0	82.3	86	0	85	89.2	85.8	98.1	17.3
40	327.3	650.1	428.5	752.5	1186.9	80.5	0	79	83.7	0	81.9	89.2	81	97.2	20.8

costs of not satisfying demand at all, i.e., $\rho_1 < \rho_2$. Hence, agent 1 places an order only for the first period.

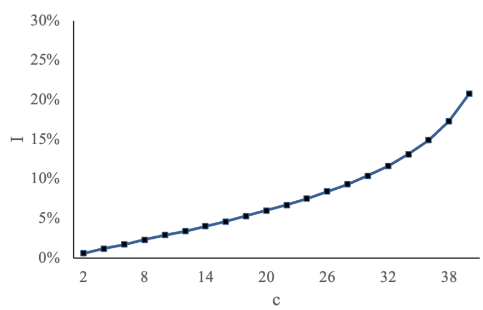
We also observe in Table (5.6) that the centralized agent splits its total order quantity among two periods for $\forall c$. Note that its order quantity for the first period remains constant at $F^{-1}(\rho_2)$ for $\forall c$ while its order quantity for the second period decrease in c, reflecting the decrease in its total order quantity.

For agent 1 in the 2-agents setting, we observe in Table (5.6) that $q_{12} = 0$ for $c \geq 34$ since agent 2 places a sufficiently large order as defined in Proposition 4.1.1. For $2 \leq c \leq 32$, both agents in the 2-agents setting decrease their order quantity for the second period as c increases. Order quantity of agent 1 for the first period is constant and determined by $F^{-1}(\rho_2)$ as in the previous settings discussed above. On the other hand, for $c \geq 34$, it is optimal for agent 1 to set $q_{12} = 0$ and decrease q_{11} beyond $F^{-1}(\rho_2)$.

Finally, observe that both P and I are positive for $\forall c$, thus both agents obtain more profits at Nash equilibrium compared to decentralized solution. Note that I is increasing in c (see Figure (5.4b)). On the other hand, as it can be shown in Figure (5.4a), P is unimodal in c.



(a) P vs c



(b) I vs c

Figure 5.4: Changes in P and I with c

For low values of c, both parties tend to order more in all settings. Hence, although the 2-agents setting close to centralized setting, we observe that P is low as profits obtained in decentralized setting is also close to that in centralized setting. For high values of c, both parties tend to order less. Hence, as c increases, although solution from under transfers bring more profits in addition to decentralized solution, resulting in higher I values, expected profits are much higher in centralized solution, explaining the decrease in P.

5.5 Effect of unit revenue, r

Table (5.7) presents the optimal solution for $r \in \{26, 40, ..., 82\}$.

An initial remark is that changes in r affect all decision makers as expected profit function of all parties depend on this parameter. Moreover, the increase in r and c have the opposite impact on optimal decisions. As r increases, it becomes more costly to not satisfy a unit. Thus, total order quantities of all decision makers increase.

As expected cost of not satisfying demand in the first period is higher than expected holding costs, i.e., $kw - h(1-k) > 0$, agent 1 places an order for the first period in all settings. Agent 1 in decentralized setting does not place an order for period 2 when unit revenues are low, i.e., $r \leq 30$, since expected backorder costs of the first period dominate all other costs and determine total order quantity. For $r > 30$, the business become profitable such that expected costs from the first period become less important than the expected cost of lost sales. Hence, we observe that it places an order for both periods and set the first period order quantity equal to $q_{11}^d = F^{-1}(\rho_2)$. However, note that this is not the case for agent 1 in the 2-agents setting. In this setting, agent 1 places an order for both periods since agent 2 places an order below the threshold q_2^T at equilibrium for $\forall r$. We also observe in Table (5.7) that the centralized agent places an order for both periods as it faces the demand $X + Y$. Also note that order quantity for the first period does not depend on the parameter r in the latter two settings thus we observe $q_{11} = q_1^c = F^{-1}(\rho_2)$ for $\forall r$.

As in the previous sections, we observe $P > 0$ and $I > 0$ for all r, which ensures that the solution at Nash equilibrium is better than that of decentralized solution. We observe that I significantly decreases in r whereas P significantly increases.

As r increases, more units will be ordered for all parties in all settings, so their ability

Table 5.7: Order decisions and optimal profits in changing r

r	$E[\pi_1^d]$	$E[\pi_2^d]$	$E[\pi_1]$	$E[\pi_2]$	$E[\pi^c]$	q_{11}^d	q_{12}^d	q_2^d	q_{11}	q_{12}	q_2	q_1^c	q_2^c	P	I
26	81.8	402.3	137.5	455.7	602.2	83.4	0	81.6	89.2	2.1	91.3	89.2	84.7	92.4	22.5
30	409.1	727.3	485.8	803.9	1296.2	89.2	0	89.2	89.2	6.3	95.6	89.2	95.5	95.9	13.5
34	751.1	1069.2	845.7	1163.9	2014	89.2	5.2	94.4	89.2	9.2	98.5	89.2	102.9	97.7	10.4
38	1103.7	1421.8	1212.9	1531.1	2747	89.2	9.1	98.4	89.2	11.5	100.7	89.2	108.4	98.6	8.7
42	1463.7	1781.9	1585.1	1903.3	3490.5	89.2	12.3	101.5	89.2	13.2	102.5	89.2	112.9	99.1	7.5
46	1829.2	2147.4	1961.1	2279.3	4241.7	89.2	14.9	104.1	89.2	14.7	104	89.2	116.6	99.5	6.6
50	2198.9	2517.1	2340	2658.1	4998.9	89.2	17.1	106.3	89.2	16	105.3	89.2	119.7	99.7	6
54	2572	2890.1	2721.1	3039.2	5760.7	89.2	19	108.3	89.2	17.1	106.4	89.2	122.5	99.9	5.5
58	2947.7	3265.9	3104	3422.2	6526.4	89.2	20.8	110	89.2	18.1	107.4	89.2	124.9	99.9	5
62	3325.7	3643.9	3488.5	3806.6	7295.3	89.2	22.3	111.5	89.2	19	108.3	89.2	127	99.9	4.7
66	3705.6	4023.7	3874.2	4192.4	8066.8	89.2	23.7	112.9	89.2	19.9	109.1	89.2	129.0	99.9	4.4
70	4087.0	4405.2	4261.1	4579.3	8840.6	89.2	24.9	114.2	89.2	20.6	109.8	89.2	130.8	99.9	4.1
74	4469.8	4788	4648.9	4967.1	9616.3	89.2	26.1	115.3	89.2	21.3	110.5	89.2	132.4	99.9	3.9
78	4853.9	5172	5037.6	5355.7	10393.7	89.2	27.1	116.4	89.2	21.9	111.2	89.2	133.9	99.9	3.7
82	5239	5557.1	5426.9	5745.1	11172.6	89.2	28.1	117.3	89.2	22.5	111.7	89.2	135.3	99.9	3.5

to manage both overage and underage risks increases. To investigate this in detail, we provide the following figure, where how much additional profits can be obtained by the central solution is provided. We compute this as $\frac{E[\pi_c]-\left(E[\pi_1^d]+E[\pi_2^d]\right)}{\left(E[\pi_1^d]+E[\pi_2^d]\right)}$. We also provide I, the measure of additional percentage profits at equilibrium, for changing values of r in the following figure.

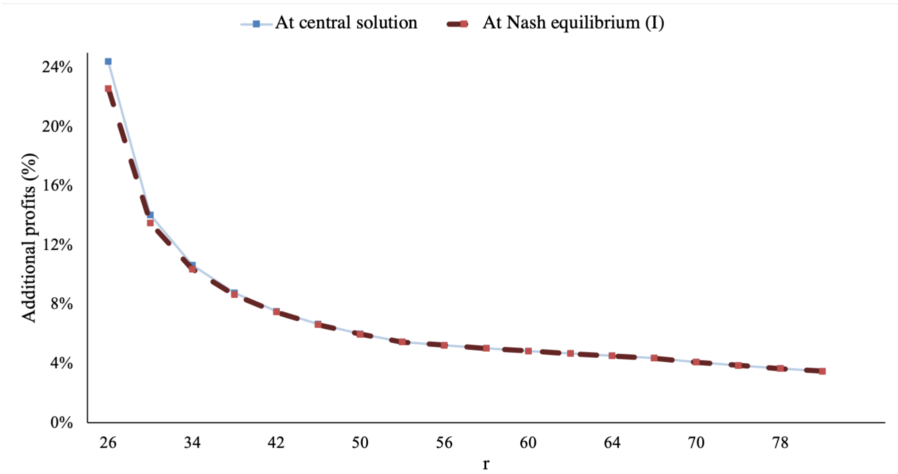


Figure 5.5: Percentage difference between total profits at equilibrium with transfers and that of the central agent in r

Observe that both figures are monotonically decreasing. Thus, the difference of total profits in all settings are decreasing in r. As a first result, the solution from the 2-agents setting cannot bring much benefits in addition to decentralized setting, which is reflected as a decrease in I. As a second result, all settings provide closer total profits as r increases, which explain the increase in P as r increases.

5.6 Effect of mean of demand X, μ_x

Holding the coefficient of variation constant at 0.25, Table (5.8) shows the optimal solutions for $\mu_x \in \{20, 60, ..., 700\}$.

An initial observation is that total order quantities of agent 1 in all settings increase as μ_x increases. On the other hand, we observe in Table (5.8) that as demands resemble, optimal order quantity of agent 2 in the 2-agents setting decreases. The

Table 5.8: Order decisions and optimal profits in changing μ_x and σ_x

μ_x	σ_x	$E[\pi_1^d]$	$E[\pi_2^d]$	$E[\pi_1]$	$E[\pi_2]$	$E[\pi^c]$	q_{11}^d	q_{12}^d	q_2^d	q_{11}	q_{12}	q_2	q_1^c	q_2^c	P	I
20	5	439.8	2517.1	483.3	2560.4	3043.9	17.8	3.4	106.3	17.8	3.3	105.8	17.8	108.6	99.8	2.9
60	15	1319.3	2517.1	1423.9	2621.5	4045.9	53.5	10.3	106.3	53.5	9.7	105.3	53.5	113.8	99.8	5.4
100	25	2198.9	2517.1	2340	2658.1	4998.9	89.2	17.1	106.3	89.2	16	105.3	89.2	119.7	99.7	6
140	35	3078.5	2517.1	3242	2680.8	5923.7	124.9	23.9	106.3	124.9	22.5	105.3	124.9	126	99.7	5.8
180	45	3958	2517.1	4136.3	2695.7	6832.9	160.6	30.8	106.3	160.6	29.1	105.4	160.6	132.4	99.7	5.5
220	55	4837.6	2517.1	5026.1	2706	7733	196.3	37.6	106.3	196.3	35.7	105.5	196.3	139	99.7	5.1
260	65	5717.2	2517.1	5913	2713.5	8627.5	232	44.5	106.3	232	42.4	105.5	232	145.6	99.8	4.8
300	75	6596.7	2517.1	6798.2	2719.1	9518.3	267.7	51.3	106.3	267.7	49.1	105.6	267.7	152.3	99.8	4.4
340	85	7476.3	2517.1	7682.2	2723.6	10406.7	303.4	58.1	106.3	303.4	55.8	105.7	303.4	159.1	99.8	4.1
380	95	8355.9	2517.1	8565.3	2727.1	11293.4	339.1	65	106.3	339.1	62.6	105.7	339.1	165.8	99.8	3.9
420	105.0	9235.4	2517.1	9447.7	2730.0	12178.7	374.8	71.8	106.3	374.8	69.4	105.8	374.8	172.6	99.8	3.6
460	115.0	10115.0	2517.1	10329.7	2732.4	13063.1	410.5	78.7	106.3	410.5	76.2	105.8	410.5	179.3	99.8	3.4
500	125	10994.6	2517.1	11211.3	2734.5	13946.7	446.2	85.5	106.3	446.2	82.9	105.8	446.2	186.1	99.8	3.2
540	135	11874.1	2517.1	12092.6	2736.2	14829.7	481.9	92.3	106.3	481.9	89.7	105.9	481.9	192.9	99.8	3
580	145	12753.7	2517.1	12973.6	2737.7	15712.3	517.6	99.2	106.3	517.6	96.6	105.9	517.6	199.7	99.8	2.9
620	155	13633.3	2517.1	13854.5	2739.1	16594.5	553.3	106	106.3	553.3	103.4	105.9	553.3	206.5	99.8	2.7
660	165	14512.8	2517.1	14735.2	2740.2	17476.4	588.9	112.9	106.3	588.9	110.2	105.9	588.9	213.3	99.8	2.6
700	175	15392.4	2517.1	15615.8	2741.2	18358	624.6	119.7	106.3	624.6	117	105.9	624.6	220.1	99.8	2.5

minimum quantity ordered by agent 2 equals to 105.25 105.3 for $\mu_x = 100$ in Table (5.8) under transfers. Moreover, we also observe that the performance indicator P follows a similar behavior, which yields to the conclusion that the solution obtained at the equilibrium become closer to that of the centralized setting as demands become asymmetric, i.e., the difference in the mean values increase.

We should highlight that as X and Y become asymmetric, the gap between expected overstock and understock quantities of agents increase. In fact, the probability of overstock and/or understock for the agent facing the demand with lower mean might decrease if transfers are allowed. In other words, we expect to observe that the agent who face demand with a lower mean benefits more from transfers. In order to investigate this, we examine the percentage increase achieved by transfers for both agents, which can be computed as $\frac{E[\pi_i]-E[\pi_i^d]}{E[\pi_i^d]}$. The following figure provides the results.

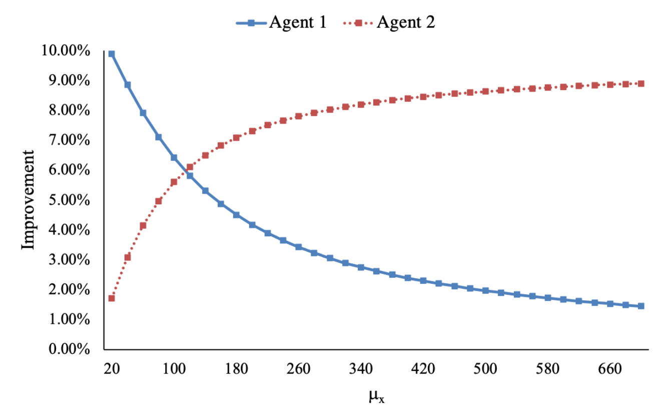


Figure 5.6: Percentage improvement achieved by transfers compared to decentralized profits with changing μ_x

In Figure (5.6), we observe that agent 1 benefits more when the demand it faces has relatively lower mean. As μ_x increase, we observe that this advantage is overtaken by agent 2. This reasoning also provides an explanation for the shape of P in μ_x.

If we look at Figure (5.6) from another perspective, while μ_y is constant, total profit of agent 2 in the decentralized solution is also constant. When $\mu_x < \mu_y$ the percentage improvement achieved by agent 2 using the transfer opportunity is around 2%. Observe that it increases as the μ_x increases. As a result, we conclude that transfers provide more benefits for the agent with lower mean demand.

Another interesting result is that as μ_x becomes closer to μ_y, we observe that I increases. When μ_x is low, total profit of the decentralized solution is low and transfers bring lower overall benefit. On the other hand, the transfers bring high overall improvement but total profit of the decentralized solution increase further while μ_x is high. As a fraction of these two terms, we conclude that the value of I is low when the asymmetry is high. Where the mean demand for both agents is equal, transfers will bring a moderate level of improvement to both agents. In this case, the profit in the decentralized solution can also be considered as moderate. In summary, the agent with smaller mean demand benefits more from asymmetry, so the total profit obtained compared to the decentralized solution is lower.

5.7 Effect of mean of demand Y, μ_y

Table (5.9) presents the optimal solution for $\mu_y \in \{20, 60, ..., 700\}$ while coefficient of variation is constant at 0.25.

Our findings in this subsection follow similar lines with the previous subsection. First of all, total order quantities of agents who face demand Y increase as μ_y increases. We also observe in Table (5.9) that as demands become symmetric, optimal order quantity of agent 1 decreases. The minimum quantity ordered by agent 1 in the 2-agents setting equals to 105.25 105.3 for $\mu_y = 100$ in Table (5.9).

Observe that P follows a similar behavior as in the previous section. Thus, we make the same conclusion: As demands become asymmetric, the solution obtained at the 2-agents setting become closer to that of the centralized setting, explaining the behavior of P.

Another interesting finding is that total order quantities in the 2-agents setting is equal for the same mean value in Tables (5.8) and (5.9). For example, observe that for $\mu_x = 160$, total order quantity at equilibrium is equal to 273.9 in (5.8). This is equal to total order quantity at equilibrium for $\mu_y = 160$ in Table (5.9).

To explain the reasoning, recall the findings on the base case that total order quantities of both players are equal if $\mu_x = \mu_y$ since $\rho_1 > \rho_2$. Also recall that we change only

Table 5.9: Order decisions and optimal profits in changing μ_y and σ_y

μ_y	σ_y	$E[\pi_1^d]$	$E[\pi_2^d]$	$E[\pi_1]$	$E[\pi_2]$	$E[\pi^c]$	q_{11}^d	q_{12}^d	q_2^d	q_{11}	q_{12}	q_2	q_1^c	q_2^c	P	I
20	5	2198.9	503.4	2242.3	546.9	2789.3	89.2	17.1	21.3	89.2	16.6	21.2	89.2	37.2	99.9	3.2
60	15	2198.9	1510.2	2303.3	1614.8	3918.6	89.2	17.1	63.8	89.2	16.1	63.2	89.2	78.2	99.8	5.6
100	25	2198.9	2517.1	2340	2658.1	4998.9	89.2	17.1	106.3	89.2	16	105.3	89.2	119.7	99.7	6
140	35	2198.9	3523.9	2362.6	3687.4	6051	89.2	17.1	148.9	89.2	16.1	147.4	89.2	161.7	99.7	5.7
180	45	2198.9	4530.7	2377.5	4709	7087.4	89.2	17.1	191.4	89.2	16.1	189.7	89.2	203.8	99.7	5.3
220	55	2198.9	5537.6	2387.8	5726	8114.8	89.2	17.1	233.9	89.2	16.2	232	89.2	246.1	99.7	4.9
260	65	2198.9	6544.4	2395.3	6740.3	9136.6	89.2	17.1	276.5	89.2	16.3	274.4	89.2	288.4	99.7	4.5
300	75	2198.9	7551.2	2401	7752.7	10154.7	89.2	17.1	319	89.2	16.4	316.8	89.2	330.8	99.8	4.1
340	85	2198.9	8558.1	2405.4	8764	11170.3	89.2	17.1	361.5	89.2	16.4	359.2	89.2	373.2	99.8	3.8
380	95	2198.9	9564.9	2408.9	9774.3	12184.2	89.2	17.1	404.1	89.2	16.5	401.7	89.2	415.7	99.8	3.6
420	105	2198.9	10571.7	2411.9	10784.0	13196.8	89.2	17.1	446.6	89.2	16.5	444.2	89.2	458.1	99.8	3.3
460	115	2198.9	11578.6	2414.3	11793.3	14208.5	89.2	17.1	489.1	89.2	16.6	486.6	89.2	500.6	99.8	3.1
500	125	2198.9	12585.4	2416.3	12802.1	15219.4	89.2	17.1	531.7	89.2	16.6	529.1	89.2	543.1	99.8	2.9
540	135	2198.9	13592.2	2418.1	13810.7	16229.7	89.2	17.1	574.2	89.2	16.6	571.6	89.2	585.5	99.8	2.8
580	145	2198.9	14599.1	2419.6	14819	17239.5	89.2	17.1	616.7	89.2	16.6	614.1	89.2	628	99.8	2.6
620	155	2198.9	15605.9	2420.9	15827.1	18249	89.2	17.1	659.3	89.2	16.7	656.6	89.2	670.5	99.8	2.5
660	165	2198.9	16612.7	2422	16835.1	19258.1	89.2	17.1	701.8	89.2	16.7	699.1	89.2	713	99.8	2.4
700	175	2198.9	17619.6	2423.1	17843	20267	89.2	17.1	744.4	89.2	16.7	741.6	89.2	755.6	99.8	2.3

the value of mean in this and previous section, hence we still have $\rho_1 > \rho_2$. Thus, total order quantity of agent 1 when $\mu_x = 160$ must also be equal to that of agent 2 when $\mu_y = 160$. In addition, since one of the mean values are always equal to 100, we observe these equalities.

Similar to the previous subsection, in order to investigate whether the increase in μ_y is beneficial for the agents under transfers, we introduce the percentage increase achieved by transfers, which is computed as $\frac{E[\pi_i]-E[\pi_i^d]}{E[\pi_i^d]}$. The following figure provides the results.

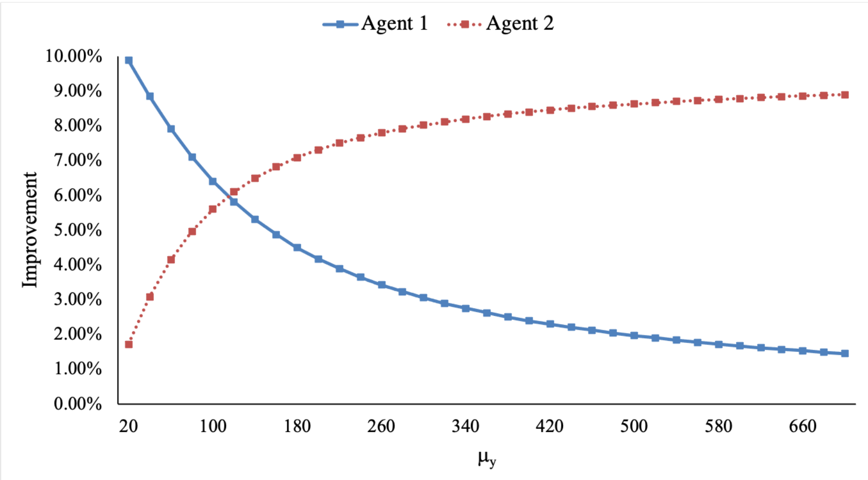


Figure 5.7: Percentage improvement achieved by transfers compared to decentralized profits for changing μ_y

Observe in Figure (5.7) that agent 2 benefits more when demand it faces a demand with a lower mean. Also observe that this advantage is overtaken by agent 1 as μ_y increase, which explains the shape of P in μ_y.

Moreover, note that total profit of agent 1 in the decentralized solution is constant as μ_x is constant. When μ_y is low, the percentage improvement achieved by agent 1 under transfers is low compared to the case when μ_y is high, which favors our conclusion in the previous section that transfers provide more benefits for the agent with lower mean demand.

We also observe that as μ_y gets closer to μ_x, I increases. The reasoning is the same as in the previous section: Transfers would help the parties to obtain more benefits compared to decentralized solution when demands become symmetric.

5.8 Effect of standard deviation of demand X, σ_x

Table (5.10) presents the optimal solution for $\sigma_x \in \{3, 6, ..., 33\}$.

Looking at Table (5.10), we observe that as the σ_x increases, the profit of agent 1 falls in all cases. Since the increase in standard deviation also causes uncertainty to increase, the profits decrease. We observe that the highest decrease in total profits is in the decentralized setting whereas the lowest decrease is in the centralized setting.

Additionally, looking at the change in total order quantities, we can observe that as σ_x increases, total order quantity of agent 1 increases in all cases, whereas its order quantity for period 1 decreases. As σ_x increases, $F^{-1}(\rho_2)$ decreases since $\rho_2 < 0.5$, while $F^{-1}(\rho_1)$ and $V^{-1}(\rho_1)$ increase because $\rho_1 > 0.5$. This causes agent 1's order quantity of period 1 to decrease and that of period 2 to increase in all scenarios.

Note that in the 2-agents setting where transfers are allowed, ρ_1 does not have a direct impact on the total order quantity. However, this relationship of rates also causes a decrease in period 1 order quantity and increase in period 2 order quantity over the profit function in this setting.

We observe that profits of agent 2 increases in σ_x in the 2-agents setting. In order to investigate the rate of this improvement, we present the percent improvement in total profits of agent 2 in terms of the profits of decentralized solution in Figure (5.8), which is computed as $\frac{E[\pi_2]-E[\pi_2^d]}{E[\pi_2^d]}$.

In Figure (5.8), observe that percent improvement in total profits increase in σ_x. Recall that decentralized total profits is constant for agent 2 since only σ_x changes. As uncertainty in demand X decreases, overage and underage quantitites of agent 1 decrease. Hence, the possibility of transfers among players decrease, which leads to a decrease in transfer earnings for agent 2. Therefore, we can conclude that when its counterpart has a demand with lower standard deviation, in other words a demand with less uncertainty, agent 2 obtains less improvements due to transfers.

Table 5.10: Order decisions and optimal profits in changing σ_x

σ_x	$E[\pi_1^d]$	$E[\pi_2^d]$	$E[\pi_1]$	$E[\pi_2]$	$E[\pi^c]$	q_{11}^d	q_{12}^d	q_2^d	q_{11}	q_{12}	q_2	q_1^c	q_2^c	P	I
3	2683.9	2517.1	2711	2544.3	5255.4	98.7	2.1	106.3	98.7	2	106	98.7	107.7	99.8	1
6	2617.7	2517.1	2668.7	2568.1	5237	97.4	4.1	106.3	97.4	4	105.7	97.4	109.1	99.8	2
9	2551.6	2517.1	2623.2	2588.6	5212.2	96.1	6.2	106.3	96.1	5.9	105.6	96.1	110.6	99.8	2.8
12	2485.5	2517.1	2574.8	2606.3	5181.6	94.8	8.2	106.3	94.8	7.8	105.4	94.8	112.2	99.8	3.6
15	2419.3	2517.1	2523.9	2621.5	5145.9	93.5	10.3	106.3	93.5	9.7	105.3	93.5	113.8	99.8	4.2
18	2353.2	2517.1	2470.7	2634.5	5105.8	92.2	12.3	106.3	92.2	11.6	105.3	92.2	115.6	99.7	4.8
21	2287.1	2517.1	2415.7	2645.7	5062	91	14.4	106.3	91	13.5	105.3	91	117.3	99.7	5.4
24	2220.9	2517.1	2359.1	2655.3	5015.1	89.7	16.4	106.3	89.7	15.4	105.3	89.7	119.1	99.7	5.8
27	2154.9	2517.1	2301.3	2663.3	4965.6	88.4	18.5	106.3	88.4	17.3	105.3	88.4	120.9	99.7	6.3
30	2088.9	2517.1	2242.4	2669.8	4913.9	87.1	20.5	106.3	87.1	19.2	105.3	87.1	122.8	99.4	6.6
33	2023.4	2517.1	2183	2674.1	4860.6	85.9	22.6	106.3	85.9	21.2	105.3	85.9	124.6	98.9	7

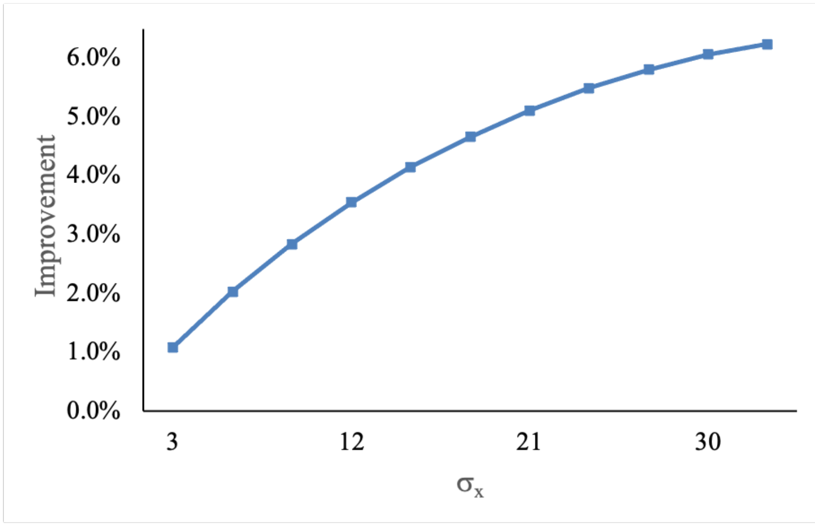


Figure 5.8: Percentage improvement of agent 2 achieved by transfers compared to decentralized profits

In the decentralized setting with transfers, looking at agent 2's order quantity, we see that it decreases as the standard deviation increases. Recall that agent 2's best response function is monotonically decreasing in Q_1. As a result, this decrease is due to the increase in the total order quantity of agent 1.

When we examine the performance indicators, we see that $P > 0$ and $I > 0$, indicating that transfer opportunity brings the total profit closer to the joint optimal solution. In addition, we observe that P decreases while I increases as σ_x increases. The reason for this is that as σ_x increases, total profit decreases the most in the decentralized solution, and the least in the centralized solution. In other words, as σ_x increases, the difference between the total profit obtained in the Nash equilibrium and that in the decentralized solution increases, which explains the increase of I. Similarly, we observe that P decreases since the increase in σ_x causes the difference between total profits of the centralized solution and the solution of 2-agents setting to increase.

5.9 Effect of standard deviation of demand Y, σ_y

Table (5.11) presents the optimal solution for $\sigma_y \in \{3, 6, ..., 33\}$.

Firstly, we observe in Table (5.11) that as the σ_y increases, the uncertainty in demand

Table 5.11: Order decisions and optimal profits in changing σ_y

σ_y	$E[\pi_1^d]$	$E[\pi_2^d]$	$E[\pi_1]$	$E[\pi_2]$	$E[\pi^c]$	q_{11}^d	q_{12}^d	q_2^d	q_{11}	q_{12}	q_2	q_1^c	q_2^c	P	I
3	2198.9	2942	2226.1	2969.2	5195.4	89.2	17.1	100.8	89.2	16.8	100.7	89.2	117.1	99.8	1
6	2198.9	2884.1	2249.9	2935.1	5185.2	89.2	17.1	101.5	89.2	16.5	101.4	89.2	117.3	99.8	2
9	2198.9	2826.1	2270.5	2897.8	5168.6	89.2	17.1	102.3	89.2	16.3	102	89.2	117.5	99.8	2.8
12	2198.9	2768.2	2288.2	2857.6	5146.2	89.2	17.1	103	89.2	16.2	102.6	89.2	117.8	99.8	3.6
15	2198.9	2710.2	2303.4	2814.8	5118.6	89.2	17.1	103.8	89.2	16.1	103.2	89.2	118.2	99.8	4.2
18	2198.9	2652.3	2316.4	2769.8	5086.8	89.2	17.1	104.6	89.2	16.1	103.8	89.2	118.6	99.7	4.8
21	2198.9	2594.3	2327.5	2723	5051.1	89.2	17.1	105.3	89.2	16	104.4	89.2	119	99.7	5.4
24	2198.9	2536.4	2337.1	2674.6	5012.4	89.2	17.1	106.1	89.2	16	105	89.2	119.5	99.7	5.8
27	2198.9	2478.5	2345.2	2624.9	4971	89.2	17.1	106.8	89.2	16	105.7	89.2	120.1	99.7	6.3
30	2198.9	2420.7	2351.6	2574.1	4927.5	89.2	17.1	107.6	89.2	16	106.4	89.2	120.7	99.4	6.6

Y increases, which results in an increase in total order quantities of agents who face demand Y. In addition, we observe that agent 1 orders for both periods in all setting. Recall that the order quantity for the first period is determined by $F^{-1}(\rho_2)$ when an order is placed for both periods. Since both the demand X and $\rho_2 = \frac{kw-h(1-k)}{k(h+w)}$ do not change, the first period order quantity of agent 1 is the same for all σ_y.

Moreover, in the 2-agents setting, we observe that the increase in σ_y is accompanied by a decrease in the total order quantity of agent 1. This is because the best response function of agent 1 decreases monotonically with respect to q_2, which increases in σ_y.

In Table (5.11), we also observe that when transfers are allowed, agent 1's total profit increases σ_y. To analyze this increase further, we examine the percentage improvement agent 1 achieved over the decentralized solution as in the previous section, which is computed as $\frac{E[\pi_1]-E[\pi_1^d]}{E[\pi_1^d]}$. We present the results in Figure (5.9).

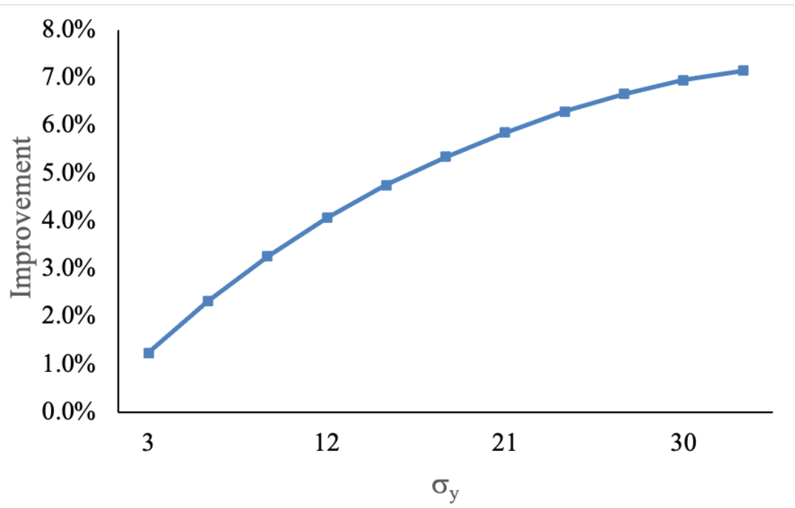


Figure 5.9: Percentage improvement of agent 1 achieved by transfers compared to decentralized profits

In Figure (5.9), similar to the previous section, observe that percent improvement in total profits increase in σ_y. As a result, we conclude that agent 1 obtains more improvements due to transfers if agent 2 has a demand with higher standard deviation.

Finally, we find that $P > 0$ and $I > 0$ in Table (5.11), meaning that the solution at Nash equilibrium is beneficial compared to decentralized solution. Similar to previous section, we observe that P decreases while I increases in σ_y. The reasoning follows similar lines with the discussions in the previous section: As uncertainty increases, total profits at Nash equilibrium decrease less compared to that in the decentralized solution whereas it decrease more compared to that in the centralized solution.

5.10 Effect of transfer prices, t

(Table 5.12) presents the results for $t \in \{12, 14, ..., 40\}$.

In the decentralized setting with transfer opportunity, we observe that both agents place non-zero orders at equilibrium as shown in Corollary 4.3.2.1. Since agent 2 does not place a sufficiently large order, i.e., $q_2 < q_2^T$, agent 1 places an order for both periods. Agent 1's order quantity for the first period depends on the critical fractile of holding and backorder costs, i.e., ρ_2.

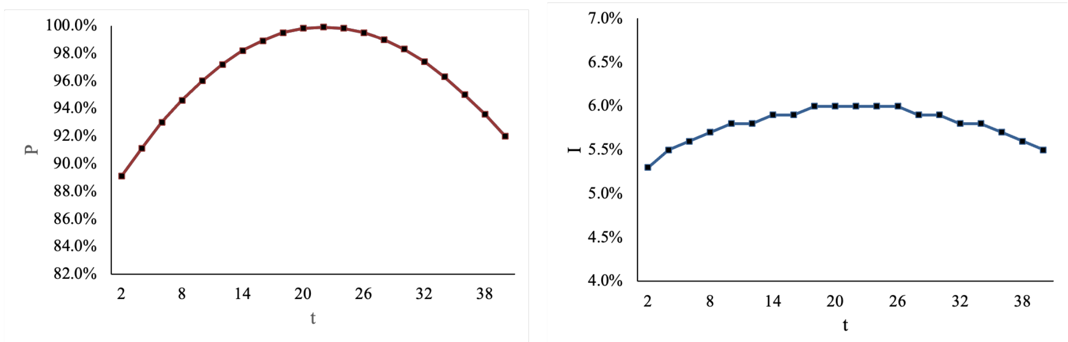
Both agents tend to increase their total order quantity as t increases with the motivation of transferring goods to the other party. This increase in total order quantity is beneficial for both parties up to $t = 22$, where both $E[\pi_1]$ and $E[\pi_2]$ attains its maximum value. Moreover, note that $P > 0$ and $I > 0$ for $\forall t$ in Table (5.12), concluding that the performance of Nash equilibrium is always better than that of decentralized game. P is also maximum for $t = 22$. For t values that are close to 12 or 40, we observe that both P and I decrease. The decrease in P shows decreasing capability of the solution at Nash equilibrium to achieve centralized solution whereas the decrease in I shows decreasing capability of solution at Nash equilibrium to achieve more than decentralized solution. As both indicators decreasing, introduction of transfers become less beneficial as t gets close to 12 or 40.

The changes in P and I for $\forall t$ can be shown in the following figures.

Figures (5.10a) and (5.10b) illustrates that both P and I is unimodal in t, and they attain the highest value for the same transfer price. Thus, we can argue that under this parameter setting, it is possible to encourage both parties by setting the most efficient transfer price. Numerical analysis show that $t = 22.0485$ gives the maximum P and

Table 5.12: Order decisions and optimal profits in changing t

t	$E[\pi_1^d]$	$E[\pi_2^d]$	$E[\pi_1]$	$E[\pi_2]$	$E[\pi^c]$	q_{11}^d	q_{12}^d	q_2^d	q_{11}	q_{12}	q_2	q_1^c	q_2^c	P	I
12	2198.9	2517.1	2336.4	2654.6	4998.9	89.2	17.1	106.3	89.2	12.6	101.8	89.2	119.7	97.2	5.8
14	2198.9	2517.1	2337.8	2656	4998.9	89.2	17.1	106.3	89.2	13.1	102.4	89.2	119.7	98.2	5.9
16	2198.9	2517.1	2338.9	2657	4998.9	89.2	17.1	106.3	89.2	13.7	102.9	89.2	119.7	98.9	5.9
18	2198.9	2517.1	2339.7	2657.8	4998.9	89.2	17.1	106.3	89.2	14.2	103.4	89.2	119.7	99.5	6
20	2198.9	2517.1	2340.1	2658.3	4998.9	89.2	17.1	106.3	89.2	14.7	103.9	89.2	119.7	99.8	6
22	2198.9	2517.1	2340.3	2658.4	4998.9	89.2	17.1	106.3	89.2	15.2	104.5	89.2	119.7	99.9	6
24	2198.9	2517.1	2340.1	2658.3	4998.9	89.2	17.1	106.3	89.2	15.8	105	89.2	119.7	99.8	6
26	2198.9	2517.1	2339.7	2657.9	4998.9	89.2	17.1	106.3	89.2	16.3	105.5	89.2	119.7	99.5	6
28	2198.9	2517.1	2339	2657.1	4998.9	89.2	17.1	106.3	89.2	16.8	106	89.2	119.7	99	5.9
30	2198.9	2517.1	2338	2656.1	4998.9	89.2	17.1	106.3	89.2	17.3	106.6	89.2	119.7	98.3	5.9
32	2198.9	2517.1	2336.7	2654.8	4998.9	89.2	17.1	106.3	89.2	17.8	107.1	89.2	119.7	97.4	5.8
34	2198.9	2517.1	2335.1	2653.3	4998.9	89.2	17.1	106.3	89.2	18.3	107.6	89.2	119.7	96.3	5.8
36	2198.9	2517.1	2333.3	2651.5	4998.9	89.2	17.1	106.3	89.2	18.8	108.1	89.2	119.7	95	5.7
38	2198.9	2517.1	2331.3	2649.5	4998.9	89.2	17.1	106.3	89.2	19.4	108.6	89.2	119.7	93.6	5.6
40	2198.9	2517.1	2329	2647.2	4998.9	89.2	17.1	106.3	89.2	19.8	109.1	89.2	119.7	92	5.5



(a) P vs t

(b) I vs t

Figure 5.10: Changes in P and I with t

I. Thus it is the value of t that brings the highest benefits.

In order to analyze the existence of a t value that will provide the total profit brought by the central solution, we expand our numerical studies. In the following analysis, we investigate the performance of solution from the 2-agents setting as overage and underage costs of both periods change. We use the base case presented in Table (5.1) by changing backorder costs, procurement costs and transfer prices. Backorder cost ,w, is evaluated at $\{0, 5, 15, 30, 50\}$ to investigate the effect of changing expected holding and backorder costs from the first period. Procurement cost, c, is evaluated at $\{5, 10, 25, 40, 45\}$ to investigate the effect of changing expected underage and overage costs of the game computed at the end of the second period. Finally, transfer price, t, is evaluated at $\{1, 2, ..., 49\}$ to observe the performance of decentralized setting with transfers.

In Table (5.13), 50 instances of the analysis are provided in 25 rows. Each row has information about two instances with two different t values: The values that the decentralized setting with transfer opportunity performs the best and the worst.

For the sake of simplicity, we use $E[\pi^d]$ and $E[\pi_N]$ as total profits from decentralized and solution from the 2-agents setting in Table (5.13), respectively. We use P and I to investigate the performance of solution from the decentralized setting with transfers as defined in previously. Recall that P measures how close the profits obtained at Nash equilibrium is to centralized solution compared to decentralized solution, whereas I measures how much additional profits obtained by transfers in addition to decentralized solution. t^b, ρ_1^b, $E[\pi_N^b]$, P^b and I^b represent the values for the best solution from the decentralized setting with transfers whereas t^w, ρ_1^w, $E[\pi_N^w]$, P^w and I^w represent that for the worst solution from the decentralized setting with transfers.

Before investigating the performance of solution from the decentralized setting with transfers, we start with analyzing how the performance indicators P and I change in t. We choose four scenarios where

- P and I are both relatively high, which corresponds to setting 22,
- P and I are both relatively low, which corresponds to setting 15,
- P is relatively high I is relatively low, which corresponds to setting 4, and
- P is relatively low I is relatively high, which corresponds to setting 20.

Remarking the discussion about unimodality of P shown for the base case, we present Figures (5.11a) and (5.11b) which show P and I changes in t for these settings in Table 5.13.

Looking at these figures, it can be seen that both performance indicators are unimodal in t. In fact, numerical analysis show that these performance measures are unimodal in all 25 settings provided in (5.13). Numerical analysis also show that both performance indicators attain the maximum value for the same transfer price. Hence, we argue that there exists a transfer price that maximizes P and I under all operable parameter settings.

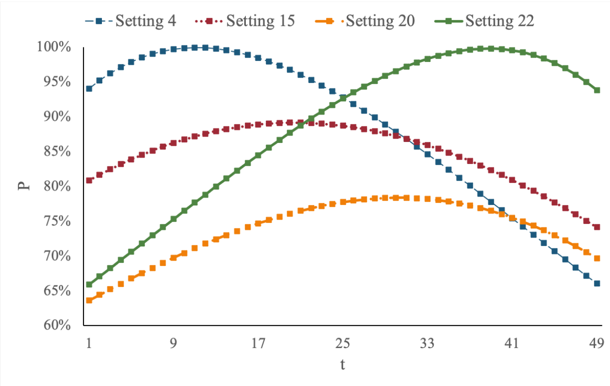
In fact, it has been shown by Kapur ([7]) that there exists a transfer price that will provide the joint optimal profits for a single-period problem. From our numerical studies, we argue that in a two-period problem with $k > 0$, there exists a transfer price for which both P and I are maximized. Moreover, we can also suggest that P and I are unimodal with respect to t. Thus, we conclude that there exists a t value for which transfers bring the highest benefits. On the other hand, this t value does not necessarily coordinate the system, i.e., achieve total joint profits, in a two-period problem. For example, in scenario 15 where $t^b = 20$, we see that total profits obtained at Nash equilibrium is at most 3824.9 whereas that at central solution is

Table 5.13: Optimal order decisions, profits and performance indicators in changing w, c and t

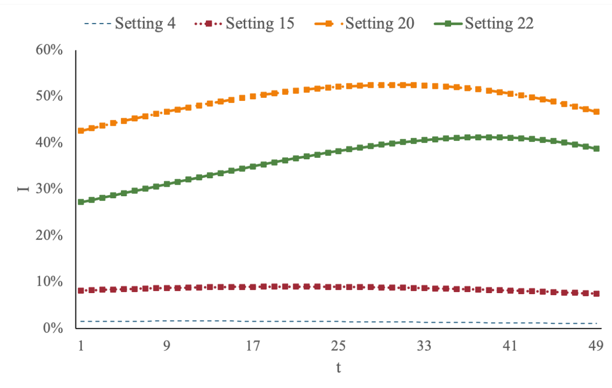
Case	c	w	$E[\pi^d]$	$E[\pi^c]$	ρ_2	t^b	ρ_1^b	$E[\pi_N^b]$	P^b	I^b	t^w	ρ_1^w	$E[\pi_N^w]$	P^w	I^w	Order Periods
1	5	0	8561.3	8689.8	-1	11	1.2	8689.7	99.9	1.5	49	45	8646.1	66	1	2
2	5	5	8311.3	8439.8	0	11	1.2	8439.7	99.9	1.5	49	45	8396.1	66	1	2
3	5	15	8211.5	8340	0.5	11	1.2	8340	99.9	1.6	49	45	8296.4	66.1	1	1, 2
4	5	30	8162.6	8291.1	0.7	11	1.2	8291	99.9	1.6	49	45	8247.4	66	1	1, 2
5	5	50	8129.7	8258.2	0.8	11	1.2	8258.2	99.9	1.6	49	45	8214.6	66.1	1	1
6	10	0	7300.1	7505.1	-1	16	1.2	7505	99.9	2.8	49	40	7453	74.6	2.1	2
7	10	5	7050.1	7255.1	0	16	1.2	7255	99.9	2.9	49	40	7203	74.6	2.2	2
8	10	15	6950.4	7155.4	0.5	16	1.2	7155.3	99.9	2.9	49	40	7103.2	74.5	2.2	1, 2
9	10	30	6901.4	7106.4	0.7	16	1.2	7106.3	99.9	3	49	40	7054.3	74.6	2.2	1, 2
10	10	50	6868.3	7073.6	0.8	13	1.1	7070.1	98.3	2.9	49	40	7021.4	74.6	2.2	1
11	25	0	4002.7	4294.8	-1	25	1	4294.6	99.9	7.3	49	25	4253.1	85.7	6.3	2
12	25	5	3752.7	4044.8	0	25	1	4044.6	99.9	7.8	49	25	4003.1	85.7	6.7	2
13	25	15	3652.9	3945	0.5	11	0.6	3944.9	99.9	8	49	25	3903.4	85.8	6.9	1, 2
14	25	30	3584.7	3896.1	0.7	23	0.9	3882.4	95.6	8.3	49	25	3839.6	81.9	7.1	1
15	25	50	3509.6	3863.2	0.8	20	0.8	3824.9	89.2	9	49	25	3771.8	74.2	7.5	1

Table 5.13: Optimal order decisions, profits and performance indicators in changing w, c and t-*continued*

Case	c	w	$E[\pi^d]$	$E[\pi^c]$	ρ_2	t^b	ρ_1^b	$E[\pi_N^b]$	P^b	I^b	t^w	ρ_1^w	$E[\pi_N^w]$	P^w	I^w	Order Periods
16	40	0	1300.1	1505.1	-1	34	0.6	1504.8	99.9	15.7	1	0.2	1452.9	74.5	11.8	2
17	40	5	1050.1	1255.1	0	34	0.6	1254.8	99.9	19.5	1	0.2	1202.9	74.5	14.6	2
18	40	15	923.4	1155.4	0.5	34	0.6	1144.9	95.5	24	1	0.2	1095.4	74.1	18.6	1
19	40	30	786.5	1106.4	0.7	32	0.6	1055.9	84.2	34.3	1	0.2	1000.6	66.9	27.2	1
20	40	50	643	1073.6	0.8	30	0.5	980.4	78.4	52.5	1	0.2	916.9	63.6	42.6	1
21	45	0	561.3	689.8	-1	39	0.5	689.5	99.8	22.8	1	0.1	645.9	65.8	15.1	2
22	45	5	311.3	439.8	0	39	0.5	439.5	99.8	41.2	1	0.1	395.9	65.8	27.2	2
23	45	15	155.9	340	0.5	38	0.4	320.4	89.4	105.5	1	0.1	271.9	63	74.4	1
24	45	30	-15.9	291.1	0.7	37	0.4	226.6	79	1525.2	1	0.1	157.7	56.5	1091.8	1
25	45	50	-190.5	258.2	0.8	36	0.4	152	76.3	179.8	1	0.1	60.8	56	131.9	1



(a) P vs t



(b) I vs t

Figure 5.11: P and I for changing t values

equal to 3863.2. We also observe that both P^b and I^b values are relatively low for this setting. Hence, there may not exist a transfer price that would achieve joint optimal profits in this setting.

Note that when $k > 0.5$ and transfers are allowed, if backorder costs are high, agent 1 has a motivation to place an order only for the first period. On the other hand, this may not be the case for the central agent. The centralized decision maker, who faces $X+Y$, might still have the flexibility to partition its total order quantity for two periods. In such cases, introduction of transfers may become incapable of achieving the central solution.

We will now investigate the performance of solution from the decentralized setting with transfers with respect to c and w. To begin with, independent of the backorder costs, we observe that the best solution from the 2-agents setting is very close to the centralized solution when purchasing costs are low, which can be observed for cases 1-10 in Table (5.13). We also observe that the percentage of profits gained in the decentralized setting with transfer opportunity in addition to decentralized solution are relatively low, i.e., at most 3%. In cases with low procurement costs, decision makers are motivated to order more in all settings, decreasing the probability of understock in all settings. Thus, the additional profits that can be obtained both by transfers and by centralization decrease. Hence, we observe that P is high whereas I is low. These explain the reason why the minimum value of P^b is 98.3% whereas the maximum value of I^b is 2.9 in the cases 1-10 in Table (5.13).

Comparing the best performances with the worst ones for the cases 1-10, we observe that $max\{I^b - I^w\} = 0.8\%$ whereas $max\{P^b - P^w\} = 33.9\%$. Selection of the best t can be regarded as less important since the possible gains by selecting the right value of t brings 0.8%, in addition to the profits obtained in worst case.

We will now continue with the analysis of cases 11-15, where c is moderate. We observe in Table (5.13) that as w increase beyond 15, P^b decreases whereas I^b increases. As c increase, total units on hand become scarce so coordination gains importance among the agents. Centralized solution brings more profits compared to the solution from the 2-agents setting, whereas solution from the 2-agents setting brings more profits compared to the decentralized solution as w increase. Hence, P^b decreases in w whereas I^b increases.

Using Table (5.13), we see that $max\{I^b - I^w\} = 1.5\%$ whereas $max\{P^b - P^w\} = 15\%$ for the cases 11-15. Selection of the best t value can be regarded as more important than it is for cases 1-10 since the possible gains by selecting the right value of t brings more profits in addition to the profits obtained in worst case.

Finally, we observe that performance of the best solution from the 2-agents setting highly depend on the backorder cost when the procurement cost is high. We use cases 16-25 in Table (5.13) to complete this part of the analysis. Observe that increase in w can decrease P^b more than 20%. Also note that if the decentralized solution brings profits, i.e., $E[\pi_d] > 0$, the solution of 2-agents setting brings profits in addition to the decentralized profits up to $I^b = 100\%$. As c increase, due to high overage costs to be charged at the end of second period, all agents tend to decrease their order quantities in all settings. With available total order quantity becoming lower, we observe that the profits that can be gained by coordination increase further, explaining why we observe the ten highest I^b values for these cases.

In order to comment on the changes in P^b values, we elaborate our investigation on the range of w: We first consider the cases 16, 17, 18, 21 and 22, where w is low compared to c, i.e., $\rho_1 > \rho_2$. Agents who face demand X has the flexibility to split their orders among two periods in these cases. On the other hand, for the cases 19, 20, 23, 24 and 25 where w is high compared to c, i.e., $\rho_2 > \rho_1$, agent 1 in the decentralized setting with transfer opportunity is forced to satisfy its demand without using the transfer opportunity in the second period. Hence, the level of coordination provided by transfer opportunity is close to that provided by the centralized solution for the cases 16, 17, 18, 21 and 22 where we observe $min\{P^b\} = 95.5\%$. On the other hand, it is lower for the cases 19, 20, 23, 24 and 25 where we observe $max\{P^b\} = 89.4\%$.

To compare these best performances with the worst performances for the cases 16-25, we compute $min\{I^b - I^w\} = 3.9\%$ whereas $min\{P^b - P^w\} = 14.8\%$. Selection of the best t can be regarded as important since both performance indicators change significantly. Moreover, expanding this analysis on the range of w discussed in the previous paragraph, we find that determination of the right transfer price is the most important if both c and w are high, i.e., ρ_1 is low whereas ρ_2 is high, since $min\{I^b - I^w\} = 7.1\%$ while $min\{P^b - P^w\}$ remains constant at 14.8%.

It should also be remarked that at the most extreme cases 24 and 25 in Table (5.13), where decentralized solution cannot bring profits, we observe that even the worst solution from the 2-agents setting brings significant profits for the decision maker, highlighting the importance of introducing transfer opportunity when w and c are high.

CHAPTER 6

CONCLUSIONS AND FUTURE WORK SUGGESTIONS

In companies working on a project basis, inventory decisions are made by different decision makers for each project, and common raw materials are transferred to production on a project basis. For this reason, although an item is overstocked for a project, it may be understocked for another project and cause the production to stop. For defense industry companies where penalty costs are higher compared to many sectors, the ignorance of such situations cause an increase in both holding and backorder costs.

In this study, we discussed the inventory management problem of common resources in a company carrying out two different projects over two-period horizon. We assumed that agents of the projects independently set order quantities at the beginning of the time horizon. Demands faced by both agents has quantitative uncertainty. Demand of the second agent is realized in the second period whereas the demand of the first agent has a probability of realization in the second period. Hence, agent 2 places an order for the second period whereas agent 1 places an order for both periods.

In order to examine the change in the expected profits of agents in case company stocks can be shared between projects, we examine the situation where agents are allowed to transfer materials at a transfer price when one has excess supply whereas the other has excess demand. We have observed that when transfers are allowed between agents, their expected profit functions depend on the opponent's total order quantity. Therefore, this problem is investigated using two-player, non-zero, continuous game theory concepts. We show that there exists a unique equilibrium and characterize it for all possible parameter settings.

We also analyze decentralized setting, where both decision makers determine their order quantities without interacting with each other, and centralized setting, where order quantities are determined by a single decision maker for all projects. The decentralized setting corresponds to the common inventory management strategy of the project-based companies.

We observe that the selection of the right transfer price becomes more important if the procurement costs are high. Numerical analysis also shows that when demands are asymmetric, introduction of transfers brings more percentage benefits to the demand with lower mean.

An immediate extension of the study in this thesis would be to consider three or more projects. With the increasing level of pooling of uncertainty, we expect total profits to increase further when transfers are allowed compared to decentralized setting. On the other hand, when the number of projects increase more than two, sub coalitions may be formed among projects. Therefore, it would be challenging to demonstrate the existence of a Nash equilibrium.

Another extension could be to investigate the situation where agents determine their order quantities at the beginning of each period. In this case, a solution to this two-period problem can be found by using dynamic programming methods. In addition, a planning horizon with more than two periods can also be considered.

Finally, a problem setting where demands of both projects have possibility to occur in the first period can be analyzed. In such a problem, one agent's demand may be realized while that of other's is still uncertain. For this reason, the problem can be examined under the condition that demand of both agents must be realized for transfers to take place.

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APPENDIX A

PROOFS OF CHAPTER 3

A.1 Proofs of Section 3.1

Proof of Lemma 3.1.1: Given a continuous and regular function of q_{11}^d and q_{12}^d, we compute the derivatives to construct the Hessian matrix of $E[\pi_1^d]$.

$$\frac{\partial E[\pi](q_{11}^d, q_{12}^d)}{\partial q_{11}^d} = -(r+l-s)F(Q^d) + r + l - c - k(h+w)F(q_{11}^d) - h(1-k) + kw \tag{A.1}$$

$$\frac{\partial E[\pi](q_{11}^d, q_{12}^d)}{\partial q_{12}^d} = -(r+l-s)F(Q^d) + r + l - c \tag{A.2}$$

$$\begin{aligned}
\frac{\partial^2 E[\pi](q_{11}^d, q_{12}^d)}{\partial {q_{11}^d}^2} &= -(r+l-s)f(Q^d) - (k(h+w))f(q_{11}^d)\\
\frac{\partial^2 E[\pi](q_{11}^d, q_{12}^d)}{\partial {q_{12}^d}^2} &= -(r+l-s)f(Q^d)\\
\frac{\partial^2 E[\pi](q_{11}^d, q_{12}^d)}{\partial q_{11}^d \partial q_{12}^d} &= -(r+l-s)f(Q^d)\\
\frac{\partial^2 E[\pi](q_{11}^d, q_{12}^d)}{\partial q_{12}^d \partial q_{11}^d} &= -(r+l-s)f(Q^d)
\end{aligned}$$

Then, the Hessian is as follows:

$$= \begin{bmatrix} -(r+l-s)f(Q^d) - (k(h+w))f(q_{11}^d) & -(r+l-s)f(Q^d) \\ -(r+l-s)f(Q^d) & -(r+l-s)f(Q^d) \end{bmatrix}$$

As $r + l > c > s$, it can be concluded that all entries of the Hessian are negative. Setting $M = -(r+l-s)f(Q^d) \leq 0$ for simplification, the determinant of the Hessian is computed as:

$$= \begin{vmatrix} M - (k(h+w))f(q_{11}^d) & M \\ M & M \end{vmatrix} = -M(k(h+w))f(q_{11}^d)$$

As the multiplication of two negative terms, we conclude that the determinant is positive. Having the second order derivatives negative and the determinant of the Hessian positive, we prove that $E[\pi_1^d]$ is jointly concave in q_{11}^d and q_{12}^d. ■

Proof of Proposition 3.1.1: Since $E[\pi_1^d]$ is jointly concave in q_{11}^d and q_{12}^d, Karush-Kuhn-Tucker conditions will yield the optimal solution. Imposing nonnegativity conditions on order quantities and using the first order derivatives in (A.1) and (A.2), the conditions are constructed as follows:

$$\begin{aligned} 0 =& -(r+l-s)F(Q^d) + r + l - c - k(h+w)F(q_{11}^d) && \text{(A.3)} \\ & - h(1-k) + kw + \mu_1 \\ 0 =& -(r+l-s)F(Q^d) + r + l - c + \mu_2 && \text{(A.4)} \end{aligned}$$

$$\begin{aligned} \mu_1 \geq& 0 && \text{(A.5)} \\ \mu_2 \geq& 0 && \text{(A.6)} \\ -q_{11}^d \leq& 0 && \text{(A.7)} \\ -q_{12}^d \leq& 0 && \text{(A.8)} \\ -q_{11}^d\mu_1 =& 0 && \text{(A.9)} \\ -q_{12}^d\mu_2 =& 0 && \text{(A.10)} \end{aligned}$$

We will construct the proof in two parts: $\rho_2 < 0$ and $\rho_2 \geq 0$. First observe that both q_{11}^d and q_{12}^d cannot be zero in an operable setting, in other words $(q_{11}^d)^* + (q_{12}^d)^* = (Q^d)^* > 0$. Since $Q^d = 0$ implies $F(Q^d) = 0$, we get $\mu_2 = -r - l + c < 0$ from A.4 which violates (A.6).

i. $\rho_2 < 0$: We show that $q_{11}^d = 0$ and $q_{12}^d > 0$. From (A.4), one can write (A.3) as follows:

$$\mu_1 = \mu_2 + (k(h+w))F(q_{11}^d) + h(1-k) - kw \qquad \text{(A.11)}$$

Since $\rho_2 < 0$, i.e. $kw < h(1-k)$, $\mu_1 > 0$ in (A.11), which implies from (A.9) that $q_{11}^d = 0$.Since $q_{11}^d + q_{12}^d = Q^d > 0$, one concludes $q_{12}^d > 0$ and from (A.10) $\mu_2 = 0$. Using this, we get $F(q_{12}^d) = \rho_1$ from (A.4).

ii. $\rho_2 \geq 0$: When $kw \geq h(1-k)$, it is possible to show that $q_{11}^d > 0$.

  - If $(q_{12}^d)^* = 0$, then $q_{11}^d > 0$ since $(Q^d)^* > 0$.
  - If $(q_{12}^d)^* > 0$, then (A.10) implies $\mu_2 = 0$. Further assume that $q_{11}^d = 0$, requiring $\mu_1 > 0$ from (A.9). Substituting (A.4) into (A.3) under these conditions, we obtain $\mu_1 = h(1-k) - kw > 0$, conflicting with the initial condition of $kw > h(1-k)$.

Hence, we conclude that $q_{11}^d > 0$ when $kw > h(1-k)$. Under this condition, $F(Q^d)$ and $F(q_{11}^d)$ can be derived from (A.3) and (A.4) as follows:

$$F(q_{11}^d) = \frac{kw - h(1-k) - \mu_2}{hk + kw} = \rho_2 - \frac{\mu_2}{hk + kw} \tag{A.12}$$

$$F(Q^d) = \frac{r + l - c + \mu_2}{r + l - s} = \rho_1 + \frac{\mu_2}{r + l - s} \tag{A.13}$$

Given $kw > h(1-k)$, we can further show that if $\rho_2 \geq \rho_1$, the optimal solution yields $q_{11}^d > 0$ and $q_{12}^d = 0$. To prove this argument, assume that $q_{12}^d > 0$, yielding $\mu_2 = 0$ from (A.10). Recall that $q_{12}^d > 0$ implies $Q^d = q_{11}^d + q_{12}^d > q_{11}^d$. Referring (A.12) and (A.13), the relation $F(Q^d) = \rho_1 + \frac{\mu_2}{r+l-s} \geq F(q_{11}^d) = \rho_2 - \frac{\mu_2}{hk+kw}$ must hold, which boils down to $F(Q^d) = \rho_1 > F(q_{11}^d) = \rho_2$, conflicting with the initial argument. Hence, $q_{12}^d = 0$ and $\mu_2 > 0$ must hold if $\rho_2 \geq \rho_1$. Setting $q_{12}^d = 0$ and $\mu_1 = 0$ in (A.3), the optimal order quantity for the first period can be found as $F(q_{11}^d) = \rho_3$.

On the other hand, given $kw > h(1-k)$, it can be proven that if $\rho_2 \leq \rho_1$, the optimal solution yields $q_{11}^d > 0$ and $q_{12}^d > 0$. We use proof by contradiction as follows: Assume that $q_{12}^d = 0$, yielding $\mu_2 > 0$ from (A.10). Having $q_{12}^d = 0$ implies $Q^d = q_{11}^d$. Recalling (A.12) and (A.13), the condition $F(Q^d) = \rho_1 + \frac{\mu_2}{r+l-s} \geq F(q_{11}^d) = \rho_2 - \frac{\mu_2}{hk+kw}$ boils down to $F(Q^d) = \rho_1 + \frac{\mu_2}{r+l-s} = \rho_2 - \frac{\mu_2}{hk+kw} = F(q_{11}^d)$. This equality cannot hold for $\mu_2 > 0$ if $\rho_2 \leq \rho_1$, completing the proof. (A.3) and (A.4) yields $q_{11}^d = F^{-1}(\rho_2)$ and $q_{12}^d = F^{-1}(\rho_1) - F^{-1}(\rho_2)$ under these conditions.

■

A.2 Proofs of Section 3.2

Proof of Lemma 3.2.1: For a continuous and regular function of two variables, concavity is controlled using the Hessian matrix. We compute the derivatives to construct the Hessian matrix of $E\left[\pi^c\right]$.

$$\frac{\partial E\left[\pi^c(q_1^c, q_2^c)\right]}{\partial q_1^c} = -(r+l-s)V(Q_c) + r + l - c - (k(h+w))F(q_1^c) + kw - h(1-k) \tag{A.14}$$

$$\frac{\partial E\left[\pi^c(q_1^c, q_2^c)\right]}{\partial q_2^c} = -(r+l-s)V(Q_c) + r + l - c \tag{A.15}$$

$$\begin{aligned}
\frac{\partial^2 E\left[\pi^c(q_1^c, q_2^c)\right]}{\partial q_1^{c2}} &= -(r+l-s)\int_0^{Q_c} f(x)g(Q_c - x)dx - f(q_1^c)(k(h+w)) \\
\frac{\partial^2 E\left[\pi^c(q_1^c, q_2^c)\right]}{\partial q_2^{c2}} &= -(r+l-s)\int_0^{Q_c} f(x)g(Q_c - x)dx \\
\frac{\partial^2 E\left[\pi^c(q_1^c, q_2^c)\right]}{\partial q_1^c \partial q_2^c} &= -(r+l-s)\int_0^{Q_c} f(x)g(Q_c - x)dx \\
\frac{\partial^2 E\left[\pi^c(q_1^c, q_2^c)\right]}{\partial q_2^c \partial q_1^c} &= -(r+l-s)\int_0^{Q_c} f(x)g(Q_c - x)dx
\end{aligned}$$

We define $M = \frac{\partial^2 E\left[\pi^c(q_1^c, q_2^c)\right]}{\partial q_2^{c2}} = \frac{\partial^2 E\left[\pi^c(q_1^c, q_2^c)\right]}{\partial q_1^c \partial q_2^c} = \frac{\partial^2 E\left[\pi^c(q_1^c, q_2^c)\right]}{\partial q_2^c \partial q_1^c}$. Furthermore, we define $M = \frac{\partial^2 E\left[\pi^c(q_1^c, q_2^c)\right]}{\partial q_1^{c2}} + f(q_1^c)k(h+w)$. For a feasible inventory problem, M is ensured to be negative as $r + l > s$. With this definition, Hessian matrix is written as

$$= \begin{bmatrix} M - (k(h+w)f(q_1) & M \\ M & M \end{bmatrix}$$

Its determinant can be computed as follows:

$$= \begin{vmatrix} M - f(q_1^c)(k(h+w)) & M \\ M & M \end{vmatrix} = -Mf(q_1^c)(k(h+w))$$

Having the second order derivatives negative and determinant of the Hessian positive, we prove that $E\left[\pi^c\right]$ is jointly concave in q_1^c and q_2^c. ■

Proof of Proposition 3.2.1: Since the expected profit function is jointly concave in q_1^c and q_2^c, the unique optimal order quantities for this function given nonnegativity constraints on order quantities can be found using Karush-Kuhn-Tucker conditions, which are constructed as below:

$$0 = -(r+l-s)V(Q_c) + r + l - c - (k(h+w))F(q_1^c) + kw - h(1-k) + \mu_1 \tag{A.16}$$

$$0 = -(r+l-s)V(Q_c) + r + l - c + \mu_2 \tag{A.17}$$

$$\mu_1 \geq 0 \tag{A.18}$$

$$\mu_2 \geq 0 \tag{A.19}$$

$$-q_1^c \leq 0 \tag{A.20}$$

$$-q_2^c \leq 0 \tag{A.21}$$

$$-q_1^c \mu_1 = 0 \tag{A.22}$$

$$-q_2^c \mu_2 = 0 \tag{A.23}$$

We will establish the proof in two parts: $\rho_2 < 0$ and $\rho_2 \geq 0$. $Q_c = 0$, i.e. $F(Q_c) = 0$, implies $\mu_2 < 0$ from (A.17). It contradicts with the KKT condition of dual feasibility in (A.19), yielding $(q_1^c)^* + (q_2^c)^* = (Q_c)^* > 0$.

- $\rho_2 < 0$: We prove that $(q_1^c)^* = 0$ and $(q_2^c)^* > 0$. From (A.17), (A.16) can be rewritten as follows:

$$\mu_1 = \mu_2 + k(h+w)F(q_1^c) + h(1-k) - kw \tag{A.24}$$

  As $\rho_2 < 0$, i.e. $kw < h(1-k)$, $\mu_1 > 0$ in (A.24), which requires from (A.22) that $q_1^c = 0$. The condition $(q_1^c)^* + (q_2^c)^* = (Q^c)^* > 0$ requires $q_2^c > 0$, which implies from (A.23) that $\mu_2 = 0$. Using this, we get $V(q_2^c) = \rho_1$ from (A.17).

- $\rho_2 \geq 0$: The inequality $kw \geq h(1-k)$ implies $q_1^c > 0$ as in the single agent case and we explain the reasoning as follows:

  - Since $(q_1^c)^* + (q_2^c)^* = (Q^c)^* > 0$, $q_2^c = 0$ implies $q_1^c > 0$.
  - If $q_2^c > 0$, we obtain $\mu_2 = 0$ from (A.23). Having $q_1^c = 0$ under this condition requires $\mu_1 > 0$ from (A.22). Substituting (A.17) into (A.16) under these conditions, we obtain $\mu_1 = h(1-k) - kw > 0$, conflicting with the initial condition of $kw > h(1-k)$.

As a result, we conclude that $(q_1^c)^* > 0$ if $kw > h(1-k)$. Then, we can derive $V(Q_c)$ and $F(q_1^c)$ from (A.16) and (A.17) as follows:

$$F(q_1^c) = \frac{kw - h(1-k) - \mu_2}{hk + kw} = \rho_2 - \frac{\mu_2}{hk + kw} \tag{A.25}$$

$$V(Q_c) = \frac{r + l - c + \mu_2}{r + l - s} = \rho_1 + \frac{\mu_2}{r + l - s} \tag{A.26}$$

Recalling (A.12) and (A.13), we should point out the similarities of the expressions for total and first period order quantities in single and central agent settings. Recall that $V(x) \leq F(x)$ in the forthcoming part of this proof.

As in the single agent case, it can be argued that if $\rho_2 \geq \rho_1$, the optimal solution requires $q_1^c > 0$ and $q_2^c = 0$ given $kw > h(1-k)$. To prove by contradiction, we assume $q_2^c > 0$, yielding $\mu_2 = 0$ from (A.23). As $q_2^c > 0$ implies $Q_c = q_1^c + q_2^c > q_1^c$, using (A.25) and (A.26), the relation $V(Q_c) = \rho_1 + \frac{\mu_2}{r+l-s} \geq F(q_1^c) = \rho_2 - \frac{\mu_2}{hk+kw}$ can be shown to hold. We can rewrite this relation as $V(Q_c) = \rho_1 > F(q_1^c) = \rho_2$ for $\mu_2 = 0$, conflicting with the initial argument. Therefore, $q_2^c = 0$ and $\mu_2 > 0$ must hold if $\rho_2 \geq \rho_1$. Optimal order quantity for the first period is found from (A.16) by substituting $q_2^c = 0$ and $\mu_1 = 0$, which corresponds to $q_1^{c\prime}$ defined former in the proposition.

The proof is completed by showing that given $kw > h(1-k)$ and $\rho_2 \leq \rho_1$, the optimal solution yields $q_1^c > 0$ and $q_2^c > 0$. To prove using contradiction, assume that $q_2^c = 0$, requiring $\mu_2 > 0$ due to (A.23). Noting $q_2^c = 0$ implies $Q_c = q_1^c$, we obtain $V(Q_c) = \rho_1 + \frac{\mu_2}{r+l-s} = \rho_2 - \frac{\mu_2}{hk+kw} = F(q_1^c)$ using (A.25) and (A.26). This equality cannot hold for $\mu_2 > 0$ if $\rho_2 \leq \rho_1$, resulting that $q_1^c = F^{-1}(\rho_2)$ and $q_2^c = V^{-1}(\rho_1) - F^{-1}(\rho_2)$ from (A.16) and (A.17) if $kw > h(1-k)$ and $\rho_2 \leq \rho_1$.

■

APPENDIX B

PROOFS OF CHAPTER 4

B.1 Proofs of Section 4.1

Proof of Lemma 4.1.1: As a continuous and regular function of q_{11} and q_{12}, the Hessian matrix of π_1 is constructed to check concavity. We assume that $r + l > t$ and $t > s$ without loss of generality as the system will not operate otherwise.

The derivatives are as follows:

$$\begin{aligned}\frac{\partial E[\pi_1(q_{11}, q_{12})]}{\partial q_{11}} =& (r + l + s - 2t)\left[\int_0^{Q_1}\int_0^{Q_1+q_2-x} f(x)g(y)dydx\right] \\ &- (r + l - t)\left(V(Q_1 + q_2) + F(Q_1)\right) + r + l - c \\ &- (hk + kw)F(q_{11}) + hk - h + kw\end{aligned} \tag{B.1}$$

$$\begin{aligned}\frac{\partial E[\pi_1(q_{11}, q_{12})]}{\partial q_{12}} =& (r + l + s - 2t)\left[\int_0^{Q_1}\int_0^{Q_1+q_2-x} f(x)g(y)dydx\right] \\ &- (r + l - t)\left(V(Q_1 + q_2) + F(Q_1)\right) + r + l - c\end{aligned} \tag{B.2}$$

$$\begin{aligned}\frac{\partial^2 E[\pi_1(q_{11}, q_{12})]}{\partial q_{11}^2} =& - (r + l - t)\left[\int_0^{Q_1+q_2} f(x)g(Q_1 + q_2 - x)dx\right] \\ &- (r + l - t)\left[-\int_0^{Q_1} f(x)g(Q_1 + q_2 - x)dx\right] \\ &- (r + l - t)\left[-f(Q_1)G(q_2) + f(Q_1)\right] \\ &- (t - s)\left[\int_0^{Q_1} f(x)g(Q_1 + q_2 - x)dx + f(Q_1)G(q_2)\right] \\ &- (hk + kw)f(q_{11})\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 E[\pi_1(q_{11},q_{12})]}{\partial q_{12}^2} =& -(r+l-t)\left[\int_0^{Q_1+q_2} f(x)g(Q_1+q_2-x)dx\right] \\
& -(r+l-t)\left[-\int_0^{Q_1} f(x)g(Q_1+q_2-x)dx\right] \\
& -(r+l-t)\left[-f(Q_1)G(q_2)+f(Q_1)\right] \\
& -(t-s)\left[\int_0^{Q_1} f(x)g(Q_1+q_2-x)dx + f(Q_1)G(q_2)\right] \\
\frac{\partial^2 E[\pi_1(q_{11},q_{12})]}{\partial q_{11}\partial q_{12}} =& -(r+l-t)\left[\int_0^{Q_1+q_2} f(x)g(Q_1+q_2-x)dx\right] \\
& -(r+l-t)\left[-\int_0^{Q_1} f(x)g(Q_1+q_2-x)dx\right] \\
& -(r+l-t)\left[-f(Q_1)G(q_2)+f(Q_1)\right] \\
& -(t-s)\left[\int_0^{Q_1} f(x)g(Q_1+q_2-x)dx + f(Q_1)G(q_2)\right] \\
\frac{\partial^2 E[\pi_1(q_{11},q_{12})]}{\partial q_{12}\partial q_{11}} =& -(r+l-t)\left[\int_0^{Q_1+q_2} f(x)g(Q_1+q_2-x)dx\right] \\
& -(r+l-t)\left[-\int_0^{Q_1} f(x)g(Q_1+q_2-x)dx\right] \\
& -(r+l-t)\left[-f(Q_1)G(q_2)+f(Q_1)\right] \\
& -(t-s)\left[\int_0^{Q_1} f(x)g(Q_1+q_2-x)dx + f(Q_1)G(q_2)\right]
\end{aligned}
$$

Noting that $\frac{\partial^2 E[\pi_1(q_{11},q_{12})]}{\partial q_{12}^2} = \frac{\partial^2 E[\pi_1(q_{11},q_{12})]}{\partial q_{12}\partial q_{11}} = \frac{\partial^2 E[\pi_1(q_{11},q_{12})]}{\partial q_{11}\partial q_{12}}$, we set $\frac{\partial^2 E[\pi_1(q_{11},q_{12})]}{\partial q_{12}^2} = M$ and remark that $\frac{\partial^2 E[\pi_1(q_{11},q_{12})]}{\partial q_{11}^2} + (hk+kw)f(q_{11}) = M$. Using this substitution, Hessian matrix of the expected profit function of player 1 is constructed as:

$$
= \begin{bmatrix} M-(kh+kw)f(q_{11}) & M \\ M & M \end{bmatrix}
$$

Its determinant can be computed as follows:

$$
\begin{aligned}
&= \begin{vmatrix} M-(kh+kw)f(q_{11}) & M \\ M & M \end{vmatrix} \\
&= -Mf(q_{11})(hk+kw)
\end{aligned}
$$

As $M \leq 0$, the second order derivatives are proven to be non-positive while the determinant is non-negative. Therefore, we conclude that the expected profit function is jointly concave in q_{11} and q_{12}. ■

Proof of Lemma 4.1.2: Setting $I_1 = \int_0^{q_{11}(q_2)} f(x)g(q_{11}(q_2) + q_2 - x)dx$, $f = f(q_{11}(q_2))$, $G = G(q_2)$ and $I_2 = \int_0^{q_{11}(q_2)+q_2} f(x)g(q_{11}(q_2) + q_2 - x)dx$, $\frac{\partial q_{11}}{\partial q_2}$ is obtained from (B.1) as

$$\frac{\partial q_{11}^F}{\partial q_2} = -\frac{(r+l-t)(I_2 - I_1) + I_1(t-s)}{(r+l-t)\left(I_2 - I_1 + f(1-G)\right) + (t-s)(I_1 + fG) + f(hk + kw)} \tag{B.3}$$

Since $r + l > t > s$, $I_2 \geq I_1$ and $f \geq fG$, (B.3) is non-positive, concluding that q_{11}^F is continuous and monotonically decreasing in q_2 under this setting. It should be remarked that $q_{11}^F(q_2)$ is strictly decreasing in q_2 for nonzero order quantities. ■

Proof of Lemma 4.1.3: Setting $I_1 = \int_0^{q_{12}^F(q_2)} f(x)g(q_{12}^F(q_2) + q_2 - x)dx$, $f = f(q_{12}^F(q_2))$, $G = G(q_2)$ and $I_2 = \int_0^{q_{12}^F(q_2)+q_2} f(x)g(q_{12}^F(q_2) + q_2 - x)dx$, we obtain $\frac{\partial q_{12}^F}{\partial q_2}$ as

$$\frac{\partial q_{12}^F}{\partial q_2} = -\frac{(r+l-t)(I_2 - I_1) + I_1(t-s)}{(r+l-t)\left(I_2 - I_1 + f(1-G)\right) + (t-s)(I_1 + fG)} \tag{B.4}$$

Having $r + l > t > s$, $I_2 \geq I_1$ and $f \geq fG$, (B.4) is ensured to be non-positive, concluding that $q_{12}^F(q_2)$ is continuous and monotonically decreasing in q_2. Moreover, the function is strictly decreasing for nonzero order quantities of q_{12} and q_2. ■

Proof of Lemma 4.1.4: We get $\frac{\partial q_{12}^T}{\partial q_2}$ as follows

$$\frac{\partial q_{12}^T}{\partial q_2} = -\frac{(r+l-t)(I_2 - I_1) + I_1(t-s)}{(r+l-t)\left(I_2 - I_1 + f(1-G)\right) + (t-s)(I_1 + fG)} \tag{B.5}$$

where $I_1 = \int_0^{q_{11}+q_{12}^T(q_2)} f(x)g(q_{11} + q_{12}^T(q_2) + q_2 - x)dx$, $f = f(q_{11} + q_{12}^T(q_2))$, $G = G(q_2)$ and $I_2 = \int_0^{q_{11}+q_{12}^T(q_2)+q_2} f(x)g(q_{11} + q_{12}^T(q_2) + q_2 - x)dx$.

Since $r + l > t > s$, $I_2 \geq I_1$ and $f \geq fG$, (B.5) is ensured to be non-positive. Hence, $q_{12}^T(q_2)$ is continuous and monotonically decreasing in q_2. Furthermore, $q_{12}^T(q_2)$ is strictly decreasing in q_2 for nonzero order quantities of q_{12} and q_2. ■

Proof of Lemma 4.1.5: Proving that the profit function of agent 1 is jointly concave in q_{11} and q_{12}, Lemma 4.1.1 also proves that (B.1) and (B.2) are continuous and monotonically decreasing in q_{11} and q_{12}. Taking the derivative of these expressions with respect to q_2, we obtain the same following expression:

$$\begin{aligned}\frac{\partial^2(E[\pi_1])}{\partial q_{11}\partial q_2} = \frac{\partial^2(E[\pi_1])}{\partial q_{12}\partial q_2} &= -(t-s)\int_0^{q_{11}+q_{12}} f(x)g(q_{11}+q_{12}+q_2-x)dx \\ &- (r+l-t)\left[\int_0^{q_{11}+q_{12}+q_2} f(x)g(q_{11}+q_{12}+q_2-x)dx\right] \\ &+ (r+l-t)\left[\int_0^{q_{11}+q_{12}} f(x)g(q_{11}+q_{12}+q_2-x)dx\right]\end{aligned} \tag{B.6}$$

In (B.6), we find that the derivative is less than 0 for all values of feasible q_{11}, q_{12} and q_2. Hence, (B.1) and (B.2) are also continuous and monotonically decreasing in q_2. Then, the existence of a feasible q_2 requires these expressions to be non-negative as $q_2 \to -\infty$ and to be non-positive as $q_2 \to \infty$ given $\rho_2 > 0$, $q_{11} = F^{-1}(\rho_2)$ and $q_{12} = 0$. With these values, (B.1) and (B.2) become identical. Substituting these values and evaluating the expression at $q_2 \to -\infty$, we get

$$-(r+l-t)F(Q_1) + r + l - c \geq 0 \tag{B.7}$$

yielding the condition $\rho_2 \leq \rho_1^n = \frac{r+l-c}{r+l-t}$.

Substituting $q_{11} = F^{-1}(\rho_2)$ and $q_{12} = 0$ for $q_2 \to \infty$, we get

$$-(t-s)F(Q_1) + t - c \leq 0 \tag{B.8}$$

yielding the condition $\rho_2 \geq \frac{t-c}{t-s}$.

As a result, $\frac{t-c}{t-s} \leq \rho_2 \leq \rho_1^n$ is a necessary and sufficient condition for a feasible value of q_2 satisfying (B.1) and (B.2) given $\rho_2 > 0$, $q_{11} = F^{-1}(\rho_2)$ and $q_{12} = 0$.

Combining the results in (B.7) and (B.8) of Lemma 4.1.5, given $\rho_2 > 0$, $q_{11} = F^{-1}(\rho_2)$ and $q_{12} = 0$, we find that (B.1) and (B.2) are negative for $\forall q_2$ if $\rho_2 > \rho_1^n$ and $\rho_2 > \frac{t-c}{t-s}$. We also prove that these expressions are positive for $\forall q_2$ if $\rho_2 < \rho_1^n$ and $\rho_2 < \frac{t-c}{t-s}$. Moreover, we observe that if $t < c$, we have $\frac{t-c}{t-s} < 0$ whereas $\rho_2 > 0$, concluding that the inequalities $\rho_2 < \rho_1^n$ and $\rho_2 < \frac{t-c}{t-s}$ cannot hold simultaneously under this parameter setting.

A final remark using Lemma 4.1.5 is that the inequality $\frac{t-c}{t-s} \geq \rho_2 \geq \rho_1^n$ cannot hold: Firstly, if $t \leq c$, we have $\frac{t-c}{t-s} \leq 0$ whereas $\rho_1^n > 0$, conflicting with the initial inequality. Secondly, if $t > c$, we have $1 > \frac{t-c}{t-s} > 0$ whereas $\rho_1^n > 1$, proving the statement. ■

Proof of Proposition 4.1.1: Since $E[\pi_1(q_{11}, q_{12})]$ is proven to be jointly concave in q_{11} and q_{12}, Karush-Kuhn-Tucker analysis will yield optimal order decisions of agent 1 given quantity ordered by the other agent, q_2. Using nonnegativity conditions on order quantities and the first order derivatives in (B.1) and (B.2) , KKT conditions are written below.

$$\begin{aligned} 0 =&(r+l+s-2t)\left[\int_0^{Q_1}\int_0^{Q_1+q_2-x} f(x)g(y)dydx\right] + r + l - c \\ &- (r+l-t)\left(V(Q_1+q_2)+F(Q_1)\right) - (hk+kw)F(q_{11}) \\ &- h(1-k) + kw + \mu_1 \end{aligned} \tag{B.9}$$

$$\begin{aligned} 0 =&(r+l+s-2t)\left[\int_0^{Q_1}\int_0^{Q_1+q_2-x} f(x)g(y)dydx\right] + r + l - c \\ &- (r+l-t)\left(V(Q_1+q_2)+F(Q_1)\right) + \mu_2 \end{aligned} \tag{B.10}$$

Plugging (B.10) into (B.9), we obtain that

$$0 = -\mu_2 - (hk+kw)F(q_{11}) - h(1-k) + kw + \mu_1 \tag{B.11}$$

$$\mu_1 \geq 0 \tag{B.12}$$

$$\mu_2 \geq 0 \tag{B.13}$$

$$-q_{11} \leq 0 \tag{B.14}$$

$$-q_{12} \leq 0 \tag{B.15}$$

$$-q_{11}\mu_1 = 0 \tag{B.16}$$

$$-q_{12}\mu_2 = 0 \tag{B.17}$$

Given $q_2 = 0$, having $Q_1 = 0$ yields $r+l-c+\mu_2 = 0$ from (B.10), implying $\mu_2 < 0$. It contradicts with (B.13). Thus, $Q_1 = q_{11} + q_{12} > 0$ if $q_2 = 0$.

i. $\rho_2 < 0$:

When $\rho_2 < 0$, all terms except μ_1 is negative in the RHS of (B.11). Hence, $\mu_1 > 0$ and $q_{11}^* = 0$.

When $\mu_2 > 0$, $q_{12} = 0$ from (B.17) which leads to $Q_1 = 0$. Then, (B.10) can be rewritten as follows:

$$V(q_2) = \frac{r+l-c+\mu_2}{r+l-t} = \rho_1^n + \frac{\mu_2}{r+l-t} \tag{B.18}$$

a. $t > c$:

When $t > c$, $\rho_1^n > 1$. Then μ_2 must be negative for (B.18) to hold, which is a contradiction. From (B.13), $\mu_2 = 0$ and from (B.17), $q_{12}^* > 0$.

Plugging $q_{11}^* = 0$ and $\mu_2 = 0$ into (B.11), we obtain $\mu_1 = h(1-k)-kw > 0$. From (B.10) and Definition 4.1.2, $q_{12}^* = q_{12}^F(q_2)$. Note that $q_{12}^F(q_2)$ must be non-negative for all q_2, to satisfy (B.14).

Observe that (B.2) is decreasing in q_{12} for all q_2. For $q_{11} = q_{12} = 0$, (B.2) is non-negative, while for $q_{11} = 0$, $q_{12} \to \infty$, it is negative. Thus, there must exist $q_{12}^F(q_2) > 0$. In conclusion, $q_{11}^* = 0$, $\mu_1 = h(1-k) - kw > 0$, $q_{12}^* = q_{12}^F(q_2) > 0$ and $\mu_2 = 0$ satisfy KKT conditions for part a, completing the proof.

b. $t \leq c$

When $t \leq c$, $\rho_1^n \leq 1$. We analyze this case in two parts: If $V(q_2) \geq \rho_1^n$, $\mu_2 = (r + l - t)\left(V(q_2) - \rho_1^n\right) > 0$ and $q_{12}^* = 0$ satisfies (B.18). Plugging μ_2 into (B.11), $\mu_1 = \mu_2 + h(1-k) - kw > 0$, satisfying the KKT conditions. If $V(q_2) < \rho_1^n$, $\mu_2 > 0$ and $q_{12} = 0$ cannot satisfy (B.18). Hence, $\mu_2 = 0$ and $q_{12}^* > 0$. Remaining of the proof follows similar lines with part a.

ii. $\rho_2 \geq 0$:

We make the following observations:

Observation 1: Since $\rho_2 > 0$, if $\mu_1 > 0$, we have $q_{11} = 0$. To satisfy (B.11), we need to have $\mu_2 > 0$ and hence $q_{12} = 0$. Equivalently, if $\mu_2 = 0$, $\mu_1 = 0$. Thus, underoptimality if $q_{12} > 0$, then $q_{11} > 0$ and if $q_{11} = 0$, then $q_{12} = 0$.

Observation 2: Suppose $\mu_1 = 0$. Then from (B.11),

$$F(q_{11}) = \rho_2 - \frac{\mu_2}{hk + kw} \tag{B.19}$$

Note $q_{12} > 0$ iff $q_{11} = F^{-1}(\rho_2)$. Moreover, $q_{12} = 0$ if $0 < q_{11} \leq F^{-1}(\rho_2)$.

Observation 3: Finally, assume $\mu_1 > 0$ and $\mu_2 > 0$, requiring $Q_1 = 0$ from (B.12) and (B.13). From (B.9)

$$V(q_2) = \rho_3^n + \frac{\mu_1}{r + l - t} \tag{B.20}$$

For $V(q_2) < \rho_3^n$, there does not exist μ_1 satisfying (B.20). Thus, $Q_1 = q_{11} + q_{12} > 0$ if $V(q_2) < \rho_3^n$.

c. $kw + t > h(1-k) + c$

   In this case, we have $\rho_3^n > 1$, which results $V(q_2) < 1 < \rho_3^n$. Since $V(q_2) < \rho_3^n$, from Observations 1 and 3 above, $\mu_1 = 0$, $q_{11}^* > 0$ must hold.

   q_{11}^* and q_{12}^* changes with respect to q_2, q_2^T and ρ_2. We identify four cases as follows:

   1. $\frac{t-c}{t-s} \leq \rho_2 \leq \rho_1^n$ and $q_2 < q_2^T$: We have $-\infty < q_2^T < \infty$ from Definition 4.1.4. For $q_{11} = F^{-1}(\rho_2)$ and $q_{12} = 0$, (B.1) and (B.2) is positive for $\forall q_2 \in (0, q_2^T)$ from Lemma 4.1.5. Furthermore, (B.1) and (B.2) equals to $s - c < 0$ for $q_{11} = F^{-1}(\rho_2)$ and $q_{12} \to \infty$. Thus, for a given $q_2 \in (0, q_2^T)$, there exists $q_{12} > 0$, which makes (B.1) and (B.2) equal to 0. Revisiting Definition 4.1.3, this value is denoted as $q_{12}^T(q_2)$.
      In conclusion, $q_{11}^* = F^{-1}(\rho_2)$, $\mu_1 = 0$, $q_{12}^* = q_{12}^T(q_2)$ and $\mu_2 = 0$ satisfy KKT conditions.

   2. $\frac{t-c}{t-s} \leq \rho_2 \leq \rho_1^n$ and $q_2 \geq q_2^T$: q_2^T is finite from Definition 4.1.4. For $q_{11} = F^{-1}(\rho_2)$ and $q_{12} = 0$, (B.1) and (B.2) are identical and non-positive for $\forall q_2 \in (q_2^T, \infty)$ from Lemma 4.1.5. In addition, for $q_{11} = 0$ and $q_{12} = 0$, (B.1) equals to $r + l - c + kw - h(1-k) > 0$ whereas (B.2) equals to $r + l - c > 0$. Thus, for a given q_2 and $q_{12} = 0$, there exists $\exists q_{11} \in (0, F^{-1}(\rho_2))$ for which (B.1)$= 0$ holds. Referring to Definition 4.1.1, this value of q_{11} corresponds to $q_{11}^F(q_2)$. $q_{11} = q_{11}^F(q_2) > 0$ requires $\mu_1 = 0$ from (B.16). From (B.11), for $q_{11}^F(q_2) \in (0, F^{-1}(\rho_2))$, $q_{12} = 0$ and $\mu_1 = 0$, we obtain $\mu_2 = kw - h(1-k) - (hk + kw)F(q_{11}^F(q_2))$, which is non-negative. Hence, we observe that $q_{11}^* = q_{11}^F(q_2)$, $q_{12}^* = 0$, $\mu_1 = 0$ and $\mu_2 = kw - h(1-k) - (hk + kw)F(q_{11}^F(q_2))$ satisfies KKT conditions for any $q_2 \in (q_2^T, \infty)$.

   3. $\frac{t-c}{t-s} > \rho_2$ and $\rho_1^n > \rho_2$: q_2^T is not finite. For $q_{11} = F^{-1}(\rho_2)$ and $q_{12} = 0$, Lemma 4.1.5 shows that RHS of (B.1) and (B.2) is positive for $\forall q_2 \in (0, \infty)$.

The analysis continues similar to case 1. As a result, we observe that $q_{11}^* = F^{-1}(\rho_2)$, $q_{12}^* = q_{12}^T(q_2)$, $\mu_1 = 0$ and $\mu_2 = 0$ satisfies KKT conditions under this case for $\forall q_2 \in (0, \infty)$.

4. $\frac{t-c}{t-s} < \rho_2$ and $\rho_1^n < \rho_2$: q_2^T is defined as $-\infty$ in Definition 4.1.4. From Lemma 4.1.5, if $q_{11} = F^{-1}(\rho_2)$ and $q_{12} = 0$, we know that (B.1) and (B.2) are negative for $\forall q_2 \in (0, \infty)$.

   The remaining analysis follows steps similar to case 2. As a result, we observe that $q_{11}^* =_{11}^F (q_2)$, $q_{12}^* = 0$, $\mu_1 = 0$ and $\mu_2 = kw - h(1 - k) - (hk + kw)F(q_{11}^F(q_2))$ satisfies KKT conditions under this case $\forall q_2 \in (0, \infty)$.

d. $kw + t < h(1 - k) + c$:

In this case, $kw + t < h(1-k) + c$ implies $t < c$. Thus, we have $\frac{t-c}{t-s} < \rho_2$ and $q_2^T < \infty$. $kw + t < h(1 - k) + c$ also implies $\rho_3^n < 1$. Since $\rho_3^n < 1$, either $V(q_2) < \rho_3^n$ or $V(q_2) \geq \rho_3^n$.

- If $V(q_2) \geq \rho_3^n$, from Observation 3, there exist $\mu_1 > 0$ and $\mu_2 > 0$ satisfying (B.20), which implies $q_{11}^* = q_{12}^* = 0$.

- If $V(q_2) < \rho_3^n$, from Observation 3, $Q_1^* > 0$. Given this condition, q_{11}^* and q_{12}^* change with respect to q_2, q_2^T and ρ_2. We identify three cases as follows:

1. $\frac{t-c}{t-s} \leq \rho_2 \leq \frac{r+l-c}{r+l-t}$ and $q_2 < q_2^T$: For $q_2 \in (0, q_2^T)$, $q_{11} = F^{-1}(\rho_2)$, $q_{12} = q_{12}^T$ and $\mu_1 = 0$ and $\mu_2 = 0$ satisfies (B.12) and (B.13). $q_{11}^* = F^{-1}(\rho_2)$, $\mu_1 = 0$, $q_{12}^* = q_{12}^T(q_2)$ and $\mu_2 = 0$ satisfy KKT conditions. The proof follows similar lines as in case 1 of part c.

2. $\frac{t-c}{t-s} \leq \rho_2 \leq \frac{r+l-c}{r+l-t}$ and $q_2^T \leq q_2 < V^{-1}(\rho_3^n)$: The proof follows similar lines as in case 2 of part c. We observe $-\infty < q_2^T < \infty$. For a given $q_2 \in (q_2^T, V^{-1}(\rho_3))$, there exists $q_{11} > 0$, i.e. $q_{11}^F(q_2)$, for which (B.1)=0 holds.

   In conclusion, $q_{11}^* = q_{11}^F(q_2)$, $\mu_1 = 0$, $q_{12}^* = 0$ and $\mu_2 = -(hk + kw)F(q_{11}^F) + kw - h(1 - k) > 0$ satisfy KKT conditions.

3. $\frac{t-c}{t-s} < \rho_2$ and $\frac{r+l-c}{r+l-t} < \rho_2$: q_2^T is not finite from Definition 4.1.4. For any q_2, the set of variables $q_{11}^* = q_{11}^F(q_2)$, $\mu_1 = 0$, $q_{12}^* = 0$ and $\mu_2 = -(hk + kw)F(q_{11}^F) + kw - h(1 - k) > 0$ satisfies KKT

conditions. The proof follows similar lines to case 4 in part c.

■

B.2 Proofs of Section 4.2

Proof of Lemma 4.2.1: As a continuous and regular function of q_2, the second order derivative of the expected profit expression is checked.

$$\begin{aligned}\frac{\partial E[\pi_2(q_2)]}{\partial q_2} =& -(r+l+s-2t)\left[\int_0^{Q_1}\int_0^{Q_1+q_2-x} f(x)g(y)dydx - F(Q_1)G(q_2)\right] \\ & -(t-s)V(Q_1+q_2)-(r+l-t)G(q_2)+r+l-c\end{aligned} \tag{B.21}$$

$$\begin{aligned}\frac{\partial^2 E[\pi_2(q_2)]}{\partial q_2^2} =& -(r+l-t)\left[\int_0^{Q_1} f(x)g(Q_1+q_2-x)dx\right] \\ & -(r+l-t)\left[-F(Q_1)g(q_2)+g(q_2)\right]-(t-s)\left[F(Q_1)g(q_2)\right] \\ & -(t-s)\left[\int_0^{Q_1+q_2} f(x)g(Q_1+q_2-x)dx - \int_0^{Q_1} f(x)g(Q_1+q_2-x)dx\right]\end{aligned}$$

Since $\int_0^{Q_1+q_2} f(x)g(Q_1+q_2-x)dx \geq \int_0^{Q_1} f(x)g(Q_1-x)dx$ and $g(q_2)F(Q_1) \leq F(Q_1)$, the second order derivative expression is ensured to be negative, yielding that the expected profit function of player 2 is concave for an operable setting. ■

Proof of Lemma 4.2.2: Taking the derivative of (B.21) with respect to Q_1, we obtain

$$\begin{aligned}\frac{\partial^2(E[\pi_2(q_2)])}{\partial q_2 \partial Q_1} =& -(t-s)\int_0^{Q_1+q_2} f(x)g(Q_1+q_2-x)dx \\ & -(r+l+s-2t)\left[\int_0^{Q_1} f(x)g(q_{11}+q_{12}+q_2-x)dx\right]\end{aligned} \tag{B.22}$$

In (B.22), we find that the derivative is defined and less than 0 for all values of feasible Q_1 and q_2. Hence, (B.21) is continuous and monotonically decreasing in Q_1. Since (B.21) is also decreasing in q_2 due to the concavity of agent 2's profit function, we can argue that the existence of a positive-valued q_2 requires (B.21) to be non-negative at $q_2 \to 0$ and non-positive at $q_2 \to \infty$. Evaluating (B.21) at $q_2 \to -\infty$, we get $s-c$, which negative for all Q_1 values. Evaluating (B.21) at $q_2 \to 0$, we obtain $-(r+l-t)V(Q_1)+r+l-c$.

Observe that for $t > c$, $-(r+l-t)V(Q_1)+r+l-c > 0$. Then, there exists $q_2^F(Q_1) > 0$ for $\forall Q_1$ given $t > c$.

Moreover, observe that for $t \leq c$, $-(r+l-t)V(Q_1)+r+l-c > 0$iff $V^{-1}(\rho_1) > Q_1$. Then, there exists $q_2^F(Q_1) > 0$ for $\forall Q_1 \in (0, V^{-1}(\rho_1))$ given $t \leq c$. ■

Proof of Lemma 4.2.3: We take q_2 as a function of Q_1 in (B.21), denoted as $q_2^F(Q_1)$.

Given $F = F(Q_1)$, $g = g(q_2^F(Q_1))$, $I_2 = \int_0^{Q_1+q_2^F(Q_1)} f(x)g(Q_1+q_2^F(Q_1)-x)dx$ and $I_1 = \int_0^{Q_1} f(x)g(Q_1+q_2^F(Q_1)-x)dx$, we obtain $\frac{\partial q_2^F}{\partial Q_1}$ as follows:

$$\frac{\partial q_2^F}{\partial Q_1} = -\frac{(t-s)(I_2-I_1)+I_1(r+l-t)}{(t-s)(gF+I_2-I_1)+(r+l-t)(g(1-F)+I_1)} \tag{B.23}$$

Since $r+l > t > s$ and $I_2 \geq I_1$, (B.23) is ensured to be non-positive. Hence, $q_2^F(Q_1)$ is continuous and monotonically decreasing in Q_1. Furthermore, $q_2^F(Q_1)$ is strictly decreasing in Q_1 for nonzero order quantities of Q_1. ■

Proof of Proposition 4.2.1: Karush-Kuhn-Tucker analysis will yield optimal ordering decision of agent 2 given quantity ordered by the other agent since $E[\pi_2(q_2)]$ is concave in q_2 given the convex set constructred with the nonnegativity condition on q_2. We write the conditions as follows using the first order expression in (B.21).

$$\begin{aligned} 0 = &-(r+l+s-2t)\left[\int_0^{Q_1}\int_0^{Q_1+q_2-x} f(x)g(y)dydx - G(q_2)F(Q_1)\right] \\ &-(t-s)V(Q_1+q_2)-(r+l-t)G(q_2)+r+l-c+\mu \end{aligned} \tag{B.24}$$

$$\mu \geq 0 \tag{B.25}$$

$$-q_2 \leq 0 \tag{B.26}$$

$$-q_2\mu = 0 \tag{B.27}$$

An initial notice is that total quantity ordered by two agents cannot be zero: For $q_2 = 0$ given $Q_1 = 0$, (B.24) can be rewritten as $\mu = -r-l+c < 0$, which conflicts with (B.25).

Assume $\mu > 0$, which requires $q_2 = 0$ from (B.27) and implies $0 < V(Q_1) \leq 1$. (B.24) can be rewritten as follows:

$$0 = -(r+l-t)V(Q_1)+r+l-c+\mu \tag{B.28}$$

a. $t > c$:

For $t > c$, (B.28) yields $\mu < 0$, conflicting with (B.25). Therefore, given $t > c$, the optimal solution requires $\mu = 0$ and $q_2^* > 0$, which corresponds to $q_2^F(Q_1)$ by definition. Using Lemma 4.2.2, $q_2^* = q_2^F(Q_1)^*$ and $\mu = 0$ satisfies KKT conditions.

b. $t \leq c$:

From (B.28), we observe that $\mu > 0$ for $Q_1 > V^{-1}(\rho_1^n)$. Hence, $q_2^* = 0$ and $\mu = (r + l - t)V(Q_1) - r - l + c$ satisfies KKT conditions if $Q_1 > V^{-1}(\rho_1^n)$. For $Q_1 < V^{-1}(\rho_1^n)$, (B.28) yields $\mu < 0$, conficting with (B.25). Thus, we conclude that given $t \leq c$ and $Q_1 < V^{-1}(\rho_1^n)$, the optimal solution requires $\mu = 0$ and $q_2^* > 0$, which corresponds to $q_2^F(Q_1)$ by definition. $q_2^* = q_2^F(Q_1)^*$ and $\mu = 0$ satisfies KKT conditions using Lemma 4.2.2.

■

B.3 Proofs of Section 3.3

Proof of Lemma 4.3.1. : Firstly, we use Proposition 4.1.1 to show continuity as follows:

- $\rho_2 < 0$ and $t > c$: We prove that $Q_1^*(q_2)$ is continuous in q_2, using Lemma 4.1.3.

- $\rho_2 < 0$ and $t < c$: For $q_2 \geq V^{(-1)}(\rho_1)$, $Q_1^*(q_2)$ is constant at $Q_1^* = 0$, ensuring continuity in q_2. For $q_2 < V^{(-1)}(\rho_1)$, $Q_1^* = q_{12}^F(q_2)$ from Proposition 4.1.1. $Q_1^*(q_2)$ is continuous and monotonically decreasing in q_2 using Lemma 4.1.3.

  We proceed with the intersection point, $q_2 = V^{-1}(\rho_1)$. From Proposition 4.1.1, we obtain $\lim_{q_2 \to V^{-1}(\rho_1)^+} Q_1(q_2) = 0$. Substituting $q_2 = V^{-1}(\rho_1)$ into (B.2), we find $\lim_{q_2 \to V^{-1}(\rho_1)^-} Q_1(q_2) = q_{12}^F(V^{-1}(\rho_1)) = 0$. Thus, we argue that $Q_1^*(q_2)$ is continuous in q_2 for $\rho_2 < 0$ and $t < c$.

- $\rho_2 > 0$ and $kw + t > c + h(1 - k)$: For $q_2 \geq q_2^T$, we have $Q_1^* = q_{11}^F(q_2)$. Using Lemma 4.1.2, we find that $Q_1^*(q_2)$ is continuous in q_2. For $q_2 < q_2^T$,

$Q_1^* = F^{-1}(\rho_2) + q_{12}^T(q_2)$. Using Lemma 4.1.4, $Q_1^*(q_2)$ is shown to be continuous in q_2.

We furthter investigate the intersection point, $q_2 = q_2^T$. From Proposition 4.1.1, we have $\lim_{q_2 \to q_2^{T+}} Q_1(q_2) = q_{11}^F(q_2)$. Substituting $q_{11} = q_{11}^F(q_2^T)$ and $q_{12} = 0$ in (B.1), we find $\lim_{q_2 \to q_2^{T+}} Q_1(q_2) = F^{-1}(\rho_2)$. Moreover, we have $\lim_{q_2 \to q_2^{T-}} Q_1(q_2) = F^{-1}(\rho_2) + q_{12}^T(q_2)$ from Proposition 4.1.1. Substituting $q_{12} = q_{12}^T(q_2^T)$ in (B.2), we find $\lim_{q_2 \to q_2^{T-}} Q_1(q_2) = F^{-1}(\rho_2)$. Having the same solution for the intersection point, we conclude that $Q_1^*(q_2)$ is continuous in q_2.

- $\rho_2 > 0$ and $kw + t < c + h(1-k)$: For $q_2 \geq V^{-1}(\rho_3)$, $Q_1^*(q_2)$ is constant at $Q_1^* = 0$, ensuring continuity in q_2. For $V^{-1}(\rho_3) \geq q_2 \geq q_2^T$, we have $Q_1^* = q_{11}^F(q_2)$, whereas for $q_2^T > q_2$, we have $Q_1^* = F^{-1}(\rho_2) + q_{12}^T(q_2)$. Similar to part c, we use Lemma 4.1.2 and Lemma 4.1.4 to ensure continuity of the best response function, respectively.

  We proceed with the intersection points at $q_2 = V^{-1}(\rho_3)$ and $q_2 = q_2^T$. Beginning with $q_2 = V^{-1}(\rho_3)$, observe that $\lim_{q_2 \to V^{-1}(\rho_3)^+} Q_1(q_2) = 0$ from Proposition 4.1.1. Substituting $q_{11} = q_{11}^F\left(V^{-1}(\rho_3)\right)$ and $q_{12} = 0$ in (B.1), we also get $\lim_{q_2 \to V^{-1}(\rho_3)^-} Q_1(q_2) = 0$, ensuring continuity at $q_2 = V^{-1}(\rho_3)$.

  The proof for the second intersection point is the same as in part c. We obtain $\lim_{q_2 \to q_2^{T+}} Q_1(q_2) = \lim_{q_2 \to q_2^{T-}} Q_1(q_2) = F^{-1}(\rho_2)$ by using Proposition 4.1.1 and (B.1), concluding that $Q_1^*(q_2)$ is continuous in q_2 for part d.

As a result, $Q_1^*(q_2)$ is continuous in q_2 for all possible parameter values.

Secondly, optimal solution structure in Proposition 4.2.1 is used to show continuity.

- $t > c$: $q_2^* = q_2^F(Q_1)$ from Proposition 4.2.1. Using Lemma 4.2.3, we conclude that $q_2^*(Q_1)$ is continuous and monotonically decreasing in Q_1.

- $t < c$: For $Q_1 > V^{-1}(\rho_1^n)$, $q_2^*(Q_1) = 0$, hence continuous. For $Q_1 < V^{-1}(\rho_1)$, we have $q_2^* = q_2^F(Q_1)$ from Proposition 4.2.1, for which we ensure continuity using Lemma 4.2.3.

  We also investigate the intersection point at $Q_1 = V^{-1}(\rho_1)$ to complete the proof. We observe that $\lim_{Q_1 \to V^{-1}(\rho_1)^+} q_2(Q_1) = 0$. Plugging $Q_1 = V^{-1}(\rho_1)$

into (B.21), we also find $\lim_{Q_1 \to V^{-1}(\rho_1)^-} q_2(Q_1) = 0$. Having the same solution at the intersection point, we argue $q_2^*(Q_1)$ is continuous in q_2.

As a result, $q_2^*(Q_1)$ is continuous in Q_1 for all possible parameter alignments.

Together with the results above, Lemmas 4.1.2, 4.1.3, 4.1.4 and 4.2.3 ensure the continuous and monotonically decreasing property of the best response functions on the (q_2, Q_1) plane. Given this property, we ensure that the upper and lower bounds are located at the values of 0 and ∞ on the plane.

Using Propositions 4.1.1 and 4.2.1, we will elaborate on possible parameter settings to investigate the bounds of the functions. We denote $\bar{a}$ as the upper bound and $\underline{a}$ as the lower bound in the following parts of the proof.

1. $\rho_2 < 0$ and $t > c$: For $q_2 = 0$, we have $\bar{Q}_1 = \bar{q}_{12} = q_{12}^F(0)$. As $V(x) \leq F(x)$, $\bar{q}_{12} \leq V^{-1}(\rho_1) = V^{-1}(\frac{r+l-c}{r+l-s})$. We obtain the lower bound on Q_1 by setting $q_2 \to \infty$, which is $\underline{Q}_1 = F^{-1}(\frac{t-c}{t-s})$.

   For $Q_1 = 0$, we have $\bar{q}_2 = q_2^F(0)$. As $V(x) \leq G(x)$, $\bar{q}_{12} \leq V^{-1}(\rho_1)$. We obtain the lower bound on q_2 by setting $Q_1 \to \infty$, which is $\underline{q}_2 = G^{-1}(\frac{t-c}{t-s})$.

2. $\rho_2 < 0$ and $t < c$: For $q_2 = 0$, we have $\bar{Q}_1 = \bar{q}_{12} = q_{12}^F(0)$. As $V(x) \leq F(x)$, $\bar{q}_{12} \leq V^{-1}(\rho_1)$. We obtain the lower bound on Q_1 by setting $q_2 \to \infty$, which is $\underline{Q}_1 = 0$.

   For $Q_1 = 0$, we have $\bar{q}_2 = q_2^F(0)$. As $V(x) \leq G(x)$, $\bar{q}_{12} \leq V^{-1}(\rho_1)$. We obtain the lower bound on q_2 by setting $Q_1 \to \infty$, which is $\underline{q}_2 = 0$.

3. $\rho_2 > 0$ and $t > c$: For $q_2 = 0$, we have $\bar{Q}_1 = F^{-1}(\rho_2) + \bar{q}_{12} = F^{-1}(\rho_2) + q_{12}^T(0)$. As $V(x) \leq F(x)$, $\bar{Q}_1 \leq V^{-1}(\rho_1)$. We obtain the lower bound on Q_1 by setting $q_2 \to \infty$, which is $\underline{Q}_1 = \underline{q}_{11} = q_{11}^F(\infty) = 0$.

   For $Q_1 = 0$, we have $\bar{q}_2 = q_2^F(0)$. As $V(x) \leq G(x)$, $\bar{q}_{12} \leq V^{-1}(\rho_1)$. We obtain the lower bound on q_2 by setting $Q_1 \to \infty$, which is $\underline{q}_2 = q_2^F(\infty) = G^{-1}(\frac{t-c}{t-s})$. A final remark is that $\underline{q}_2 > 0$ as $t > c$ under this parameter setting.

4. $\rho_2 > 0, t < c$ and $kw + t > h(1-k) + c$: For $q_2 = 0$, we have $\bar{Q}_1 = F^{-1}(\rho_2) + \bar{q}_{12} = F^{-1}(\rho_2) + q_{12}^T(0)$. As $V(x) \leq F(x)$, $\bar{Q}_1 \leq V^{-1}(\rho_1)$. We

obtain the lower bound on Q_1 by setting $q_2 \to \infty$, which is $\underline{Q}_1 = q_{11}^F(\infty) = F^{-1}(\frac{t-c+kw-h(1-k)}{kw+hk})$. Note that this value is positive since $kw+t > h(1-k)+c$ under this parameter setting.

For $Q_1 = 0$, we have $\bar{q}_2 = q_2^F(0)$. As $V(x) \leq G(x)$, $\bar{q}_{12} \leq V^{-1}\left(\frac{r+l-c}{r+l-s}\right)$. We obtain the lower bound on q_2 by setting $Q_1 \to \infty$, which is $\underline{q}_2 = 0$.

5. $\rho_2 > 0, t < c$ and $kw + t < h(1-k) + c$: For $q_2 = 0$, we have $\bar{Q}_1 = F^{-1}(\rho_2) + \bar{q}_{12} = F^{-1}(\rho_2) + q_{12}^T(0)$. As $V(x) \leq F(x)$, $\bar{Q}_1 \leq V^{-1}(\rho_1)$. We obtain the lower bound on Q_1 by setting $q_2 \to \infty$, which is $\underline{Q}_1 = 0$.

   For $Q_1 = 0$, we have $\bar{q}_2 = q_2^F(0)$. As $V(x) \leq G(x)$, $\bar{q}_{12} \leq V^{-1}(\rho_1)$. We obtain the lower bound on q_2 by setting $Q_1 \to \infty$, which is $\underline{q}_2 = 0$.

As a result, we prove that both best response functions are bounded on the (q_2, Q_1) plane. ■

Proof of Theorem 4.3.1. Lemma 4.3.1 proves that best response functions are continuous and bounded, which is the necessary and sufficient condition to prove the existence of a Nash equilibrium. ■

Proof of Lemma 4.3.2. We begin with comparing q_{11}^F and q_2^F. We can compute the slopes of functions in (q_2, Q_1) plane, $\frac{\mathrm{d}q_{11}}{\mathrm{d}q_2}$ using implicit derivation from the first order conditions in (B.1) and (B.21) with $q_{12} = 0$. Using (B.3), since $(r+l-t)\left(f(1-G)\right) + fG(t-s) + f(hk+kw) \geq 0$, we prove that the slope of $q_{11}^F(q_2)$ is always greater than -1.

To compare this to the slope of q_2^F, we obtain $\frac{\mathrm{d}q_{11}}{\mathrm{d}q_2}$ of q_2^F using (B.21) as

$$\frac{\mathrm{d}q_{11}}{\mathrm{d}q_2} = -\frac{(r+l-t)(I_1 + g(1-F)) + (t-s)(I_2 - I_1 + gF)}{I_1(r+l-t) + (t-s)(I_2 - I_1)} < -1 \qquad \text{(B.29)}$$

where $F = F(q_{11})$, $g = g(q_2)$, $I_2 = \int_0^{q_{11}+q_2} f(x)g(q_{11}+q_2-x)dx$ and $I_1 = \int_0^{q_{11}} f(x)g(q_{11}+q_2-x)dx$.

Note that the slope of q_2^F is always less than -1 using (B.29). This completes the proof that q_2^F is steeper than q_{11}^F in (q_2, Q_1) plane.

We continue with q_{12}^F and q_2^F. Slopes of the functions in (q_2, Q_1) plane, $\frac{\mathrm{d}q_{12}^F}{\mathrm{d}q_2}$, can be obtained from the first order conditions in (B.2) and (B.21) by means of implicit derivation. Recall the equation (B.3) where $f = f(q_{12})$, $G = G(q_2)$, $I_1 = \int_0^{q_{12}} f(x)g(q_{12}+q_2-x)dx$ and $I_2 = \int_0^{q_{12}+q_2} f(x)g(q_{12}+q_2-x)dx$, since $(r+l-t)\left(f(1-G)\right) + fG(t-s) \geq 0$, we prove that the slope of q_{12}^F is always greater than -1.

Moreover, using (B.21), we obtain the slope of q_2^F on the (q_2, Q_1) plane as

$$\frac{\mathrm{d}q_{12}^F}{\mathrm{d}q_2} = -\frac{(r+l-t)(I_1+g(1-F)) + (t-s)(I_2-I_1+gF)}{I_1(r+l-t)+(t-s)(I_2-I_1)} \tag{B.30}$$

where $F = F(q_{12})$, $g = g(q_2)$, $I_2 = \int_0^{q_{12}+q_2} f(x)g(q_{12}+q_2-x)dx$ and $I_1 = \int_0^{q_{12}} f(x)g(q_{12}+q_2-x)dx$. Since $(r+l-t)\left(g(1-F)\right) + gF(t-s) \geq 0$, we ensure $\frac{\mathrm{d}q_{12}}{\mathrm{d}q_2} < -1$.

As a result, we prove that q_2^F is steeper than q_{12}^F in (q_2, Q_1) plane.

Finally, we compare q_{12}^T and q_2^F. With $q_{11} = F^{-1}(\rho_2)$, we obtain the slopes from the first order conditions in (B.2) and (B.21).

We use (B.5), where $I_1 = \int_0^{F^{-1}(\rho_2)+q_{12}} f(x)g(F^{-1}(\rho_2)+q_{12}+q_2-x)dx$, $f = f(F^{-1}(\rho_2)+q_{12})$ $G = G(q_2)$ and $I_2 = \int_0^{F^{-1}(\rho_2)+q_{12}+q_2} f(x)g(F^{-1}(\rho_2)+q_{12}+q_2-x)dx$ to investigate the slope of q_{12}^T. As $(r+l-t)\left(f(1-G)\right)+fG(t-s) \geq 0$, its slope is always greater than -1.

We obtain $\frac{\mathrm{d}q_{12}^T}{\mathrm{d}q_2}$ from (B.21) as

$$\frac{\mathrm{d}q_{12}^T}{\mathrm{d}q_2} = -\frac{(r+l-t)(I_1+g(1-F)) + (t-s)(I_2-I_1+gF)}{I_1(r+l-t)+(t-s)(I_2-I_1)} < -1 \tag{B.31}$$

where $I_2 = \int_0^{F^{-1}(\rho_2)+q_{12}+q_2} f(x)g(F^{-1}(\rho_2)+q_{12}+q_2-x)dx$, $F = F(F^{-1}(\rho_2)+q_{12})$, $g = g(q_2)$ and $I_1 = \int_0^{F^{-1}(\rho_2)+q_{12}} f(x)g(F^{-1}(\rho_2)+q_{12}+q_2-x)dx$.

Comparing the slopes, we prove that q_2^F is steeper than q_{11}^F on the (q_2, Q_1) plane. ■

Proof of Theorem 4.3.2. Given the existence of the equilibrium in Theorem 4.3.1, uniqueness of the equilibrium requires that the slope of the best response function of one agent is always strictly smaller than that of the other.

If $Q_1^*(q_2)$ is constant, $\frac{\mathrm{d}Q_1^*(q_2)}{\mathrm{d}q_2} = 0$. If $q_2^*(Q_1)$ is constant, $\frac{\mathrm{d}q_2^*(Q_1)}{\mathrm{d}q_2} = -\infty$. Together with Lemma 4.3.2, we prove that $\frac{\mathrm{d}Q_1^*(q_2)}{\mathrm{d}q_2} > -1 > \frac{\mathrm{d}q_2^*(Q_1)}{\mathrm{d}q_2}$.

As a result, the slope of $q_2^*(Q_1)$ is always strictly smaller than that of $Q_1^*(q_2)$, completing the proof. ■

Proof of Corollary 4.3.2.1. We will elaborate on five possible parameter alignments.

1. $\rho_2 < 0$ and $t > c$: For $t > c > s$, $\underline{Q}_1 = \underline{q}_{12} = F^{-1}(\frac{t-c}{t-s}) > 0$ and $\underline{q}_2 = G^{-1}(\frac{t-c}{t-s}) > 0$, which ensure $Q_1^* = q_{12}^* > 0$ and $q_2^* > 0$ at equilibrium.

2. $\rho_2 < 0$ and $t < c$: It is shown that $\bar{Q}_1$ and $\bar{q}_2$ are bounded by $V^{-1}(\rho_1)$. $Q_1^* = 0$ requires $q_2^* > V^{-1}(\rho_1^n)$ from Proposition 4.1.1, whereas $q_2^* = 0$ requires $Q_1^* > V^{-1}(\rho_1^n)$ from Proposition 4.2.1. Recall that $t > s$, yielding $V^{-1}(\rho_1^n) > V^{-1}(\rho_1)$. Hence, both agents have positive order quantities at the unique equilibrium point.

3. $\rho_2 > 0$ and $t > c$: Having $\underline{Q}_1 = q_{11}^F(\infty) = F^{-1}(\frac{t-c+kw-h(1-k)}{kw+hk}) > 0$ and $\underline{q}_2 = G^{-1}(\frac{t-c}{t-s}) > 0$, the agents are ensured to place positive order quantities.

4. $\rho_2 > 0, t < c$ and $kw + t > h(1-k) + c$: We found that $\bar{Q}_1 = V^{-1}(\rho_1)$. However, the value of Q_1 that makes $q_2^* = 0$ is $V^{-1}(\rho_1^n) > \bar{Q}_1 = V^{-1}(\rho_1)$. Hence, $Q_1^* > 0$. In addition, the lower bound on agent 1's order quantity is shown to be positive, completing the proof.

5. $\rho_2 > 0, t < c$ and $kw + t < h(1-k) + c$: From Proposition 4.1.1, the value of q_2 that makes $Q_1^* = 0$ is $V^{-1}(\rho_3^n)$. However, $V^{-1}(\rho_3^n) > V^{-1}(\rho_1) \geq \bar{q}_2$ since $kw - h(1-k) > 0$ and $t > s$. Thus, $Q_1^* > 0$ at equilibrium.

   Moreover, $q_2^* = 0$ requires $Q_1^* > V^{-1}(\rho_1^n)$ from Proposition 4.2.1. However, $V^{-1}(\rho_1^n) > \bar{Q}_1 = V^{-1}(\rho_1)$. Hence, $q_2^* > 0$ at equilibrium.

■