LIMIT MONOMIAL GROUPS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

MARCH 2021

Approval of the thesis:

LIMIT MONOMIAL GROUPS

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ABSTRACT

LIMIT MONOMIAL GROUPS

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March 2021, 79 pages

In this thesis, monomial groups, which are obtained by using strictly diagonal embeddings of complete monomial groups over a group of finite degree, are studied. The direct limit of complete monomial groups of finite degree with strictly diagonal embeddings is called **limit monomial group** .

Normal subgroup structure of limit monomial groups over abelian groups is studied. We classified all subgroups of rational numbers, containing integers, by using base subgroup of limit monomial groups.

We also studied the splitting problem in limit monomial groups. Using the facts in the case of complete monomial groups of finite degree, we prove that limit monomial group splits over its base group and there are uncountably many complements of the base group in limit monomial group, depending on the group and Steinitz number. Moreover, to classify all complements of the base group up to conjugacy, we prove that the group, over which limit monomial group set, contains diagonal direct limit of finite symmetric groups.

Keywords: monomial group, direct limit, strictly diagonal embedding, splitting...

ÖZ

LİMİT MONOMİAL GRUPLAR

Bostan, Sezen
Doktora, Matematik Bölümü
Tez Yöneticisi: Prof. Dr. Mahmut Kuzucuoğlu

Mart 2021 , 79 sayfa

Bu tezde, sonlu dereceli tam monomial gruplardan kati köşegen tipindeki gömmelerle elde edilen monomial gruplar çalışılmıştır. Sonlu dereceli tam monomial gruplardan kati köşegen tipindeki gömmelerle elde edilen direkt limit grubuna **limit monomial grup** denir.

Değişmeli gruplar üzerindeki limit monomial grupların normal altgrup yapısı çalışılmıştır. Rasyonel sayıların tamsayıları içeren altgrupları, limit monomial grubun taban altgrubu kullanılarak sınıflandırılmıştır.

Limit monomial gruplarda ayrışma problemi çalışılmıştır. Sonlu dereceli tam monomial gruplar kullanılarak, limit monomial grupların taban altgrubu üzerinde ayrıştığı gösterilmiştir ve seçilen gruba ve Steinitz sayısına bağlı olarak, taban altgrubunun sayılamaz sonsuz sayıda tümleyeni olduğu gösterilmiştir. Ayrıca, bütün tümleyen altgrupları bulmak için, limit monomial grubun üzerine kurulduğu grubun sonlu simetrik gruplardan köşegen tipindeki gömmelerle oluşturulmuş simetrik grubu içermesi gerektiği gösterilmiştir.

Anahtar Kelimeler: monomial grup, direkt limit, kati köşegen gömme, ayrışma...

To my dad and my 'little' sister

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Mahmut Kuzucuoğlu, with deep respect for all his guidance. Besides his expertise, he is like a father to me. Sometimes compassionate or angry, but always patient. I have learned so much from him and hope that will enlighten my way through both in my academic and social life. I hope I can be at least as passionate as he is as I continue my way on mathematics.

I would also like to thank my Thesis Monitoring Committee, Prof. Ali Erdoğan and Assoc. Gökhan Benli for their ideas coming through when I felt stuck. Moreover, I send my gratitudes to Examining Committee.

It would not have been possible without my father, Levent Bostan, who never withheld his moral support. I thank my -so called- little sister, Dilek Bostan, who always supports me, who I love, without whom I could not see the light. She is more than me. I would like to thank my mother, Ayşe Bostan, who I sincerely hope to be proud of me, who loves me in her own way, maybe not as I wanted.

I especially thank my best friend Deniz Onur Yeşilyurt for his endless support and patience in times when I cry and in times when I thought it was the end. Perhaps, he is the one who taught me that everything is just beginning. He is, most probably, the best friend ever.

I am grateful to my friends Gizem Güneş, Seda Yıldız, Baran Kaya for making me believe that I am a genius except the fact that I am not. They are true friends no matter where they are. I also thank my neighbour, Şule Şen Keskin, for her joy and cheerful conversations on every break. My academic sister, Dr. Büşra Güven, deserves special thanks for always helping me day and night, patiently answering all my questions, big and small, and sharing her academic experience with me.

Lastly, I would like to thank Olgun Yılmaz and he will understand as he usually does for the last five years.

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ξ	A sequence of natural numbers
$char(\xi)$	Characteristic of the sequence ξ
$\Sigma_n(H)$	Complete monomial group over the group H of degree n
$\Sigma_\lambda(H)$	Limit monomial group over the group H with respect to the Steinitz number λ
$B(n, H)$	Base group of $\Sigma_n(H)$
$B(\lambda, H)$	Base group of $\Sigma_\lambda(H)$

CHAPTER 1

INTRODUCTION

In the theory of groups, embedding a group in a larger group is one of the most useful tools to understand the group structure and its properties. There are three main types of representing a group. These are permutation representation, monomial representation and linear representation. For a given group, homomorphisms into a symmetric group, into a complete monomial group and a general linear group give the representations, respectively. Permutation and linear representations of groups are studied in detail by many mathematicians. Monomial representation are in the middle of these two types of representing a group.

Monomial groups were studied in 1930s by Turkin [19] in 1935 and by Ore [18] in 1942. The structure and properties of monomial groups were given by many authors, see [17], [19], [18]. In 60s, monomial groups are considered as the permutational wreath product, see [1],[15]. After that, it is natural to consider these groups as a natural generalization of symmetric groups.

Complete monomial groups of finite degree were studied in detail by Ore in [18]. All definitions, basic properties such as conjugation, centralizer of elements, decomposition of these groups, subgroup structure and monomial representations into complete monomial groups of finite degree were given in [18]. Moreover, Crouch adapt some results of Ore to the case of monomial groups of infinite degree in [4].

Since taking direct limit of groups is one of the tools to obtain new groups, we aim to construct monomial groups obtained by direct limits and investigate the properties of these groups. We are mainly interested in direct limits of complete monomial groups of finite degree with strictly diagonal embeddings.

Kuzucuoğlu, Olynyk and Suschanskyy constructed limit monomial groups by using a direct system consisting of complete monomial groups of finite degree and strictly diagonal embeddings, in [12]. They gave all the definitons and basic properties of these groups. They found the structure of centralizers of elements in limit monomial groups. Moreover, they classified limit monomial groups using Steinitz numbers and splitting of these groups.

Let H be a group and $n \in \mathbb{N}$. *A monomial substitution* is a linear transformation

$$\rho = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ h_1 x_{i_1} & h_2 x_{i_2} & \dots & h_n x_{i_n} \end{pmatrix}$$

where each variable x_j is changed into some other variable x_{i_j} in a one-to-one correspondence with a multiple h_j from the group H and the product $h_j x_{i_j}$ of a group element by a variable is a formal product, which satisfies the associative property. Namely, for $h, k \in H$ and a variable $x \in \mathbb{N}$, $(hk)x = h(kx)$. The elements $h \in H$ are called the *multipliers* of ρ.

If $\kappa = \begin{pmatrix} x_1 & \dots & x_n \\ k_1 x_{j_1} & \dots & k_n x_{j_n} \end{pmatrix}$ is another monomial substitution, then the product of ρ and κ is defined by

$$\rho\kappa = \begin{pmatrix} x_1 & \dots & x_n \\ h_1 k_{i_1} x_{j_{i_1}} & \dots & h_n k_{i_n} x_{j_{i_n}} \end{pmatrix}.$$

The inverse of κ is

$$\kappa^{-1} = \begin{pmatrix} x_{j_1} & \dots & x_{j_n} \\ k_1^{-1} x_1 & \dots & k_n^{-1} x_n \end{pmatrix}.$$

The identity monomial substitution $\begin{pmatrix} x_1 & \dots & x_n \\ 1_H x_1 & \dots & 1_H x_n \end{pmatrix}$ is denoted by E.

The set of all monomial substitution on n variables with the above multiplication forms a group $\Sigma_n(H)$, called *the complete monomial group of degree n* or *the symmetry of degree n of H*.

Definition 1.0.1. *A (monomial) permutation is a monomial substitution whose factors are all identity. Moreover, the set of all (monomial) permutations forms a subgroup, isomorphic to finite symmetric group S_n on n letters.*

Definition 1.0.2. *A monomial substitution of the form*

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ h_1x_1 & h_2x_2 & \dots & h_nx_n \end{pmatrix} = [h_1, h_2, \dots, h_n]$$

is called a (monomial) multiplication. The set of all multiplications forms a normal subgroup of $\Sigma_n(H)$, denoted by $B(n, H)$ and called the base group of $\Sigma_n(H)$.

Ore gives the construction and basic properties of a complete monomial group of finite degree. In [18], the centralizer of an element is found, splitting problem over the base group is solved up o conjugacy, normal subgroups are searched in detail and monomial representations are studied.

Limit monomial groups obtained by strictly diagonal embeddings are investigated by Kuzucuoğlu, Olynyk and Sushchanskyy in [12]. They found some basic properties of the limit monomial group, centralizer structure of elements. Moreover, they classified limit monomial groups using Steinitz numbers and splitting of the group.

Definition 1.0.3. *Let $\mathbb{P}$ be the set of all prime numbers and $r_p \in \mathbb{N} \cup \{0, \infty\}$. A $Steinitz\ number$ (or a supernatural number) is a formal product of the form $\prod_{p \in \mathbb{P}} p^{r_p}$.*

Definition 1.0.4. *Let $\xi = (p_1, p_2, \dots)$ be a sequence of, not necessarily distinct, primes. The **characteristic** of ξ is a Steinitz number, $char(\xi) = \prod_{i \in \mathbb{N}} p_i^{r_i}$, where r_i is the number of p_i appearing in ξ.*

For a given sequence $\xi = (p_1, p_2, \dots, p_i, \dots)$, where p_i's are not necessarily distinct primes, for an element $\rho = \begin{pmatrix} x_1 & x_2 & \dots & x_{n_i} \\ h_1x_{i_1} & h_2x_{i_2} & \dots & h_{n_i}x_{i_n} \end{pmatrix} \in \Sigma_{n_i}(H)$, we consider the embeddings

$$d^{p_{i+1}} : \Sigma_{n_i}(H) \to \Sigma_{n_i p_{i+1}}(H)$$

defined by

$$d^{p_{i+1}}(\rho) =$$

$$\left(\begin{array}{ccc|ccc|c|ccc} x_1 & \dots & x_{n_i} & x_{n_i+1} & \dots & x_{n_i+n_i} & \dots & x_{(p_{i+1}-1)n_i+1} & \dots & x_{(p_{i+1}-1)n_i+n_i} \\ h_1x_{i_1} & \dots & h_{n_i}x_{i_{n_i}} & h_1x_{n_i+i_1} & \dots & h_{n_i}x_{n_i+i_{n_i}} & \dots & h_1x_{(p_{i+1}-1)n_i+i_1} & \dots & h_{n_i}x_{(p_{i+1}-1)n_i+i_{n_i}} \end{array}\right).$$

With these embeddings, we have a direct system and the direct limit group is the limit monomial group $\Sigma_\lambda(H)$ of degree $char\xi = \lambda$ over the group H.

As notation, we use parenthesis to denote the images of subgroups under embeddings; such as, the image of the permutation subgroup $S(n) = d(S_n)$, the image of base group $B((n), H) = d(B(n, H))$.

In Chapter 2, we studied the properties of the complete monomial group of finite degree. We gave the definition of Stenitz numbers and then using Steinitz numbers, we construct limit monomial groups via strictly diagonal embeddings. We considered the complete monomial group of finite degree n as a subgroup of general linear group $GL(n, \mathbb{Z}H)$ over the integral ring $\mathbb{Z}H$ and studied the monomial representations in detail. Lastly, using the correspondence with the matrices, we showed that the embedding of the complete monomial groups of finite degree is strictly diagonal embedding.

In Chapter 3, we found uncountably many non-conjugate complements of the base group of limit monomial group depending on the group chosen and Steinitz number. We gave a description of all complements of the base group using diagonal embeddings.

Definition 1.0.5. *Let G be a group and K be a normal subgroup of G. G is said to split over K, if there exists a subgroup H of G such that*

$$G = KH \quad \textit{and} \quad K \cap H = 1.$$

H is called a complement of K in G.

G splits regularly over K if all complements of K are conjugate.

Let T_λ be a complement of the base group $B(\lambda, H)$ in a limit monomial group $\Sigma_\lambda(H)$. We consider the subgroup

$$T_i = \{v_\pi \pi \mid v_\pi \in B(\lambda, H) \text{ and } \pi \in S(n_i)\} \leq \Sigma_\lambda(H)$$

where $S(n_i)$ denotes the image of the finite symmetric group S_{n_i} under strictly diagonal embedding.

Lemma 1.0.1. *Let T_λ be a complement of $B(\lambda, H)$ in $\Sigma_\lambda(H)$ and for each i, $T_i = \{v_\pi \pi \mid \pi \in S(n_i)\}$. Then a conjugate of T_i is a complement of $B((n_i), H)$ in $\Sigma_{(n_i)}(H)$.*

Theorem 1.0.6. *Let T_λ be a complement of $B(\lambda, H)$ in $\Sigma_\lambda(H)$. Then T_λ is a direct limit of complements of $B(n_i, H)$ in $\Sigma_{n_i}(H)$.*

Theorem 1.0.7. *Limit monomial group $\Sigma_\lambda(H)$ splits regularly over its base group $B(\lambda, H)$ if and only if H has no subgroup which is the homomorphic image of the diagonal symmetric group S_χ where $\chi =< (1, p_1 - 1), (p_2, p_2 - 1), \ldots >$ and $\lambda = char(p_1, p_2, \ldots) = char(\chi)$.*

In Chapter 4, we studied limit monomial group $\Sigma_\lambda(H)$ when H is abelian. We investigated a special subgroup $B_0(\lambda, H)$ of the base group $B(\lambda, H)$ where

$$B_0(\lambda, H) = \{[h_1, h_2, \ldots, h_{n_i}, h_1, h_2, \ldots] \in B(\lambda, H) \mid h_1 h_2 \ldots h_{n_i} = 1 \text{ for some } n_i | \lambda\}.$$

In [18], Ore stated that the quotient group $B(n, H)/B_0(n, H)$ is isomorphic to H where H is an abelian group. We found that depending on the choice of Steinitz number and the abelian group H, the quotient is not always isomorphic to H. For H being infinite cyclic group C, we proved the following:

Theorem 1.0.8. *Let $\lambda = \prod_{p \in \mathbb{P}} p^{r_p}$ be a Steinitz number and let $\xi = (p_1, p_2, \ldots, p_i, \ldots)$ be a sequence of primes with $char\xi = \lambda$ and $n_1 = p_1$, $n_i = p_1 p_2 \ldots p_i$. Then*

$$B(\lambda, C) \diagup B_0(\lambda, C) \cong \mathbb{Q}_\lambda$$

where $\mathbb{Q}_\lambda =< \dfrac{1}{p^{t_p}} \mid 0 \le t_p \le r_p, \;\; p|\lambda >$.

Moreover, we found the structure of the quotient for all finitely generated abelian groups and divisible abelian groups.

Lemma 1.0.2. *[2, Lemma 3] Let C_p be the cyclic group of order p where p is a prime. If $p^\infty | \lambda$, then $B_0(\lambda, C_p) = B(\lambda, C_p)$. Otherwise,*

$$B(\lambda, C_p) \diagup B_0(\lambda, C_p) \cong C_p.$$

Lemma 1.0.3. *Let C_{p^k} be a cyclic group of order p^k where p is a prime, $k \in \mathbb{N}$. If $p^\infty | \lambda$, then $B_0(\lambda, C_{p^k}) = B(\lambda, C_{p^k})$. If not,*

$$B(\lambda, C_{p^k}) \diagup B_0(\lambda, C_{p^k}) \cong C_{p^k}.$$

Theorem 1.0.9. *Let λ be a Steinitz number and $\Sigma_\lambda(H)$ be the limit monomial group where H is a finitely generated abelian group and $H \cong C^r \times \prod_{i=1}^{k} C_{{p_i}^{t_i}}$. Then*

$$B(\lambda, H) \diagup B_0(\lambda, H) \cong \mathbb{Q}_\lambda^r \times \prod_{i=1}^{k} B(\lambda, C_{{p_i}^{t_i}}) \diagup B_0(\lambda, C_{{p_i}^{t_i}})$$

where ${}^{B(\lambda, C_{{p_i}^{t_i}})}\!\diagup_{B_0(\lambda, C_{{p_i}^{t_i}})} \cong C_{{p_i}^{t_i}}$ *if* ${p_i}^{\infty} \nmid \lambda$ *and* ${}^{B(\lambda, C_{{p_i}^{t_i}})}\!\diagup_{B_0(\lambda, C_{{p_i}^{t_i}})}$ *is trivial if* ${p_i}^{\infty}|\lambda$.

Lemma 1.0.4. *Let C_{p^∞} be the Prüfer-p-group where p is a prime. If $p^\infty|\lambda$, then $B_0(\lambda, C_{p^\infty}) = B(\lambda, C_{p^\infty})$. Otherwise,*

$$ {}^{B(\lambda, C_{p^\infty})}\!\diagup_{B_0(\lambda, C_{p^\infty})} \cong C_{p^\infty}. $$

Lemma 1.0.5. *Let H be a torsion free divisible abelian group. Then*

$$ {}^{B(\lambda, H)}\!\diagup_{B_0(\lambda, H)} \cong H. $$

Since a divisible abelian group is a direct sum of groups each of which is isomorphic to a Prüfer p-group or $\mathbb{Q}$, the structure of the quotient is determined for all divisible abelian groups.

CHAPTER 2

PRELIMINARIES

In this chapter, some basic definitions and properties, that are used in the chapters, are given.

2.1 Complete monomial group over a group of finite degree

Definition 2.1.1. *Let H be an arbitrary group and $\Omega = \{x_1, x_2, \ldots, x_n\}$.* ***A monomial substitution over*** *H is a linear transformation*

$$\rho = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ r_1 x_{i_1} & r_2 x_{i_2} & \ldots & r_n x_{i_n} \end{pmatrix}$$

where each variable x_k is changed into some other variable x_{i_k} formally multiplied by an element of H. The elements r_i are called the ***factors*** *or* ***multipliers*** *in ρ. The multiplication rx has the associative property $r(sx) = (rs)x$ where $r, s \in H$.*

If $\kappa = \begin{pmatrix} x_1 & \ldots & x_n \\ k_1 x_{j_1} & \ldots & k_n x_{j_n} \end{pmatrix}$ is another monomial substitution, then the product of ρ and κ is defined by

$$\rho\kappa = \begin{pmatrix} x_1 & \ldots & x_n \\ r_1 k_{i_1} x_{j_{i_1}} & \ldots & r_n k_{i_n} x_{j_{i_n}} \end{pmatrix}.$$

The inverse of κ is

$$\kappa^{-1} = \begin{pmatrix} x_{j_1} & \ldots & x_{j_n} \\ k_1^{-1} x_1 & \ldots & k_n^{-1} x_n \end{pmatrix}.$$

The identity monomial substitution $\begin{pmatrix} x_1 & \ldots & x_n \\ 1_H x_1 & \ldots & 1_H x_n \end{pmatrix}$ is denoted by E.

The set of all monomial substitution on n variables with the above multiplication forms a group $\Sigma_n(H)$, called *the complete monomial group of degree n* or *the symmetry of degree n of H*. Ore gives the construction and some basic properties of a complete monomial group of finite degree in [18].

Definition 2.1.2. *A permutation in $\Sigma_n(H)$ is a monomial substitution of the form*

$$\pi = \begin{pmatrix} x_1 & \dots & x_n \\ 1_H x_{i_1} & \dots & 1_H x_{i_n} \end{pmatrix} = \begin{pmatrix} 1 & \dots & n \\ i_1 & \dots & i_n \end{pmatrix}$$

The permutations form a subgroup S_n of $\Sigma_n(H)$, isomorphic to the ordinary symmetric group on n letters.

Moreover, if the group H is the trivial group $\{e\}$, then the complete monomial group $\Sigma_n(\{e\})$ is a group isomorphic to the finite symmetric group on n letters. Hence, the complete monomial groups of finite degree can be easily considered as a natural generalization of finite symmetric groups.

Definition 2.1.3. *A monomial substitution of the form*

$$\mu = \begin{pmatrix} x_1 & \dots & x_n \\ r_1 x_1 & \dots & r_n x_n \end{pmatrix} = [r_1, \dots, r_n]$$

is called a multiplication.

The multiplications form a normal subgroup $B(n, H)$ of $\Sigma_n(H)$, called *the base (or basis) group*. This subgroup is isomorphic to the direct product $H \times \ldots \times H$ of n-copies of the group H.

Every monomial substitution can be written uniquely as a product of a multiplication and a permutation :

$$\rho = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ r_1 x_{i_1} & r_2 x_{i_2} & \dots & r_n x_{i_n} \end{pmatrix} = [r_1, r_2, \dots, r_n] \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_{i_1} & x_{i_2} & \dots & x_{i_n} \end{pmatrix}.$$

The action of a monomial permutation on the base group is to permute the factors in the multiplication since for $\pi = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_{i_1} & x_{i_2} & \dots & x_{i_n} \end{pmatrix}$ and $\mu = [k_1, \dots, k_n]$,

$$\pi\mu\pi^{-1} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_{i_1} & x_{i_2} & \dots & x_{i_n} \end{pmatrix} [k_1, \dots, k_n] \begin{pmatrix} x_{i_1} & x_{i_2} & \dots & x_{i_n} \\ x_1 & x_2 & \dots & x_n \end{pmatrix}$$
$$= [k_{i_1}, k_{i_2}, \dots, k_{i_n}] = [k_{1^\pi}, k_{2^\pi}, \dots, k_{n^\pi}].$$

Hence, the complete monomial group over H of degree n is a semi-direct product of groups and moreover, it is the permutational wreath product of H by S_n.

$$\Sigma_n(H) \cong B(n, H) \rtimes S_n \cong H \wr S_n.$$

Definition 2.1.4. *A multiplication of the form $\nu = [a, \dots, a] = [a]$ where all factors are equal, is called a scalar.*

The set of all scalars is a subgroup of $\Sigma_n(H)$, which is isomorphic to H. Moreover, the scalars are the only elements that commute with all monomial permutations. Using these, we can conclude that the center of $\Sigma_n(H)$ is the set of all scalars $[z]$ where $z \in Z(H)$ and so $Z(\Sigma_n(H))$ is isomorphic to $Z(H)$.

Definition 2.1.5. *A monomial substitution of the form*

$$\begin{pmatrix} x_{i_1} & x_{i_2} & \dots & x_{i_m} \\ h_1 x_{i_2} & h_2 x_{i_3} & \dots & h_m x_{i_1} \end{pmatrix}$$

is called a ***monomial cycle of length m****.*

As in symmetric groups, monomial cycles can also be written in the cycle form

$$(h_1 x_{i_2}, h_2 x_{i_3}, \dots, h_m x_{i_1})$$

and every monomial substitution can be written uniquely as a product of disjoint commuting monomial cycles, see [18, Chapter 2].

For a monomial cycle $\gamma = \begin{pmatrix} x_{i_1} & x_{i_2} & \dots & x_{i_m} \\ h_1 x_{i_2} & h_2 x_{i_3} & \dots & h_m x_{i_1} \end{pmatrix}$,

$$\gamma^m = \begin{pmatrix} x_{i_1} & x_{i_2} & \dots & x_{i_m} \\ \Delta_1 x_{i_1} & \Delta_2 x_{i_2} & \dots & \Delta_m x_{i_m} \end{pmatrix}$$

where $\Delta_1 = h_1h_2 \dots h_m$, $\Delta_2 = h_2 \dots h_m h_1$, ..., $\Delta_m = h_m h_1 \dots h_{m-1}$, which are called *determinants of* γ. As

$$\Delta_2 = h_1^{-1}\Delta_1 h_1,\ \Delta_3 = h_2^{-1}\Delta_2 h_2, \dots, \Delta_m = h_{m-1}^{-1}\Delta_{m-1}h_{m-1},\ \Delta_1 = h_m^{-1}\Delta_m h_m,$$

they are all conjugate to each other and so each determinant class of a given cycle is contained in a conjugacy class of H and there is a unique associated *determinant class* to each cycle.

If $n \times n$ matrices over a field are considered, then the determinant of a matrix is unique. However, if, instead of a field, a non-abelian group is taken, not the determinant but the determinant class of a matrix is unique up to conjugacy as in the example of a 2×2 diagonal matrix over a non-abelian group H with entries a and b, the determinants are ab and ba which are conjugate to each other.

Theorem 2.1.6. *[18, Chapter 1, Theorem 6] Two monomial cycles are conjugate if and only if they have the same length and the same determinant class.*

Proof. Let $\gamma = \begin{pmatrix} x_1 & x_2 & \dots & x_m \\ h_1x_2 & h_2x_3 & \dots & h_mx_1 \end{pmatrix}$ be a monomial cycle and let $\rho = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ r_1x_{i_1} & r_2x_{i_2} & \dots & r_nx_{i_n} \end{pmatrix}$ be a monomial substitution. Then

$$\rho^{-1}\gamma\rho = \begin{pmatrix} x_{i_1} & x_{i_2} & \dots & x_{i_m} \\ r_1^{-1}h_1r_2x_{i_2} & r_2^{-1}h_2r_3x_{i_3} & \dots & r_m^{-1}h_mr_1x_{i_1} \end{pmatrix}.$$

The length of $\rho^{-1}\gamma\rho$ is the same as the length of γ. Moreover, the first determinant $r_1^{-1}h_1h_2 \dots h_m r_1^{-1}$ of $\rho^{-1}\gamma\rho$ is a conjugate of the first determinant $h_1h_2 \dots h_m$ of γ. As the determinants of a monomial cycle are conjugate to each other, the determinant class of $\rho^{-1}\gamma\rho$ and γ are the same. As the permutation part of ρ is arbitrary, any monomial cycle of length m with the same determinant class of γ can be obtained by conjugation.

Now, let $\gamma' = \begin{pmatrix} x_{i_1} & x_{i_2} & \dots & x_{i_m} \\ h'_1x_{i_2} & h'_2x_{i_3} & \dots & h'_mx_{i_1} \end{pmatrix}$ be another monomial cycle, which has the same length and the same determinant class with γ. Then the determinants of γ' are

$$\Delta'_1 = h'_1h'_2 \dots h'_m, \Delta'_2 = h'_2 \dots h'_m h_1, \dots, \Delta'_m = h'_m h'_1 \dots h_{m-1}.$$

As γ and γ' have the same determinant class, there exists $r_1, r_2, \ldots, r_m$ in H such that $r_1\Delta_1 r_1^{-1} = \Delta_1'$, $r_2\Delta_2 r_2^{-1} = \Delta_2^{-1}$, …, $r_m\Delta_m r_m^{-1} = \Delta_m'$ where $\Delta_1 = h_1h_2\ldots h_m$, $\Delta_2 = h_2\ldots h_mh_1$, …, $\Delta_n = h_mh_1\ldots h_{m-1}$ are determinants of γ.

Then for $\rho = \begin{pmatrix} x_1 & x_2 & \ldots & x_m \\ r_1x_{i_1} & r_2x_{i_2} & \ldots & r_mx_{i_m} \end{pmatrix}$, $\rho^{-1}\gamma\rho = \gamma'$.

□

Moreover, as every monomial substitution can be written uniquely as a product of disjoint commuting cycles, two monomial substitutions are conjugate if and only if their cycle decomposition has the same number of monomial cycles of the same length and the same determinant class.

Now, we introduce *a normal form* of a cycle in monomial groups for simplification in calculations in monomial groups.

Theorem 2.1.7. *[18, Chapter 1, Theorem 7] Any monomial cycle of length m is conjugate to a monomial cycle in a* ***normal form***

$$\begin{pmatrix} x_{i_1} & x_{i_2} & \ldots & x_{i_m} \\ x_{i_2} & x_{i_3} & \ldots & ax_{i_1} \end{pmatrix}$$

where a is any element from the determinant class of the monomial cycle. Any monomial substitution is conjugate to a product of disjoint commuting monomial cycles, each of which is in a normal form.

From theorem above, it is seen that for every cycle, we can write a normal form and if the length and the determinant class of the cycle are given, a normal form of the cycle is determined easily. This simplification is useful for calculations.

Definition 2.1.8. *A group G is said to* ***split*** *over its normal subgroup N if there exists a subgroup M of G with*

$$G = MN \quad , \quad M \cap N = 1.$$

The subgroup M is called ***a complement*** *of N in G. We say G* ***splits regularly*** *over N if all complements are conjugate.*

If M is a complement of the normal subgroup N in G, then

$$M \cong M/M \cap N \cong MN/N = G/N$$

which implies that all complements of N are isomorphic to G/N and so they are isomorphic to each other.

Let g be an element in G. Then

$$G = G^g = (MN)^g = M^g N^g = M^g N.$$

The last equality holds as N is a normal subgroup of G. This shows that every conjugate of a complement is also a complement of N as $M^g \cap N = 1$. However, the normal subgroup N in G may have complements which are not conjugate to each other. As an example, consider the symmetric group S_6 on 6 letters. We know that

$$S_6 \cong A_6 \rtimes \langle (1\,2) \rangle \cong A_6 \rtimes \langle (1\,2)(3\,4)(5\,6) \rangle$$

where A_6 is the alternating group on 6 letters, $\langle (1\,2) \rangle$ and $\langle (1\,2)(3\,4)(5\,6) \rangle$ are the subgroups generated by $(1\,2)$ and $(1\,2)(3\,4)(5\,6)$, respectively. As the cycle types of $(1\,2)$ and $(1\,2)(3\,4)(5\,6)$ are different, the respective subgroups are not conjugate but they, both, are complements the normal subgroup A_6.

Hence, it is reasonable to classify all complements of a group up to conjugacy.

For a complete monomial group of finite degree n over H, $\Sigma_n(H)$, Ore classified all complements of the base group $B(n, H)$ in $\Sigma_n(H)$ up to conjugacy with the following theorem in [18]:

Theorem 2.1.9. *[18, Theorem 10] The symmetry $\Sigma_n(H)$ splits over its base group,*

$$\Sigma_n(H) = B(n, H) \rtimes T\,,\ T \cap B(n, H) = E.$$

Any group T in this decomposition is the conjugate of some group T_0 obtained by the following construction.

Let $\varphi : S_{n-1}^{(1)} \to H$ be a homomorphism taking π_1 to a_{π_1} where $S_{n-1}^{(1)}$ is the subgroup of S_n, which contains all permutations fixing 1. In particular, we write

$$(i, j) \mapsto a_{i,j}\,,\ (i, i) \mapsto 1 = a_{i,i}.$$

Then the elements of T_0 are obtained from the permutations of the symmetric group S_n by the isomorphism $\theta : S_n \to T$ defined by

$$\pi_1 \mapsto \pi_1[a_{\pi_1}] \textit{ for } \pi_1 \in S_{n-1}^{(1)}$$

$$(1, i) \mapsto [1, a_{2,i}, \ldots, a_{n,i}](1, i).$$

Proof. Let T be a complement of $B(n, H)$ in $\Sigma_n(H)$. Then T is isomorphic to S_n as S_n is also a complement of $B(n, H)$. Let

$$\theta : S_n \to T$$

$$\pi \mapsto v_\pi \alpha_\pi$$

be an isomorphism since every element of $\Sigma_n(H)$ is written uniquely as a product of a multiplication and a permutation, we can write the elements of T in this form. Take $\pi, \sigma \in S_n$. As θ is an isomorphism,

$$\theta(\pi\sigma) = v_{\pi\sigma}\alpha_{\pi\sigma} = v_\pi \alpha_\pi v_\sigma \alpha_\sigma = \theta(\pi)\theta(\sigma).$$

As every monomial substitution is uniquely written as a product of a multiplication and a permutation, we have the equalities

$$v_{\pi\sigma} = v_\pi(\alpha_\pi v_\sigma \alpha_\pi^{-1})$$

and

$$\alpha_{\pi\sigma} = \alpha_\pi \alpha_\sigma.$$

The second equality gives an automorphism of S_n. As every automorphism of S_n, for $n \neq 2, 6$, $\alpha : S_n \to S_n$ taking π to α_π is an inner automoprhism induced by an element γ of S_n.

Hence, in a conjugate T' of the complement T, we have the isomorphism

$$\tilde{\theta} : S_n \to T'$$

$$\pi \mapsto w_\pi \pi$$

and so we can consider T as T' since we investigate all complements up to conjugacy.

Now the isomorphism between S_n and T is as follows:

$$\theta : S_n \to T$$

$$\pi \mapsto v_\pi \pi.$$

As S_n is generated by all transpositions of the form $(1\,i)$, for $i = 2, \ldots, n$, T is generated by $\lambda_i = v_{(1\,i)}(1\,i) = [a_{1,i}, \ldots, a_{n,i}](1\,i)$.

Consider the conjugate of T by the multiplication

$$\kappa = [1, a_{1,2}, \ldots, a_{1,n}].$$

Then we have

$$\kappa\lambda_i\kappa^{-1} = [1, a_{1,2}, \ldots, a_{1,n}][a_{1,i}, \ldots, a_{n,i}](1\,i)[1, a_{1,2}^{-1}, \ldots, a_{1,n}^{-1}]$$

$$= [1, a_{1,2}, \ldots, a_{1,n}][a_{1,i}, \ldots, a_{n,i}][a_{1,i}^{-1}, a_{1,2}^{-1}, \ldots, 1, \ldots, a_{1,n}^{-1}](1\,i)$$

$$= [1, \ldots, a_{1,i}a_{i,i}, \ldots, a_{1,j}a_{j,i}a_{1,j}^{-1}, \ldots](1\,i) \text{ for } j \neq 1, i.$$

This means that by a conjugation, the first factors of the multiplication parts of the generators of T can be changed into the identity of H and so we may assume that the first factors are all identity in T and

$$\lambda_i = [1, a_{2,i}, \ldots, a_{i,i}, \ldots, a_{n,i}](1\,i).$$

Then

$$\lambda_i^2 = [a_{i,i}, \ldots, a_{i,i}, \ldots, a_{j,i}^2, \ldots] \in B(n, H) \cap T$$

and as T is a complement of $B(n, H)$, $T \cap B(n, H) = \{E\}$. Hence, $\lambda_i^2 = E$ and so

$$a_{i,i} = a_{j,i}^2 = 1.$$

From this, we can directly say that the complete monomial group of degree 2 over any group H splits regularly.

For $n \geq 3$, we consider the third power of $\lambda_i\lambda_j$, which is also identity and gives the following equalities:

$$a_{i,j}a_{j,i} = 1 \text{ and } (a_{k_i}a_{k,j})^3 = 1 \text{ for } k \neq i, j.$$

As $a_{j,i}^2 = 1$ and $a_{i,j}a_{j,i} = 1$, we have $a_{i,j} = a_{j,i}$.

Lastly, for $n \geq 4$, we form the product $\lambda_j\lambda_i\lambda_j$ which corresponds to the transposition $(i\,j)$ under the isomorphism from S_n to T. Using the the fact in S_n that disjoit cycles commute, we have

$$\lambda_h.\lambda_j\lambda_i\lambda_j = \lambda_j\lambda_i\lambda_j.\lambda_h \quad \text{for different indices} \quad 1, i, j, h.$$

Comparing the factors, we get

$$a_{i,j} = a_{h,j}a_{h,i}a_{h,j} \quad \text{and} \quad a_{i,j}a_{k,h} = a_{k,h}a_{i,j}.$$

Applying all the equalities to the product $\lambda_j\lambda_i\lambda_j$, we get

$$\lambda_j\lambda_i\lambda_j = [a_{i,j}](i\,j).$$

Now, the map φ from $S_{n-1}^{(1)} = Stab_{S_n}(1)$ to H taking the permutations π_1 fixing 1 to a_{π_1} is a homomorphism as, for $\pi_1, \pi_2 \in S_{n-1}^{(1)}$,

$$\varphi(\pi_1\pi_2) = [a_{\pi_1\pi_2}]\pi_1\pi_2$$

$$\varphi(\pi_1)\varphi(\pi_2) = [a_{\pi_1}]\pi_1[a_{\pi_2}]\pi_2 = [a_{\pi_1}][a_{\pi_2}]\pi_1\pi_2 = [a_{\pi_1}a_{\pi_2}]\pi_1\pi_2$$

and the expression of an element in T is unique and the scalars commute with all permutations. Hence, $a_{i,j}$'s generate a subgroup of H, which is a homomorphic image of $S_{n-1}^{(1)} = Stab_{S_n}(1)$.

□

2.2 Steinitz Numbers

For the construction and the classification of limit monomial groups, definitions and some related concepts about Steinitz numbers are given.

Every natural number is written as a product of finitely many prime numbers. The set of natural numbers forms a partially ordered set with respect to division. Moreover, it has a least element, which is 1. The set of *Steinitz numbers* or *supernatural numbers* is a generalization of natural numbers $\mathbb{N}$.

Definition 2.2.1. *Let $\mathbb{P}$ be the set of all prime numbers and $r_p \in \mathbb{N} \cup \{0, \infty\}$. A $Steinitz\ number$ (or a supernatural number) is a formal product of the form $\prod_{p \in \mathbb{P}} p^{r_p}$.*

Recall that a complete lattice is a partially ordered set $(S, \leq)$ if every subset of S has a least upper bound and a greatest lower bound. Hence, this means that S has a greatest and a least element.

Let $u = \prod_{p \in \mathbb{P}} p^{r_p}$ and $v = \prod_{p \in \mathbb{P}} p^{k_p}$ be two Steinitz numbers. The multiplication of these numbers are as follows:

$$uv = \prod_{p \in \mathbb{P}} p^{r_p + k_p}$$

where $\infty + \infty = \infty + n = n + \infty = \infty$ for every $n \in \mathbb{N} \cup \{0\}$.

Denote the set of all Steinitz numbers by $\mathbb{SN}$. We say that the Steinitz number v divides the Steinitz number u, denoted by $v|u$, if there exists $w \in \mathbb{SN}$ such that $u = vw$. With this divisibility relation $|$, $\mathbb{SN}$ is a partially ordered set with the greatest element $I = \prod_{p \in \mathbb{P}} p^{\infty}$ and the least element 1.

Lemma 2.2.1. *[7, Lemma 2.6] The set of all Steinitz numbers is a partially ordered set with respect to division and forms a complete lattice.*

Proof. Define the meet and join of two Steinitz numbers as follows:

$$u \wedge v = \prod_{p \in \mathbb{P}} p^{\min\{r_p, k_p\}},$$

$$u \vee v = \prod_{p \in \mathbb{P}} p^{\max\{r_p, k_p\}}.$$

With these, the partially ordered set $\mathbb{SN}$ is a complete lattice with respect to division.

□

Definition 2.2.2. *Let $\xi = (p_1, p_2, \ldots)$ be a sequence of, not necessarily distinct, primes. The **characteristic** of ξ is a Steinitz number, $char(\xi) = \prod_{i \in \mathbb{N}} p_i^{r_i}$, where r_i is the number of p_i appearing in ξ.*

Definition 2.2.3. *A sequence of positive integers $\xi = (n_i)_{i \in \mathbb{N}}$ is called a divisible sequence if $n_i | n_{i+1}$ for each $i \in \mathbb{N}$.*

Any divisible sequence $\xi = (n_i)_{i\in\mathbb{N}}$ corresponds to a unique Steinitz number, the characteristic of ξ, $char(\xi) = \prod_{i\in\mathbb{N}} k_i$ where

$$k_1 = n_1 \ , k_{i+1} = \frac{n_{i+1}}{n_i} \text{ for each } i \in \mathbb{N}.$$

Then, we write each k_i as a product of prime numbers. Using all the primes in the decompositions, we can obtain a sequence of primes ξ' and $char\xi = char\xi'$. Hence, instead of a divisible sequence, we can use a sequence of prime numbers $\xi = (p_1, p_2, \ldots)$ where p_i's are not necessarily distinct.

Moreover, given a sequence of, not necessarily distinct, primes, $\xi = (p_1, p_2, \ldots)$, we may obtain a divisible sequence $\xi' = (n_1, n_2, \ldots, n_i, \ldots)$ by defining

$$n_1 = p_1 \quad \text{and} \quad n_i = p_i n_{i-1} \quad \text{for} \quad i \geq 2.$$

2.3 Monomial Representations of Finite Degree

Let G be an arbitrary group and H be a subgroup of G of finite index n. Let X be the set of all right cosets of H in G and $\Omega = \{x_1, x_2, \ldots, x_n\}$ be a transversal for H. Then G acts on X by right multiplication:

$$X \to X$$

$$Hx_i \mapsto Hx_{i\cdot\pi(g)}$$

where $\pi(g)$ is a permutation on the set $\{1, 2, \ldots, n\}$, which comes from the action of G on X.

Since each x_i is an element of G, for any $g \in G$,

$$x_i g = h_i(g) x_{i\cdot\pi(g)} \text{ where } h_i(g) \in H.$$

Then there are maps

$$\Phi : G \to \Sigma_n(H)$$

$$g \mapsto [h_1(g), h_2(g), \ldots, h_n(h)]\pi(g)$$

and

$$T : G \to GL(n, \mathbb{Z}H)$$

$$g \mapsto T_g = \begin{bmatrix} h_1(g) & 0 & \dots & 0 \\ 0 & h_2(g) & \dots & \vdots \\ \vdots & \dots & \ddots & 0 \\ 0 & \dots & \dots & h_n(g) \end{bmatrix} \hat{\pi}(g)$$

where $\hat{\pi}(g)$ is the matrix representation of $\pi(g)$ with respect to the basis $\{v_1, v_2, \dots, v_n\}$ where

$$v_i \cdot \hat{\pi}(g) = v_{i \cdot \pi(g)}$$

and extended to $\mathbb{Z}H$ linearly.

Let $g, t \in G$. Then as

$$x_i(gt) = (x_i g)t = h_i(g) x_{i \cdot \pi(g)} t = h_i(g) h_{i \cdot \pi(g)}(t) x_{(i \cdot \pi(g)) \cdot \pi(t)}$$

$$= h_i(g) h_{i \cdot \pi(g)}(t) x_{i \cdot (\pi(g)\pi(t))},$$

$$\Phi(gt) = [h_1(g) h_{1 \cdot \pi(g)}(t), h_2(g) h_{2 \cdot \pi(g)}(t), \dots, h_n(g) h_{n \cdot \pi(g)}(t)] \hat{\pi}(g) \hat{\pi}(t)$$

$$= [h_1(g), h_2(g), \dots, h_n(g)][h_{1 \cdot \pi(g)}(t), h_{2 \cdot \pi(g)}(t), \dots, h_{n \cdot \pi(g)}(t)] \hat{\pi}(g) \hat{\pi}(t)$$

$$= [h_1(g), h_2(g), \dots, h_n(g)] \hat{\pi}(g) (\hat{\pi}(g)^{-1} [h_{1 \cdot \pi(g)}(t), h_{2 \cdot \pi(g)}(t), \dots, h_{n \cdot \pi(g)}(t)] \hat{\pi}(g)) \hat{\pi}(t)$$

$$= [h_1(g), h_2(g), \dots, h_n(g)] \hat{\pi}(g) [h_1(t), h_2(t), \dots, h_n(t)] \hat{\pi}(t)$$

$$= \Phi(g)\Phi(t)$$

and so T is a homomorphism

$$T_{gt} = T_g T_t$$

with kernel

$$\ker T = \{g \in G \mid T_g = I_n\}.$$

Indeed,

$$\ker T = \{g \in G \mid \begin{bmatrix} h_1(g) & 0 & \dots & 0 \\ 0 & h_2(g) & \dots & \vdots \\ \vdots & \dots & \ddots & 0 \\ 0 & \dots & \dots & h_n(g) \end{bmatrix} \hat{\pi}(g) = \begin{bmatrix} 1_H & 0 & \dots & 0 \\ 0 & 1_H & \dots & \vdots \\ \vdots & \dots & \ddots & 0 \\ 0 & \dots & \dots & 1_H \end{bmatrix} \}$$

$$\ker T = \{g \in G \mid h_i(g) = 1_H,\ i = 1, \dots, n \text{ and } \pi(g) = id_{\{1,2,\dots,n\}}\}$$

$$\ker T = \{1_G\}.$$

Hence, T is an embedding of G into $GL(n, \mathbb{Z}H)$ since the diagonal matrices

$$diag(h_1(g), h_2(g), \ldots, h_n(g))$$

and permutation matrices are invertible matrices, T_g is invertible for each $g \in G$ and so T is a monomial representation of G. Hence, if a group G has a subgroup H of finite index n, then G has a monomial representation and so it can be seen as a subgroup of the complete monomial group $\Sigma_n(H)$.

Let $\mathbb{Z}H$ be the integral group ring of H with the usual addition and multiplication. Consider $[\mathbb{Z}H]^n = \mathbb{Z}H \times \mathbb{Z}H \times \ldots \times \mathbb{Z}H$ the free $\mathbb{Z}H$-module. Then $\mathbb{Z}H$ acts on $[\mathbb{Z}H]^n$ as

$$r \cdot (v_1, v_2, \ldots, v_n) = (rv_1, rv_2, \ldots, rv_n)$$

where $\{v_1, v_2, \ldots, v_n\}$ is a basis of the free $\mathbb{Z}H$-module $[\mathbb{Z}H]^n$ and $r \in \mathbb{Z}H$.

The action of G on the set X of cosets of H in G can be carried to the action of G on the free module $[\mathbb{Z}H]^n = \mathbb{Z}H \times \mathbb{Z}H \times \ldots \times \mathbb{Z}H$ by defining its action on the basis $\{v_1, v_2, \ldots, v_n\}$ of $[\mathbb{Z}H^n]$ as follows:

$$v_i \cdot g = v_{i \cdot \pi(g)}$$

The action of G on $[\mathbb{Z}H]^n$ is a module automorphism of $[\mathbb{Z}H]^n$ as it permutes the basis of $\mathbb{Z}H$.

The complete monomial group of degree n over an arbitrary group H corresponds to the subgroup of $GL(n, \mathbb{Z}H)$ generated by all diagonal matrices of the form $diag(h_1, h_2, \ldots, h_n)$ for $h_i \in H$ and the permutation matrices.

Definition 2.3.1. *Let G be a group acting on a set X transitively. A non-empty subset Δ of X is called a* block *for G if for each $g \in G$, either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$ where $\Delta^g = \{g \cdot x \mid x \in X\}$.*

We know that a permutation group G acting on a non-empty set X is called *primitive* if G acts transitively on X and G preserves no nontrivial partition of X, where non-trivial partition means a partition that is not a partition into singleton sets or partition into one set X. In other words, if G is a group acting on a non-empty set X, we say that G is *primitive* if G has no non-trivial blocks on X. Otherwise, it is called *imprimitive*.

The notion of transitivity and imprimitivity of a permutation group can be adapted to the theory of monomial groups.

Definition 2.3.2. *Let $\Gamma \leq \Sigma_n(H)$ be a monomial group.*

(i) Γ is transitive if the permutation subgroup of Γ is transitive.

(ii) Γ is a primitive monomial group if the permutation subgroup of Γ is primitive.

Since every 2-transitive group is primitive, S_n is primitive. Hence, with the above definition, the complete $\Sigma_n(H)$ is a primitive monomial group.

Example 2.3.3. Consider the subgroup of S_4 generated by the permutation $\sigma = (1\ 2)(3\ 4)$. Then $\langle\sigma\rangle$ also acts on $\{1, 2, 3, 4\}$. The orbits of its action are

$$\mathcal{O}_1 = \{1, 2\} \text{ and } \mathcal{O}_2 = \{3, 4\}$$

and so the action is not transitive.

Example 2.3.4. Consider the subgroup of S_4 generated by $\alpha = (1\ 2\ 3\ 4)$.

$$\langle\alpha\rangle = \{(1), (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)\}$$

$\langle\alpha\rangle$ is a transitive permutation group but not primitive. Consider the partition (Δ_1, Δ_2) of $\{1, 2, 3, 4\}$ where $\Delta_1 = \{1, 3\}$ and $\Delta_2 = \{2, 4\}$. Then

$$\Delta_1^\alpha = \Delta_2 \text{ and } \Delta_2^\alpha = \Delta_1$$

and

$$\Delta_1^\alpha \cap \Delta_1 = \emptyset \text{ and } \Delta_2^\alpha \cap \Delta_2 = \emptyset$$

which implies that Δ_1 and Δ_2 are non-trivial blocks of imprimitivity.

We also know this from Theorem 1.6A in [5] which says that the orbits of a normal subgroup form a system of blocks for G. In our example, the subgroup generated by $(1\,3)(2\,4)$ is a subgroup of index 2 in $\langle\alpha\rangle$ and hence, a normal subgroup of $\langle\alpha\rangle$. Hence, its orbits give blocks of imprimitivity for G.

2.4 Construction of Limit Monomial Groups

Definition 2.4.1. *Let G and H be two permutation groups on sets X and Y with actions ϕ_1 and ϕ_2, respectively. Then G and H are said to be* ***permutation isomorphic*** *if there exists a bijective map $\lambda : X \to Y$ and a group isomorphism $\psi : G \to H$ such that*

$$\lambda(\phi_1(g, x)) = \phi_2(\psi(g), \lambda(x))$$

for all $g \in G$ and $x \in X$.

Definition 2.4.2. *Let G be a transitive permutation group on a set X and H be a permutation group on Y. If we have an embedding d from G into H such that $(d(G), O)$ is permutational isomorphic to (G, X), for any orbit $\mathcal{O}$ of $d(G)$ on Y of length greater than 1, then d is called* ***diagonal****. On the other hand, if all the orbits have length greater than 1, then the embedding is called* ***strictly diagonal****.*

If $\rho = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ h_1 x_{i_1} & h_2 x_{i_2} & \dots & h_n x_{i_n} \end{pmatrix} \in \Sigma_n(H)$, then we define a map

$$d^p : \Sigma_n(H) \to \Sigma_{np}(H)$$

by

$$d^p(\rho) = \left(\begin{array}{ccc|ccc|c|ccc} x_1 & \dots & x_n & x_{n+1} & \dots & x_{n+n} & \dots & x_{(p-1)n+1} & \dots & x_{(p-1)n+n} \\ h_1 x_{i_1} & \dots & h_n x_{i_n} & h_1 x_{n+i_1} & \dots & h_n x_{n+i_n} & \dots & h_1 x_{(p-1)n+i_1} & \dots & h_n x_{(p-1)n+i_n} \end{array}\right).$$

Lemma 2.4.1. *d^p is a strictly diagonal embedding.*

Proof. As the complete monomial group $\Sigma_{n_i}(H)$ corresponds to the subgroup of $GL(n_i, \mathbb{Z}H)$ generated by all diagonal matrices of the form

$$diag(h_1, h_2, \dots, h_n)$$

and permutation matrices, d^p can be viewed as the embedding of this subgroup into $GL(n_{i+1}, \mathbb{Z}H)$ diagonally: For $A \in GL(n_i, \mathbb{Z}H)$,

$$\begin{pmatrix} A & 0_{n_i} & \dots & 0_{n_i} \\ 0_{n_i} & A & \dots & 0_{n_i} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n_i} & 0_{n_i} & \dots & A \end{pmatrix}_{n_{i+1} \times n_{i+1}}$$

there are p_{i+1}-copies of A in the diagonal.

Moreover, by transitivity definition 2.3.2 of monomial groups, the embedding is strictly diagonal.

□

Using these embeddings, we can construct limit monomial groups [12].

Let $\xi = (p_i)_{i \in \mathbb{N}}$ be a given sequence of primes (not necessarily distinct) and H be a group. Consider the embeddings $d^{p_{i+1}} : \Sigma_{n_i}(H) \to \Sigma_{n_{i+1}}(H)$ where $n_1 = p_1$ and $n_i = p_1 \dots p_i$. The limit group generated by d^{p_i}'s is **limit monomial group**, denoted by $\Sigma_\xi(H)$.

The construction and classification of these groups are done by Kuzucuoğlu, Olynyk and Sushchansky in [12].

$$\Sigma_\xi(H) = \bigcup_{i=1}^{\infty} \Sigma_{n_i}(H).$$

If $H = \{e\}$, then $\Sigma_\xi(\{e\}) \cong S_\xi$, limit (homogeneous) symmetric group, see [11]. We can say that limit (homogeneous) symmetric group is the limit of finite symmetric groups obtained by strictly diagonal embeddings and

$$S_\xi = \varinjlim S_{n_i}.$$

Similarly, if we take the limit of alternating subgroups of finite symmetric groups, we get limit (homogeneous) alternating group

$$A_\xi = \varinjlim A_{n_i}.$$

Kroshko and Suschansky in [11] proved the following result about limit (homogeneous) symmetric groups:

Theorem 2.4.3. *[11, Page 175, Theorem 1] Let $\xi = (p_1, p_2, \dots)$ be a sequence of not necessarily distinct primes.*

- *if $2^\infty | char(\xi)$, then $S_\xi = A_\xi$.*
- *If there are only finitely many 2 in ξ, then $|S_\xi : A_\xi| = 2$.*

- A_ξ *is simple.*

They also classified limit (homogeneous) symmetric groups using Steinitz numbers:

Theorem 2.4.4. *[11, Lemma 3.3] Let ξ_1 and ξ_2 be two sequences of prime nubers, not necessarily distinct. Then*

$$S_{\xi_1} \leq S_{\xi_2} \text{ if and only if } char(\xi_1)|char(\xi_2).$$

Using this classification theorem on limit (homogeneous) symmetric groups and Steinitz numbers, Kuzucuoğlu, Olynyk and Sushchansky proved the following [12]:

Lemma 2.4.2. *[12, Lemma 2.8] Let ξ and ν be two sequences of prime numbers. Then*

$$\Sigma_\xi(H) \leq \Sigma_\nu(H) \text{ if and only if } char(\xi)|char(\nu).$$

This lemma tells that limit monomial group is uniquely determined by a Steinitz number which is the characteristic of related divisible sequence. Hence, instead of $\Sigma_\xi(H)$, it is convenient to use the notation $\Sigma_\lambda(H)$ where $\lambda = char(\xi)$.

2.5 Some basic definitions and properties of limit monomial groups

In [4], Crouch studied monomial groups over a group H on an infinite set of variables. The notions in monomial groups of finite degree that Ore presented in [18] are adapted to infinite degree monomial groups.

Let B be an infinite cardinal and Ω be a set of cardinality B. Let C and D be cardinal numbers such that $\omega \leq C, D \leq B^+$ where ω is the cardinality of natural numbers and B^+ is the successor of B. The monomial group $\Sigma(H; B, C, D)$ is the group consisting of all monomial substitutions of the form

$$\begin{pmatrix} \dots & x_\epsilon & \dots \\ \dots & h_\epsilon x_{i_\epsilon} & \dots \end{pmatrix}$$

where the map taking x_ϵ to x_{i_ϵ} is a bijection on Ω, whose support has cardinality less than C and there are less than D non-identity h_ϵ. The product of two monomial substitutions are defined similarly as in finite degree case.

We use this to define elements in limit monomial groups as follows:

Definition 2.5.1. *Let $\xi = (p_1, p_2, \ldots)$ be a set of (not necessarily distinct) primes, $n_1 = p_1$, $n_i = p_1 \ldots p_i$ and $\lambda = char(\xi)$. A monomial substitution $\kappa \in \Sigma(H; \omega, \omega^+, \omega^+)$ is said to be **λ-periodic** if it is of the form*

$$\left(\begin{array}{ccc|c|ccc|c} x_1 & \ldots & x_{n_s} & \ldots & x_{(k_{s+1}-1)n_s+1} & \ldots & x_{(k_{s+1}-1)n_s+n_s} & \ldots \\ h_1 x_{i_1} & \ldots & h_{n_s} x_{i_{n_s}} & \ldots & h_1 x_{(k_{s+1}-1)n_s+i_1} & \ldots & h_{n_s} x_{(k_{s+1}-1)n_s+i_{n_s}} & \ldots \end{array}\right)$$

*for some $s \in \mathbb{N}$ where $k_1 = n_1$ and $k_{i+1} = \frac{n_{i+1}}{n_i}$ for all $i \geq 2$. Then the set of all λ-periodic substitutions forms a group, which is **limit monomial group over** H, $\Sigma_\lambda(H)$.*

*A λ-periodic monomial substitution with all factors 1_H is a **permutation**.*

A λ-periodic monomial substitution of the form

$$\left(\begin{array}{ccc|c|ccc|c} x_1 & \ldots & x_{n_s} & \ldots & x_{(k_{s+1}-1)n_s+1} & \ldots & x_{(k_{s+1}-1)n_s+n_s} & \ldots \\ h_1 x_1 & \ldots & h_{n_s} x_{n_s} & \ldots & h_1 x_{(k_{s+1}-1)n_s+1} & \ldots & h_{n_s} x_{(k_{s+1}-1)n_s+n_s} & \ldots \end{array}\right)$$

*is a **multiplication**, denoted by $[h_1, \ldots, h_{n_s} \,|\, \ldots \,|\, h_1, \ldots, h_{n_s} \,|\, \ldots]$.*

*A λ-periodic multiplication with all factors equal is called a **scalar**, denoted by $[h]$.*

The set of all λ-periodic monomial permutations is a subgroup of limit monomial group and this subgroup is isomorphic to the limit symmetric group S_λ. Moreover, all multiplications form a normal subgroup $B(\lambda, H)$ of $\Sigma_\lambda(H)$. As in complete monomial groups of finite degree, every monomial substitution is written uniquely as a product of a λ-periodic multiplication and a permutation as we take the limit of complete monomial groups of finite degree with strictly diagonal embeddings. For a monomial substitution

$$\rho = \left(\begin{array}{ccc|c|ccc|c} x_1 & \ldots & x_{n_s} & \ldots & x_{(k_{s+1}-1)n_s+1} & \ldots & x_{(k_{s+1}-1)n_s+n_s} & \ldots \\ h_1 x_{i_1} & \ldots & h_{n_s} x_{i_{n_s}} & \ldots & h_1 x_{(k_{s+1}-1)n_s+i_1} & \ldots & h_{n_s} x_{(k_{s+1}-1)n_s+i_{n_s}} & \ldots \end{array}\right),$$

$$\rho = \mu\pi$$

where $\mu = [h_1, \ldots, h_{n_s} \,|\, \ldots \,|\, h_1, \ldots, h_{n_s} \,|\, \ldots] \in B(\lambda, H)$ and

$$\pi = \left(\begin{array}{ccc|c|ccc|c} x_1 & \ldots & x_{n_s} & \ldots & x_{(k_{s+1}-1)n_s+1} & \ldots & x_{(k_{s+1}-1)n_s+n_s} & \ldots \\ 1_H x_{i_1} & \ldots & 1_H x_{i_{n_s}} & \ldots & 1_H x_{(k_{s+1}-1)n_s+i_1} & \ldots & 1_H x_{(k_{s+1}-1)n_s+i_{n_s}} & \ldots \end{array}\right) \in S_\lambda.$$

Moreover, for a permutation $\pi \in S_\lambda$ and a multiplication

$$\kappa = (k_i)_{i\in\mathbb{N}} = [k_1, \ldots, k_{n_i} | k_{n_i+1}, \ldots, k_{2n_i} | \ldots] \in B(\lambda, H),$$

$$\kappa^\pi = \pi^{-1}\kappa\pi = (k_{\pi^{-1}(i)})_{i\in\mathbb{N}}.$$

Limit monomial group is the permutational wreath product of H with limit symmetric group S_λ. Limit monomial group $\Sigma_\lambda(H)$ splits over its base group $B(\lambda, H)$ and S_λ is a complement of $B(\lambda, H)$ in $\Sigma_\lambda(H)$,

$$\Sigma_\lambda(H) = B(\lambda, H) \rtimes S_\lambda.$$

We know that every λ-periodic substitution can be written uniquely as a product of a multiplication and a permutation. As in symmetric groups, we can define the type of a monomial substitution. The factors of the monomial substitution should be taken into consideration and the type is defined accordingly.

For a permutation $\pi \in S_n$, the type of π is defined to be $T(\pi) = (r_1, r_2, \ldots, r_n)$ where r_i is the number of cycles of length i in the cycle decomposition of π as a product of disjoint cycles. We know that two permutations are conjugate if and only if they have the same cycle type.

For $\rho \in \Sigma_n(H)$ a monomial substitution, the type of ρ is defined to be $T(\rho) = (a_{11}r_1, \ldots, a_{1i_1}r_1, a_{21}r_2, \ldots, a_{2i_2}r_2, \ldots, a_{n1}r_n, \ldots, a_{ni_n}r_n)$ where a_{ij} is a representative of a determinant class in H and r_i is the number of cycles of length i in the cycle decomposition of ρ with determinant class a_{ij}. Two monomial substitutions are conjugate if and only if the cycles in their cycle decomposition may be made to correspond in such a manner that corresponding cycles have the same length and the same determinant class. i.e. They have the same type, see [12].

As an example, consider two monomial substitution in $\Sigma_6(S_3)$

$$\rho_1 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ (1\,2)x_2 & (1\,2)x_3 & (1\,2\,3)x_1 & (2\,3)x_5 & (1)x_4 & (1\,3\,2)x_6 \end{pmatrix}$$

and

$$\rho_2 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ (1\,3\,2)x_3 & (1)x_6 & (1)x_5 & (1\,2\,3)x_4 & (1)x_1 & (1\,2)x_2 \end{pmatrix}.$$

To see whether we can find a monomial substitution which conjugates ρ_1 to ρ_2, we take a monomial substitution $\rho = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ h_1x_1 & h_2x_3 & h_3x_5 & h_4x_2 & h_5x_6 & h_6x_4 \end{pmatrix}$ whose permutation part conjugates the permutation part of ρ_1 to the permutation part of ρ_2 and try to find the factors h_i's. Consider the product

$$\rho^{-1}\rho_1\rho =$$

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ h_1^{-1}(1\,2)h_2x_3 & h_4^{-1}(2\,3)h_5x_6 & h_2^{-1}(1\,2)h_3x_5 & h_6^{-1}(1\,3\,2)h_6x_4 & h_3^{-1}(1\,2\,3)h_1x_1 & h_5^{-1}h_4x_2 \end{pmatrix}.$$

If $\rho^{-1}\rho_1\rho = \rho_2$, then we have a system of equations to solve, which is the following:

$$h_1^{-1}(1\,2)h_2 = (1\,3\,2),$$

$$h_4^{-1}(2\,3)h_5 = (1),$$

$$h_2^{-1}(1\,2)h_3 = (1),$$

$$h_6^{-1}(1\,3\,2)h_6 = (1\,2\,3),$$

$$h_3^{-1}(1\,2\,3)h_1 = (1),$$

$$h_5^{-1}h_4 = (1\,2).$$

From the second and the last equation, we have

$$h_5(1\,2)h_5^{-1} = h_4$$

and we choose $h_5 = (1\,3\,2)$ and so $h_4 = (1\,3)$.

From the 4^{th} equation, we choose $h_6 = (2\,3)$.

From the 3^{rd} and 5^{th} equation, we have $h_2 = (1\,2)h_3$ and $h_3 = (1\,2\,3)h_1$. Substituting these into the 1^{st} equation, we get $h_1^{-1}(1\,2\,3)h_1 = (1\,3\,2)$. After choosing $h_1 = (2\,3)$, we have $h_2 = (1\,2\,3)$ and $h_3 = (1\,3)$. Hence,

$$\rho = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ (2\,3)x_1 & (1\,2\,3)x_3 & (1\,3)x_5 & (1\,3)x_2 & (1\,3\,2)x_6 & (2\,3)x_4 \end{pmatrix}$$

satisfies $\rho^{-1}\rho_1\rho = \rho_2$. A conjugator for ρ_1 and ρ_2 is found by calculation.

By looking at the types of the monomial substitutions, we can determine whether these two monomial substitutions are conjugate without trying to find a conjugator. ρ_1 and ρ_2 can be written as a product of disjoint monomial cycles as follows:

$$\rho_1 = ((1\,2)x_2\ (1\,2)x_3\ (1\,2\,3)x_1)((2\,3)x_5\ (1)x_4)((1\,3\,2)x_6)$$

and

$$\rho_2 = ((1\,3\,2)x_3\ (1)x_5\ (1)x_1)((1\,2)x_2\ (1)x_6)((1\,2\,3)x_4).$$

Then type of ρ_1 is $T(\rho_1) = ((1\,3\,2)1,\ (2\,3)1,\ (1\,2\,3)1, 0, 0, 0)$ and type of ρ_2 is $T(\rho_2) = ((1\,2\,3)1,\ (1\,2)1,\ (1\,3\,2)1, 0, 0, 0)$. As, in symmetric groups, cycles of same length are conjugate, $T(\rho_1) = T(\rho_2)$ and hence, ρ_1 and ρ_2 are conjugate.

In limit monomial groups, we have the following for two monomial substitution to be conjugate:

Lemma 2.5.1. *[12, Lemma 2.5] Two elements of $\Sigma_\lambda(H)$ are conjugate in $\Sigma_\lambda(H)$ if and only if they have the same cycle type in Σ_{n_j} for some n_j dividing λ.*

We know that every monomial substitution can be written uniquely as a product of a multiplication and a permutation both in complete monomial groups of finite degree and limit monomial groups. Moreover, permutation subgroups S_n and S_λ are complements of the base groups $B(n, H)$ and $B(\lambda, H)$ in $\Sigma_n(H)$ and $\Sigma_\lambda(H)$, respectively. It is mentioned that Ore in [18] classified all complements of the base group up to conjugacy in complete monomial groups of finite degree and so it is a natural question to ask whether all complements of the base group in limit monomial groups can be found and expressed properly.

We recall that a group G is said to *split regularly* over a normal subgroup N if all complements of N in G are conjugate to each other.

The regular splitting of these limit groups are also essential to the complete classification. Kuzucuoğlu, Olynyk and Sushchansky proved the following theorem in [12]:

Theorem 2.5.2. *[12, Theorem 3.6] Let λ and μ be two Steinitz numbers. The limit monomial groups $\Sigma_\lambda(G)$ and $\Sigma_\mu(H)$ are isomorphic if and only if $\lambda = \mu$ and $H \cong G$ provided that the splittings of $\Sigma_\lambda(G)$ and $\Sigma_\mu(H)$ are regular.*

CHAPTER 3

SPLITTING OF LIMIT MONOMIAL GROUPS

In this chapter, the splitting problem in limit monomial groups is studied. In the first section, some examples of non-conjugate complements of base group of limit monomial group are given. For some specific limit monomial groups, uncountably infinite number of non-conjugate complements of base group are found. In the second section, the definition of the symmetric groups obtained by using diagonal embeddings and some properties of these groups are given. These groups have an essential role in classifying complements of the base group. In the last section, using symmetric groups of diagonal type, all complements of base group in limit monomial groups are classified.

3.1 Examples of non-conjugate complements

There are complements of $B(\lambda, H)$ in $\Sigma_\lambda(H)$ which are not conjugate to S_λ, depending on H.

Proposition 3.1.1. *Let H be a group and a be an involution in H. Assume that $2^\infty \nmid \lambda$. Then $T = A_\lambda \cup \{[a]\pi | \pi \in S_\lambda - A_\lambda\}$ is a complement of $B(\lambda, H)$, which is not conjugate to S_λ.*

Proof. Let H be a group which has an element a of order 2. Assume that 2^∞ does not divide λ. Since 2^∞ does not divide λ, limit alternating group A_λ is a proper subgroup of S_λ, see [11, Theorem 1]. Consider the subset

$$T = A_\lambda \cup \{[a]\pi | \pi \in S_\lambda - A_\lambda\}.$$

Our claim is that T is a complement of $B(\lambda, H)$ in $\Sigma_\lambda(H)$ and it is not conjugate to S_λ.

Let t_1, t_2 be in T. Then $t_1 = [x]\pi_1$ and $t_2 = [y]\pi_2$ where x, y are 1 or a depending on whether π_1 and π_2 lie in A_λ. Then

$$t_1 t_2^{-1} = [x]\pi_1([y]\pi_2)^{-1} = [xy]\pi_1\pi_2^{-1}$$

as scalars commute with all permutations.

If both π_1 and π_2 lie in A_λ , then $x = y = 1$ and so, $[xy]\pi_1\pi_2^{-1} = [1]\pi_1\pi_2^{-1}$ lies in T as $A_\lambda \leq S_\lambda$ and $\pi_1\pi_2^{-1} \in A_\lambda$.

If only one of them lies in A_λ, then $\pi_1\pi_2^{-1}$ is not in A_λ and $xy = a$ as one of x or y is a and the other one is 1. So $t_1t_2^{-1} = [a]\pi_1\pi_2^{-1}$ is in T.

If both π_1 and π_2 are outside of A_λ, then $t_1 = [a]\pi_1$ and $t_2 = [a]\pi_2$, $t_1t_2^{-1} = [1]\pi_1\pi_2^{-1}$ is in T as the product of two odd permutations is even.

Hence, T is a subgroup of $\Sigma_\lambda(H)$.

As every monomial substitution is uniquely written as a product of a multiplication and a permutation, the intersection of T and $B(\lambda, H)$ is trivial.

Let $\rho = \mu\pi$ be a monomial substitution where μ is in the base group and π is a permutation. Then

$$\rho = \mu\pi = \mu[x]^{-1}[x]\pi = (\mu[x])([x]\pi)$$

where $\mu[x]$ is a multiplication, $[x]\pi$ is in T and $x = 1$ or a depending on the parity of π.

T is isomorphic to S_λ with the isomorphism

$$\theta : S_\lambda \to T$$

$$\pi \to \begin{cases} \pi & \text{if } \pi \in A_\lambda \\ [a]\pi & \text{if } \pi \notin A_\lambda \end{cases}$$

Lastly, we need to show that this complement T is not conjugate to S_λ.

Suppose to the contrary that there exists $\rho = [k_1, k_2, \ldots, k_{n_i} | \ldots]\sigma \in \Sigma_\lambda(H)$ such that $\rho T \rho^{-1} = S_\lambda$. ρ is in $\Sigma_{(n_i)}(H)$ for some $n_i|\lambda$. Here $\Sigma_{(n_i)}(H) = d(\Sigma_{n_i}(H))$, the image of the complete monomial group of degree n_i under the strictly diagonal embedding. Let π be an even permutation in $S(n_i)$. Then $\pi \in T$ and

$$\rho\pi\rho^{-1} = [k_1, \ldots, k_{n_i} | \ldots]\sigma\pi\sigma^{-1}[k_1^{-1}, \ldots, k_{n_i}^{-1}]$$

$$= [k_1 k^{-1}_{1^{\sigma\pi\sigma^{-1}}}, \ldots, k_{n_i} k^{-1}_{n_i^{\sigma\pi\sigma^{-1}}} | \ldots]\sigma\pi\sigma^{-1} \in S_\lambda$$

and so

$$k_t = k_{t^{\sigma\pi\sigma^{-1}}} \text{ for } t = 1, 2, \ldots, n_i. \qquad (*)$$

As A_{n_i} is a normal subgroup of S_{n_i} and A_{n_i} is transitive on $\{1, \ldots, n_i\}$, $(*)$ gives that all factors k_t are equal and so $\rho = [k]\sigma$.

Now, we take $[a]\pi \in T$ such that $\pi \in S(n_i)$ and $\pi \notin A_\lambda$. Then

$$\rho[a]\pi\rho^{-1} = [k]\sigma[a]\pi\sigma^{-1}[k^{-1}] = [kak^{-1}]\sigma\pi\sigma^{-1} \in S_\lambda$$

which implies that

$$kak^{-1} = 1$$

and so $a = 1$, which is not the case. Hence, we reach a contradiction. □

Remark 3.1.1. We notice that, in the complete monomial group of degree n over a group H, the subset $T = A_n \cup \{[a]\pi \,|\, \pi \in S_n - A_n\}$ is also a complement of $B(n, H)$ in $\Sigma_n(H)$. Consider the multiplication $\kappa = [1, a, \ldots, a] \in B(n, H)$ and for $i = 2, \ldots, n$,

$$\kappa[a](1\,i)\kappa^{-1} = [1, a, \ldots, a, \ldots, a][a](1\,i)[1, a^{-1}, \ldots, a^{-1}, \ldots, a^{-1}]$$

$$= [1, a, \ldots, a, \underbrace{1}_{i^{th}-component}, a, \ldots, a](1\,i).$$

Hence, $\kappa T \kappa^{-1}$ is the complement generated by

$$\lambda_i = [1, a, \ldots, a, 1, a, \ldots, a](1\,i) \text{ for } i = 2, \ldots, n,$$

$$[a]\pi_1 \text{ for } \pi_1 \in S^{(1)}_{n-1}.$$

In this case, the subgroup A of H, which is the homomorphic image of $S^{(1)}_{n-1}$, mentioned in the splitting theorem of Ore 2.1.9,[18, Theorem 10], is the cyclic group of order 2.

As an immediate consequence of the proposition, we have the following:

Proposition 3.1.2. Let H be a group such that there are infinitely many non-conjugate elements of order 2 and assume that 2^∞ does not divide λ. Then $B(\lambda, H)$ has infinitely many non-conjugate complements in $\Sigma_\lambda(H)$.

Proof. Let a and b be two non-conjugate elements of order 2 in H and set the complements T_a and T_b as in the previous proposition. Then as a and b are not conjugate, the complements T_a and T_b are not conjugate to each other and to S_λ. $\square$

Remark: Depending on H and λ, there may be infinitely many pairwise non-conjugate complements of $B(\lambda, H)$ in $\Sigma_\lambda(H)$.

There may be uncountably many non-conjugate complements of $B(\lambda, H)$ in $\Sigma_\lambda(H)$.

Example 3.1.1. Consider $\Sigma_\lambda(S_\lambda)$ and the subset

$$T = \{[\pi]\pi \mid \pi \in S_\lambda\}.$$

Let $[\pi]\pi$ and $[\sigma]\sigma$ be in T. Then as scalars commute with all permutations,

$$[\pi]\pi([\sigma]\sigma)^{-1} = [\pi\sigma^{-1}]\pi\sigma^{-1} \in T.$$

The intersection of $B(\lambda, S_\lambda)$ and T is trivial as the expression of a monomial substitution as a product of a multiplication and a permutation is unique. Also, T is isomorphic to S_λ with the isomorphism $\theta : S_\lambda \to T$, taking π to $[\pi]\pi$. Similarly, as T contains substitutions with cycles of non-identity determinant, T is not conjugate to S_λ.

Our aim is to obtain uncountably infinite number of non-conjugate complements of $B(\lambda, H)$ in $\Sigma_\lambda(H)$.

It is known that all automorphisms of limit (homogeneous) symmetric group S_λ are locally inner, see [14, Proposition 10]. In [8, Theorem 2], it is proved that there are uncountably many automorphims of S_λ which does not preserve any level n_i dividing λ. By the construction of these automorphisms, it is easily seen that they are not inner automorphisms.

Let α be an outer automorphism of S_λ and consider the subset

$$T_\alpha = \{[\pi]\pi^\alpha \mid \pi \in S_\lambda\}.$$

Clearly, T_α is a subgroup whose intersection with the base group is trivial as α is an automorphism of S_λ and scalars commute with all permutations. Let $\rho = \mu\pi$ be a monomial substitution where $\mu \in B(\lambda, H)$ and $\pi \in S_\lambda$. Then as α is an automorphism of S_λ, there exists unique σ_π in S_λ such that $\sigma_\pi^\alpha = \pi$. So

$$\rho = \mu\pi = \mu[\sigma_\pi^{-1}][(\sigma_\pi)]\sigma_\pi^\alpha$$

and $\mu[\sigma_\pi^{-1}] \in B(\lambda, H)$, $[(\sigma_\pi)]\sigma_\pi^\alpha \in T_\alpha$. Hence, T_α is a complement of $B(\lambda, H)$ in $\Sigma_\lambda(H)$.

First, we need to show that T and T_α are not conjugate:

Suppose that T and T_α are conjugate subgroups. Then there exists $\rho = \mu\theta$ in $\Sigma_\lambda(H)$ such that $\rho T_\alpha \rho^{-1} = T$. Let $[(\pi)]\pi^\alpha$ be an arbitrary element of T_α. Then $([\pi]\pi^\alpha)^{\rho^{-1}} \in T$. So

$$([\pi]\pi^\alpha)^{(\mu\theta)^{-1}} = [\gamma_\pi]\gamma_\pi \quad \text{for some unique } \gamma_\pi \in S_\lambda.$$

We have

$$\begin{aligned} ([\pi]\pi^\alpha)^{(\mu\theta)^{-1}} &= \mu\theta[\pi]\pi^\alpha(\mu\theta)^{-1} \\ &= \mu\theta[\pi]\pi^\alpha\theta^{-1}\mu^{-1} \\ &= \mu[\pi]\theta\pi^\alpha\theta^{-1}\mu^{-1} \qquad (*) \\ &= \mu[\pi]((\theta\pi^\alpha\theta^{-1})\mu^{-1}(\theta\pi^\alpha\theta^{-1})^{-1})(\theta\pi^\alpha\theta^{-1}) \end{aligned}$$

and so $\mu[\pi]((\theta\pi^\alpha\theta^{-1})\mu^{-1}(\theta\pi^\alpha\theta^{-1})^{-1}) = [\gamma_\pi]$ with $\theta\pi^\alpha\theta^{-1} = \gamma_\pi$ for every $\pi \in S_\lambda$. $(*)$ holds as scalars commute with all permutations.

Let $\mu = [k_1, k_2, \ldots, k_{n_i}, |k_1, k_2, \ldots.]$. Then from the first equation, we have

$$k_t \pi k_{t^{((\pi^\alpha)^{\theta^{-1}})}} = \gamma_\pi = (\pi^\alpha)^{\theta^{-1}}.$$

Since α is a locally inner automorphism of S_λ, for every finite subset F of S_λ, there exists $g_F \in S_\lambda$ such that $g^\alpha = g^{g_F}$ for $g \in F$. So for $F = S(n_j)$, there exists $g_{S(n_j)} \in S_\lambda$ such that for $\pi \in S_\lambda$, $\pi^\alpha = \pi^{g_{S(n_j)}}$.

As α and conjugation with θ^{-1} are automorphisms of S_λ, consider π with $(\pi^\alpha)^{\theta^{-1}} = (n_i+1\ n_i+2)(n_{i+1}+n_i+1\ n_{i+1}+n_i+2)(2n_{i+1}+n_i+1\ 2n_{i+1}+n_i+1)\ldots.$ in $S(n_{i+1})$. For this π,

$$k_t \pi k_t^{-1} = (\pi^\alpha)^{\theta^{-1}} \text{ for } t = 1, \ldots, n_i$$

as $(\pi^\alpha)^{\theta^{-1}}$ fixes $1, \ldots, n_i$ and

$$k_2 \pi k_1^{-1} = k_1 \pi k_2^{-1} = (\pi^\alpha)^{\theta^{-1}} \text{ from the second block of } \mu.$$

Combining these equations, we obtain $k_1 = k_2$.

So, consider

$$(\pi^\alpha)^{\theta^{-1}} = (n_i{+}1\ n_i{+}3)(n_{i+1}{+}n_i{+}1\ n_{i+1}{+}n_i{+}3)(2n_{i+1}{+}n_i{+}1\ 2n_{i+1}{+}n_i{+}3)\ldots.$$

and by above, we get $k_1 = k_3$. If we continue like this up to

$$(\pi^\alpha)^{\theta^{-1}} = (n_i{+}1\ n_i{+}n_i)(n_{i+1}{+}n_i{+}1\ n_{i+1}{+}n_i{+}n_i)(2n_{i+1}{+}n_i{+}1\ 2n_{i+1}{+}n_i{+}n_i)\ldots.,$$

we get

$$k_1 = k_2 = \ldots. = k_{n_i} \text{ and hence } \mu = [k] \text{ for some } k \in H.$$

So we have

$$\sigma_\pi = (\pi^\alpha)^{\theta^{-1}} = k^{-1}\pi k$$

$$(\pi^\alpha)^{\theta^{-1}k^{-1}} = \pi \text{ for every } \pi \in S_\lambda$$

which implies that

$$\alpha i_{(k\theta)^{-1}} = id_{S_\lambda} \text{ where } i_{\theta k} \text{ denotes the inner automorphism by the element } \theta k.$$

As all inner automorphisms of a group form a subgroup of automorphism group, α must be an inner automorphism which contradicts with the choice of α.

Hence, T and T_α are non-conjugate complements of the base group and they are not conjugate to limit symmetric group.

□

3.2 Symmetric groups of diagonal type

In the next section, we use symmetric groups of diagonal type to describe the complements of the base group in limit monomial groups.

As mentioned in the previous chapters, Ore described all complements of the base group in complete monomial groups of finite degree in [18]. In Theorem (2.1.9), a complement is found up to conjugacy by using a 1-point stabilizer of finite symmetric group, $S_n^{(1)}$, which is the subgroup of S_n consisting of all permutations fixing 1 and it is said that the group H has a subgroup which is a homomorphic image of this group $S_n^{(1)}$.

Similar to this, it is reasonable to ask whether there is a corresponding description of complements using a 1-point stabilizer related to limit (homogeneous) symmetric group S_λ. For this purpose, we need to look deeper into some special properties of limit (homogeneous) symmetric group S_λ.

3.2.1 Trees

Definition 3.2.1.
- *A **graph** T is a pair of sets (V, E), where V is the set of vertices and E is the set of edges, formed by pairs of vertices.*
- *The vertices v_1 and v_2 are said to be **adjacent** if there is an edge $e = \{v_1, v_2\} \in E$.*
- *The **degree** of a vertex v is the number of vertices which are adjacent to v.*
- *A **path** is a sequence of distinct vertices $\{v_1, v_2, \ldots, v_n\}$ such that each $\{v_i, v_{i+1}\}$, $1 \leq i \leq n-1$, is an edge.*
- *If the initial and the end vertices of a path are the same, then the path is called a **cycle**.*
- *A graph is **connected** if there is a path between any two vertices.*
- *A **tree** is a connected graph without any cycles.*
- *A **rooted tree** (T, v_0) is a tree with a fixed vertex v_0, which is called the **root**.*

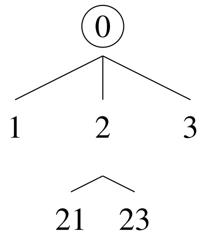

Figure 3.1: A rooted tree

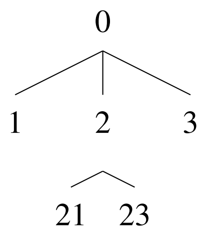

Figure 3.2: A non-rooted tree

Some examples of rooted and non-rooted trees are shown in the Figure 3.1 and 3.2, respectively.

In Figure 3.1, since the vertex 0 is specified, it is a rooted tree and the second one is not a rooted tree as none of the vertices is specified.

Definition 3.2.2.
- *A tree is **infinite** if the cardinality of V is infinite.*
- *An **end** of a rooted tree (T, v_0) is an infinite path $\{v_0, v_1, \ldots\}$ starting from the root v_0 where $\{v_i, v_{i+1}\} \in E$, $i \in \{0\} \cup \mathbb{N}$.*
- *The set of all ends of a rooted tree (T, v_0) is called the **boundary** ∂T of T.*

Let $\xi = (p_1, p_2)$ be a finite sequence of prime numbers such that p_1 and p_2 are not necessarily distinct. We form a rooted tree with respect to this finite prime sequence as follows:

- The degree of the root v_0 is p_1.
- The degree of all vertices $v_1, v_2, \ldots v_{p_1}$ adjacent to v_0 is $p_2 + 1$.

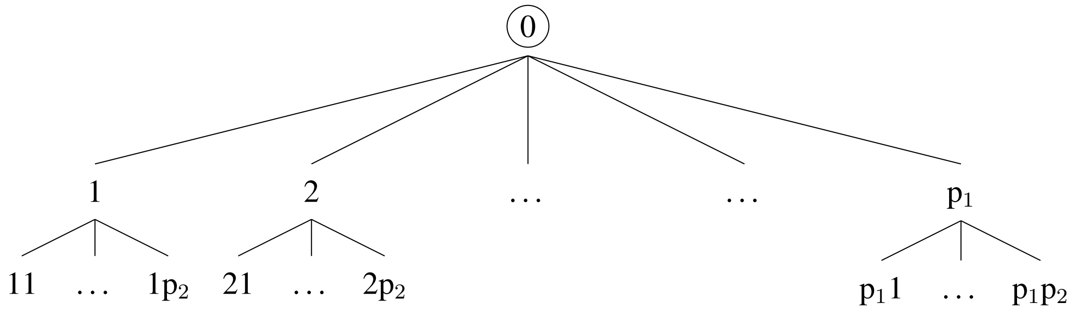

Figure 3.3: Rooted tree for the sequence $\xi = (p_1, p_2)$

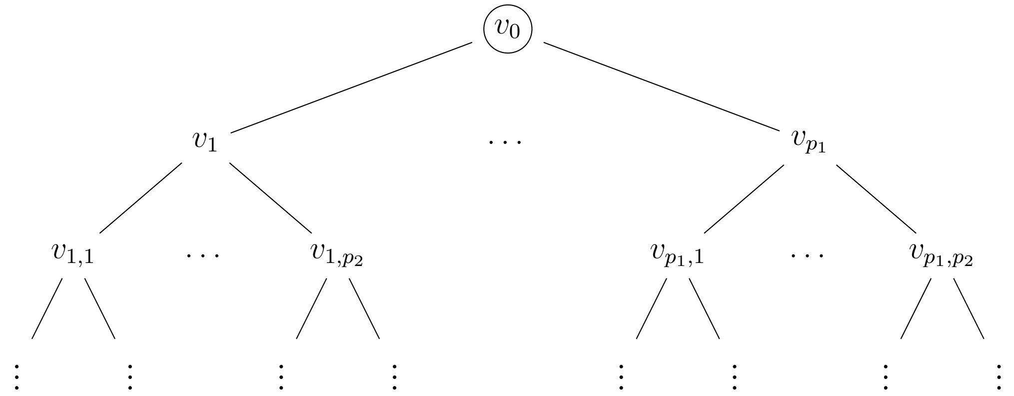

Figure 3.4: An infinite rooted tree for the sequence $\xi = (p_1, p_2, \ldots, p_i, \ldots)$

Similarly, as seen in Figure 3.4, if $\xi = (p_1, p_2, \ldots)$ is an infinite sequence of not necessarily distinct prime numbers, we form a rooted tree (T, v_0) with respect to ξ as follows:

- The degree of the root v_0 is p_1.
- The degree of all vertices $v_1, v_2, \ldots v_{p_1}$ adjacent to v_0 is $p_2 + 1$.
- The degree of all vertices adjacent to v_i, $1 \leq i \leq p_1$ is $p_3 + 1$.

  $\vdots$
- The degree of all vertices in the n-th level is $p_{n+1} + 1$.

  $\vdots$

3.2.2 Action of limit symmetric group on a rooted tree

Consider limit (homogeneous) symmetric group S_λ with respect to the sequence $\xi = (p_1, p_2, \ldots)$ with $char(\xi) = \lambda$. S_λ acts on the boundary of this rooted tree with the action of permuting the sub-trees on the same level, see [13] and [7] for details.

With this action, we consider the stabilizer of an end of the tree. The stabilizer corresponds to a different type of symmetric group obtained by diagonal embeddings related to the same sequence of primes $\xi = (p_1, p_2, \ldots)$.

Consider the following embeddings of finite symmetric groups:

$$d(p,s) : S_n \to S_{np+s}$$

$$d(p,s)(\begin{pmatrix} 1 & 2 & \ldots & n \\ i_1 & i_2 & \ldots & i_n \end{pmatrix}) =$$

$$\left(\begin{array}{cccc|c|ccc|ccc} 1 & 2 & \ldots & p & \ldots & (n-1)p+1 & \ldots & (n-1)p+p & np+1 & \ldots & np+s \\ (i_1-1)p+1 & (i_1-1)p+2 & \ldots & (i_1-1)p+p & \ldots & (i_n-1)p+1 & \ldots & (i_n-1)p+p & np+1 & \ldots & np+s \end{array}\right)$$

for $\alpha = \begin{pmatrix} 1 & 2 & \ldots & n \\ i_1 & i_2 & \ldots & i_n \end{pmatrix} \in S_n$.

By Lemma 5.1 in [7], $d(p,s)$ is a diagonal embedding.

For an infinite sequence of integer tuples $\chi =< (1, k_0), (n_1, k_1), (n_2, k_2), \ldots >$, we obtain a direct system and direct limit group by the following:

$$S_{k_0} \xrightarrow{d(n_1,k_1)} S_{n_1k_0+k_1} \xrightarrow{d(n_2,k_2)} S_{(n_1k_0+k_1)n_2+k_2} \xrightarrow{d(n_3,k_3)} \ldots .$$

This limit group is denoted by S_χ [13]. These kind of groups have similar properties as symmetric groups obtained by strictly diagonal embeddings and we have a corresponding theorem related to its alternating subgroup as follows:

Theorem 3.2.3 ([13], Theorem 3.1)**.** *For an infinite sequence of integer tuples* $\chi =< (1, k_0), (n_1, k_1), (n_2, k_2), \ldots >$*, we have the following:*

(i) $S_\chi = A_\chi$ *if and only if* $2^\infty | char(\chi)$.

(ii) *If* $2^\infty \nmid char(\chi)$, $[S_\chi : A_\chi] = 2$.

(iii) A_χ *is a simple group.*

Similar to limit (homogeneous) symmetric groups, the only normal subgroups of symmetric groups of diagonal type are (1), alternating subgroup A_χ and itself S_χ.

Consider the diagonal direct limit group S_χ where $\chi =< (1, p_1-1), (p_2, p_2-1), \ldots >$ and $\lambda = char(p_1, p_2, \ldots) = char(\chi)$. Let $\gamma \in \partial T$ be an end of the rooted tree (T, v_0). By Theorem 5.11 in [7],

$$S_\chi \cong Stab_{S_\lambda}(\gamma).$$

3.3 Complements of base group

We know that limit monomial group splits over its base group as

$$\Sigma_\lambda(H) \cong B(\lambda, H) \rtimes S_\lambda.$$

Let T be another complement of $B(\lambda, H)$. Then $\Sigma_\lambda(H) \cong B(\lambda, H) \rtimes T$. Using second isomorphism theorem,

$$S_\lambda \cong \Sigma_\lambda(H)/B(\lambda, H) \cong B(\lambda, H)T/B(\lambda, H) \cong T/(B(\lambda, H) \cap T) \cong T$$

implies that any two complements are isomorphic.

If T is a complement of $B(\lambda, H)$ in $\Sigma_\lambda(H)$, then the elements of T are of the form $v_\pi \pi$ for each $\pi \in S_\lambda$ and the multiplication part v_π is uniquely determined. If not, let $v_\pi \pi$ and $w_\pi \pi$ be two elements in the complement T with the permutation part π. Then $v_\pi \pi (w_\pi \pi)^{-1}$ is also an element of T as T is a subgroup. However, $v_\pi \pi (w_\pi \pi)^{-1} = v_\pi w_\pi^{-1}$ is a multiplication. As T is a complement of $B(\lambda, H)$, the intersection of T and $B(\lambda)$ is trivial and so $v_\pi = w_\pi$.

As $\Sigma_\lambda(H)$ is a direct limit group, it is reasonable to ask whether the complements of the base group $B(\lambda, H)$ are also direct limit groups and there are subgroups of limit monomial group whose limit is a complement of the base group.

As mentioned before, Ore in [18] described all complements of the base group up to conjugacy in the complete monomial groups of finite degree. Another question to ask for limit monomial groups is whether it is possible to describe all complements of the base group.

For these purposes, we use the information about the structure of the complements of the base group $B(n, H)$ in the complete monomial group $\Sigma_n(H)$ of degree n. As notation, we use $B((n), H) = d(B(n, H))$, $S(n) = d(S_n)$ and $\Sigma_{(n)}(H) = d(\Sigma_n(H))$, which are the images of the groups under the strictly diagonal embedding d.

Lemma 3.3.1. *Let T_λ be a complement of $B(\lambda, H)$ in $\Sigma_\lambda(H)$ and for each i, $T_i = \{v_\pi \pi \mid \pi \in S(n_i)\}$. Then a conjugate of T_i is a complement of $B((n_i), H)$ in $\Sigma_{(n_i)}(H)$.*

Proof. We adapt the proof of the splitting theorem 2.1.9 of Ore on page 12.

The group $S(n_i)$ is generated by $\alpha_t = (1\,t)^d$ where $(1\,t) \in S_{n_i}$ and this implies that T_λ has λ-periodic substitutions of the form

$$\lambda_t = v_{\alpha_t} \alpha_t$$

where v_{α_t} is a multiplication and

$$\lambda_t = [a^{(1)}_{1,t}, a^{(1)}_{2,t}, \ldots, a^{(1)}_{n_i,t}, |a^{(2)}_{1,t}, a^{(2)}_{2,t}, \ldots, a^{(2)}_{n_i,t}| \ldots |a^{(n_j/n_i)}_{1,t}, a^{(n_j/n_i)}_{2,t}, \ldots, a^{(n_j/n_i)}_{n_i,t}| \ldots](1\,t)^d.$$

Apply conjugation with the suitable multiplication κ_i,

$$\kappa_i = [1, a^{(1)}_{1,2}, a^{(1)}_{1,3}, \ldots, a^{(1)}_{1,n_i}, \ldots, 1, a^{(n_j/n_i)}_{1,2}, a^{(n_j/n_i)}_{1,3}, \ldots, a^{(n_j/n_i)}_{1,n_i}| \ldots\ldots]$$

in $B((n_j), H)$, for each $s = 1, \ldots, n_j/n_i$, we obtain

$$\kappa_i \lambda_t \kappa_i^{-1} = [\ldots, \underbrace{1, a^{(s)}_{1,2} a^{(s)}_{2,t} (a^{(s)}_{1,2})^{-1}, \ldots, a^{(s)}_{1,t} a^{(s)}_{t,t}, \ldots, a^{(s)}_{1,n_i} a^{(s)}_{n_i,t} (a^{(s)}_{1,n_i})^{-1}}_{n_i}, \ldots | \ldots](1\,t)^d.$$

Hence, in each n_i-block of the multiplication part, the 1^{st} components are all identity, t^{th} components are $a^{(s)}_{1,t} a^{(s)}_{t,t}$ and the rest are just the conjugates of the original corresponding components. With this calculation, wihtout loss of generality, we may assume that T_λ contains substitutions

$$\lambda_t = [\ldots, \underbrace{1, a^{(s)}_{2,t}, \ldots, a^{(s)}_{n_i,t}}_{n_i}, \ldots | \ldots](1\,t)^d.$$

Then

$$\lambda_t^2 = [\ldots, \underbrace{a^{(s)}_{t,t}, \ldots, a^{(s)}_{t,t}, \ldots, (a^{(s)}_{k,t})^2, \ldots}_{n_i}, \ldots | \ldots]$$

where $k \neq 1,\, t$ and λ_t^2 is a multiplication. As T_λ is a complement of $B(\lambda, H)$, the intersection $T_\lambda \cap B(\lambda, H)$ is trivial. This implies that $\lambda_t^2 = [1_H]$. This leads to the following:

$$a_{t,t}^{(s)} = 1 \quad \text{and} \quad (a_{r,t}^{(s)})^2 = 1_H \quad \text{for each} \quad r \neq 1,\, t. \tag{3.1}$$

Now, consider the product for $t \neq r$,

$$\lambda_t\lambda_r = \left(\begin{array}{c|cccccc|c} \cdots & x_{(s-1)p_i+1} & x_{(s-1)p_i+t} & x_{(s-1)p_i+r} & \cdots & x_{(s-1)p_i+k} & \cdots & \cdots \\ \cdots & a_{t,r}^{(s)}x_{(s-1)p_i+t} & x_{(s-1)p_i+r} & a_{r,t}^{(s)}x_{(s-1)p_i+1} & \cdots & a_{k,t}^{(s)}a_{k,r}^{(s)}x_{(s-1)p_i+k} & \cdots & \cdots \end{array}\right)$$

and

$$(\lambda_t\lambda_r)^3 =$$

$$\left(\begin{array}{c|cccccc|c} \cdots & x_{(s-1)p_i+1} & x_{(s-1)p_i+t} & x_{(s-1)p_i+r} & \cdots & x_{(s-1)p_i+k} & \cdots & \cdots \\ \cdots & a_{t,r}^{(s)}a_{r,t}^{(s)}x_{(s-1)p_i+1} & a_{r,t}^{(s)}a_{t,r}^{(s)}x_{(s-1)p_i+t} & a_{r,t}^{(s)}a_{t,r}^{(s)}x_{(s-1)p_i+r} & \cdots & (a_{k,t}^{(s)}a_{k,r}^{(s)})^3x_{(s-1)p_i+k} & \cdots & \cdots \end{array}\right)$$

which is a multiplication and so it has to be identity as it is both contained in $B(\lambda, H)$ and its complement T_λ. This gives the following equations on $a_{r,t}$'s:

$$a_{t,r}^{(s)}a_{r,t}^{(s)} = 1_H \quad \text{and} \quad (a_{k,t}^{(s)}a_{k,r}^{(s)})^3 = 1_H \quad \text{for} \quad k \neq 1,\, t,\, r. \tag{3.2}$$

Using (3.1) and (3.2), we get $a_{r,t}^{(s)} = a_{t,r}^{(s)}$ for every t, r.

Notice that the element $\lambda_t\lambda_r$ is the monomial substitution in T_λ corresponding to the permutation $((1\,t)(1\,r))^d$.

As the next step, we perform the product $\lambda_r\lambda_t\lambda_r$ which is

$$\left(\begin{array}{c|cccccc|c} \cdots & x_{(s-1)p_i+1} & x_{(s-1)p_i+t} & x_{(s-1)p_i+r} & \cdots & x_{(s-1)p_i+k} & \cdots & \cdots \\ \cdots & a_{r,t}^{(s)}x_{(s-1)p_i+1} & a_{t,r}^{(s)}x_{(s-1)p_i+r} & a_{t,r}^{(s)}x_{(s-1)p_i+t} & \cdots & a_{k,r}^{(s)}a_{k,t}^{(s)}a_{k,r}^{(s)}x_{(s-1)p_i+k} & \cdots & \cdots \end{array}\right)$$

for $k \neq 1, t, r$. The permutation part of $\lambda_r\lambda_t\lambda_r$ is $(x_t\, x_r)^d \in S(n_i)$.

For different indices $1, k, t, r$, in finite symmetric groups, we have

$$(1\,k)(t\,r) = (t\,r)(1\,k).$$

Since T_λ is isomorphic to S_λ, for each permutation π in S_λ, there exists a unique element with permutaton part π in T_λ. Using the equation for symmetric groups,

$$\lambda_k(\lambda_r\lambda_t\lambda_r) = (\lambda_r\lambda_t\lambda_r)\lambda_k$$

in T_λ and equating the corresponding components, we obtain from the first and the k^{th} components

$$a_{k,r}^{(s)}a_{k,t}^{(s)}a_{k,r}^{(s)} = a_{r,t}^{(s)} \quad \text{and} \quad a_{l,k}^{(s)}a_{r,t}^{(s)} = a_{r,t}^{(s)}a_{l,k}^{(s)} \quad \text{for} \quad l \neq k,\, t,\, r. \tag{3.3}$$

By using (3.3), we obtain a simpler form for the product $\lambda_r\lambda_t\lambda_r$ as

$$\lambda_r\lambda_t\lambda_r = [\ldots, \underbrace{a_{r,t}^{(s)}, \ldots, a_{r,t}^{(s)}}_{n_i}, \ldots | \ldots](t\,r)^d.$$

The transpositions $(t\,r)$ generate the subgroup of S_{n_i}, which fixes 1 and so this kind of elements in T_λ generate all elements in T_i with the permutation part fixing $1, n_i + 1, 2n_i + 2, \ldots$ For a permutation π_1 fixing $1, n_i + 1, 2n_i + 2, \ldots$ in $S(n_i)$, T_i has elements

$$[\ldots, \underbrace{a_{\pi_1}^{(s)}, \ldots, a_{\pi_1}^{(s)}}_{n_i}, \ldots | \ldots]\pi_1.$$

Denote $S_{n_i-1}^{(1)}$ as the stablizer of 1 in S_{n_i}. Then the maps

$$\begin{aligned} \varphi_s \colon S_{n_i-1}^{(1)} &\to H \\ \pi_1 &\mapsto a_{\pi_1}^{(s)}. \end{aligned}$$

are homomorphisms, since for $\pi_1, \sigma_1 \in S(n_i)$ fixing $1, n_i + 1, \ldots$,

$$\begin{aligned} &[\ldots, \underbrace{a_{\pi_1}^{(s)}, \ldots, a_{\pi_1}^{(s)}}_{n_i}, \ldots | \ldots]\pi_1[\ldots, \underbrace{a_{\sigma_1}^{(s)}, \ldots, a_{\sigma_1}^{(s)}}_{n_i}, \ldots | \ldots]\sigma_1 \\ &= [\ldots, \underbrace{a_{\pi_1}^{(s)}, \ldots, a_{\pi_1}^{(s)}}_{n_i}, \ldots | \ldots][\ldots, \underbrace{a_{\sigma_1}^{(s)}, \ldots, a_{\sigma_1}^{(s)}}_{n_i}, \ldots | \ldots]\pi_1\sigma_1 \\ &= [\ldots, \underbrace{a_{\pi_1}^{(s)}a_{\sigma_1}^{(s)}, \ldots, a_{\pi_1}^{(s)}a_{\sigma_1}^{(s)}}_{n_i}, \ldots | \ldots]\pi_1\sigma_1 \end{aligned}$$

and so

$$a_{\pi_1}^{(s)}a_{\sigma_1}^{(s)} = a_{\pi_1\sigma_1}^{(s)} \iff \varphi_s(\pi_1)\varphi_s(\sigma_1) = \varphi_s(\pi_1\sigma_1).$$

By these homomorphisms, seperately in each n_i-block, the elements $a_{\pi_1}^{(s)}$ generate a subgroup $G_{n_i}^{(s)}$ of H, which is a homomorphic image of $S_{n_i-1}^{(1)}$, for $s = 1, \ldots n_j/n_i$.

Claim: $G_{n_i}^{(s)}$, $s = 1, 2, \ldots, n_j/n_i$, are conjugate subgroups.

Proof of claim: Apply the same procedure for $\kappa_j T_j \kappa_j^{-1}$. Let

$$\alpha_t = [b_{1,t}^{(1)}, b_{2,t}^{(1)}, \ldots, b_{n_j,t}^{(1)}, |b_{1,t}^{(2)}, b_{2,t}^{(2)}, \ldots, b_{n_j,t}^{(2)}, \ldots, b_{1,t}^{(n_k/n_j)}, b_{2,t}^{(n_k/n_j)}, \ldots, b_{n_j,t}^{(n_k/n_j)}| \ldots.](1\ t)^d$$

for $(1\ t)^d \in S(n_j)$ and

$$\kappa_j = [1, b_{1,2}^{(1)}, b_{1,3}^{(1)}, \ldots, b_{1,n_j}^{(1)}, \ldots., 1, b_{1,2}^{(n_k/n_j)}, b_{1,3}^{(n_k/n_j)}, \ldots, b_{1,n_j}^{(n_k/n_j)}| \ldots\ldots] \in B((n_k), H).$$

Consider a permutation π_1 fixing $1, n_i+1, 2n_i+1, \ldots$ in $S(n_i)$. Then π_1 fixes $1, n_j+1, 2n_j+1, \ldots$ and the image of π_1 in $\kappa_j T_j \kappa_j^{-1}$ is γ_i

$$\underbrace{[\underbrace{a_{\pi_1}^{(1)}, \ldots, a_{\pi_1}^{(1)}, a_{\pi_1}^{(2)}, \ldots, a_{\pi_1}^{(2)}, \ldots, a_{\pi_1}^{(n_j/n_i)}, \ldots, a_{\pi_1}^{(n_j/n_i)}}_{n_j}, \ldots, a_{\pi_1}^{(1)}, \ldots, a_{\pi_1}^{(1)}, \ldots, a_{\pi_1}^{(n_j/n_i)}, \ldots, a_{\pi_1}^{(n_j/n_i)}}_{n_k}, \ldots.]\pi_1$$

Then

$$\begin{aligned}
\kappa_j \gamma_i \kappa_j^{-1} &= \kappa_j [a_{\pi_1}^{(1)}, \ldots a_{\pi_1}^{(1)}, \ldots, a_{\pi_1}^{(n_j/n_i)}, \ldots, a_{\pi_1}^{(n_j/n_i)}, \ldots.]\pi_1 \kappa_j^{-1} \\
&= \kappa_j [a_{\pi_1}^{(1)}, \ldots a_{\pi_1}^{(1)}, \ldots, a_{\pi_1}^{(n_j/n_i)}, \ldots, a_{\pi_1}^{(n_j/n_i)}, \ldots.](\pi_1 \kappa_j^{-1} \pi_1^{-1})\pi_1
\end{aligned}$$

$$= [\underbrace{a_{\pi_1}^{(1)}, ***, k_{n_i+1}^{-1} a_{\pi_1}^{(2)} k_{n_i+1}, ***, \ldots, k_{(n_j/n_i-1)n_i+1}^{-1} a_{\pi_1}^{(n_j/n_i)} k_{(n_j/n_i-1)n_i+1}, ***}_{n_j}, \ldots.]\pi_1$$

$$= [\underbrace{b_{\pi_1}^{(1)}, \ldots, b_{\pi_1}^{(1)}}_{n_j}, \ldots, b_{\pi_1}^{(n_k/n_j)}, \ldots, b_{\pi_1}^{(n_k/n_j)}, \ldots.]\pi_1.$$

which implies that

$$b_{\pi_1}^{(1)} = a_{\pi_1}^{(1)} = k_{n_i+1}^{-1} a_{\pi_1}^{(2)} k_{n_i+1} = \ldots = k_{(n_j/n_i-1)n_i+1}^{-1} a_{\pi_1}^{(n_j/n_i)} k_{(n_j/n_i-1)n_i+1} \qquad (3.4)$$

and the subgroups $G_{n_i}^{(s)}$ are conjugate to each other. Hence, by the splitting theorem 2.1.9 of Ore on page 12, and the above argument, there exists $\rho_i \in \Sigma_\lambda(H)$ such that $\rho_i T_i \rho_i^{-1}$ is a subgroup of $\Sigma_{(n_i)}(H)$ and a complement of $B((n_i), H)$ in $\Sigma_{(n_i)}(H)$.

□

To see whether it is possible to diagonally embed a complement of $B(n, H)$ in $\Sigma_n(H)$ into a complement of $B(nm, H)$ in $\Sigma_{nm}(H)$, we give the following example:

Example 3.3.1. Consider $\Sigma_3(S_6)$ and embed this group into $\Sigma_6(S_6)$ by using strictly diagonal embedding d^p with $p = 2$.

By the splitting theorem of Ore 2.1.9, the subgroup T_3 generated by

$$\alpha_2 = [1, 1, (2\,3)(5\,6)](1\,2) \text{ and } \alpha_3 = [1, (2\,3)(5\,6), 1](1\,3)$$

is a complement of $B(3, S_6)$ in $\Sigma_3(S_6)$ and the homomorphic image subgroup of S_6 is $S_{3-1}^{(1)}$. Then $T_3^{d^2} \leq \Sigma_6(S_6)$ is generated by

$$\overline{\alpha_2} = \alpha_2^{d^2} = [1, 1, (2\,3)(5\,6), 1, 1, (2\,3)(5\,6)](1\,2)(4\,5)$$

and

$$\overline{\alpha_3} = \alpha_3^{d^2} = [1, (2\,3)(5,6), 1, 1, (2\,3)(5,6), 1](1\,3)(4\,6).$$

In $\Sigma_6(S_6)$, we consider a complement T_6 where the homomorphic image subgroup is $S_{6-1}^{(1)}$, which is generated by the following elements:

$$\lambda_2 = [1, 1, (2\,3), (2\,4), (2\,5), (2\,6)](1\,2),$$

$$\lambda_3 = [1, (2\,3), 1, (3\,4), (3\,5), (3\,6)](1\,3),$$

$$\lambda_4 = [1, (2\,4), (3\,4), 1, (4\,5), (4\,6)](1\,4),$$

$$\lambda_5 = [1, (2\,5), (3\,5), (4\,5), 1, (5\,6)](1\,5),$$

$$\lambda_6 = [1, (2\,6), (3\,6), (4\,6), (5\,6), 1](1\,6)$$

and for every $\pi_1 \in S_{6-1}^{(1)}$, we have $[\pi_1]\pi_1 \in T_6$. Now, our question becomes if there is an element $\rho \in \Sigma_6(S_6)$ such that $\rho T_3^{d^2} \rho^{-1} \leq T_6$.

Consider the elements in T_6, whose permutation parts are the same with $\overline{\alpha_2}$ and $\overline{\alpha_3}$, which are

$$\beta_2 = [1, 1, (2\,3), (2\,4), (2\,5), (2\,6)](1\,2)[(4\,5)](4\,5)$$

$$= [(4\,5), (4\,5), (2\,3)(4\,5), (2\,5\,4), (2\,4\,5), (2\,6)(4\,5)](1\,2)(4\,5),$$

$$\beta_3 = [1, (2\,3), 1, (3\,4), (3\,5), (3\,6)](1\,3)[(4\,6)](4\,6)$$

$$= [(4\,6), (2\,3)(4\,6), (4\,6), (3\,6\,4), (3\,5)(4\,6), (3\,4\,6)](1\,3)(4\,6).$$

Is it possible to find a multiplication $\kappa = [k_1, k_2, k_3, k_4, k_5, k_6]$ in $\Sigma_6(S_6)$ such that

$$\kappa\overline{\alpha_2}\kappa^{-1} = \beta_2 \text{ and } \kappa\overline{\alpha_3}\kappa^{-1} = \beta_3?$$

$$\kappa\overline{\alpha_2}\kappa^{-1} = [k_1k_2^{-1}, k_2k_1^{-1}, k_3(2\,3)(5\,6)k_3^{-1}, k_4k_5^{-1}, k_5k_4^{-1}, k_6(2\,3)(5\,6)k_6^{-1}](1\,2)(4\,6),$$

$$\kappa\overline{\alpha_3}\kappa^{-1} = [k_1k_3^{-1}, k_2(2\,3)(5\,6)k_2^{-1}, k_3k_1^{-1}, k_4k_6^{-1}, k_5(2\,3)(5\,6)k_5^{-1}, k_6k_4^{-1}](1\,3)(4\,6).$$

We need to solve the following system:

$$\left.\begin{aligned} &k_1k_2^{-1} = (4\,5), \\ &k_1k_3^{-1} = (4\,6), \\ &k_2(2\,3)(5\,6)k_2^{-1} = (2\,3)(4\,6), \\ &k_3(2\,3)(5\,6)k_3^{-1} = (2\,3)(4\,5) \end{aligned}\right\} \qquad (3.5)$$

$$\left.\begin{aligned} k_4k_5^{-1} &= (2\,5\,4), \\ k_4k_6^{-1} &= (3\,6\,4), \\ k_5(2\,3)(5\,6)k_5^{-1} &= (3\,5)(4\,6), \\ k_6(2\,3)(5\,6)k_6^{-1} &= (2\,6)(4\,5) \end{aligned}\right\} \tag{3.6}$$

The system 3.5 is to solve for k_1, k_2 and k_3 and the system 3.6 is to solve for k_4, k_5 and k_6. They are seperated in 2 blocks of length 3.

From the first system of equations 3.5, substituting $k_2 = (4\,5)k_1$, $k_3 = (4\,6)k_1$ into the 3^{rd} and the 4^{th} equation, we obtain

$$k_1(2\,3)(5\,6)k_1^{-1} = (2\,3)(5\,6)$$

and similarly, from the second system 3.6, substituting $k_5 = (2\,4\,5)k_4$, $k_6 = (3\,4\,6)k_4$, we get

$$k_4(2\,3)(5\,6)k_4^{-1} = (2\,3)(5\,6).$$

Hence, k_1 and k_4 are elements that centralizes $(2\,3)(5\,6)$. Choose $k_1 = k_4 = (1)$. Then

$$k_2 = (4\,5),\ \ k_3 = (4\,6),\ \ k_5 = (2\,4\,5),\ \ k_6 = (3\,4\,6).$$

Since we can solve the above system, we have the conjugator

$$\kappa = [1, (4\,5), (4\,6), 1, (2\,4\,5), (3\,4\,6)]$$

such that

$$\kappa T_3^{d^2}\kappa^{-1} \leq T_6.$$

From the example, it is understood that we need to deal with systems of equation on seperated blocks to conjugate a complement in $\Sigma_n(H)$ into a complement in $\Sigma_{nm}(H)$.

We need the following lemma to prove that the group H contains a subgroup, which is a homomorphic image of a symmetric group of diagonal type:

Lemma 3.3.2. *[3, Lemma 2.3] Let H and G be direct limit groups of subgroups H_i and G_i via the embeddings θ_i and ψ_i, respectively. Let α_i be an isomorphism between*

H_i and G_i for all $i \geq 1$. If the following diagram

$$\begin{array}{ccc} H_i & \xrightarrow{\theta_i} & H_{i+1} \\ \alpha_i \downarrow & & \downarrow \alpha_{i+1} \\ G_i & \xrightarrow[\psi_i]{} & G_{i+1} \end{array}$$

is commutative for all $i \geq 1$, then the groups H and G are isomorphic.

Proposition 3.3.1. *Let T be a complement of $B(\lambda, H)$ in $\Sigma_\lambda(H)$. Then H contains a subgroup G which is a homomorphic image of the diagonal direct limit group S_χ where $\chi =< (1, p_1 - 1), (p_2, p_2 - 1), \ldots >$ and $\lambda = char(p_1, p_2, \ldots) = char(\chi)$.*

Proof. In the same manner of the proof of Lemma 3.3.1, there exists $\rho_j \in \Sigma_\lambda(H)$ such that $\rho_j \rho_i T_j \rho_i^{-1} \rho_j^{-1}$ is a complement of $B((n_j), H)$. In each conjugation, we obtain a subgroup G_{n_i} of H, which is a homomorphic image of $S^{(1)}_{n_i - 1}$ where $S^{(1)}_{n_i-1}$ is the subgroup of S_{n_i} fixing 1 and by (3.4), G_{n_i} is a subgroup of G_{n_j} for $n_i | n_j$.

We divide the proof into 2 main parts as $2^\infty \nmid \lambda$ and $2^\infty | \lambda$.

__**First:**__ Assume that $2^\infty \nmid \lambda$.

Case 1: Assume that G_{n_i} is trivial for every i. Then the identity subgroup of H is the homomorphic image of the diagonal symmetric group.

Case 2: Assume that $G_{n_i} \cong C_2$. Let $\phi_i : S^{(1)}_{n_i-1} \to G_{n_i}$ be the homomorphism

$$\phi_i(\pi_1) = \begin{cases} 1_H & \pi_1 \text{ is even} \\ a & \pi_1 \text{ is odd} \end{cases}$$

for some $a \in H$ of order 2.

By the above argument, since $2^\infty \nmid \lambda$, $G_{n_j} = G_{n_i}$ for $n_i | n_j$ and then $\bigcup_i G_{n_i} = G = \langle a \rangle$. In this case,

$$T = A_\lambda \cup \{[a]\pi \mid \pi \notin A_\lambda\}$$

is a complement of the base group.

Hence, in these cases, H contains a subgroup which is a homomorphic image of S_χ.

Case 3: Our claim is that if G_{n_i} is isomorphic to $S^{(1)}_{n_i-1}$ for each i, then G_{n_i} is a diagonal subgroup of G_{n_j} for $n_i|n_j$.

Proof of claim: Since G_{n_i}'s are homomorphic images of symmetric groups $S^{(1)}_{n_i-1}$, they act on $\{1, 2, \ldots, n_i\}$ the same as the finite symmetric group $S^{(1)}_{n_i-1}$ does. By the above argument and (3.4), we see that $G_{n_i} \leq G_{n_{i+1}}$. Hence, the embeddings from G_{n_i} into $G_{n_{i+1}}$ are the inclusion maps

$$\theta_i : G_{n_i} \to G_{n_{i+1}}$$

$$a_{\pi_1} \mapsto a_{\pi_1}$$

Hence, from this direct system, we have a direct limit group $G = \varinjlim G_{n_i}$. We need to show that $G \cong S_\chi$.

For this purpose, we first need to show that the embedding of $S^{(1)}_{n_i-1}$ into $S^{(1)}_{n_{i+1}-1}$ is diagonal and hence, limit group obtained from these finite symmetric groups is S_χ.

Since $S^{(1)}_{n_i-1} = Stab_{S_{n_i}}(1)$, it fixes the point 1 and acts as full permutation group on $\{2, 3, \ldots, n_i\}$. Hence, the orbits of $S^{(1)}_{n_i-1}$ is of length 1 or $n_i - 1$. For $1 \leq k \leq p_{i+1} - 1$ and $2 \leq j \leq n_i$,

$$(kn_i + j)^\alpha = kn_i + j^\alpha$$

and

$$\mathcal{O}_k = \{kn_i + 2, kn_i + 3, \ldots, kn_i + n_i\}$$

Define the bijection

$$\lambda : \{2, 3, \ldots, n_i\} \to \mathcal{O}_k$$

$$j \mapsto kn_i + j$$

Then for $\alpha \in S^{(1)}_{n_i-1}$ and $j \in \{2, 3, \ldots, n_i\}$,

$$\lambda(\alpha \cdot j) = \lambda(j^\alpha) = kn_i + j^\alpha = d^{p_{i+1}}(\alpha) \cdot \lambda(j)$$

Hence, the embedding of $S^{(1)}_{n_i-1}$ into $S^{(1)}_{n_{i+1}-1}$ is diagonal and so

$$\varinjlim S^{(1)}_{n_i-1} = S_\chi$$

where $\chi =< (1, p_1 - 1), (p_2, p_2 - 1), \ldots . >$.

In this case, we have two direct systems. The first one is $\{(G_{n_i}, \theta_i) \mid i \in \mathbb{N}\}$ where θ_i is inclusion map and $\{(S^{(1)}_{n_i-1}, d^{p_{i+1}}) \mid i \in \mathbb{N}\}$. We also have the isomorphisms $\alpha_i : S^{(1)}_{n_i-1} \to G_{n_i}$ defined by $\alpha_i(\pi) = a_{\pi^d}$ and so we have the following diagram:

$$\begin{array}{ccc} S^{(1)}_{n_i-1} & \xrightarrow{d^{p_{i+1}}} & S^{(1)}_{n_{i+1}-1} \\ \alpha_i \downarrow & & \downarrow \alpha_{i+1} \\ G_{n_i} & \xrightarrow[\theta_i]{} & G_{n_{i+1}} \end{array}$$

Let $\pi \in S^{(1)}_{n_i-1}$. Then

$$\alpha_{i+1} \circ d^{(p_{i+1})}(\pi) = \alpha_{i+1}(\pi^{d^{p_{i+1}}}) = a_{(\pi^{d^{p_{i+1}}})^d} = a_{\pi^d} = \theta_i(a_{\pi^d}) = \alpha_i(a_{\pi^d})$$

and hence, the diagram is commutative. By Lemma 3.3.2 above,

$$\varinjlim G_{n_i} \cong \varinjlim S^{(1)}_{n_i} = S_\chi$$

where $\chi =< (1, p_1 - 1), (p_2, p_2 - 1), \ldots . >$.

Second: Assume that $2^\infty | \lambda$.

Then $S_\lambda = A_\lambda$ and $S_\chi = A_\chi$. Since A_n is simple for $n \geq 5$ and A_λ and A_χ are simple groups as being direct limit of simple groups, there is no non-trivial homomorphic image of A_χ.

If each G_{n_i} is trivial, then $\bigcup_{i \in \mathbb{N}} G_{n_i}$ is the trivial subgroup, and hence, a homomorphic image of S_χ.

In this case, it is not possible to have G_{n_i}'s as groups of order 2. Assume that there exists n_i such that $G_{n_i} = \langle a \rangle$ is cyclic of order 2. By the same procedure in the proof of Lemma 3.3.1 and the splitting theorem of Ore 2.1.9, $T_i = \{v_\pi \pi \mid \pi \in S(n_i)\} \leq T$ is conjugate to $A = A(n_i) \cup \{[a]\pi, \mid \pi \in S(n_i) - A(n_i)\}$ since this subgroup A is a complement of $B((n_i), H)$ in $\Sigma_{(n_i)}(H)$ and the corresponding factor subgroup $\langle a \rangle$ is G_{n_i}. Without loss of generality, we can assume that $T_i = A$.

As $2^\infty | \lambda$, we may assume that $n_{i+1} = 2n_i$. Now, $T_i \leq T_{i+1}$. By Lemma 3.3.1, $G_{n_i} = \langle a \rangle \leq G_{n_{i+1}}$ and so $G_{n_{i+1}}$ is either $G_{n_i} = \langle a \rangle$ or isomorphic to $S^{(1)}_{n_{i+1}-1}$.

Assume that $G_{n_{i+1}} = \langle a \rangle$. Then, by remark 3.1.1, T_{i+1} is conjugate to $T_0 = A(n_{i+1}) \cup \{[a]\pi \mid \pi \in S(n_{i+1}) - A(n_{i+1})\}$, say with ρ and $\rho T_{i+1} \rho^{-1} = T_0$. As $n_{i+1} = 2n_i$, the permutation parts of all elements of T_i are even. and hence, after the conjugation, $\rho T_i \rho^{-1} \leq A(n_{i+1})$, which is not possible as some of the elements of T_i has non-identity determinant classes. Hence, $G_{n_{i+1}}$ can not be cyclic of order 2.

Assume that $G_{n_{i+1}} \cong S^{(1)}_{n_{i+1}-1}$. Consider $\pi \in A(n_i) \leq T_i$, which fixes 1 in S_{n_i}. As $G_{n_{i+1}} \cong S^{(1)}_{n_{i+1}-1}$, after the conjugation, the image of an even permutation γ fixing 1 becomes $[b_\gamma]\gamma$ where b_γ is the image of γ under the isomorphism between $G_{n_{i+1}}$ and $S^{(1)}_{n_{i+1}-1}$. However, if we consider an even permutation $\pi \in A(n_i)$ and its image inside T_{i+1}, the determinants of its cycles are all identity, which gives the contradiction.

Lastly, we assume that $G_{n_i} \cong S^{(1)}_{n_i-1}$. Then as $G_{n_i} \leq G_{n_{i+1}}$, $G_{n_{i+1}}$ have to be isomorphic to $S^{(1)}_{n_{i+1}-1}$. Then T_i is conjugate to the subgroup generated by

$$[1, a_{2,j}, \ldots, a_{n_i,j} | \ldots](1\, i)^d \text{ for } i = 2, \ldots, n_i.$$

By the same procedure in the example 3.3.1, there exists a multiplication κ in $B((n_{i+1}), H)$ such that $\kappa T_i \kappa^{-1} \leq T_{i+1}$. By the third case above, H contains a subgroup which is isomorphic to $S_\chi = A_\chi$.

Hence, in each case, we reach the result.

□

By using Lemma 3.3.1 and Proposition 3.3.1, we obtain the following about complements of the base group:

Theorem 3.3.2. *Let T_λ be a complement of $B(\lambda, H)$ in $\Sigma_\lambda(H)$. Then T_λ is a direct limit of complements of $B((n_i), H)$ in $\Sigma_{(n_i)}(H)$.*

Proof. For each i, $T_i = \{v_\pi \pi \mid \pi \in S(n_i)\}$ is a subgroup of T_λ and by Lemma 3.3.1, a conjugate of T_i is a complement of $B((n_i), H)$ in $\Sigma_{(n_i)}(H)$. Let $\rho_i \in \Sigma_\lambda(H)$ be the monomial substitution such that $T_{n_i} = \rho_i T_i \rho_i^{-1}$ is a complement of $B((n_i), H)$ in $\Sigma_{(n_i)}(H)$ for each $i \in \mathbb{N}$.

Define homomorphisms

$$\varphi_{ij}\colon T_{n_i} \to T_{n_j}$$
$$\rho_i v_\pi \pi \rho_i^{-1} \mapsto \rho_j v_\pi \pi \rho_j^{-1}.$$

φ_{ij}'s are monomorphisms induced by the conjugation by the elements $\rho_i \rho_j^{-1}$ for each i, j. Then

$$\varphi_{ii}\colon T_{n_i} \to T_{n_i}$$
$$\rho_i v_\pi \pi \rho_i^{-1} \mapsto \rho_i v_\pi \pi \rho_i^{-1}$$

and $\varphi_{ii} = id_{T_{n_i}}$. Moreover, for $k > j > i$,

$$\varphi_{jk} \circ \varphi_{ij}(\rho_i v_\pi \pi \rho_i^{-1}) = \varphi_{jk}(\rho_j v_\pi \pi \rho_j^{-1}) = \rho_k v_\pi \pi \rho_k^{-1}$$
$$= \rho_k \rho_i^{-1}(\rho_i v_\pi \pi \rho_i^{-1})\rho_i \rho_k^{-1} = \varphi_{ik}(\rho_i v_\pi \pi \rho_i^{-1}).$$

Hence, $\{T_{n_i}\}_{i\in\mathbb{N}}$ and the morphisms $\{\varphi_{ij} : T_{n_i} \to T_{n_j}\}_{i\leq j}$ form a direct system.

Now, for each $i \in \mathbb{N}$, define

$$\varphi_i\colon T_{n_i} \to T_\lambda$$
$$\rho_i v_\pi \pi \rho_i^{-1} \mapsto v_\pi \pi$$

as the conjugation with ρ_i. Then for any $i \leq j$, the following diagram commutes

$$\begin{array}{ccc} T_{n_i} & & \\ {\scriptstyle \varphi_{ij}}\downarrow & \searrow^{\varphi_i} & \\ T_{n_j} & \xrightarrow{\varphi_j} & T_\lambda \end{array}$$

as

$$\varphi_j \circ \varphi_{ij}(\rho_i v_\pi \pi \rho_i^{-1}) = \varphi_j(\rho_j \rho_i^{-1}(\rho_i v_\pi \pi \rho_i^{-1})\rho_i \rho_j^{-1}) = \varphi_j(\rho_j v_\pi \pi \rho_j^{-1})$$
$$= \rho_j^{-1}\rho_j v_\pi \pi \rho_j^{-1}\rho_j = v_\pi \pi = \rho_i^{-1}(\rho_i v_\pi \pi \rho_i^{-1})\rho_i = \varphi_i(\rho_i v_\pi \pi \rho_i^{-1}).$$

Hence, $T_\lambda = \varinjlim T_{n_i}$.

□

Theorem 3.3.3. *Limit monomial group $\Sigma_\lambda(H)$ splits regularly over its base group $B(\lambda, H)$ if and only if H has no non-trivial subgroup which is the homomorphic image of the diagonal symmetric group S_χ where $\chi =< (1, p_1-1), (p_2, p_2-1), \ldots >$ and $\lambda = char(p_1, p_2, \ldots) = char(\chi)$.*

Proof. (⇒) Assume that H contains no subgroup which is a non-trivial homomorphic image of S_χ, then the factors $a_{i,j}$'s are all identity and so the only complement of $B(\lambda, H)$ in $\Sigma_\lambda(H)$ is S_λ.

(⇐) Assume that $\Sigma_\lambda(H)$ splits regularly over $B(\lambda, H)$. Then for any complement T, there exist $\rho \in \Sigma_\lambda(H)$ such that $\rho T \rho^{-1} = S_\lambda$. Since a permutation only permutes the factors of a monomial substitution, we can consider ρ as a multiplication, say κ;

$$\kappa = [k_1, \ldots, k_{n_i} | k_1, \ldots].$$

Then consider $T_i = \{v_\pi \pi \,|\, \pi \in S(n_i)\}$. By Lemma 3.3.1, we may assume that T_i is generated by

$$[1, a_{2,j}, \ldots, a_{n_i,j} | \ldots](1\,j)^d.$$

Then setting the product for every $i = 2, \ldots, n_i$

$$\kappa[1, a_{2,j}, \ldots, a_{n_i,j} | \ldots](1\,j)^d \kappa^{-1} = (1\,j)^d$$

gives

$$k_1 = k_2 = \ldots = k_{n_i}$$

and so

$$a_{t,j} = 1.$$

□

Corollary 3.3.1. $\Sigma_\lambda(H)$ *splits regularly over* $B(\lambda, H)$ *if and only if* H *has no elements of order 2.*

Proof. This immediately follows from the above theorem 3.3.3. □

CHAPTER 4

LIMIT MONOMIAL GROUPS OVER ABELIAN GROUPS

In this chapter, limit monomial groups over abelian groups are studied. In the first section, base group and a special subgroup of the base group are investigated. The quotient of the base group by this special subgroup over various abelian groups is studied. The lattice structure of the subgroups of the rational numbers containing integers is found using the lattice structure of Steinitz numbers and the quotient of the base group over infinite cyclic group.

4.1 Base Group

Lemma 4.1.1. *[12, Theorem 3.6] Let λ be a Steinitz number, different from 2. Then $B(\lambda, H)$ is a characteristic subgroup of the limit monomial group $\Sigma_\lambda(H)$.*

Proof. Let λ be a Steinitz number, different from 2. Assume that $B(\lambda, H)$ is not a characteristic subgroup of $\Sigma_\lambda(H)$. Then there exists an automorphism α of $\Sigma_\lambda(H)$ and a subgroup M such that $B(\lambda, H)^\alpha = M$ and M is not contained in $B(\lambda, H)$. So we have $M \triangleleft \Sigma_\lambda(H)$.

So since $M \not\subseteq B(\lambda, H)$, there exists n_i such that

$$M_i = M \cap \Sigma_{n_i}(H) \not\subseteq B(n_i, H).$$

Also, as $M \triangleleft \Sigma_\lambda(H)$, we have $M_i \triangleleft \Sigma_{n_i}(H)$.

Consider the subgroup $N = M \cap B(\lambda, H)$. Since $M \triangleleft \Sigma_\lambda(H)$, N is a normal subgroup of the symmetry.

For every i, if $N_i = N \cap \Sigma_{n_i}(H)$, we have

$$N_i = M \cap B(n_i, H) \quad \text{and} \quad N \cap B(n_i, H) = N \cap M_i$$

N_i is the multiplication part of the normal subgroup M_i of $\Sigma_{n_i}(H)$. So by Theorem 8 of section 2 of [18], the elements of N_i are of the form

$$\eta = [h_1, h_2, \ldots, h_{n_i}, |h_1, h_2, \ldots\ldots]$$

such that $h_1 \ldots . h_{n_i}$ is in S where $S \triangleleft H$ with ${H}/{S}$ is abelian. Then we have

$$B(n_i, H)/N_i \cong H/S.$$

As $N_i = N \cap \Sigma_{n_i}(H)$ for each i, N is the limit of the subgroups N_i.

By second isomorphism theorem, we get

$$B(n_i, H)/N_i = B(n_i, H)/N \cap B(n_i, H) \cong B(n_i, H)N/N.$$

Also since

$$\bigcup_{i=1}^{\infty} B(n_i, H) = B(\lambda, H),$$

we have

$$\bigcup_{i=1}^{\infty} B(n_i, H)N/N = B(\lambda, H)/N.$$

For every $n_j \geq n_i$, since the quotient group $B(n_j, H)N/N$ is abelian, $B(\lambda, H)/N$ is abelian.

As M and $B(\lambda, H)$ are isomorphic via an automorphism of the symmetry, we have

$$\Sigma_\lambda(H)/M \cong \Sigma_\lambda(H)/B(\lambda, H) \cong S_\lambda$$

by the isomorphism

$$\bar{\alpha} : \Sigma_\lambda(H)/B(\lambda, H) \to \Sigma_\lambda(H)/M$$

which takes $\rho B(\lambda, H)$ to $\rho^\alpha M$.

By applying the second isomorphism theorem, we have

$$B(\lambda, H)\big/_N = B(\lambda, H)\big/_{M \cap B(\lambda, H)} \cong B(\lambda, H)M\big/_M$$

and since

$$B(\lambda, H)M\big/_M \lhd \Sigma_\lambda(H)\big/_M \cong S_\lambda,$$

$B(\lambda, H)\big/_N$ is isomorphic to an abelian normal subgroup of S_λ . But S_λ has only two non-trivial normal subgroups, depending on λ, namely, A_λ and S_λ.

$B(\lambda, H)\big/_N$ is non-trivial since N is a proper subgroup of $B(\lambda, H)$. Otherwise, if $N = M \cap B(\lambda, H) = B(\lambda, H)$, then $B(\lambda, H) \subset M$, which is not the case. As S_λ has no non-trivial abelian normal subgroup, we get a contradiction.

Hence, $B(\lambda, H)$ is a characteristic subgroup of $\Sigma_\lambda(H)$ for every λ except for $\lambda = 2$.

□

For $\lambda = 2$ and $H = C_2 = \langle a \rangle$, cyclic group of order 2, $\Sigma_2(C_2)$ has an automorphism which does not leave the base group invariant, as Ore did in page 44, Theorem 2 of Chapter 3 [18].

The complete monomial group $\Sigma_2(C_2)$ is a group of order 8. As, up to isomorphism, there are 3 groups of order 8 and the quaternion group of order 8 is not a semi-direct product, $\Sigma_2(C_2)$ is the dihedral group of order 8 and it is generated by the elements $x = (x_1\, x_2)$ and $y = (x_1\, x_2)[a, 1]$ with the relations

$$x^2 = 1 \ , \ y^4 = 1 \ \text{and} \ xyx^{-1} = y^3.$$

The center of $\Sigma_2(C_2)$ is $\{[1, 1], [a, a]\}$ and y and y^3 are the only elements of order 4.

As the base group is a normal subgroup of the complete monomial group, we need to find an outer automorphism of the group to show that the base group is not a characteristic subgroup. Let α be an outer automorphism of $\Sigma_2(C_2)$. Since y and y^3 are only elements of order 4, α takes y to y^3. As we have $xyx^{-1} = y^3$, we multiply the outer automorphism α by the inner automorphism induced by x and assume

$$y^\alpha = y.$$

Moreover, since the center of a group is a characteristic subgroup, $([a,a])^\alpha = [a,a]$.

As x is of order 2, α takes x to one of $[a,1]$, $[1,a]$ and $(x_1\,x_2)[a,a]$. Notice that $y^{-1}xy = (x_1\,x_2)[a,a]$ and so the last correspondence gives an inner automorphism of the group. Hence,

$$x^\alpha = [1,a] \;\text{ or }\; x^\alpha = [a,1].$$

Since $[1,a]^{(x_1\,x_2)} = [a,1]$, two possibilities for α differ by an inner automorphism and they define outer automorphisms of $\Sigma_2(C_2)$ by

$$(x_1\,x_2)^\alpha = [1,a], \;\; ([1,a])^\alpha = (x_1\,x_2)[a,a],$$

$$((x_1\,x_2)[a,a])^\alpha = [a,1], \;\; ([1,a])^\alpha = (x_1\,x_2)$$

which does not leave the base group invariant and hence, in this case, $B(2,C_2)$ is not a characteristic subgroup of $\Sigma_2(C_2)$.

Now, we introduce a special subgroup of the base group $B(\lambda,H)$, which is a normal subgroup of limit monomial group $\Sigma_\lambda(H)$.

Definition 4.1.1. *Let H be an abelian group. The set of all λ-periodic multiplications*

$$\{[h_1,h_2,\ldots,h_{n_i},h_1,h_2,\ldots] \in B(\lambda,H) \mid h_1h_2\ldots h_{n_i} = 1 \textit{ for some } n_i|\lambda\}$$

is a subgroup of the base group $B(\lambda,H)$, denoted by $B_0(\lambda,H)$.

Proposition 4.1.1. *$B_0(\lambda,H)$ is a normal subgroup of $\Sigma_\lambda(H)$.*

Proof. Since H is abelian, $B(\lambda,H)$ is abelian. As a subgroup of abelian group, $B_0(\lambda,H)$ is normalized by the base group. Moreover, since the action of S_λ is to permute the components of a multiplication and $B_0(\lambda,H)$ is abelian, $B_0(\lambda,H)$ is also normalized by S_λ. Hence, $B_0(\lambda,H)$ is a normal subgroup of $\Sigma_\lambda(H)$.

□

This subgroup is not a characteristic subgroup of $B(\lambda, H)$. As an example, take $\lambda = 2$ and $H = \mathbb{Z}_7$. Consider the monomial group $\Sigma_2(\mathbb{Z}_7)$ and its subgroups $B(2, \mathbb{Z}_7)$, $B_0(2, \mathbb{Z}_7)$. Since $B(2, \mathbb{Z}_7) \cong \mathbb{Z}_7 \times \mathbb{Z}_7$, we may consider this subgroup as a vector space over $\mathbb{F}_7$ and so $Aut(B(2, \mathbb{Z}_7)) \cong GL(2, \mathbb{F}_7)$. Consider the automorphism

$$T : \mathbb{F}_7 \oplus \mathbb{F}_7 \to \mathbb{F}_7 \oplus \mathbb{F}_7$$
$$\vec{x} \to \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \vec{x}.$$

Take the vector $\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ in $\mathbb{F}_7 \oplus \mathbb{F}_7$. This vector corresponds to the multiplication $[3, 4]$ in $B(2, \mathbb{Z}_7)$ with $3 + 4 = 0$ and hence, in $B_0(2, \mathbb{Z}_7)$. Then

$$T(\begin{bmatrix} 3 \\ 4 \end{bmatrix}) = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

and since 3+6 $\neq$ 0, $B_0(2, \mathbb{Z}_7)$ is not a characteristic subgroup of $B(2, \mathbb{Z}_7)$.

Our aim is to determine the structure of this special normal subgroup $B_0(\lambda, H)$ of $B(\lambda, H)$ and $\Sigma_\lambda(H)$ by considering the quotient group $B(\lambda, H)/B_0(\lambda, H)$. For a complete monomial group $\Sigma_n(H)$ of finite degree over an abelian group H, consider the epimorphism

$$\varphi \colon B(n, H) \to H$$
$$[h_1, h_2, \ldots, h_n] \mapsto h_1 h_2 \ldots h_n.$$

with $\ker \varphi = B_0(n, H)$. Hence, $B(n, H)/B_0(n, H)$ is isomorphic to H as Ore mentioned in [18].

The question is to ask whether this is the case in limit monomial groups and we investigate this quotient over various abelian groups.

Lemma 4.1.2. *[2, Lemma 5.2] Let C_p be the cyclic group of order p where p is a prime. If $p^\infty | \lambda$, then $B_0(\lambda, C_p) = B(\lambda, C_p)$. Otherwise,*

$$B(\lambda, C_p) \big/ B_0(\lambda, C_p) \cong C_p.$$

Proof. The group of λ-periodic multiplications $B(\lambda, C_p)$ is abelian and every element has order p. Therefore, it is isomorphic to a vector space over the field $\mathbb{F}_p$ and

$B_0(\lambda, C_p)$ is a subgroup of $B(\lambda, C_p)$. So $B(\lambda, C_p)/B_0(\lambda, C_p)$ is a vector space of dimension greater than 2.

Then there exists elements $v_1, v_2, \ldots, v_{p^2}$ in $B(\lambda, C_p)$ such that

$$\mathbb{Z}_p \times \mathbb{Z}_p \cong \langle \overline{v_1}, \overline{v_2}, \ldots, \overline{v_{p^2}} \rangle \leq B(\lambda, C_p)/B_0(\lambda, C_p)$$

Since $B(\lambda, C_p) = \bigcup B((n_i), C_p)$ where $B((n_i), C_p) = d(B(n_i, C_p))$, there exists n_j such that

$$\langle v_1, v_2, \ldots, v_{p^2} \rangle \leq B((n_j), C_p).$$

Then there exists v_l, v_s such that they are linearly independent and $\langle v_l, v_s \rangle = \langle v_1, v_2, \ldots, v_{p^2} \rangle$ and $v_l, v_s \notin B_0(\lambda, C_p)$. But then

$$B((n_j), C_p)/B_0((n_j), C_p) \cong B((n_j), C_p)/B_0((n_j), C_p)$$

contains a subspace of dimension 2. However, by [18], $B((n_j), C_p)/B_0((n_j), C_p)$ is isomorphic to C_p.

Hence, $B(\lambda, C_p)/B_0(\lambda, C_p)$ is either isomorphic to C_p or $B(\lambda, C_p) = B_0(\lambda, C_p)$.

Indeed, if $p^\infty | \lambda$, given any element $b \in B(\lambda, C_p) = \bigcup B(n_i, C_p)$, $b \in B(n_i, C_p)$ for some n_i. But as $p^\infty | \lambda$, there exists $n_j > n_i$ such that $d^{p_{i+1} \cdots p_j}(b)$ lies in $B(n_j, C_p)$. Hence, in this case,

$$B_0(\lambda, C_p) = B(\lambda, C_p).$$

If $p^\infty \nmid \lambda$, say $p^i | \lambda$ and $p^{i+1} \nmid \lambda$, then the element

$$b_1 = [g, 1, \ldots, 1_{n_{i+1}}, g, 1, \ldots, 1_{n_{i+1}}, \ldots] \notin B_0(\lambda, C_p).$$

Hence, by the above proof, we have

$$B(\lambda, C_p)/B_0(\lambda, C_p) \cong C_p.$$

□

Lemma 4.1.3. *The only normal subgroup of $\Sigma_\lambda(C_p)$ contained in $B(\lambda, C_p)$ is $B_0(\lambda, C_p)$.*

Proof. Assume that N is a normal subgroup of $\Sigma_\lambda(C_p)$ contained in the base group and $N \neq B_0(\lambda, C_p)$. Let

$$\mu = [h_1, h_2, \ldots, h_{n_i} | h_1, h_2, \ldots]$$

be a non-identity element of N. Then

$$\mu^{-1}\mu^{(1\ j)(n_i+1\ n_i+j)\ldots} = [h_1^{-1}h_j, 1, \ldots, 1, h_j^{-1}h_1, 1, \ldots, 1 | h_1^{-1}h_j, 1, \ldots, 1, h_j^{-1}h_1, 1, \ldots, 1, \ldots]$$

is an element of N for $j = 2, \ldots, n_i$. As $h_1^{-1}h_jh_j^{-1}h_1 = 1$, $\mu^{-1}\mu^{(1\ j)(n_i+1\ n_i+j)\ldots}$ is also an element of $B_0(\lambda, C_p)$ and hence,

$$N \cap B_0(\lambda, C_p) \neq E.$$

By the above product, N contains elements of the form

$$[1, \ldots, 1, s, 1 \ldots, 1, \ldots, 1, s^{-1}, 1, \ldots 1_{n_i} | \ldots].$$

As N is a normal subgroup of $\Sigma_\lambda(H)$, it is invariant under the conjugation by permutations. Let $\pi \in S_\lambda$ and $[h_1, h_2, \ldots, h_{n_i}, | \ldots] \in N$. Then

$$\pi[h_1, h_2, \ldots, h_{n_i}, | \ldots]\pi^{-1} = [h_{1^\pi}, h_{2^\pi}, \ldots, h_{n_i^\pi}, | \ldots].$$

As $h_1h_2 \ldots h_{n_i} = 1$ and H is abelian, $h_{1^\pi}h_{2^\pi} \ldots h_{n_i^\pi} = 1$ and so it is always possible to re-place the factors of the elements in N. Hence, these factors form a subgroup S of C_p and N consists of elements of the form

$$[s_1, s_2, \ldots, s_{n_i-1}, (s_1s_2 \ldots s_{n_i-1})^{-1}].$$

As N is non-trivial, S is non-trivial, which implies that $S = C_p$. Hence, we have

$$B_0(\lambda, C_p) \subset N.$$

As $B_0(\lambda, C_p)$ is of index p in $B(\lambda, C_p)$ and $B_0(\lambda, C_p) \subset N$, $N = B(\lambda, C_p)$.

□

Now, let C_n be a cyclic group of odd order. We can consider $B(\lambda, C_n)$ as an S_λ-module and $B_0(\lambda, C_n)$ as a submodule. Then we have the following:

Proposition 4.1.2. *[2, Proposition 5.4] Let C_n be a cyclic group of odd order. $B_0(\lambda, C_n)$ is a maximal cyclic S_λ-submodule of $B(\lambda, C_n)$ and the element*

$$[\underbrace{a, 1, 1, \ldots a^{-1}}_{p}, a, 1, 1, \ldots a^{-1} \ldots]$$

generates $B_0(\lambda, C_n)$ where $C_n = \langle a \rangle$ and p is the smallest prime dividing λ.

Proof. Let C_n be a cyclic group of odd order generated by a. Let $\xi = (n_1, n_2, \ldots.)$ be a divisible sequence with $char(\xi) = \lambda$. Let

$$\bar{a} = [a, a^{-1}, 1, \ldots, 1, a, a^{-1}, 1, \ldots, 1, \ldots]$$

be an element of $B_0(\lambda, C_n)$ satisfying $\bar{a}$ has period p and $p = n_1$ is the smallest prime dividing λ. We may choose such an element because λ is given and divisible sequences is uniquely determined by $char(\xi) = \lambda$.

If necessary, we may change the order of primes in ξ. So $\bar{a} \in B_0((n_1), C_n) \leq B_0(\lambda, C_n)$ where $B_0((n_1), C_n) = d(B_0(n_1, C_n))$.

Consider

$$\bar{a}_{n_i} = [\underbrace{a, a^{-1}, 1, \ldots, 1}_{n_i}, \ldots.]$$

for some $n_i | \lambda$. Then

$$\bar{a}' = (\bar{a})^{d((n_1+1\ n_1+2)(2n_1+1\ 2n_1+2)\ldots.((\frac{n_i}{n_1}-1)n_1+1\ (\frac{n_i}{n_1}-1)n_1+2))}$$

$$= [\underbrace{\underbrace{a, a^{-1}, 1, \ldots, 1}_{n_1}, a^{-1}, a, 1, \ldots, 1, \ldots., a^{-1}, a, 1, \ldots, 1}_{n_i}, \ldots.]$$

and

$$\bar{a}\bar{a}' = [\underbrace{a^2, a^{-2}, 1, \ldots., 1}_{n_i}, a^2, a^{-2}, 1, \ldots., 1, \ldots.].$$

As $(2, |C|) = 1$, a^2 is also a generator of C and so $a = (a^2)^t$ for some $t \in \mathbb{Z}$. Hence, we get

$$(\bar{a}\bar{a}')^t = \bar{a}_{n_i}.$$

Let $\bar{b} = [b_1, b_2, \ldots, b_{n_i}, b_1, b_2, \ldots]$ be an arbitrary element in $B_0(\lambda, C)$ with $b_1 b_2 \ldots b_{n_i} = 1$. As $\langle a \rangle = C$, $b_j = a^{k_j}$ for some $k_j \in \mathbb{Z}$, $1 \leq j \leq n_i - 1$. Our aim is to write $\bar{b}$ in terms of $\bar{a}_{n_i}$ and hence, in terms of $\bar{a}$. We have

$$\bar{a}'' = (\bar{a}_{n_i})^{d((2\ n_i))} = [\underbrace{a, 1, \ldots, 1, a^{-1}}_{n_i}, a, 1, \ldots, 1, a^{-1}, \ldots].$$

To obtain the first factor in $\bar{b}$, we take the k_1^{th}-power of $\bar{a}_{n_i}$ as $b_1 = a^{k_1}$. For the second component, we first apply the permutation $d(1\ 2)$ to $\bar{a}_{n_i}$ to obtain a in the second factor of $\bar{a_{n_i}}$ and then take the k_2^{th}-power. We continue like this until $(n_i - 1)^{st}$ factor. Since $\bar{b}$ is in $B_0(\lambda, C_n)$ and $b_1 b_2 \ldots b_{n_i} = 1$, we have $b_{n_i} = (b_1)^{-1}(b_2)^{-1} \ldots (b_{n_{i-1}})^{-1}$ and so the last component is uniquely determined by the first $n_i - 1$ components.

Thus,

$$\bar{b} = (\bar{a}'')^{k_1} (\bar{a}''^{d(1\ 2)})^{k_2} \ldots (\bar{a}''^{d(1\ n_i-1)})^{k_{n_i-1}}.$$

and $B_0(\lambda, C)$ is a cyclic S_λ-submodule of $B(\lambda, C)$, generated by

$$\bar{a} = [\underbrace{a, a^{-1}, 1, \ldots, 1}_{n_1}, a, a^{-1}, 1, \ldots, 1, \ldots].$$

□

Lemma 4.1.4. *Let C_{p^k} be a cyclic group of order p^k where p is a prime, $k \in \mathbb{N}$. If $p^\infty | \lambda$, then $B_0(\lambda, C_{p^k}) = B(\lambda, C_{p^k})$. If not,*

$$B(\lambda, C_{p^k}) \big/ B_0(\lambda, C_{p^k}) \cong C_{p^k}.$$

Proof. If p^∞ divides λ, then $B(\lambda, C_{p^k}) = B_0(\lambda, C_{p^k})$ as the order of every element of C_{p^k} is a power of p.

Assume that p^∞ does not divide λ. Let g be a generator of C_{p^k}. Let

$$\mu = [h_1, h_2, \ldots, h_{n_i} | h_1, h_2, \ldots] B_0(\lambda, C_{p^k})$$

be a non-identity element of $B(\lambda, C_{p^k}) \big/ B_0(\lambda, C_{p^k})$. As μ is non-identity,

$$(h_1 h_2 \ldots h_{n_i})^{\frac{n_k}{n_i}} \neq 1$$

for any $n_k \geq n_i$.

As g is a generator of C_{p^k}, $h_1 h_2 \dots h_{n_i} = g^t$ for some $t = 1, 2, \dots, p-1$. Consider

$$\eta = [g, 1, \dots, 1|g, 1, \dots, 1, \dots]B_0(\lambda, C_p)$$

of period n_j where n_j is the smallest integer in ξ such that $p \nmid \frac{\lambda}{n_j}$. Our claim is that η is a generator of ${B(\lambda, C_{p^k})}/{B_0(\lambda, C_{p^k})}$. As g is of order p^k, η is also of order p^k.

Consider $\beta_1 = [g_1, g_2, \dots, g_{n_j}, |g_1, g_2, \dots]B_0(\lambda, C_{p^k})$. Then $g_1 g_2 \dots g_{n_j} = g^t$ for some $t = 1, 2, \dots, p^k$. Then $\beta_1 = \eta^t$ by the same argument in the case of C_p.

Now, consider $\beta_2 = [g_1, g_2, \dots, g_{n_{j+1}}|g_1, g_2, \dots]B_0(\lambda, C_{p^k})$ with

$$g_1 g_2 \cdots g_{n_{j+1}} = g^t$$

for some $t = 1, 2, \dots, p^k$. In the $n_{j+1} - th$ step, the product of factors in η is $g^{p_{j+1}}$ where p_{j+1} is a prime different from p by the choice of n_j. So the greatest common divisor of p and p_{j+1} is 1 and hence, there exist $m, n \in \mathbb{Z}$ such that

$$mp^k + np_{j+1} = 1.$$

So we have

$$g = g^{mp^k + np_{j+1}} = (g^{p^k})^m (g^{p_{j+1}})^n = (g^{p_{j+1}})^n,$$

$$g^t = (g^{p_{j+1}})^{nt}$$

which implies that $\eta^{nt} = \beta_2$.

□

With the following lemma, we give an example of the fact that the quotient group $B(\lambda, H)/B_0(\lambda, H)$ is not always isomorphic to H.

Lemma 4.1.5. *[2, Lemma 6.1]*

(i) Let $\Sigma_\lambda(C)$ be the symmetry for any Steinitz number λ and C be the infinite cyclic group generated by c. Then the quotient group ${B(\lambda, C)}/{B_0(\lambda, C)}$ is not isomorphic to C.

(ii) $B(\lambda, C)/B_0(\lambda, C)$ is an abelian, torsion free locally cyclic group.

Proof. As C is the infinite cyclic group generated by c, the multiplication

$$[c, 1, \ldots, 1, c, 1, \ldots, 1, \ldots]$$

of period n_1 is not in the subgroup $B_0(\lambda, C)$ and so $B(\lambda, C) \neq B_0(\lambda, C)$.

Suppose to the contrary that $\alpha = [c^{k_1}, c^{k_2}, \ldots, c^{k_{n_i}}, \ldots]B_0(\lambda, C)$ is a generator of $B(\lambda, C) \big/ B_0(\lambda, C)$. Then

$$\alpha = [\underbrace{c^t, 1, \ldots, 1}_{n_i}, c^t, 1, \ldots, 1, \ldots]B_0(\lambda, C)$$

where $t = k_1 + k_2 + \ldots + k_{n_i}$. Consider the element

$$\beta = [\underbrace{c, 1, \ldots, 1}_{n_i}, c, 1, \ldots, 1, \ldots]B_0(\lambda, C).$$

Then since α is a generator, $\beta = \alpha^k$ for some $k \in \mathbb{Z}$ and this implies that

$$[c^{kt-1}, 1, \ldots, 1, \ldots] \in B_0(\lambda, C)$$

and so

$$(c^{kt-1})^{p_{i+1} \cdots p_s} = 1$$

for some $s \geq i$. As c is infinite order, $kt = 1$ and so $k = t = \pm 1$,

$$\alpha = [\underbrace{c, 1, \ldots, 1}_{n_i}, c, 1, \ldots, 1, \ldots]B_0(\lambda, C).$$

Moreover, we take the element $\beta' = [\underbrace{c, 1, \ldots, 1}_{n_{i+1}}, c, 1, \ldots, 1, \ldots]B_0(\lambda, C)$ and so $\beta' = \alpha^k$ for some $k \in \mathbb{Z}$. Similarly,

$$[\underbrace{c^k, 1, \ldots, 1}_{n_i}, c^k, 1, \ldots, 1, \ldots][\underbrace{c^{-1}, 1, \ldots, 1}_{n_{i+1}}, \ldots] \in B_0(\lambda, C)$$

which implies that

$$(c^{k-1}(c^k)^{p_i - 1})^{p_{i+1} \cdots p_s} = 1,$$

$$k - 1 + k(p_i - 1) = 0 \quad \text{and} \quad kp_i = 1$$

as c is infinite order. Since p_i is a prime, this is a contradiction.

Hence, this quotient is not isomorphic to C.

We know that as c is infinite order, ${B(\lambda, C)}\big/{B_0(\lambda, C)}$ is torsion-free. Also, this group is locally cyclic since

$$ {B(\lambda, C)}\big/{B_0(\lambda, C)} = \bigcup_i {B(n_i, C)B_0(\lambda, C)}\big/{B_0(\lambda, C)}, $$

by second isomorphism theorem,

$$ {B(n_i, C)B_0(\lambda, C)}\big/{B_0(\lambda, C)} \cong {B(n_i, C)}\big/{B_0(n_i, C)} $$

and

$$ {B(n_i, C)}\big/{B_0(n_i, C)} \cong C $$

by theorem 4 in Chapter 2 in [18].

□

Theorem 4.1.2. *[2, Theorem 6.2] Let* $\lambda = \Pi_{p\in\mathbb{P}} p^{r_p}$ *be a Steinitz number and let* $\xi = (p_1, p_2, \ldots, p_i, \ldots)$ *be a sequence of primes with* $char\xi = \lambda$ *and* $n_1 = p_1$, $n_i = p_1 p_2 \ldots p_i$. *Then*

$$ {B(\lambda, C)}\big/{B_0(\lambda, C)} \cong \mathbb{Q}_\lambda $$

where $\mathbb{Q}_\lambda = \langle \frac{1}{p^{t_p}} \;\mid\; 0 \leq t_p \leq r_p, \;\; p|\lambda \rangle$.

Proof. For each $n_i|\lambda$, by Second Isomorphism Theorem and [18],

$$ {B(n_i, C)B_0(\lambda, C)}\big/{B_0(\lambda, C)} \cong {B(n_i, C)}\big/{B_0(n_i, C)} \cong C $$

and generated by $\mu_i = [\underbrace{c, 1, \ldots, 1}_{n_i}, \underbrace{c, 1, \ldots, 1}_{n_i}, \ldots]B_0(\lambda, C)$.

Define $\theta : {B(\lambda, C)}\big/{B_0(\lambda, C)} \to \mathbb{Q}_\lambda$ by $\theta(\mu_i^k) = \dfrac{k}{n_i}$ for each $n_i|\lambda$ and $k \in \mathbb{Z}$.

<u>θ is well-defined:</u> Let

$$ \eta_1 = [c_1, c_2, \ldots, c_{n_i}|c_1, c_2, \ldots]B_0(\lambda, C) $$

and

$$ \eta_2 = [d_1, d_2, \ldots, d_{n_j}|d_1, d_2, \ldots]B_0(\lambda, C) $$

be elements of $B(\lambda,C)/_{B_0(\lambda,C)}$ such that $\eta_1 = \eta_2$. We have $c_1c_2\dots c_{n_i} = c^{k_i}$ and $\underbrace{(c_1c_2\dots c_{n_i})\dots(c_1c_2\dots c_{n_i})}_{p_ip_{i+1}\dots p_j} = c^{k_ip_ip_{i+1}\dots p_j}$ and $d_1d_2\dots d_{n_j} = c^{k_j}$ for some $k_i, k_j \in \mathbb{Z}$ and so $\eta_1 = \mu_i^{k_i}$ and $\eta_2 = \mu_j^{k_j}$ and $k_ip_ip_{i+1}\dots p_j = k_j$ as c is of infinite order. Then

$$\theta(\eta_1) = \theta(\mu_i^{k_i}) = \frac{k_i}{n_i} = \frac{k_ip_ip_{i+1}\dots p_j}{n_ip_ip_{i+1}\dots p_j} = \frac{k_ip_ip_{i+1}\dots p_j}{n_j} = \frac{k_j}{n_j} = \theta(\mu_j^{k_j}) = \theta(\eta_2).$$

<u>θ is a homomorphism:</u> Let η_1 and η_2 as above.Then

$$\theta(\eta_1\eta_2) = \theta(\mu_i^{k_i}\mu_j^{k_j}) = \theta(\mu_j^{k_ip_ip_{i+1}\dots p_j+k_j}) = \frac{k_ip_ip_{i+1}\dots p_j + k_j}{n_j}$$

$$= \frac{k_ip_ip_{i+1}\dots p_j}{n_j} + \frac{k_j}{n_j} = \frac{k_i}{n_i} + \frac{k_j}{n_j} = \theta(\eta_1) + \theta(\eta_2).$$

<u>θ is an injection:</u>

$$\ker\theta = \{[c_1, c_2, \dots, c_{n_i}, \dots]B_0(\lambda,C) \in B(\lambda,C)/_{B_0(\lambda,C)} \mid c_1c_2\dots c_{n_i} = 1 \text{ for some } n_i|\lambda\}$$

$$\ker\theta = B_0(\lambda,C).$$

<u>θ is onto:</u> Let q be an arbitrary element of $\mathbb{Q}_\lambda$. Then there exists $p_1, p_2, \dots, p_l$ and $t_{p_1}, t_{p_2}, \dots, t_{p_l}$ such that $p_j^{t_{p_j}}$ divides λ and

$$q = \frac{k_1}{p_1^{t_{p_1}}} + \dots + \frac{k_l}{p_l^{t_{p_l}}} = \frac{k}{p_1^{t_{p_1}}p_2^{t_{p_2}}\dots p_l^{t_{p_l}}}.$$

There exists $n_i|\lambda$ such that $p_1^{t_{p_1}}p_2^{t_{p_2}}\dots p_l^{t_{p_l}}|n_i$ and $n_i = p_1^{t_{p_1}}p_2^{t_{p_2}}\dots p_l^{t_{p_l}}s$ for some $s \in \mathbb{Z}$. Then

$$q = \frac{ks}{n_i} \quad \text{and so} \quad \theta(\mu_i^{ks}) = q.$$

Hence, θ is an isomorphism.

□

Remark: Let $\{G_i|i \in I\}$ be set of groups and $\{N_i|i \in I\}$ be the set of normal subgroups, respectively. Then

$$Dr_{i\in I}G_i/Dr_{i\in I}N_i \cong Dr_{i\in I}G_i/N_i$$

by the epimorphism taking $(g_i)_{i\in I}$ to $(g_iN_i)_{i\in I}$ with kernel $Dr_{i\in I}N_i$.

It is easy to see that $B(\lambda, Dr_{i\in I}G_i) \cong Dr_{i\in I}B(\lambda, G_i)$. Using the remark above, we get

$$B(\lambda, Dr_{i\in I}G_i)/B_0(\lambda, Dr_{i\in I}G_i) \cong Dr_{i\in I}B(\lambda, G_i)/Dr_{i\in I}B_0(\lambda, G_i)$$

and so

$$B(\lambda, Dr_{i\in I}G_i)/B_0(\lambda, Dr_{i\in I}G_i) \cong Dr_{i\in I}(B(\lambda, G_i)/B_0(\lambda, G_i)).$$

We know that an abelian group G is finitely generated if and only if it is a direct sum of finitely many cyclic groups of infinite or prime-power orders.

By using the above observations, we prove the following theorem to determine the structure of the quotient over finitely generated abelian groups.

Theorem 4.1.3. *Let λ be a Steinitz number and $\Sigma_\lambda(H)$ be the limit monomial group where H is a finitely generated abelian group and $H \cong C^r \times \Pi_{i=1}^k C_{{p_i}^{t_i}}$. Then*

$$B(\lambda, H)\big/B_0(\lambda, H) \cong \mathbb{Q}_\lambda^r \times \Pi_{i=1}^k B(\lambda, C_{{p_i}^{t_i}})\big/B_0(\lambda, C_{{p_i}^{t_i}})$$

where $B(\lambda, C_{{p_i}^{t_i}})\big/B_0(\lambda, C_{{p_i}^{t_i}}) \cong C_{{p_i}^{t_i}}$ if ${p_i}^\infty \nmid \lambda$ and $B(\lambda, C_{{p_i}^{t_i}})\big/B_0(\lambda, C_{{p_i}^{t_i}})$ is trivial if ${p_i}^\infty|\lambda$.

Proof. The proof follows from the above remark, Lemma (4.1.4) and Theorem (4.1.5).

□

Example 4.1.4. Let λ be an infinite Steinitz number and $\Sigma_\lambda(F_{ab}(S))$ be the limit monomial group where $F_{ab}(S)$ is free abelian group generated by the set S. Since $F_{ab}(S)$ is free abelian group generated by S, $F_{ab}(S) \cong \bigoplus_{s\in S}\mathbb{Z}$. Then

$$B(\lambda, F_{ab}(S))\big/B_0(\lambda, F_{ab}(S)) \cong \bigoplus_{s\in S}\mathbb{Q}_\lambda$$

by the above remark.

Lastly, the quotient $B(\lambda, H)/B_0(\lambda, H)$ is investigated over divisible abelian groups.

Lemma 4.1.6. *Let C_{p^∞} be the Prüfer-p-group where p is a prime. If $p^\infty|\lambda$, then $B_0(\lambda, C_{p^\infty}) = B(\lambda, C_{p^\infty})$. Otherwise,*

$$B(\lambda, C_{p^\infty})\big/B_0(\lambda, C_{p^\infty}) \cong C_{p^\infty}.$$

Proof. If $p^\infty | \lambda$, since C_{p^∞} is a periodic abelian p-group , every element of $B(\lambda, C_{p^\infty})$ is of a p-power order and hence $B_0(\lambda, C_{p^\infty}) = B(\lambda, C_{p^\infty})$.

Assume that $p^\infty \nmid \lambda$. Let $\xi = (n_1, n_2, \ldots)$ be a sequence satisfying $n_i | n_{i+1}$ for all $i \in \mathbb{N}$ with $char(\xi) = \lambda$. Let the maximum p-power in λ be r_p and n_i be the smallest natural number in ξ such that $p^{r_p} | n_i$ and $p^{r_p+1} \nmid n_{i+1}$. Consider the non-identity element

$$\alpha = [h, 1, \ldots, 1_{n_i}, h, 1, \ldots, 1_{n_i}, \ldots] \in B(\lambda, C_{p^\infty}).$$

Since $p \nmid \dfrac{\lambda}{n_i}$ and $h \in C_{p^\infty}$ is an element of p-power order, α is not in the subgroup $B_0(\lambda, C_{p^\infty})$ and so $B(\lambda, C_{p^\infty}) \neq B_0(\lambda, C_{p^\infty})$.

Since ${B(n_i, C_{p^\infty})}/{B_0(n_i, C_{p^\infty})} \cong C_{p^\infty}$ by [18], ${B(\lambda, C_{p^\infty})}/{B_0(\lambda, C_{p^\infty})}$ is a locally cyclic periodic p-group. As locally cyclic periodic abelian p-groups are unique up to isomorphism [10, p.47], this quotient is isomorphic to C_{p^∞}.

$\square$

Lemma 4.1.7. *Let H be a torsion free divisible abelian group. Then*

$$ {B(\lambda, H)}/{B_0(\lambda, H)} \cong H. $$

Proof. Since H is an torsion free divisible abelian group, for every h in H and $n \in \mathbb{N}$, there exists unique $g \in H$ such that $h = g^n$.

Define the map $\theta : {B(\lambda, H)}/{B_0(\lambda, H)} \to H$ by

$$\theta([h_1, h_2, \ldots, h_{n_i}, h_1, h_2, \ldots]B_0(\lambda, H)) = h$$

where h is the unique n_i^{th}-root of $h_1 h_2 \ldots h_{n_i}$.

Note that $h_1 h_2 \ldots h_{n_i} = h^{n_i}$ implies

$$(h_1 h_2 \ldots h_{n_i})^{p_{i+1}p_{i+2}\ldots p_j} = h^{n_i p_{i+1} p_{i+2} \ldots p_j} = h^{n_j}.$$

If $h^{n_i p_{i+1} p_{i+2} \ldots p_j} = g^{n_j}$, then $(hg^{-1})^{n_j} = 1$ and since H is aperiodic, $h = g$.

<u>θ is well-defined:</u> Let μ, η in ${B(\lambda, H)}/{B_0(\lambda, H)}$ such that $\mu = \eta$ where $\mu = [h_1, h_2, \ldots, h_{n_i}, h_1, h_2, \ldots]B_0(\lambda, H)$ and $\eta = [k_1, k_2, \ldots, k_{n_j}, \ldots]B_0(\lambda, H)$ such

that $h_1 h_2 \dots h_{n_i} = h^{n_i}$ and $k_1 k_2 \dots k_{n_j} = k^{n_j}$. Such h and k exist and unique since H is aperiodic divisible. Since $\mu = \eta$,

$$[h_1, h_2, \dots, h_{n_i}, h_1, h_2, \dots][k_1, k_2, \dots, k_{n_j}, \dots]^{-1} \in B_0(\lambda, H)$$

and so $(h_1 h_2 \dots h_{n_i})^{p_{i+1}p_{i+2}\dots p_j}(k_1 k_2 \dots k_{n_j})^{-1} = 1$ which implies that

$$h^{p_{i+1}p_{i+2}\dots p_j} = k.$$

Then

$$\theta(\eta) = k = h^{p_{i+1}p_{i+2}\dots p_j} = \theta([\underbrace{h_1, h_2, \dots, h_{n_i}, \dots, h_1, h_2, \dots, h_{n_i}}_{n_j}, \dots]B_0(\lambda, H)) = \theta(\mu).$$

<u>θ is a homomorphism:</u> Let μ and η be as above. Then

$$\theta(\mu\eta) = h^{p_{i+1}p_{i+2}\dots p_j}k = \theta(\mu)\theta(\eta).$$

<u>θ is one-to-one:</u>

$$\ker\theta = \{[h_1, h_2, \dots, h_{n_i}, h_1, h_2, \dots]B_0(\lambda, H) \mid h = 1, \text{ where } h_1 h_2 \dots h_{n_i} = h^{n_i}\}$$

$$= \{[h_1, h_2, \dots, h_{n_i}, h_1, h_2, \dots]B_0(\lambda, H) \mid h_1 h_2 \dots h_{n_i} = 1\} = B_0(\lambda, H).$$

<u>θ is onto:</u> Let $h \in H$ arbitrary. Then

$$\theta([\underbrace{h^{n_i}, 1, 1, \dots, 1}_{n_i}, h^{n_i}, 1, 1, \dots, 1, \dots]B_0(\lambda, H)) = h.$$

□

Since a divisible abelian group is a direct sum of groups each of which is isomorphic to a Prüfer p-group or $\mathbb{Q}$ [9, Theorem 9.1.6], the structure of the quotient is determined for all divisible abelian groups.

Theorem 4.1.5. *Let H be a divisible abelian group. Then*

$$B(\lambda, H)/B_0(\lambda, H) \cong \bigoplus_{p \in P'} C_{p^\infty} \oplus \bigoplus_{i \in I} \mathbb{Q}$$

where $H \cong \bigoplus_{p \in P} C_{p^\infty} \oplus \bigoplus_{i \in I} \mathbb{Q}$, P is a set of primes and $P' \subset P$ is a set of primes such that $p^\infty \nmid \lambda$.

Proof. The proof follows from remark above, Lemma 4.1.6 and Lemma 4.1.7 immediately. □

Example 4.1.6. As an example, we know that the additive group of a field $\mathbb{F}$ of characteristic 0 is a torsion free divisible abelian group since it contains a subfield isomorphic to $\mathbb{Q}$ and it is isomorphic to a direct sum of additive group of rationals $\mathbb{Q}$. Hence, the quotient is determined for a field of characteristic 0.

By using above observation, we consider the polynomial ring $\mathbb{F}[x]$ over a field $\mathbb{F}$ of characteristic 0. We know that

$$\mathbb{F}[x] \cong \bigoplus_{i\in\mathbb{N}} \mathbb{F}$$

and

$$\mathbb{F} \cong \bigoplus_{i\in I} \mathbb{Q} \text{ for some index set } I.$$

These imply that

$$B(\lambda, \mathbb{F}[x]) \big/ B_0(\lambda, \mathbb{F}[x]) \cong \bigoplus_{i\in N} \bigoplus_{j\in I} \mathbb{Q}.$$

4.2 Lattice Structures on Some Subgroups

In this section, we consider the subgroups $\mathbb{Q}_\lambda$, $\lambda \in \mathbb{SN}$, of the rational numbers $\mathbb{Q}$, which is mentioned in Theorem (4.1.2). We prove that there is a lattice isomorphism between the lattice of subgroups $\mathbb{Q}_\lambda$ of $\mathbb{Q}$ and the lattice of Steinitz numbers $\mathbb{SN}$.

Proposition 4.2.1. *The set of subgroups of $\mathbb{Q}$ of the form $\mathbb{Q}_\lambda$, $\lambda \in \mathbb{SN}$, is exactly the set of subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$.*

Proof. For every $\lambda \in \mathbb{SN}$ and for a prime $p|\lambda$, since $1 = \dfrac{1}{p^0} \in \mathbb{Q}_\lambda$, $\mathbb{Z}$ is contained in $\mathbb{Q}_\lambda$. Let A be a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ and $\dfrac{n}{m} \in A$. We can take n and m relatively prime. Then there exists $r, s \in \mathbb{Z}$ such that

$$rn + sm = 1$$

and so

$$r\frac{n}{m} + s = \frac{1}{m} \in A.$$

We need to show that every $\frac{n}{m} \in A$ lies in $\mathbb{Q}_\lambda$ for some $\lambda \in \mathbb{SN}$. Use induction on the number of primes in the prime factorization of the denominators of the elements in A. Hence, we need to show that

$$\frac{1}{p_1^{t_1} \dots p_m^{t_m}} = \frac{k_1}{p_1^{t_1}} + \dots + \frac{k_m}{p_m^{t_m}}.$$

Let $\frac{1}{p_1^{t_1} p_2^{t_2}} \in A$. Since $(p_1^{t_1}, p_2^{t_2}) = 1$, there exists $r, s \in \mathbb{Z}$ such that $rp_1^{t_1} + sp_2^{t_2} = 1$ and so $\frac{1}{p_1^{t_1} p_2^{t_2}} = \frac{r}{p_2^{t_2}} + \frac{s}{p_1^{t_1}}$.

Assume that there exists $k_1, \dots, k_m \in \mathbb{Z}$ such that

$$\frac{1}{p_1^{t_1} \dots p_m^{t_m}} = \frac{k_1}{p_1^{t_1}} + \dots + \frac{k_m}{p_m^{t_m}}.$$

Let $\frac{1}{p_1^{t_1} \dots p_{m+1}^{t_{m+1}}} \in A$. As $(p_1^{t_1} \dots p_m^{t_m}, p_{m+1}^{t_{m+1}}) = 1$, there exist $r, s \in \mathbb{Z}$ such that $rp_1^{t_1} \dots p_m^{t_m} + sp_{m+1}^{t_{m+1}} = 1$ and so

$$\frac{r}{p_{m+1}^{t_{m+1}}} + \frac{s}{p_1^{t_1} \dots p_m^{t_m}} = \frac{1}{p_1^{t_1} \dots p_m^{t_m} p_{m+1}^{t_{m+1}}}.$$

By induction hypothesis,

$$\frac{1}{p_1^{t_1} \dots p_m^{t_m} p_{m+1}^{t_{m+1}}} = \frac{r}{p_{m+1}^{t_{m+1}}} + \frac{s_1}{p_1^{t_1}} + \dots + \frac{s_m}{p_m^{t_m}}.$$

Hence, every subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ is of the form $\mathbb{Q}_\lambda$ for some $\lambda \in \mathbb{SN}$. $\square$

These are not all subgroups of $\mathbb{Q}$ and there are subgroups of $\mathbb{Q}$ which does not contain $\mathbb{Z}$. For example, consider the subgroup generated by $\frac{2}{3}$, $\langle \frac{2}{3} \rangle$ as a subgroup not containing $\mathbb{Z}$.

Proposition 4.2.2. *The set* $\{\mathbb{Q}_\lambda \mid \lambda \in \mathbb{SN}\}$ *of subgroups of* $\mathbb{Q}$ *is a complete lattice with respect to inclusion.*

Proof. Let $\mathbb{Q}_\lambda$ and $\mathbb{Q}_\mu$ be subgroups of $\mathbb{Q}$ corresponding to the Steinitz numbers λ and μ, respectively. Assume that λ divides μ. Let q_λ be an element of $\mathbb{Q}_\lambda$. Then there exist $p_1, p_2, \dots, p_j$ in $\mathbb{P}$ and $t_{p_1}, t_{p_2}, \dots, t_{p_j} \in \mathbb{N} \cup \{0\}$ and $k_1, k_2, \dots, k_j \in \mathbb{Z}$ such that

$$q_\lambda = \frac{k_1}{p_1^{t_{p_1}}} + \frac{k_2}{p_2^{t_{p_2}}} + \dots + \frac{k_j}{p_j^{t_{p_j}}}$$

and each $p_i^{t_{p_i}}|\lambda$. Since $\lambda|\mu$, each $p_i^{t_{p_i}}|\mu$ and so q_λ is also an element of $\mathbb{Q}_\mu$.

By this observation, if $\lambda|\mu$, then $\mathbb{Q}_\lambda$ is a subgroup of $\mathbb{Q}_\mu$.

The set of subgroups of $\mathbb{Q}$ is a complete lattice with respect to inclusion with the greatest element $\mathbb{Q}$ and the least element as trivial subgroup $\{0\}$ where, for subgroups A and B,

$$A \wedge B = A \cap B \quad \text{and} \quad A \vee B = \langle A, B\rangle.$$

The lattice structure on the subgroups of $\mathbb{Q}$ gives a lattice structure on the subgroups of the form $\mathbb{Q}_\lambda$ with the greatest element $\mathbb{Q}_I$, where $I = \Pi_{p\in\mathbb{P}}p^\infty$, and the least element $\mathbb{Q}_1 = \mathbb{Z}$ with

$$\mathbb{Q}_\lambda \wedge \mathbb{Q}_\mu = \mathbb{Q}_\lambda \cap \mathbb{Q}_\mu$$

and

$$\mathbb{Q}_\lambda \vee \mathbb{Q}_\mu = \langle \mathbb{Q}_\lambda, \mathbb{Q}_\mu\rangle$$

where

$$\mathbb{Q}_\lambda \cap \mathbb{Q}_\mu = \mathbb{Q}_\eta, \ \ \eta = \prod_{p\in\mathbb{P}} p^{t_p}, \ \ t_p = \min\{r_p, s_p\} \tag{4.1}$$

and

$$\langle \mathbb{Q}_\lambda, \mathbb{Q}_\mu\rangle = \mathbb{Q}_\eta, \ \ \eta = \prod_{p\in\mathbb{P}} p^{t_p}, \ \ t_p = \max\{r_p, s_p\}. \tag{4.2}$$

For (4.1), by the observation above, since η divides both λ and μ, $\mathbb{Q}_\eta$ is a subgroup of $\mathbb{Q}_\lambda \cap \mathbb{Q}_\mu$. For the other side of the inclusion, let q be an element from the intersection. Then there exist primes $p_1, p_2, \ldots, p_n$ dividing both λ and μ and $0 \leq t_{p_1}, t_{p_2}, \ldots, t_{p_n} \leq \min\{r_p, s_p\} = t_p$ such that

$$q = \frac{k}{p_1^{t_{p_1}} \ldots p_n^{t_{p_n}}}$$

and so q lies in $\mathbb{Q}_\eta$.

For (4.2), again by the observation above, since both λ and μ divides η, $\mathbb{Q}_\lambda$ and $\mathbb{Q}_\mu$ are subgroups of $\mathbb{Q}_\eta$ and hence, $\langle \mathbb{Q}_\lambda, \mathbb{Q}_\mu\rangle$ is a subgroup of $\mathbb{Q}_\eta$.

Let q be an element in $\mathbb{Q}_\eta$. Then there exist primes $p_1, p_2, \ldots, p_n$ dividing λ, primes $q_1, q_2, \ldots, q_m$ dividing μ and $1 \leq t_{p_1}, t_{p_2}, \ldots, t_{p_n}, t_{q_1}, t_{q_2}, \ldots, t_{q_m} \leq \max\{r_p, s_p\} =$

t_p such that

$$q = \frac{k}{p_1^{t_{p_1}} \dots p_n^{t_{p_n}} q_1^{t_{q_1}} \dots q_m^{t_{q_m}}} = \frac{k_1}{p_1^{t_{p_1}} \dots p_n^{t_{p_n}}} + \frac{k_2}{q_1^{t_{q_1}} \dots q_m^{t_{q_m}}} \in \langle \mathbb{Q}_\lambda, \mathbb{Q}_\mu \rangle$$

and so $\mathbb{Q}_\eta$ is contained in $\langle \mathbb{Q}_\lambda, \mathbb{Q}_\mu \rangle$.

Then $\{\mathbb{Q}_\lambda \mid \lambda \in \mathbb{SN}\}$ is a complete lattice with respect to inclusion. □

Moreover, we consider the subgroup lattice of the group ${}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}$.

Recall that $(\mathbb{SN}, |)$ forms a complete lattice with the greatest element $I = \prod_{p \in \mathbb{P}} p^\infty$ and the least element 1 and the definitions of the meet and join of $\lambda = \prod_{p \in \mathbb{P}} p^{r_p}$ and $\mu = \prod_{p \in \mathbb{P}} p^{s_p}$ where $r_p, s_p \in \mathbb{N} \cup \{0, \infty\}$ as follows:

$$\lambda \wedge \mu = \prod_{p \in \mathbb{P}} p^{t_p}, \quad t_p = \min\{r_p, s_p\}$$

and

$$\lambda \vee \mu = \prod_{p \in \mathbb{P}} p^{t_p}, \quad t_p = \max\{r_p, s_p\}.$$

Lemma 4.2.1. *The set $\{\mathbb{Q}_\lambda \mid \lambda \in \mathbb{SN}\}$ of subgroups of $\mathbb{Q}$ and the set $\mathbb{SN}$ of Steinitz numbers are isomorphic as lattices.*

Proof. The map

$$\theta : \mathbb{SN} \to \{\mathbb{Q}_\lambda \mid \lambda \in \mathbb{SN}\} \quad \text{defined by} \quad \theta(\lambda) = \mathbb{Q}_\lambda$$

is a lattice isomorphism. □

Lastly, we study the relation between these subgroups $\mathbb{Q}_\lambda$ of $\mathbb{Q}$ and the quotient group $\mathbb{Q}/\mathbb{Z}$ and their lattice structures. First, we claim the following:

Lemma 4.2.2. *The set of subgroups of $\mathbb{Q}/\mathbb{Z}$ and the set of subgroups $\mathbb{Q}_\lambda$ of $\mathbb{Q}$ are isomorphic as lattices.*

Proof. Let A be a subgroup of ${}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}$ and $\dfrac{n}{m} + \mathbb{Z}$ be an element of A. We can take the representative $\dfrac{n}{m}$ as n and m being relatively prime. As n and m are relatively prime, there exist $r, s \in \mathbb{Z}$ such that

$$rn + sm = 1$$

which implies that

$$r\frac{n}{m}+s=\frac{1}{m}$$

and so

$$\frac{1}{m}+\mathbb{Z}\in A.$$

If $m=p_1^{t_1}p_2^{t_2}\ldots p_k^{t_k}$ for distinct primes p_i and $t_i\geq 1$,

$$\frac{1}{{p_i}^{t_i-j}}+\mathbb{Z}\in A \text{ for } j=0,1,\ldots,t_i.$$

Let A, A' be subgroups of $\mathbb{Q}/\mathbb{Z}$ and $P_A, P_{A'}$ be the sets of all primes such that $\frac{1}{p}+\mathbb{Z}$ is an element of A and A', respectively. Let $r_p=\sup\{t_p \mid \frac{1}{p^{t_p}}+\mathbb{Z}\in A\}$ and $s_p=\sup\{t'_p \mid \frac{1}{p^{t'_p}}+\mathbb{Z}\in A'\}$. Assume that $P_A\subseteq P_{A'}$ and $r_p\leq s_p$ for every prime p. Then A is a subgroup of A'.

For every subgroup A of ${}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}$ with the prime set P_A, there exists a unique Steinitz number $\lambda=\Pi_{p\in P_A}p^{r_p}$ where $r_p=\sup\{t_p \mid \frac{1}{p^{t_p}}+\mathbb{Z}\in A\}$. Moreover, for every Steinitz number $\lambda=\Pi_{p\in\mathbb{P}}p^{r_p}$ where $r_p\in\mathbb{N}\cup\{0,\infty\}$, there corresponds a subgroup of ${}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}$ generated by $\frac{1}{p^{t_p}}$ with $0\leq t_p\leq r_p$.

The meet and the join of two subgroups A and A' of $\mathbb{Q}/\mathbb{Z}$ in terms of Steinitz numbers are as follows:

$$A\wedge A'=A_\eta \quad \text{where} \quad \eta=\prod_{p\in\mathbb{P}}p^{t_p} \quad ,t_p=\min\{r_p,s_p\}, \tag{4.3}$$

$$A\vee A'=A_\eta \quad \text{where} \quad \eta=\prod_{p\in\mathbb{P}}p^{t_p} \quad ,t_p=\max\{r_p,s_p\}. \tag{4.4}$$

In (4.3), A_η is the intersection of the subgroups A and A' and in (4.4), A_η is the subgroup generated by A and A'.

If we denote each subgroup A with A_λ according to the Steinitz number λ, which corresponds to the subgroup A, there is a lattice isomorphism between the lattice of

Steinitz numbers, $(\mathbb{SN}, |)$ and the subgroup lattice of ${}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}$ taking λ to the subgroup A_λ .

Hence, the subgroup lattice of ${}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}$ is isomorphic to the lattice of subgroups of $\mathbb{Q}$ of the form $\mathbb{Q}_\lambda$. $\square$

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