GENERATING THE SURFACE MAPPING CLASS GROUP BY TORSION ELEMENTS

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ABSTRACT

GENERATING THE SURFACE MAPPING CLASS GROUP BY TORSION ELEMENTS

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In this thesis, we study generators of the mapping class group $\mathrm{Mod}(\Sigma_g)$ of a closed connected orientable surface Σ_g of genus g. The mapping class group $\mathrm{Mod}(\Sigma_g)$ is known to be generated by three involutions if $g \geq 8$ and by two torsion elements of order $4g + 2$.

We show that the mapping class group is also generated by three involutions for surfaces with smaller genera and by two torsion elements of smaller orders.

Keywords: Mapping Class Groups, Generators.

ÖZ

YÜZEY GÖNDERİM SINIFLARI GRUBUNUN TORSİYON ELEMANLARLA ÜRETİMİ

Yıldız, Oğuz
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Bu tezde, cinsi g olan kapalı bağlantılı yönlendirilebilir bir yüzeyin Σ_g gönderim sınıfı grubunun $\mathrm{Mod}(\Sigma_g)$ üreteçlerini araştırdık. Gönderim sınıfı grubunun $4g+2$ mertebeli iki torsiyon eleman tarafından ve eğer $g \geq 8$ ise üç involüsyon tarafından üretildiği bilinmektedir.

Bu çalışmanın amacı, gönderim sınıfı grubunun daha düşük mertebeli iki torsiyon eleman tarafından ve daha düşük cinsli yüzeyler için de üç involüsyon tarafından üretildiğini göstermektir.

Anahtar Kelimeler: Gönderim Sınıfı Grupları, Üreteçler.

To Merve

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CHAPTER 1

INTRODUCTION

In this chapter, we briefly introduce the mapping class group of a surface and give an outline of the thesis. Throughout the thesis, we mainly deal with closed (i.e. compact without boundary) orientable surfaces and our aim is to define minimal generating sets of their mapping class groups.

Let Σ_g denote the closed orientable surface of genus g. Define $\mathrm{Homeo}(\Sigma_g)$ as the group of self-homeomorphisms of Σ_g and $\mathrm{Homeo}^+(\Sigma_g)$ as the subgroup of $\mathrm{Homeo}(\Sigma_g)$ consisting of orientation-preserving elements and $\mathrm{Homeo}_0(\Sigma_g)$ as the subgroup of $\mathrm{Homeo}(\Sigma_g)$ consisting of the elements which are isotopic to the identity. Clearly, $\mathrm{Homeo}_0(\Sigma_g)$ is a normal subgroup of $\mathrm{Homeo}^+(\Sigma_g)$.

The mapping class group of Σ_g is the group of orientation-preserving homemorphisms of Σ_g up to isotopy and is denoted by $\mathrm{Mod}(\Sigma_g)$:

$$\begin{aligned} \mathrm{Mod}(\Sigma_g) &= \mathrm{Homeo}^+(\Sigma_g)/\mathrm{Homeo}_0(\Sigma_g) \\ &= \pi_0(\mathrm{Homeo}^+(\Sigma_g)). \end{aligned}$$

The results obtained from the studies on the mapping class groups are related with many fields of mathematics, such as hyperbolic geometry, 3 and 4-manifolds, algebraic geometry and symplectic geometry.

In Chapter 2, we give a review of background information about mapping class groups. The chapter contains some examples of simple calculations of mapping class groups. We introduce Dehn twists which are the simplest type of elements of infinite orders and the chapter ends with their actions on the simple closed curves and their relations containing braid relation, commutativity and lantern relation.

In Chapter 3, we define some explicit generating sets for the mapping class group. We first start with the generating sets consisting of elements of infinite orders. Next, we give the minimal generating set consisting of elements of infinite orders. After that, we show the generating sets consisting of torsion elements which are the elements of finite orders. We then give the minimal generating set consisting of torsion elements. The mapping class group $\mathrm{Mod}(\Sigma_g)$ is known to be generated by two torsion elements of order $4g+2$ [8]. Finally, we introduce the generating sets consisting of involutions which are the elements of order 2. The mapping class group $\mathrm{Mod}(\Sigma_g)$ is known to be generated by three involutions if $g \geq 8$ [9].

In Chapter 4, we prove the main results of this thesis. We devide this chapter into two sections, torsion generators and involution generators.

In the first section, we construct the minimal generating set consisting of involutions for the mapping class group of closed orientable surfaces of genus $g \geq 6$. We prove that the mapping class group of a closed connected orientable surface of genus g is generated by three involutions for $g \geq 6$.

Theorem 1.0.1. *The mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by three involutions if* $g \geq 6$.

Since it is not possible to generate $\mathrm{Mod}(\Sigma_g)$ for $g = 1$ or $g = 2$ by involutions, the only missing cases are $g = 3, 4$ and 5, see Section 3.3. We also prove Theorem 1.0.2 and Theorem 1.0.3.

Theorem 1.0.2. *The mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by three elements of order* g *if* $g \geq 3$.

Theorem 1.0.3. *The mapping class group* $\mathrm{Mod}(\Sigma_4)$ *is generated by three elements of order* 4, 4 *and* 2.

In the second section, we construct a minimal generating set consisting of elements of small orders for the mapping class group. We prove that the mapping class group of a closed connected orientable surface of genus g is generated by two elements of order g if $g \geq 6$. Moreover, for $g \geq 7$ we obtain a generating set of two elements, of order g and g' which is the least divisor of g such that $g' > 2$. Furthermore, we prove

that it is generated by two elements of order $g/\gcd(g,k)$ for $g \geq 3k^2+4k+1$ and any positive integer k.

Theorem 1.0.4. *The mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by two elements of order* g *if* $g \geq 6$.

Theorem 1.0.5. *For* $g \geq 7$*, the mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by two elements of order* g *and order* g'*, where* g' *is the least divisor of* g *such that* $g' > 2$ *.*

Theorem 1.0.6. *If* k *is any positive integer and if* $g \geq 3k^2+4k+1$*, then the mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by two elements of order* $g/\gcd(g,k)$.

In the last section, we introduce other possible results that can be achieved by using the idea in Sections 4.1 and 4.2 with little modifications to generate the mapping class group of closed connected nonorientable surfaces, the twist subgroup of the mapping class group of closed connected nonorientable surfaces and the extended mapping class group of connected surfaces possibly with punctures.

CHAPTER 2

BACKGROUND

In this chapter, we give basic facts about mapping class groups and we introduce Dehn twists as important elements of the mapping class groups and relations among them.

2.1 Basic Facts and Examples

In this section, we state preliminary definitions and results. Then, we give some examples of simple calculations of mapping class groups.

2.1.1 Preliminaries

Elements of the mapping class group are distinguished by their actions on the homotopy classes of simple closed curves. We first classify surfaces and define simple closed curves. For the proof of the theorems and more details on basics of mapping class groups, refer to [17].

Theorem 2.1.1. *There is a bijection between the set $\{g \in \mathbb{Z} : g \geq 0\}$ and the set of homeomorphism classes of closed, connected, orientable surfaces.*

Definition 2.1.2. *A simple closed curve on Σ_g is the image of a injective continuous map $S^1 \to \Sigma_g$ from the unit circle S^1.*

Definition 2.1.3. *A simple closed curve c on Σ_g is called nonseparating if its complement $\Sigma_g \setminus c$ is connected. Otherwise, it is called separating.*

Definition 2.1.4. *Let c_1 and c_2 be simple closed curves on Σ_g. The least possible number of intersection points of the curves in the homotopy classes c_1 and c_2 is called the geometric intersection number and denoted by $i(c_1, c_2)$.*

Theorem 2.1.5. *There exists an orientation-preserving homeomorphism of Σ_g mapping a simple closed curve a to a simple closed curve b if and only if $\Sigma_g - a$ and $\Sigma_g - b$ are homeomorphic. In particular, there exists a homeomorphism of Σ_g for every pair of nonseparating simple closed curves taking one to another.*

2.1.2 Some Basic Examples

We give basic examples of mapping class groups.

(i) The mapping class group of the closed disk is trivial.

(ii) The mapping class group of the sphere is trivial, $\mathrm{Mod}(\Sigma_0) = 1$.

(iii) The mapping class group of the annulus is isomorphic to $\mathbb{Z}$. The generator of the group is called a Dehn twist.

(iv) The mapping class group of the closed disk with n marked points is the braid group on n strands, B_n.

2.2 Dehn Twists

The simplest elements of the mapping class group are called Dehn twists. We first define Dehn twists and then introduce relations between them.

Let $I = [0, 1]$, $S = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\} \subseteq \mathbb{C}$ and define $t : I \times S \to I \times S$ be the map as $t(x, e^{i\theta}) = (x, e^{i(\theta - 2\pi t}))$.

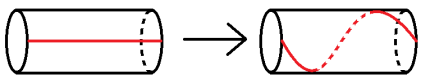


Figure 2.1: Dehn twist

Let Σ_g be the closed orientable surface of genus g and N be a closed regular neighbourhood of a simple closed curve a on Σ_g. Take an orientation-preserving homeomorphism h from N to $I \times S$. The homeomorphism $t_a : \Sigma_g \to \Sigma_g$

$$t_a(x) = \begin{cases} h^{-1} \circ t \circ h(x) & \text{if} \quad x \in N, \\ x & \text{if} \quad x \in \Sigma_g \setminus N, \end{cases}$$

is called a right (positive) Dehn twist.

We denote simple closed curves by lowercase letters possibly with subscriptions a, a_i, b_i, c_i and the corresponding right Dehn twists by uppercase letters A, A_i, B_i, C_i or by $t_a, t_{a_i}, t_{b_i}, t_{c_i}$, respectively.

Throughout this thesis, homeomorphisms and simple closed curves are considered up to isotopy.

Some Basic Facts on Dehn Twists

Let c_1 and c_2 be simple closed curves on Σ_g and $f : \Sigma_g \to \Sigma_g$ be a homeomorphism. Then, the following hold:

(i) The curve c_1 is isotopic to c_2 if and only if C_1 is isotopic to C_2.

(ii) The equality $f t_{c_1} f^{-1} = t_{f(c_1)}$ holds.

(iii) If $i(c_1, c_2) = 0$, then $C_1(c_2) = c_2$ or equivalently $C_1 C_2 = C_2 C_1$. This relation is called *the commutativity*.

(iv) If $i(c_1, c_2) = 1$, then $C_1 C_2(c_1) = c_2$ or equivalently $C_1 C_2 C_1 = C_2 C_1 C_2$. This relation is called *the braid relation*.

(v) Let Σ_0' be a sphere with four holes, boundary components, and a, b, c, d, x, y and z be the simple closed curves on Σ_0' as shown on Figure 2.2. Suppose that Σ_0' is embedded in Σ_g.

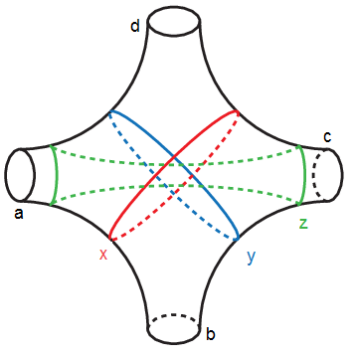


Figure 2.2: Lantern relation

Then, we have $ABCD = XYZ$ in $\mathrm{Mod}(\Sigma_g)$. This relation is called *the lantern relation*.

(vi) Let Σ_1'' be a torus with two holes, boundary components, and a, b, c, d and e be the simple closed curves on Σ_1'' as shown on Figure 2.3. Suppose that Σ_1'' is embedded in Σ_g.

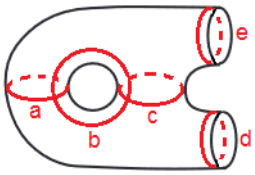


Figure 2.3: Two-holed torus relation

Then, we have $(ABC)^4 = DE$ in $\mathrm{Mod}(\Sigma_g)$. This relation is called *two-holed torus relation*. It can be generalized to $(A_1A_2\cdots A_{2h}A_{2h+1})^{2h+2} = DE$ with suitable choice of curves for an orientable closed surface of genus h with two holes.

(vii) Let Σ_1' be a torus with a boundary component, and a, b, c and d be the simple closed curves on Σ_1' as shown on Figure 2.4. Suppose that Σ_1' is embedded in Σ_g.

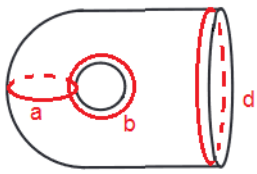


Figure 2.4: One-holed torus relation

Then, we have $(AB)^6 = D$ in $\mathrm{Mod}(\Sigma_g)$. It can be deduced directy from two-holed torus relation. This relation is called *one-holed torus relation*. It can also be generalized to $(A_1A_2 \cdots A_{2h-1}A_{2h})^{4h+2} = D$ with suitable choice of curves for an orientable closed surface of genus h with one hole.

CHAPTER 3

EXPLICIT SETS OF GENERATORS

In this chapter, we introduce a list of generating sets for mapping class groups $\mathrm{Mod}(\Sigma_g)$ of orientable closed surfaces of genus g. Mapping class groups was first generated by using infinite order elements called Dehn twist. Humphries found a generating set consisting of minimum possible number of Dehn twists. The next problem was to generate mapping class groups with finite order elements called torsions. Korkmaz introduced a minimal generating set of torsions. Another problem was to generate mapping class groups with order 2 elements, called involutions. Korkmaz proved that for $g \geq 8$ mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by three involutions which is the minimum possible number.

3.1 Dehn Twists Generators

We give a bunch of historical results on Dehn twist generators of mapping class groups.

(i) Dehn [3] showed in 1938 that $\mathrm{Mod}(\Sigma_g)$ is generated by $2g^2 - 2g$ Dehn twists.

(ii) Mumford proved in 1967 that $\mathrm{Mod}(\Sigma_g)$ is generated by Dehn twists about non-separating simple closed curves among Dehn's generating set in 1938.

(iii) Lickorish [12] showed in 1964 that $\mathrm{Mod}(\Sigma_g)$ is generated by $3g - 1$ Dehn twists about nonseparating simple closed curves.

(iv) Humphries [6] proved in 1979 that $\mathrm{Mod}(\Sigma_g)$ is generated by $2g + 1$ Dehn twists about nonseparating simple closed curves. This is the least number of Dehn twists to generate $\mathrm{Mod}(\Sigma_g)$.

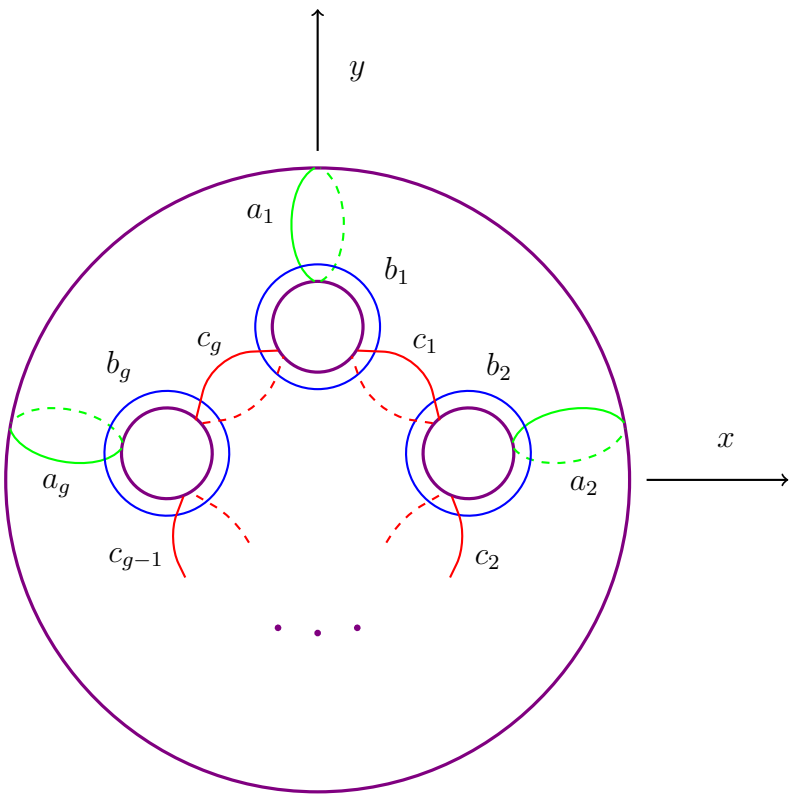


Figure 3.1: The curves a_i, b_i, c_i on the surface Σ_g

While the set $\{A_1, A_2, \ldots, A_g, B_1, B_2, \ldots B_g, C_1, C_2, \cdots, C_{g-1}\}$ is called Lickorish generators, the set $\{A_1, A_2, B_1, B_2, \ldots B_g, C_1, C_2, \cdots, C_{g-1}\}$ is known as Humphries generators.

Humphries showed also that the mapping class group $\mathrm{Mod}(\Sigma_g)$ cannot be generated by less than $2g+1$ Dehn twists.

3.2 Torsion Generators

Lu [14] obtained a generating set consisting of three elements, two of which are of finite order. Wajnryb [23] found a presentation for the mapping class group of an orientable surface. Wajnryb [24] also proved that $\mathrm{Mod}(\Sigma_g)$ can be generated by two elements; one is of order $4g+2$ and the other is a product of opposite Dehn twists. After that, Korkmaz [8] showed that $\mathrm{Mod}(\Sigma_g)$ is generated by an element of order $4g+2$ and a Dehn twist, improving Wajnryb's result.

Maclachlan [16] proved that $\mathrm{Mod}(\Sigma_g)$ can be generated by using only torsions. Ko-

rkmaz is the first person to find a generating set consisting of two torsions, elements of finite order, for the mapping class group $\mathrm{Mod}(\Sigma_g)$. He [8] proved that the group $\mathrm{Mod}(\Sigma_g)$ is generated by two elements of order $4g+2$. Note that, since $\mathrm{Mod}(\Sigma_g)$ is not a cyclic group, it cannot be generated by a single element.

Moreover, Monden [19] proved that $\mathrm{Mod}(\Sigma_g)$ can be generated by three elements of order 3 and by four elements of order 4. Lanier [10] showed $\mathrm{Mod}(\Sigma_g)$ is generated by three elements of any order $k \geq 6$ if $g \geq (k-1)^2+1$. See [4] for generating set of torsion elements for the extended mapping class group.

3.3 Involution Generators

McCarthy and Papadopoulos [18] proved that $\mathrm{Mod}(\Sigma_g)$ can also be generated by involutions. Note that for homological reasons $\mathrm{Mod}(\Sigma_g)$ is not generated by involutions if $g=1$ or $g=2$. Stukow [21] showed that the index five subgroup of $\mathrm{Mod}(\Sigma_2)$ is generated by involutions. So, we consider the case of $g \geq 3$. Luo [15] found an upper bound $12g+6$ for the number of involutions needed to generate $\mathrm{Mod}(\Sigma_g)$. Brendle and Farb [1] decreased this number to 6. Kassabov [7] showed that four involutions are enough for $g \geq 7$. Recently, Korkmaz [9] found a generating set of three involutions if $g \geq 8$ and of four involutions if $g \geq 3$.

Note that two involutions generate a dihedral group. If their multiplication is a torsion element, then the group is a finite dihedral group, otherwise it is the infinite dihedral group. In both cases, the group is not isomorphic to a mapping class group. Therefore, mapping class groups cannot be generated by two involutions.

Dihedral group of order $2n$ has the presentation:

$$\begin{aligned} D_n &= \langle\, a, b \mid a^2 = b^n = (ab)^2 = 1 \,\rangle \\ &= \langle\, a, ab \mid a^2 = (ab)^2 = b^n = 1 \,\rangle \end{aligned}$$

Infinite dihedral group has the presentation:

$$D_\infty = \langle\, a, b \mid a^2 = b^2 = 1 \,\rangle$$

On the other hand, any group generated by involutions has the first homology group

as multiplications of $\mathbb{Z}_2$s. For $g = 1$ and $g = 2$, it is known that first homology group of $\mathrm{Mod}(\Sigma_g)$ is $\mathbb{Z}_{12}$ and $\mathbb{Z}_{10}$, respectively. Hence, the mapping class groups cannot be generated by involutions if $g = 1$ or $g = 2$.

CHAPTER 4

MINIMAL GENERATORS

In this chapter, we prove the main results of this thesis. We devide this chapter into two sections, involution generators and torsion generators.

In the first section, we prove that the mapping class group $\mathrm{Mod}(\Sigma_g)$ of a closed connected orientable surface of genus g is generated by three involutions for $g \geq 6$. Moreover, we form generating sets for $g = 3$, 4 and 5.

In the second section, we prove that the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by two elements of order g if $g \geq 6$. Moreover, for $g \geq 7$ we obtain a generating set of two elements, of order g and g' which is the least divisor of g such that $g' > 2$. Furthermore, we prove that $\mathrm{Mod}(\Sigma_g)$ is generated by two elements of order $g/\gcd(g,k)$ for $g \geq 3k^2 + 4k + 1$ and any positive integer k.

Let R be the rotation by $2\pi/g$ about the z–axis represented in Figure 4.1 which is perpendicular to the page. Then, $R(a_k) = a_{k+1}$, $R(b_k) = b_{k+1}$ and $R(c_k) = c_{k+1}$. Korkmaz found a generating set of four elements, one of which is the rotation element R and each of the other elements is a product of two opposite Dehn twists. Korkmaz [9] proved the following useful theorem as a corollary to Humphries generators.

Theorem 4.0.1. *[9] If $g \geq 3$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the four elements $R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}$.*

The next result is easily deduced from Theorem 4.0.1.

Corollary 4.0.2. *If $g \geq 3$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the four elements $R, A_1B_1^{-1}, B_1C_1^{-1}, C_1B_2^{-1}$.*

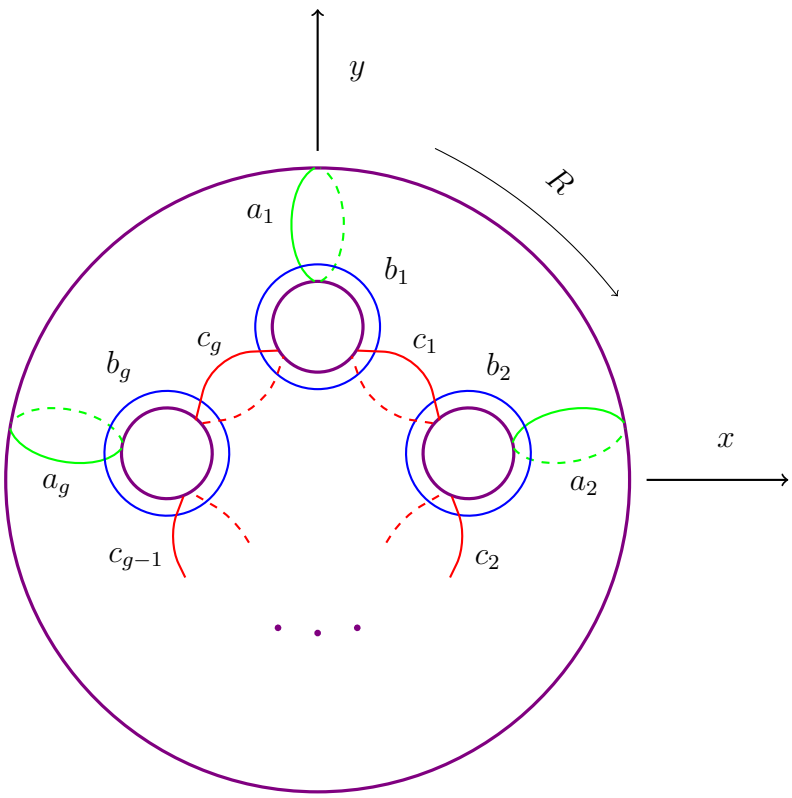


Figure 4.1: The curves a_i, b_i, c_i and the rotation R on the surface Σ_g

Proof. Let H be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, A_1B_1^{-1}, B_1C_1^{-1}, C_1B_2^{-1}\}$.

We have $B_2A_2^{-1} = R(B_1A_1^{-1})R^{-1} \in H$ and $B_2C_2^{-1} = R(B_1C_1^{-1})R^{-1} \in H$.

Hence, $B_1B_2^{-1} = (B_1C_1^{-1})(C_1B_2^{-1}) \in H$, $C_1C_2^{-1} = (C_1B_2^{-1})(B_2C_2^{-1}) \in H$ and $A_1A_2^{-1} = (A_1B_1^{-1})(B_1B_2^{-1})(B_2A_2^{-1}) \in H$.

It follows from Theorem 4.0.1 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the corollary. □

Lemma 4.0.3. *If K is an element of order k in a group G and if x and y are elements in G satisfying $KxK^{-1} = y$, then the order of Kxy^{-1} is also k. In particular, if ρ is an involution element in a group G and if x and y are elements in G satisfying $\rho x \rho = y$, then the element $\rho x y^{-1}$ is also an involution.*

Proof. Since $Kxy^{-1} = KxK^{-1}Ky^{-1} = yKy^{-1}$, the order of Kxy^{-1} is equal to the order of K.

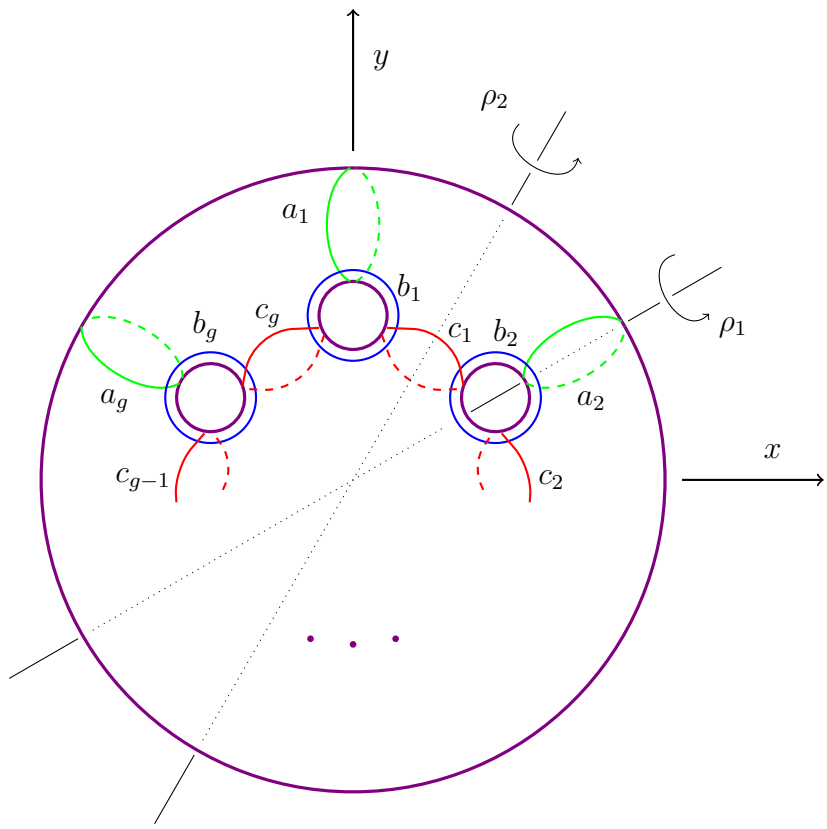


Figure 4.2: The involutions ρ_1 and ρ_2 on the surface Σ_g

□

4.1 Involution Generators

In this section, we introduce a new generating set for the mapping class group of Σ_6 and a new generating set for the mapping class group of Σ_g for $g \geq 7$. We use these sets to generate the mapping class group by three involutions.

Define involutions ρ_1 and ρ_2 as rotations about the given axis on Figure 4.2 by π degrees.

Recall that R denotes the $2\pi/g$–rotation of Σ_g represented in Figure 4.1. It is a torsion element of order g in the group $\mathrm{Mod}(\Sigma_g)$. We follow the idea of Korkmaz in [9] to create our generating sets in the next lemmas as corollaries to Theorem 4.0.1. Our idea is to use ρ_1, ρ_2 as the first two elements and products of Dehn twists as the last element. Observe that $R = \rho_1\rho_2$.

4.1.1 Two new generating sets

In this subsection, we introduce two new generating sets for mapping class groups.

We use the next lemma (Lemma 4.1.1) to create a generating set of three involutions for $g = 6$ and the second next lemma (Lemma 4.1.2) to create a generating set of three involutions for $g \geq 7$.

Lemma 4.1.1. *If $g = 6$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the three elements ρ_1, ρ_2 and $A_1C_1B_3B_4^{-1}C_5^{-1}A_6^{-1}$.*

Proof. Let $F_1 = A_1C_1B_3B_4^{-1}C_5^{-1}A_6^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_6)$ generated by the set $\{\rho_1, \rho_2, F_1\}$. Clearly, $R \in H$ since $R = \rho_1\rho_2$.

It suffices to show that H contains $A_1A_2^{-1}$, $B_1B_2^{-1}$ and $C_1C_2^{-1}$. Now, we define a list of elements to prove the desired result.

Let

$$\begin{aligned} F_2 &= RF_1R^{-1} \\ &= R(A_1C_1B_3B_4^{-1}C_5^{-1}A_6^{-1})R^{-1} \\ &= RA_1R^{-1}RC_1R^{-1}RB_3R^{-1}RB_4^{-1}R^{-1}RC_5^{-1}R^{-1}RA_6^{-1}R^{-1} \\ &= A_2C_2B_4B_5^{-1}C_6^{-1}A_1^{-1}. \end{aligned}$$

Note that $RA_iR^{-1} = Rt_{a_i}R^{-1} = t_{R(a_i)} = t_{a_{i+1}} = A_{i+1}$, similarly $RB_iR^{-1} = B_{i+1}$ and $RC_iR^{-1} = C_{i+1}$ for all i mod g.

We have $F_2F_1(a_2, c_2, b_4, b_5, c_6, a_1) = (a_2, b_3, b_4, c_5, c_6, a_1)$ so that $F_3 = A_2B_3B_4C_5^{-1}C_6^{-1}A_1^{-1} \in H$. Note that $F_2F_1(c_2) = b_3$ since

$$\begin{aligned} t_{F_2F_1(c_2)} &= (F_2F_1)t_{c_2}(F_2F_1)^{-1} \\ &= F_2F_1C_2F_1^{-1}F_2^{-1} \\ &= C_2B_3C_2B_3^{-1}C_2^{-1} \\ &= (t_{c_2}t_{b_3})t_{c_2}(t_{c_2}t_{b_3})^{-1} \\ &= t_{t_{c_2}t_{b_3}(c_2)} \\ &= t_{b_3}. \end{aligned}$$

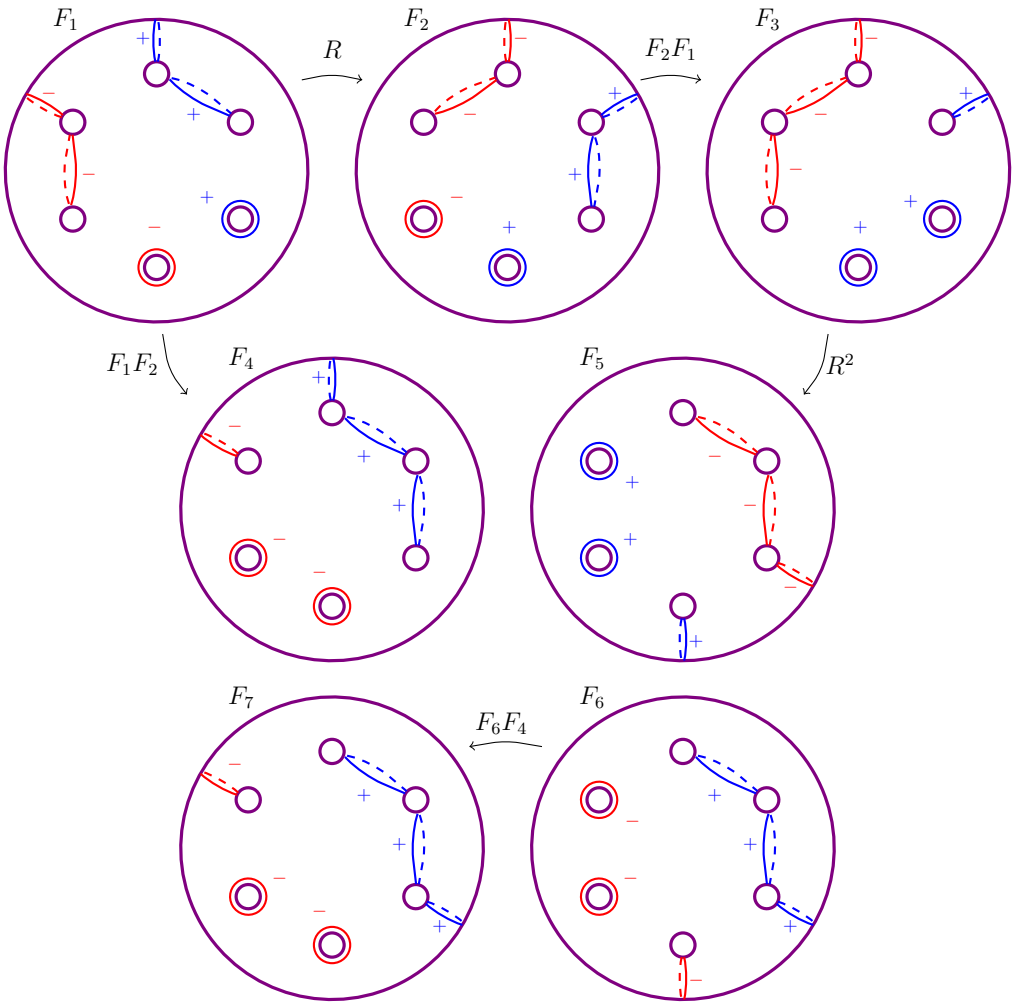


Figure 4.3: Proof of Lemma 4.1.1

Similarly, $F_1F_2(a_1, c_1, b_3, b_4, c_5, a_6) = (a_1, c_1, c_2, b_4, b_5, a_6)$ so that
$F_4 = A_1C_1C_2B_4^{-1}B_5^{-1}A_6^{-1} \in H$.

Let

$$\begin{aligned} F_5 &= R^2F_3R^{-2} \\ &= A_4B_5B_6C_1^{-1}C_2^{-1}A_3^{-1} \end{aligned}$$

and

$$\begin{aligned} F_6 &= F_5^{-1} \\ &= A_3C_2C_1B_6^{-1}B_5^{-1}A_4^{-1}. \end{aligned}$$

We then get $F_6F_4(a_3, c_2, c_1, b_6, b_5, a_4) = (a_3, c_2, c_1, a_6, b_5, b_4)$ so that
$F_7 = A_3C_2C_1A_6^{-1}B_5^{-1}B_4^{-1} \in H$.

Let

$$\begin{aligned} F_8 &= F_4F_7^{-1} \\ &= A_1A_3^{-1}. \end{aligned}$$

$F_8^{-1}F_3(a_1, a_3) = (a_1, b_3)$ so that $A_1B_3^{-1} \in H$ and then by conjugating $A_1B_3^{-1}$ with R iteratively, we get $A_iB_{i+2}^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_9 &= B_3B_5^{-1} \\ &= (B_3A_1^{-1})(A_1A_3^{-1})(A_3B_5^{-1}). \end{aligned}$$

$F_9F_3(b_3, b_5) = (b_3, c_5)$ so that $B_3C_5^{-1} \in H$ and then $B_iC_{i+2}^{-1} \in H$ for all i.
$\rho_1(B_2C_4^{-1})\rho_1 = B_2C_5^{-1} \in H$ and then $B_iC_{i+3}^{-1} \in H$ for all i.

Finally, $B_1B_2^{-1} = (B_1C_4^{-1})(C_4B_2^{-1}) \in H$ and then $B_3B_4^{-1} \in H$,
$A_1A_2^{-1} = (A_1B_3^{-1})(B_3B_4^{-1})(B_4A_2^{-1}) \in H$ and
$C_1C_2^{-1} = (C_1B_5^{-1})(B_5C_2^{-1}) \in H$.

It follows from Theorem 4.0.1 that $H = \mathrm{Mod}(\Sigma_6)$, completing the proof of the lemma. $\square$

Lemma 4.1.2. *If $g \geq 7$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the three elements ρ_1, ρ_2 and $A_1C_1B_3B_5^{-1}C_6^{-1}A_7^{-1}$.*

Proof. We follow a similar pattern to Lemma 4.1.1.

Let $F_1 = A_1C_1B_3B_5^{-1}C_6^{-1}A_7^{-1}$ and H be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{\rho_1, \rho_2, F_1\}$. It suffices to show that H contains $A_1A_2^{-1}$, $B_1B_2^{-1}$ and $C_1C_2^{-1}$. Assume $g = 7$. (Respectively, $g \geq 8$.)

Let

$$\begin{aligned} F_2 &= RF_1R^{-1} \\ &= A_2C_2B_4B_6^{-1}C_7^{-1}A_1^{-1}. \end{aligned}$$

(Resp., $F_2 = A_2C_2B_4B_6^{-1}C_7^{-1}A_8^{-1}$.)

We get $F_2F_1(a_2, c_2, b_4, b_6, c_7, a_1) = (a_2, b_3, b_4, c_6, c_7, a_1)$ so that $F_3 = A_2B_3B_4C_6^{-1}C_7^{-1}A_1^{-1} \in H$. (Resp., $F_3 = A_2B_3B_4C_6^{-1}C_7^{-1}A_8^{-1}$.)

Let

$$\begin{aligned} F_4 &= R^{-1}F_3R \\ &= A_1B_2B_3C_5^{-1}C_6^{-1}A_7^{-1}. \end{aligned}$$

Then, we have $F_4F_3(a_1, b_2, b_3, c_5, c_6, a_7) = (a_1, a_2, b_3, c_5, c_6, a_7)$ so that $F_5 = A_1A_2B_3C_5^{-1}C_6^{-1}A_7^{-1} \in H$.

Hence, $F_4F_5^{-1} = B_2A_2^{-1}$ so that $B_2A_2^{-1} \in H$ and then by conjugating $B_2A_2^{-1}$ with R iteratively, we get $B_iA_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_6 &= (A_3B_3^{-1})F_1 \\ &= A_1C_1A_3B_5^{-1}C_6^{-1}A_7^{-1} \end{aligned}$$

and

$$\begin{aligned} F_7 &= RF_6R^{-1} \\ &= A_2C_2A_4B_6^{-1}C_7^{-1}A_1^{-1}. \end{aligned}$$

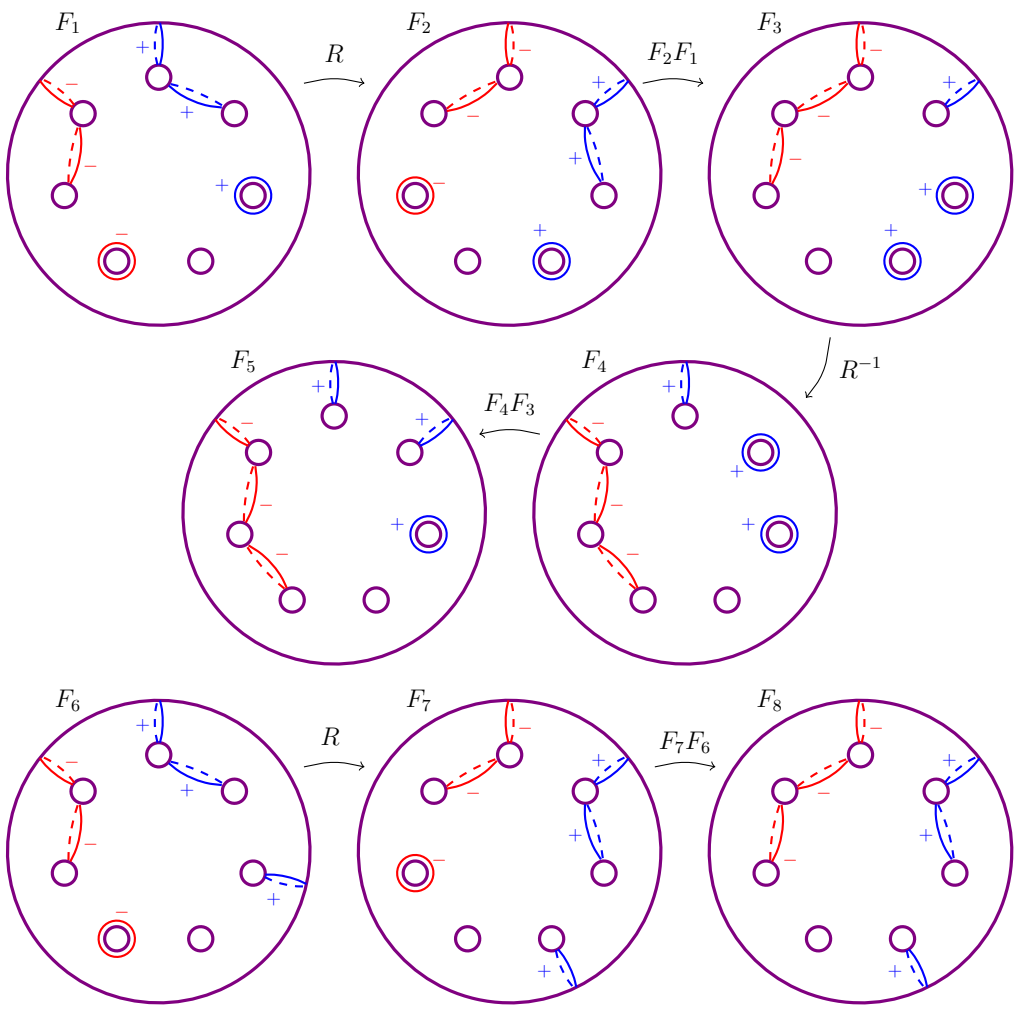


Figure 4.4: Proof of Lemma 4.1.2 for $g = 7$

(Resp., $F_7 = A_2C_2A_4B_6^{-1}C_7^{-1}A_8^{-1}$.)

Similarly, $F_7F_6(a_2, c_2, a_4, b_6, c_7, a_1) = (a_2, c_2, a_4, c_6, c_7, a_1)$ so that $F_8 = A_2C_2A_4C_6^{-1}C_7^{-1}A_1^{-1} \in H$. (Resp., $F_8 = A_2C_2A_4C_6^{-1}C_7^{-1}A_8^{-1}$.)

Hence, $F_7^{-1}F_8 = B_6C_6^{-1}$ so that $B_6C_6^{-1} \in H$ and then $B_iC_i^{-1} \in H$ for all i.
$\rho_2(B_1C_1^{-1})\rho_2 = B_2C_1^{-1} \in H$ and then $B_{i+1}C_i^{-1} \in H$ for all i.

Finally, we have $C_1C_2^{-1} = (C_1B_2^{-1})(B_2C_2^{-1}) \in H$,
$B_1B_2^{-1} = (B_1C_1^{-1})(C_1C_2^{-1})(C_2B_2^{-1}) \in H$ and
$A_1A_2^{-1} = (A_1B_1^{-1})(B_1B_2^{-1})(B_2A_2^{-1}) \in H$.

It follows from Theorem 4.0.1 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the lemma. $\square$

Now, we can prove Theorem 1.0.1.

4.1.2 Proof of Theorem 1.0.1

Proof. Let $\rho_3 = R^2\rho_1R^{-2}$ and $\rho_4 = R^2\rho_2R^{-2}$.

Then, by Lemma 4.0.3 we have if $g \geq 7$, then $\rho_3A_1C_1B_3B_5^{-1}C_6^{-1}A_7^{-1}$ is an involution since $\rho_3A_1C_1B_3\rho_3 = A_7C_6B_5$ and if $g \geq 6$, then $\rho_4A_1C_1B_3B_4^{-1}C_5^{-1}A_6^{-1}$ is an involution since $\rho_4A_1C_1B_3\rho_4 = A_6C_5B_4$.

The mapping class group $\mathrm{Mod}(\Sigma_6)$ of closed connected orientable surface of genus 6 is generated by three involutions ρ_1, ρ_2 and $\rho_4A_1C_1B_3B_4^{-1}C_5^{-1}A_6^{-1}$ by Lemma 4.1.1.

If $g \geq 7$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by three involutions ρ_1, ρ_2 and $\rho_3A_1C_1B_3B_5^{-1}C_6^{-1}A_7^{-1}$ by Lemma 4.1.2. □

We have proved that the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by three involutions if $g \geq 6$. Since it is not possible to generate $\mathrm{Mod}(\Sigma_g)$ by involutions for $g = 1$ or $g = 2$, the only missing cases are $g = 3, 4$ and 5. Now, we deal with that cases on Theorem 1.0.2 and Theorem 1.0.3. Note that Korkmaz proved [9] that $\mathrm{Mod}(\Sigma_g)$ is generated by 4 involutions for $g = 3, 4$ and 5.

4.1.3 Proof of Theorem 1.0.2

In this subsection, we introduce two new generating sets for the mapping class group Mod(Σ_g). We show that $\mathrm{Mod}(\Sigma_3)$ is generated by three elements of order 3. We show that for $g \geq 4$ $\mathrm{Mod}(\Sigma_g)$ is generated by three elements of order g. Theorem 1.0.2 states that the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by three elements of order g if $g \geq 3$. Proof directly follows from the next two theorems.

Theorem 4.1.3. *The mapping class group* $\mathrm{Mod}(\Sigma_3)$ *is generated by the three elements of order* 3*,* R*,* $RA_1A_2^{-1}$*,* $RA_1B_1C_2^{-1}C_3B_2^{-1}A_2^{-1}$*.*

Proof. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_3)$ generated by the set $\{R, RA_1A_2^{-1}, RA_1B_1C_2^{-1}C_3B_2^{-1}A_2^{-1}\}$.

Clearly, $A_1A_2^{-1} = (R)^2(RA_1A_2^{-1}) \in H$.

Since $R(a_1, a_2) = (a_2, a_3)$, we have $A_2A_3^{-1} \in H$ and
$A_1A_3^{-1} = (A_1A_2^{-1})(A_2A_3^{-1}) \in H$.

We also have $A_1B_1C_2^{-1}C_3B_2^{-1}A_2^{-1} = (R)^2(RA_1B_1C_2^{-1}C_3B_2^{-1}A_2^{-1}) \in H$ and
$A_1B_1C_2^{-1}C_3B_2^{-1}A_2^{-1}(a_1, a_3) = (b_1, a_3)$ so that $B_1A_3^{-1} \in H$.

Moreover, $R(b_1, a_3) = (b_2, a_1)$ so that $B_2A_1^{-1} \in H$ and
$B_1B_2^{-1} = (B_1A_3^{-1})(A_3A_1^{-1})(A_1B_2^{-1}) \in H$.

Hence, $A_1B_1C_2^{-1}C_3B_2^{-1}A_2^{-1}(b_1, a_3) = (c_3, a_3)$ so that $C_3A_3^{-1} \in H$.

We see that $R(c_3, a_3) = (c_1, a_1)$ so that $C_1A_1^{-1} \in H$.

Similarly, $R(c_1, a_1) = (c_2, a_2)$ so that $C_2A_2^{-1} \in H$.

Finally, $C_1C_2^{-1} = (C_1A_1^{-1})(A_1A_2^{-1})(A_2C_2^{-1}) \in H$.

Note that, R is an element of order 3 and by Lemma 4.0.3, $RA_1A_2^{-1}$ and
$RA_1B_1C_2^{-1}C_3B_2^{-1}A_2^{-1}$ are elements of order 3 since $R(A_1)R^{-1} = A_2$ and
$R(A_1B_1C_2^{-1})R^{-1} = A_2B_2C_3^{-1}$.
Since H contains R, $A_1A_2^{-1}$, $B_1B_2^{-1}$ and $C_1C_2^{-1}$, it follows from Theorem 4.0.1 that
$H = \mathrm{Mod}(\Sigma_3)$. □

Theorem 4.1.4. *For $g \geq 3$, the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the three elements of order g, R, $RA_1A_2^{-1}$, $RA_1B_1C_{g-1}^{-1}C_gB_2^{-1}A_2^{-1}$.*

Proof. We are done by Theorem 4.1.3 if $g = 3$. Now, assume $g \geq 4$.

Let H be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set
$\{R, RA_1A_2^{-1}, RA_1B_1C_{g-1}^{-1}C_gB_2^{-1}A_2^{-1}\}$.

$A_1A_2^{-1} = (R)^{g-1}(RA_1A_2^{-1}) \in H$ and then $A_iA_{i+1}^{-1} \in H$ for all i.
So, $A_1A_3^{-1} = (A_1A_2^{-1})(A_2A_3^{-1}) \in H$.

We have $A_1B_1C_{g-1}^{-1}C_gB_2^{-1}A_2^{-1} = (R)^{g-1}(RA_1B_1C_{g-1}^{-1}C_gB_2^{-1}A_2^{-1}) \in H$.
Then, $A_1B_1C_{g-1}^{-1}C_gB_2^{-1}A_2^{-1}(a_1, a_3) = (b_1, a_3)$ so that $B_1A_3^{-1} \in H$.

We also have $R(b_1, a_3) = (b_2, a_4)$ so that $B_2A_4^{-1} \in H$ and

$B_1B_2^{-1} = (B_1A_3^{-1})(A_3A_4^{-1})(A_4B_2^{-1}) \in H$.

Since $A_1B_1C_{g-1}^{-1}C_gB_2^{-1}A_2^{-1}(b_1, a_3) = (c_g, a_3)$ so that $C_gA_3^{-1} \in H$, we have $C_iA_{i+3}^{-1} \in H$ for all i.

Finally, $C_1C_2^{-1} = (C_1A_4^{-1})(A_4A_5^{-1})(A_5C_2^{-1}) \in H$ if $g \geq 5$ and $C_1C_2^{-1} = (C_1A_4^{-1})(A_4A_1^{-1})(A_1C_2^{-1}) \in H$ if $g = 4$.

Note that, clearly R is an element of order g and by Lemma 4.0.3 we have $RA_1A_2^{-1}$ is an element of order g since $R(A_1)R^{-1} = A_2$ and $RA_1B_1C_{g-1}^{-1}C_gB_2^{-1}A_2^{-1}$ is an element of order g since $R(A_1B_1C_{g-1}^{-1})R^{-1} = A_2B_2C_g^{-1}$. Since H contains R, $A_1A_2^{-1}$, $B_1B_2^{-1}$ and $C_1C_2^{-1}$, it follows from Theorem 4.0.1 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the theorem. □

4.1.4 Proof of Theorem 1.0.3

Next, we show that $\mathrm{Mod}(\Sigma_4)$ is generated by three elements, one of which is an involution and each of the other elements is order of 4.

Theorem 4.1.5. *The mapping class group* $\mathrm{Mod}(\Sigma_4)$ *is generated by the three elements,* R, $RA_1A_2^{-1}$, $R^2A_1B_1C_1C_3^{-1}B_3^{-1}A_3^{-1}$.

Proof. Let H be the subgroup of $\mathrm{Mod}(\Sigma_4)$ generated by the set $\{R, RA_1A_2^{-1}, R^2A_1B_1C_1C_3^{-1}B_3^{-1}A_3^{-1}\}$.

First, we get $A_1A_2^{-1} = (R)^3(RA_1A_2^{-1}) \in H$.

We have $R(a_1, a_2) = (a_2, a_3)$ so that $A_2A_3^{-1} \in H$.

After that we get $A_1B_1C_1C_3^{-1}B_3^{-1}A_3^{-1} = (R)^2(R^2A_1B_1C_1C_3^{-1}B_3^{-1}A_3^{-1}) \in H$ and $A_1B_1C_1C_3^{-1}B_3^{-1}A_3^{-1}(a_1, a_2) = (b_1, a_2)$ so that $B_1A_2^{-1} \in H$.

We also have $R(b_1, a_2) = (b_2, a_3)$ so that $B_2A_3^{-1} \in H$.
$B_1B_2^{-1} = (B_1A_2^{-1})(A_2A_3^{-1})(A_3B_2^{-1}) \in H$.

Similarly, $A_1B_1C_1C_3^{-1}B_3^{-1}A_3^{-1}(b_1, a_2) = (c_1, a_2)$ so that $C_1A_2^{-1} \in H$.

$R(c_1, a_2) = (c_2, a_3)$ so that $C_2A_3^{-1} \in H$.

Finally, $C_1C_2^{-1} = (C_1A_2^{-1})(A_2A_3^{-1})(A_3C_2^{-1}) \in H$.

Note that, clearly R is an element of order 4 and by Lemma 4.0.3 we have $RA_1A_2^{-1}$ is an element of order 4 since $R(A_1)R^{-1} = A_2$ and $R^2A_1B_1C_1C_3^{-1}B_3^{-1}A_3^{-1}$ is an element of order 2 since $R^2(A_1B_1C_1)R^2 = A_3B_3C_3$.

Since H contains the elements R, $A_1A_2^{-1}$, $B_1B_2^{-1}$ and $C_1C_2^{-1}$, it follows from Theorem 4.0.1 that $H = \mathrm{Mod}(\Sigma_4)$, completing the proof of the theorem. □

4.2 Torsion Generators

4.2.1 Twelve new generating sets

In this section, we obtain twelve new generating sets consisting of two elements of small orders for the mapping class group. We follow the idea of Korkmaz in [9] to create our generating sets in the corollaries to Theorem 4.0.1. Our idea is to use the rotation element R, a torsion element of order g in the group $\mathrm{Mod}(\Sigma_g)$, as the first element and products of Dehn twists as the second element.

We recall Theorem 4.0.1.

Theorem 4.0.1. *[9] If $g \geq 3$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the four elements $R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}$.*

We use the first four corollaries to create generating sets of elements of order g. In order to follow Theorem 4.0.1, we need the rotation element R, which is of order g, in our generating set. So, we cannot decrease order g by following this method. But indeed, we can reduce the order of the second element. Applying Lemma 4.0.3, we see that most suitable candidates are the divisors of g. Unfortunately, it is not possible to find an order 2 element, an involution, as the second element in our generating sets by following this method. Instead, we found the least divisor of g greater than 2 as the minimal order for the second element. We use the next six corollaries after the

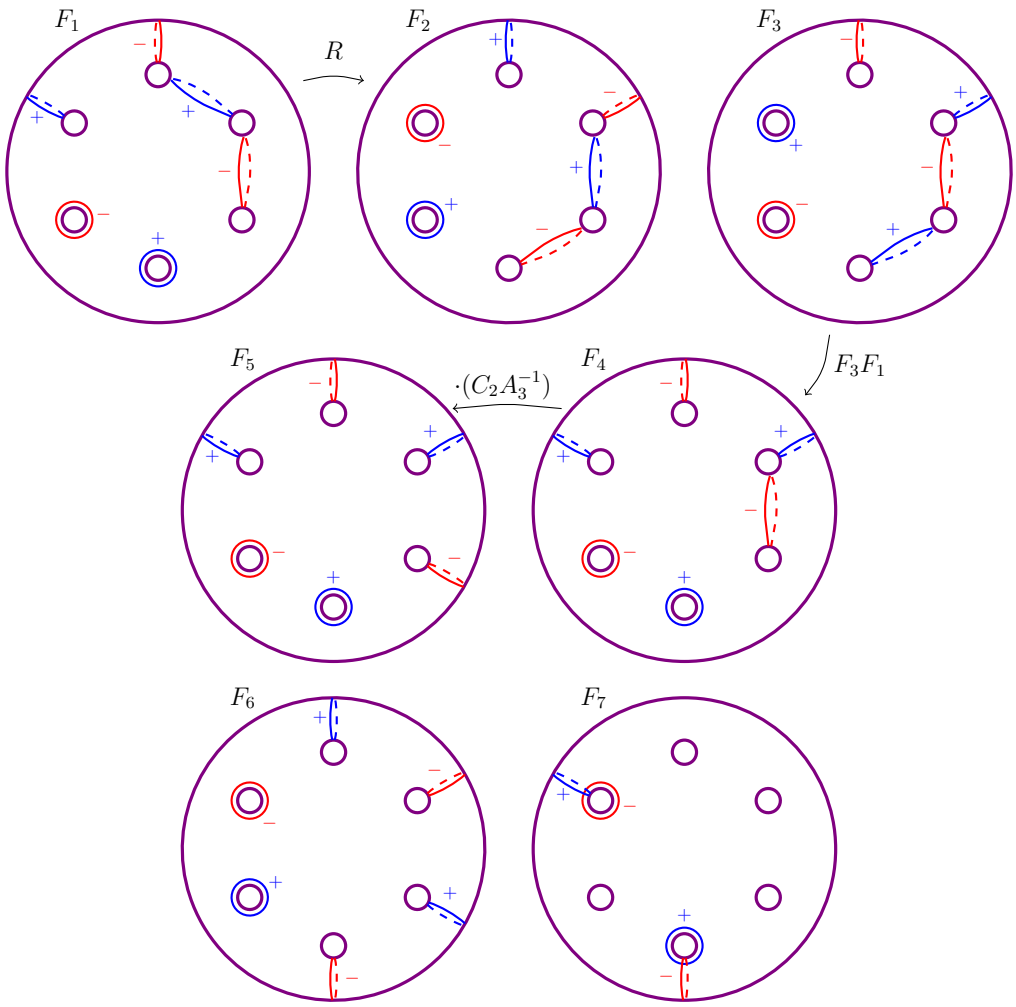


Figure 4.5: Proof of Corollary 4.2.1

first four ones to create generating sets of elements of order g and g' where g' is the least divisor of g such that $g' > 2$.

Corollary 4.2.1. *If $g = 6$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $C_1B_4A_6A_1^{-1}B_5^{-1}C_2^{-1}$.*

Proof. Let $F_1 = C_1B_4A_6A_1^{-1}B_5^{-1}C_2^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_6)$ generated by the set $\{R, F_1\}$.

If H contains the elements $A_1A_2^{-1}$, $B_1B_2^{-1}$ and $C_1C_2^{-1}$, then we are done by Theorem 4.0.1.

Let

$$\begin{aligned} F_2 &= RF_1R^{-1} \\ &= R(C_1B_4A_6A_1^{-1}B_5^{-1}C_2^{-1})R^{-1} \\ &= RC_1R^{-1}RB_4R^{-1}RA_6R^{-1}RA_1^{-1}R^{-1}RB_5^{-1}R^{-1}RC_2^{-1}R^{-1} \\ &= C_2B_5A_1A_2^{-1}B_6^{-1}C_3^{-1} \end{aligned}$$

and

$$\begin{aligned} F_3 &= F_2^{-1} \\ &= C_3B_6A_2A_1^{-1}B_5^{-1}C_2^{-1}. \end{aligned}$$

We have $F_3F_1(c_3, b_6, a_2, a_1, b_5, c_2) = (b_4, a_6, a_2, a_1, b_5, c_2)$ so that $F_4 = B_4A_6A_2A_1^{-1}B_5^{-1}C_2^{-1} \in H$. Note that $F_3F_1(c_3) = b_4$ since

$$\begin{aligned} t_{F_3F_1(c_3)} &= (F_3F_1)t_{c_3}(F_3F_1)^{-1} \\ &= F_3F_1C_3F_1^{-1}F_3^{-1} \\ &= C_3B_4C_3B_4^{-1}C_3^{-1} \\ &= (t_{c_3}t_{b_4})t_{c_3}(t_{c_3}t_{b_4})^{-1} \\ &= t_{t_{c_3}t_{b_4}(c_3)} \\ &= t_{b_4}. \end{aligned}$$

We get $F_1F_4^{-1} = C_1A_2^{-1} \in H$ and then by conjugating $C_1A_2^{-1}$ with R iteratively, we get $C_iA_{i+1}^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_5 &= F_4(C_2A_3^{-1}) \\ &= B_4A_6A_2A_1^{-1}B_5^{-1}A_3^{-1} \end{aligned}$$

$$\begin{aligned} F_6 &= RF_5R^{-1} \\ &= B_5A_1A_3A_2^{-1}B_6^{-1}A_4^{-1} \end{aligned}$$

and

$$\begin{aligned} F_7 &= F_5F_6 \\ &= B_4A_6B_6^{-1}A_4^{-1}. \end{aligned}$$

$(C_4A_5^{-1})F_7(c_4, a_5) = (b_4, a_5)$ so that $B_4A_5^{-1}\in H$ and then $B_iA_{i+1}^{-1}\in H$ for all i. Hence, $B_iC_i^{-1} = (B_iA_{i+1}^{-1})(A_{i+1}C_i^{-1})\in H$ for all i.

$(A_4B_3^{-1})F_7(a_4, b_3) = (b_4, b_3)$ so that $B_4B_3^{-1}\in H$ and then $B_{i+1}B_i^{-1}\in H$ for all i. In particular, $B_1B_2^{-1}\in H$.

Finally, $C_1C_2^{-1} = (C_1B_1^{-1})(B_1B_2^{-1})(B_2C_2^{-1})\in H$ and
$A_1A_2^{-1} = (A_1B_6^{-1})(B_6B_1^{-1})(B_1A_2^{-1})\in H$.

It follows from Theorem 4.0.1 that $H = \mathrm{Mod}(\Sigma_6)$, completing the proof of the corollary. □

Corollary 4.2.2. *If $g = 7$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $C_1B_4A_6A_7^{-1}B_5^{-1}C_2^{-1}$.*

Proof. Let $F_1 = C_1B_4A_6A_7^{-1}B_5^{-1}C_2^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_7)$ generated by the set $\{R, F_1\}$.

Let

$$\begin{aligned} F_2 &= RF_1R^{-1} \\ &= C_2B_5A_7A_1^{-1}B_6^{-1}C_3^{-1} \end{aligned}$$

and

$$\begin{aligned} F_3 &= F_2^{-1} \\ &= C_3B_6A_1A_7^{-1}B_5^{-1}C_2^{-1}. \end{aligned}$$

First, we have $F_3F_1(c_3, b_6, a_1, a_7, b_5, c_2) = (b_4, a_6, a_1, a_7, b_5, c_2)$ so that
$F_4 = B_4A_6A_1A_7^{-1}B_5^{-1}C_2^{-1} \in H$.

Let

$$\begin{aligned} F_5 &= RF_4R^{-1} \\ &= B_5A_7A_2A_1^{-1}B_6^{-1}C_3^{-1} \end{aligned}$$

and

$$\begin{aligned} F_6 &= F_5^{-1} \\ &= C_3B_6A_1A_2^{-1}A_7^{-1}B_5^{-1}. \end{aligned}$$

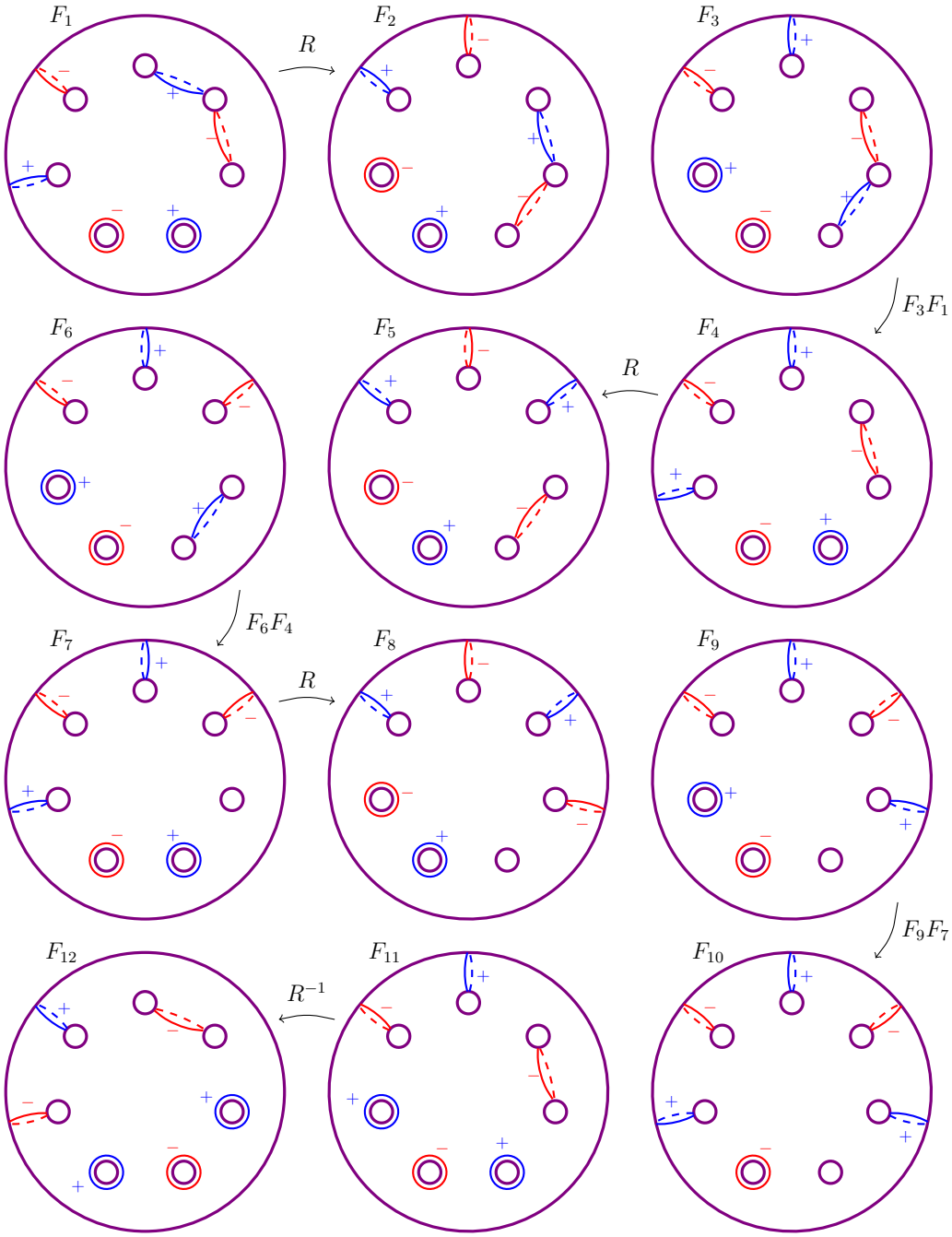


Figure 4.6: Proof of Corollary 4.2.2

Then, we get $F_6F_4(c_3, b_6, a_1, a_2, a_7, b_5) = (b_4, a_6, a_1, a_2, a_7, b_5)$ so that $F_7 = B_4A_6A_1A_2^{-1}A_7^{-1}B_5^{-1} \in H$.

Let

$$\begin{aligned} F_8 &= RF_7R^{-1} \\ &= B_5A_7A_2A_3^{-1}A_1^{-1}B_6^{-1} \end{aligned}$$

and

$$\begin{aligned} F_9 &= F_8^{-1} \\ &= B_6A_1A_3A_2^{-1}A_7^{-1}B_5^{-1}. \end{aligned}$$

Similarly, $F_9F_7(b_6, a_1, a_3, a_2, a_7, b_5) = (a_6, a_1, a_3, a_2, a_7, b_5)$ so that $F_{10} = A_6A_1A_3A_2^{-1}A_7^{-1}B_5^{-1} \in H$.

$F_{10}F_8 = A_6B_6^{-1} \in H$ and then by conjugating $A_6B_6^{-1}$ with R iteratively, we get $A_iB_i^{-1} \in H$ for all i.

Now, we have $F_1F_4^{-1} = C_1A_1^{-1} \in H$.
$B_1C_1^{-1} = (B_1A_1^{-1})(A_1C_1^{-1}) \in H$ and then $B_iC_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_{11} &= (B_6A_6^{-1})F_4 \\ &= B_4B_6A_1A_7^{-1}B_5^{-1}C_2^{-1} \end{aligned}$$

and

$$\begin{aligned} F_{12} &= R^{-1}F_{11}R \\ &= B_3B_5A_7A_6^{-1}B_4^{-1}C_1^{-1}. \end{aligned}$$

Finally, $F_{12}F_1 = B_3C_2^{-1} \in H$ and then $B_{i+1}C_i^{-1} \in H$ for all i.

It follows from Corollary 4.0.2 that $H = \mathrm{Mod}(\Sigma_7)$, completing the proof of the corollary. □

Corollary 4.2.3. *If $g = 8$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $B_1C_4A_7A_8^{-1}C_5^{-1}B_2^{-1}$.*

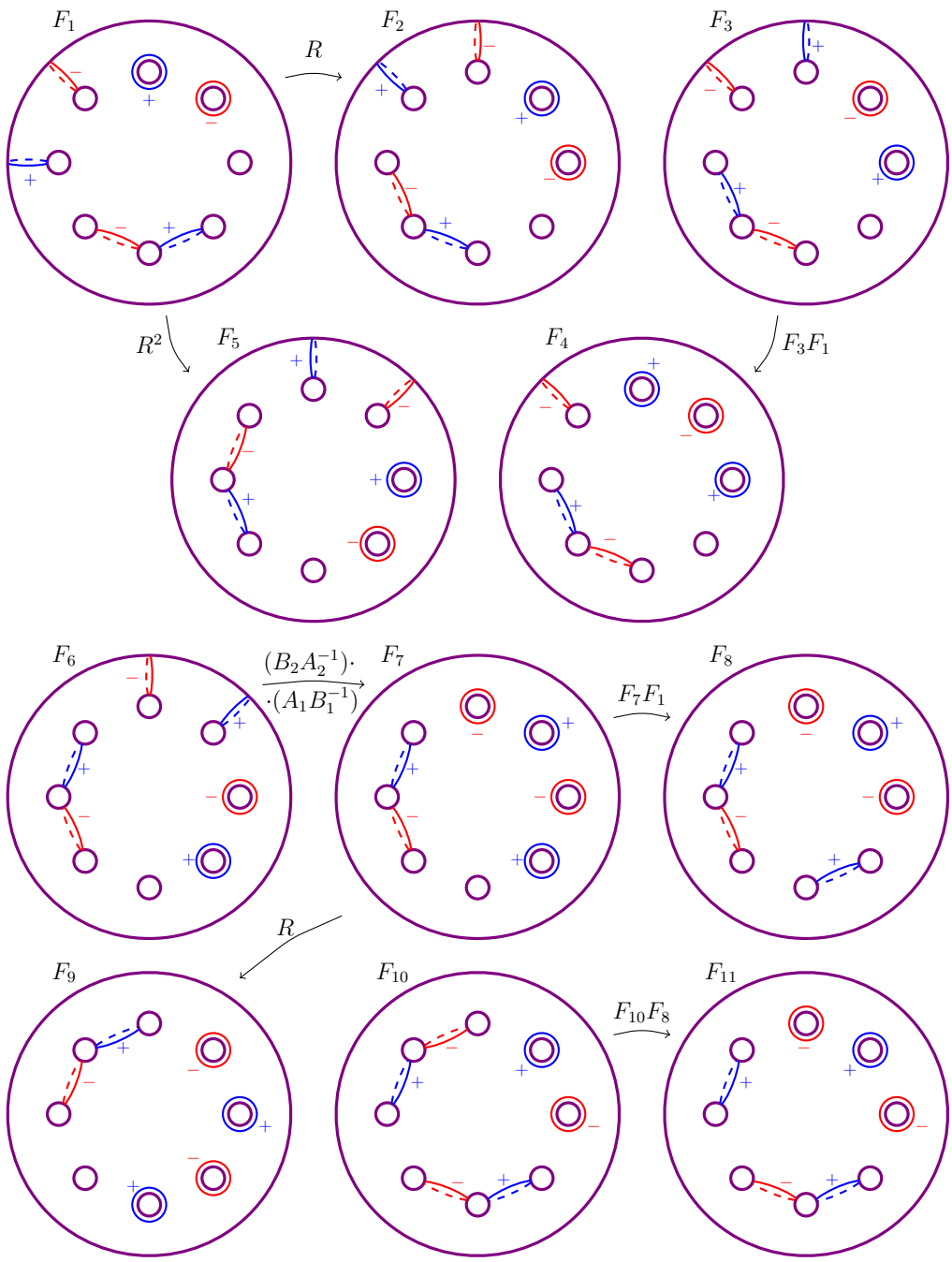


Figure 4.7: Proof of Corollary 4.2.3

Proof. Let $F_1 = B_1C_4A_7A_8^{-1}C_5^{-1}B_2^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_8)$ generated by the set $\{R, F_1\}$.

Let

$$\begin{aligned} F_2 &= RF_1R^{-1} \\ &= B_2C_5A_8A_1^{-1}C_6^{-1}B_3^{-1} \end{aligned}$$

and

$$\begin{aligned} F_3 &= F_2^{-1} \\ &= B_3C_6A_1A_8^{-1}C_5^{-1}B_2^{-1}. \end{aligned}$$

We have $F_3F_1(b_3, c_6, a_1, a_8, c_5, b_2) = (b_3, c_6, b_1, a_8, c_5, b_2)$ so that $F_4 = B_3C_6B_1A_8^{-1}C_5^{-1}B_2^{-1} \in H$.

Hence, $F_4F_3^{-1} = B_1A_1^{-1} \in H$ and then by conjugating $B_1A_1^{-1}$ with R iteratively, we get $B_iA_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_5 &= R^2F_1R^{-2} \\ &= B_3C_6A_1A_2^{-1}C_7^{-1}B_4^{-1} \end{aligned}$$

$$\begin{aligned} F_6 &= F_5^{-1} \\ &= B_4C_7A_2A_1^{-1}C_6^{-1}B_3^{-1} \end{aligned}$$

and

$$\begin{aligned} F_7 &= (B_2A_2^{-1})F_6(A_1B_1^{-1}) \\ &= B_4C_7B_2B_1^{-1}C_6^{-1}B_3^{-1}. \end{aligned}$$

Now, we get $F_7F_1(b_4, c_7, b_2, b_1, c_6, b_3) = (c_4, c_7, b_2, b_1, c_6, b_3)$ so that $F_8 = C_4C_7B_2B_1^{-1}C_6^{-1}B_3^{-1} \in H$.

Hence, $F_8F_7^{-1} = C_4B_4^{-1} \in H$ and then we get $C_iB_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_9 &= RF_7R^{-1} \\ &= B_5C_8B_3B_2^{-1}C_7^{-1}B_4^{-1} \end{aligned}$$

and

$$\begin{aligned} F_{10} &= (C_4B_4^{-1})F_9^{-1}(B_5C_5^{-1}) \\ &= C_4C_7B_2B_3^{-1}C_8^{-1}C_5^{-1}. \end{aligned}$$

Then, $F_{10}F_8(c_4, c_7, b_2, b_3, c_8, c_5) = (c_4, c_7, b_2, b_3, b_1, c_5)$ so that $F_{11} = C_4C_7B_2B_3^{-1}B_1^{-1}C_5^{-1} \in H$.

Finally, $F_{10}^{-1}F_{11} = C_8B_1^{-1} \in H$ and then we get $C_iB_{i+1}^{-1} \in H$ for all i.

It follows from Corollary 4.0.2 that $H = \mathrm{Mod}(\Sigma_8)$, completing the proof of the corollary. □

Corollary 4.2.4. *If $g \geq 9$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $C_1B_4A_7A_8^{-1}B_5^{-1}C_2^{-1}$.*

Proof. Let $F_1 = C_1B_4A_7A_8^{-1}B_5^{-1}C_2^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

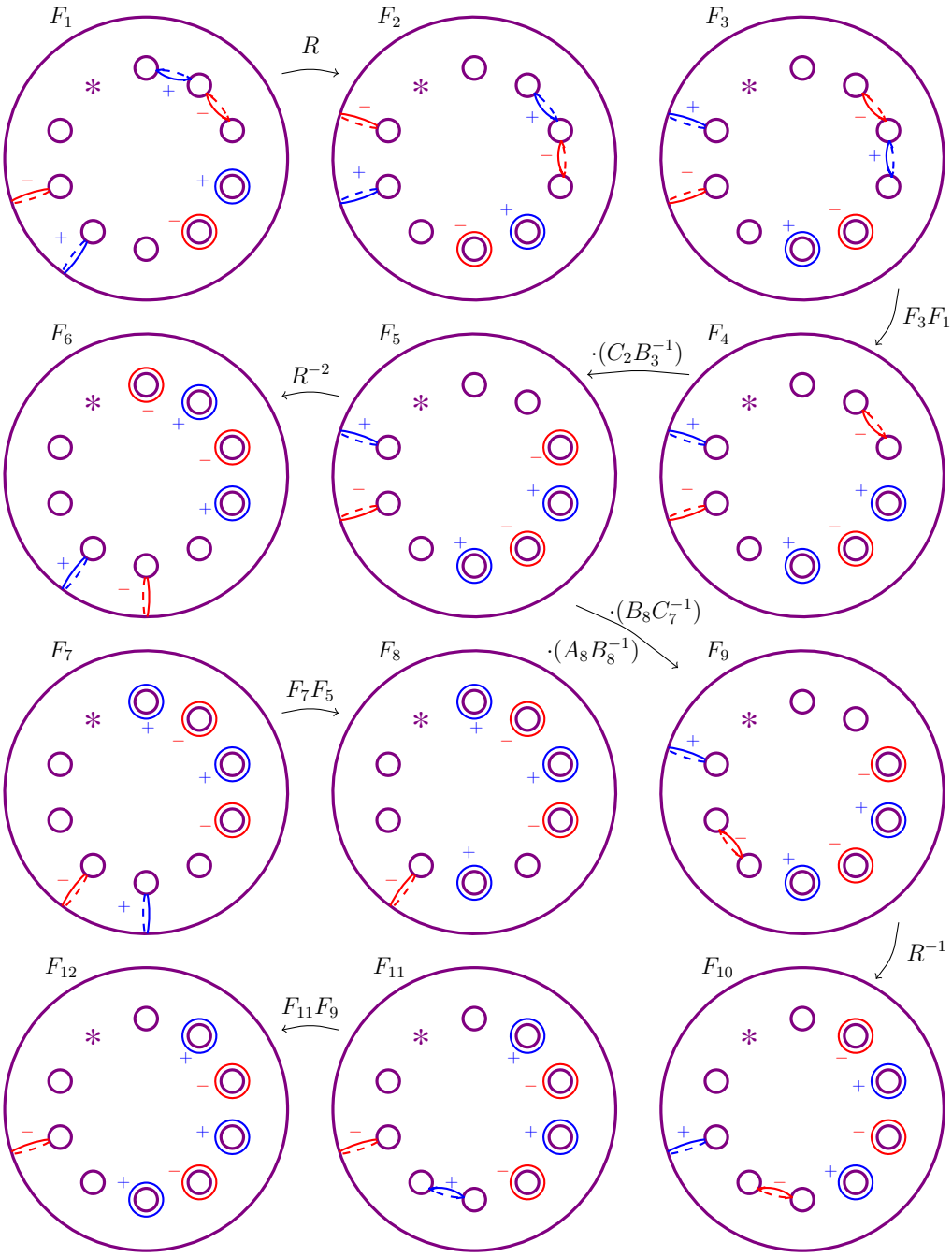
$$\begin{aligned} F_2 &= RF_1R^{-1} \\ &= C_2B_5A_8A_9^{-1}B_6^{-1}C_3^{-1} \end{aligned}$$

and

$$\begin{aligned} F_3 &= F_2^{-1} \\ &= C_3B_6A_9A_8^{-1}B_5^{-1}C_2^{-1}. \end{aligned}$$

We get $F_3F_1(c_3, b_6, a_9, a_8, b_5, c_2) = (b_4, b_6, a_9, a_8, b_5, c_2)$ so that $F_4 = B_4B_6A_9A_8^{-1}B_5^{-1}C_2^{-1} \in H$.

Hence, $F_4F_3^{-1} = B_4C_3^{-1} \in H$ and then by conjugating $B_4C_3^{-1}$ with R iteratively, we get $B_{i+1}C_i^{-1} \in H$ for all i.



Note that $*$ represents $g-9$ genera.

Figure 4.8: Proof of Corollary 4.2.4

Let

$$\begin{aligned} F_5 &= F_4(C_2B_3^{-1}) \\ &= B_4B_6A_9A_8^{-1}B_5^{-1}B_3^{-1} \end{aligned}$$

$$\begin{aligned} F_6 &= R^{-2}F_5R^2 \\ &= B_2B_4A_7A_6^{-1}B_3^{-1}B_1^{-1} \end{aligned}$$

and

$$\begin{aligned} F_7 &= F_6^{-1} \\ &= B_1B_3A_6A_7^{-1}B_4^{-1}B_2^{-1}. \end{aligned}$$

Then, we have $F_7F_5(b_1, b_3, a_6, a_7, b_4, b_2) = (b_1, b_3, b_6, a_7, b_4, b_2)$ so that $F_8 = B_1B_3B_6A_7^{-1}B_4^{-1}B_2^{-1} \in H$.

Hence, $F_8F_7^{-1} = B_6A_6^{-1} \in H$ and then $B_iA_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_9 &= F_5(A_8B_8^{-1})(B_8C_7^{-1}) \\ &= B_4B_6A_9C_7^{-1}B_5^{-1}B_3^{-1} \end{aligned}$$

$$\begin{aligned} F_{10} &= R^{-1}F_9R \\ &= B_3B_5A_8C_6^{-1}B_4^{-1}B_2^{-1} \end{aligned}$$

and

$$\begin{aligned} F_{11} &= F_{10}^{-1} \\ &= B_2B_4C_6A_8^{-1}B_5^{-1}B_3^{-1}. \end{aligned}$$

Now, we have $F_{11}F_9(b_2, b_4, c_6, a_8, b_5, b_3) = (b_2, b_4, b_6, a_8, b_5, b_3)$ so that $F_{12} = B_2B_4B_6A_8^{-1}B_5^{-1}B_3^{-1} \in H$.

Finally, $F_{12}F_{11}^{-1} = B_6C_6^{-1} \in H$ and then $B_iC_i^{-1} \in H$ for all i.

It follows from Corollary 4.0.2 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the corollary. $\square$

Corollary 4.2.5. *If $g = 8$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $B_1A_5C_5C_7^{-1}A_7^{-1}B_3^{-1}$.*

Proof. Let $F_1 = B_1A_5C_5C_7^{-1}A_7^{-1}B_3^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

$$\begin{aligned} F_2 &= RF_1R^{-1} \\ &= B_2A_6C_6C_8^{-1}A_8^{-1}B_4^{-1} \end{aligned}$$

and

$$\begin{aligned} F_3 &= F_2^{-1} \\ &= B_4A_8C_8C_6^{-1}A_6^{-1}B_2^{-1}. \end{aligned}$$

We have $F_3F_1(b_4, a_8, c_8, c_6, a_6, b_2) = (b_4, a_8, b_1, c_6, a_6, b_2)$ so that $F_4 = B_4A_8B_1C_6^{-1}A_6^{-1}B_2^{-1} \in H$.

$F_4F_3^{-1} = B_1C_8^{-1} \in H$ and then by conjugating $B_1C_8^{-1}$ with R iteratively, we get $B_{i+1}C_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_5 &= RF_4R^{-1} \\ &= B_5A_1B_2C_7^{-1}A_7^{-1}B_3^{-1}. \end{aligned}$$

Hence, $F_5F_4(b_5, a_1, b_2, c_7, a_7, b_3) = (b_5, b_1, b_2, c_7, a_7, b_3)$ so that $F_6 = B_5B_1B_2C_7^{-1}A_7^{-1}B_3^{-1} \in H$.

Now, we have $F_6F_5^{-1} = B_1A_1^{-1} \in H$ and then $B_iA_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_7 &= (C_4B_5^{-1})F_6(C_7B_8^{-1})(A_7B_7^{-1}) \\ &= C_4B_1B_2B_3^{-1}B_8^{-1}B_7^{-1} \end{aligned}$$

$$\begin{aligned} F_8 &= RF_7R^{-1} \\ &= C_5B_2B_3B_4^{-1}B_1^{-1}B_8^{-1} \end{aligned}$$

and

$$\begin{aligned} F_9 &= F_8^{-1} \\ &= B_8 B_1 B_4 B_3^{-1} B_2^{-1} C_5^{-1}. \end{aligned}$$

Hence, $F_9 F_7(b_8, b_1, b_4, b_3, b_2, c_5) = (b_8, b_1, c_4, b_3, b_2, c_5)$ so that
$F_{10} = B_8 B_1 C_4 B_3^{-1} B_2^{-1} C_5^{-1} \in H$.

Finally, $F_{10} F_9^{-1} = C_4 B_4^{-1} \in H$ and then $C_i B_i^{-1} \in H$ for all i.

It follows from Corollary 4.0.2 that $H = \mathrm{Mod}(\Sigma_8)$, completing the proof of the corollary. □

Corollary 4.2.6. *If $g = 9$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $B_1 A_3 C_5 C_8^{-1} A_6^{-1} B_4^{-1}$.*

Proof. Let $F_1 = B_1 A_3 C_5 C_8^{-1} A_6^{-1} B_4^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_9)$ generated by the set $\{R, F_1\}$.

Let

$$\begin{aligned} F_2 &= R F_1 R^{-1} \\ &= B_2 A_4 C_6 C_9^{-1} A_7^{-1} B_5^{-1} \end{aligned}$$

and

$$\begin{aligned} F_3 &= F_2^{-1} \\ &= B_5 A_7 C_9 C_6^{-1} A_4^{-1} B_2^{-1}. \end{aligned}$$

We have $F_3 F_1(b_5, a_7, c_9, c_6, a_4, b_2) = (c_5, a_7, b_1, c_6, b_4, b_2)$ so that
$F_4 = C_5 A_7 B_1 C_6^{-1} B_4^{-1} B_2^{-1} \in H$.

Let

$$\begin{aligned} F_5 &= R F_4 R^{-1} \\ &= C_6 A_8 B_2 C_7^{-1} B_5^{-1} B_3^{-1} \end{aligned}$$

and

$$\begin{aligned} F_6 &= F_5^{-1} \\ &= B_3 B_5 C_7 B_2^{-1} A_8^{-1} C_6^{-1}. \end{aligned}$$

Then, we get $F_6F_4(b_3, b_5, c_7, b_2, a_8, c_6) = (b_3, c_5, c_7, b_2, a_8, c_6)$ so that $F_7 = B_3C_5C_7B_2^{-1}A_8^{-1}C_6^{-1} \in H$.

Hence, $F_7F_6^{-1} = C_5B_5^{-1} \in H$ and then by conjugating $C_5B_5^{-1}$ with R iteratively, we get $C_iB_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_8 &= (B_7C_7^{-1})F_6(C_6B_6^{-1}) \\ &= B_3B_5B_7B_2^{-1}A_8^{-1}B_6^{-1} \end{aligned}$$

$$\begin{aligned} F_9 &= RF_8R^{-1} \\ &= B_4B_6B_8B_3^{-1}A_9^{-1}B_7^{-1} \end{aligned}$$

and

$$\begin{aligned} F_{10} &= F_9^{-1} \\ &= B_7A_9B_3B_8^{-1}B_6^{-1}B_4^{-1}. \end{aligned}$$

Now, we get $F_{10}F_8(b_7, a_9, b_3, b_8, b_6, b_4) = (b_7, a_9, b_3, a_8, b_6, b_4)$ so that $F_{11} = B_7A_9B_3A_8^{-1}B_6^{-1}B_4^{-1} \in H$.

Hence, $F_{11}^{-1}F_{10} = A_8B_8^{-1} \in H$ and then $A_iB_i^{-1} \in H$ for all i.

Finally, $F_4(B_4A_4^{-1})F_2(B_5C_5^{-1}) = B_1C_9^{-1} \in H$ and then $B_{i+1}C_i^{-1} \in H$ for all i.

It follows from Corollary 4.0.2 that $H = \mathrm{Mod}(\Sigma_9)$, completing the proof of the corollary. □

Corollary 4.2.7. *If $g = 10$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $A_1C_1B_3B_7^{-1}C_5^{-1}A_5^{-1}$.*

Proof. Let $F_1 = A_1C_1B_3B_7^{-1}C_5^{-1}A_5^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_{10})$ generated by the set $\{R, F_1\}$.

Let

$$\begin{aligned} F_2 &= RF_1R^{-1} \\ &= A_2C_2B_4B_8^{-1}C_6^{-1}A_6^{-1}. \end{aligned}$$

Then, we get $F_2F_1(a_2, c_2, b_4, b_8, c_6, a_6) = (a_2, b_3, b_4, b_8, b_7, a_6)$ so that $F_3 = A_2B_3B_4B_8^{-1}B_7^{-1}A_6^{-1} \in H$.

Let

$$\begin{aligned} F_4 &= R^4F_3R^{-4} \\ &= A_6B_7B_8B_2^{-1}B_1^{-1}A_{10}^{-1} \end{aligned}$$

and

$$\begin{aligned} F_5 &= F_4^{-1} \\ &= A_{10}B_1B_2B_8^{-1}B_7^{-1}A_6^{-1}. \end{aligned}$$

We also have $F_5F_3(a_{10}, b_1, b_2, b_8, b_7, a_6) = (a_{10}, b_1, a_2, b_8, b_7, a_6)$ so that $F_6 = A_{10}B_1A_2B_8^{-1}B_7^{-1}A_6^{-1} \in H$.

Hence, $F_6F_5^{-1} = A_2B_2^{-1} \in H$ and then by conjugating $A_2B_2^{-1}$ with R iteratively, we get $A_iB_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_7 &= (B_2A_2^{-1})(A_3B_3^{-1})F_3(B_7A_7^{-1})(A_6B_6^{-1}) \\ &= B_2A_3B_4B_8^{-1}A_7^{-1}B_6^{-1} \end{aligned}$$

$$\begin{aligned} F_8 &= RF_2F_3^{-1}R^{-1}F_7 \\ &= B_2A_3C_3C_7^{-1}A_7^{-1}B_6^{-1} \end{aligned}$$

$$\begin{aligned} F_9 &= F_8^{-1} \\ &= B_6A_7C_7C_3^{-1}A_3^{-1}B_2^{-1} \end{aligned}$$

and

$$\begin{aligned} F_{10} &= R^4F_9R^{-4} \\ &= B_{10}A_1C_1C_7^{-1}A_7^{-1}B_6^{-1}. \end{aligned}$$

Now, we get $F_{10}F_8(b_{10}, a_1, c_1, c_7, a_7, b_6) = (b_{10}, a_1, b_2, c_7, a_7, b_6)$ so that $F_{11} = B_{10}A_1B_2C_7^{-1}A_7^{-1}B_6^{-1} \in H$.

Hence, $F_{11}F_{10}^{-1} = B_2C_1^{-1} \in H$ and then $B_{i+1}C_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_{12} &= (B_2A_2^{-1})F_3(A_6B_6^{-1})(B_6C_5^{-1})(B_7A_7^{-1})(B_8A_8^{-1}) \\ &= B_2B_3B_4A_8^{-1}A_7^{-1}C_5^{-1} \end{aligned}$$

$$\begin{aligned} F_{13} &= F_{12}^{-1} \\ &= C_5A_7A_8B_4^{-1}B_3^{-1}B_2^{-1} \end{aligned}$$

and

$$\begin{aligned} F_{14} &= RF_{13}R^{-1} \\ &= C_6A_8A_9B_5^{-1}B_4^{-1}B_3^{-1}. \end{aligned}$$

Hence, $F_{14}F_{12}(c_6, a_8, a_9, b_5, b_4, b_3) = (c_6, a_8, a_9, c_5, b_4, b_3)$ so that $F_{15} = C_6A_8A_9C_5^{-1}B_4^{-1}B_3^{-1} \in H$.

Finally, $F_{15}^{-1}F_{14} = C_5B_5^{-1} \in H$ and then $C_iB_i^{-1} \in H$ for all i.

It follows from Corollary 4.0.2 that $H = \mathrm{Mod}(\Sigma_{10})$, completing the proof of the corollary. □

Corollary 4.2.8. *If $g \geq 13$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $A_1B_4C_8C_{10}^{-1}B_6^{-1}A_3^{-1}$.*

Proof. Let $F_1 = A_1B_4C_8C_{10}^{-1}B_6^{-1}A_3^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

$$\begin{aligned} F_2 &= RF_1R^{-1} \\ &= A_2B_5C_9C_{11}^{-1}B_7^{-1}A_4^{-1} \end{aligned}$$

and

$$\begin{aligned} F_3 &= F_2^{-1} \\ &= A_4B_7C_{11}C_9^{-1}B_5^{-1}A_2^{-1}. \end{aligned}$$

First, we have $F_3F_1(a_4, b_7, c_{11}, c_9, b_5, a_2) = (b_4, b_7, c_{11}, c_9, b_5, a_2)$ so that $F_4 = B_4B_7C_{11}C_9^{-1}B_5^{-1}A_2^{-1} \in H$.

Hence, $F_4F_3^{-1} = B_4A_4^{-1} \in H$ and then by conjugating $B_4A_4^{-1}$ with R iteratively, we get $B_iA_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_5 &= R^2F_1R^{-2} \\ &= A_3B_6C_{10}C_{12}^{-1}B_8^{-1}A_5^{-1} \end{aligned}$$

and

$$\begin{aligned} F_6 &= F_5^{-1} \\ &= A_5B_8C_{12}C_{10}^{-1}B_6^{-1}A_3^{-1}. \end{aligned}$$

Then, $F_6F_1(a_5, b_8, c_{12}, c_{10}, b_6, a_3) = (a_5, c_8, c_{12}, c_{10}, b_6, a_3)$ so that $F_7 = A_5C_8C_{12}C_{10}^{-1}B_6^{-1}A_3^{-1} \in H$.

Now, we have $F_7F_6^{-1} = C_8B_8^{-1} \in H$ and then $C_iB_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_8 &= (A_4B_4^{-1})F_1(A_3B_3^{-1}) \\ &= A_1A_4C_8C_{10}^{-1}B_6^{-1}B_3^{-1} \end{aligned}$$

$$\begin{aligned} F_9 &= R^3F_8R^{-3} \\ &= A_4A_7C_{11}C_{13}^{-1}B_9^{-1}B_6^{-1} \end{aligned}$$

and

$$\begin{aligned} F_{10} &= F_9^{-1} \\ &= B_6B_9C_{13}C_{11}^{-1}A_7^{-1}A_4^{-1}. \end{aligned}$$

We get $F_{10}F_8(b_6, b_9, c_{13}, c_{11}, a_7, a_4) = (b_6, c_8, c_{13}, c_{11}, a_7, a_4)$ so that $F_{11} = B_6C_8C_{13}C_{11}^{-1}A_7^{-1}A_4^{-1} \in H$.

Hence, $F_{11}F_{10}^{-1} = C_8B_9^{-1} \in H$ and then $C_iB_{i+1}^{-1} \in H$ for all i.

It follows from Corollary 4.0.2 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the corollary. $\square$

Corollary 4.2.9. *If $g \geq 12$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the two elements R and $B_1A_3C_6C_{10}^{-1}A_7^{-1}B_5^{-1}$.*

Proof. Let $F_1 = B_1A_3C_6C_{10}^{-1}A_7^{-1}B_5^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

$$\begin{aligned} F_2 &= RF_1R^{-1} \\ &= B_2A_4C_7C_{11}^{-1}A_8^{-1}B_6^{-1} \end{aligned}$$

and

$$\begin{aligned} F_3 &= F_2^{-1} \\ &= B_6A_8C_{11}C_7^{-1}A_4^{-1}B_2^{-1}. \end{aligned}$$

Hence, $F_3F_1(b_6, a_8, c_{11}, c_7, a_4, b_2) = (c_6, a_8, c_{11}, c_7, a_4, b_2)$ so that $F_4 = C_6A_8C_{11}C_7^{-1}A_4^{-1}B_2^{-1} \in H$.

We have $F_4F_3^{-1} = C_6B_6^{-1} \in H$ and then by conjugating $C_6B_6^{-1}$ with R iteratively, we get $C_iB_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_5 &= F_1(C_{10}B_{10}^{-1})(B_5C_5^{-1}) \\ &= B_1A_3C_6B_{10}^{-1}A_7^{-1}C_5^{-1} \end{aligned}$$

and

$$\begin{aligned} F_6 &= R^2F_5R^{-2} \\ &= B_3A_5C_8B_{12}^{-1}A_9^{-1}C_7^{-1}. \end{aligned}$$

Now, we get $F_6F_5(b_3, a_5, c_8, b_{12}, a_9, c_7) = (a_3, a_5, c_8, b_{12}, a_9, c_7)$ so that $F_7 = A_3A_5C_8B_{12}^{-1}A_9^{-1}C_7^{-1} \in H$.

Hence, $F_7F_6^{-1} = A_3B_3^{-1} \in H$ and then $A_iB_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_8 &= (C_1B_1^{-1})(B_3A_3^{-1})F_1(B_5C_5^{-1}) \\ &= C_1B_3C_6C_{10}^{-1}A_7^{-1}C_5^{-1} \end{aligned}$$

and

$$\begin{aligned} F_9 &= RF_8R^{-1} \\ &= C_2B_4C_7C_{11}^{-1}A_8^{-1}C_6^{-1}. \end{aligned}$$

We also have $F_9F_8(c_2, b_4, c_7, c_{11}, a_8, c_6) = (b_3, b_4, c_7, c_{11}, a_8, c_6)$ so that $F_{10} = B_3B_4C_7C_{11}^{-1}A_8^{-1}C_6^{-1} \in H$.

Finally, $F_{10}F_9^{-1} = B_3C_2^{-1} \in H$ and then $B_{i+1}C_i^{-1} \in H$ for all i.

It follows from Corollary 4.0.2 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the corollary. □

Lemma 4.2.10. *If $g \geq 11$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by two elements R and $A_1B_2C_4C_{g-1}^{-1}B_{g-3}^{-1}A_{g-4}^{-1}$.*

Proof. Let $F_1 = A_1B_2C_4C_{g-1}^{-1}B_{g-3}^{-1}A_{g-4}^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

$$F_2 = RF_1R^{-1} = A_2B_3C_5C_g^{-1}B_{g-2}^{-1}A_{g-3}^{-1}.$$

We have $F_2F_1(a_2, b_3, c_5, c_g, b_{g-2}, a_{g-3}) = (b_2, b_3, c_5, c_g, b_{g-2}, b_{g-3})$ so that $F_3 = B_2B_3C_5C_g^{-1}B_{g-2}^{-1}B_{g-3}^{-1} \in H$.

Let

$$\begin{aligned} F_4 &= R^{-1}F_3R \\ &= B_1B_2C_4C_{g-1}^{-1}B_{g-3}^{-1}B_{g-4}^{-1} \end{aligned}$$

and

$$\begin{aligned} F_5 &= F_3^{-1} \\ &= B_{g-3}B_{g-2}C_gC_5^{-1}B_3^{-1}B_2^{-1}. \end{aligned}$$

Hence, $F_5F_4(b_{g-3}, b_{g-2}, c_g, c_5, b_3, b_2) = (b_{g-3}, b_{g-2}, b_1, c_5, b_3, b_2)$ so that $F_6 = B_{g-3}B_{g-2}B_1C_5^{-1}B_3^{-1}B_2^{-1} \in H$.

We also have $F_6F_5^{-1} = B_1C_g^{-1} \in H$ and then by conjugating $B_1C_g^{-1}$ with R iteratively, we get $B_{i+1}C_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_7 &= (C_{g-3}B_{g-2}^{-1})(C_{g-4}B_{g-3}^{-1})F_6 \\ &= C_{g-3}C_{g-4}B_1C_5^{-1}B_3^{-1}B_2^{-1} \end{aligned}$$

and

$$\begin{aligned} F_8 &= R^2F_7R^{-2} \\ &= C_{g-1}C_{g-2}B_3C_7^{-1}B_5^{-1}B_4^{-1}. \end{aligned}$$

Hence, $F_8F_7(c_{g-1}, c_{g-2}, b_3, c_7, b_5, b_4) = (c_{g-1}, c_{g-2}, b_3, c_7, c_5, b_4)$ so that $F_9 = C_{g-1}C_{g-2}B_3C_7^{-1}C_5^{-1}B_4^{-1} \in H$.

Now, we get $F_9F_8^{-1} = C_5B_5^{-1} \in H$ and then $C_iB_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_{10} &= F_1(B_{g-3}C_{g-3}^{-1}) \\ &= A_1B_2C_4C_{g-1}^{-1}C_{g-3}^{-1}A_{g-4}^{-1} \end{aligned}$$

and

$$\begin{aligned} F_{11} &= RF_{10}R^{-1} \\ &= A_2B_3C_5C_g^{-1}C_{g-2}^{-1}A_{g-3}^{-1}. \end{aligned}$$

Hence, $F_{11}F_{10}(a_2, b_3, c_5, c_g, c_{g-2}, a_{g-3}) = (b_2, b_3, c_5, c_g, c_{g-2}, a_{g-3})$ so that $F_{12} = B_2B_3C_5C_g^{-1}C_{g-2}^{-1}A_{g-3}^{-1} \in H$.

Finally, $F_{12}F_{11}^{-1} = B_2A_2^{-1} \in H$ and then $B_iA_i^{-1} \in H$ for all i.

It follows from Corollary 4.0.2 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the lemma. $\square$

Lemma 4.2.11. *If $g \geq 13$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by two elements R and $A_1B_2C_4C_{g-2}^{-1}B_{g-4}^{-1}A_{g-5}^{-1}$.*

Proof. Let $F_1 = A_1B_2C_4C_{g-2}^{-1}B_{g-4}^{-1}A_{g-5}^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

$$\begin{aligned} F_2 &= RF_1R^{-1} \\ &= A_2B_3C_5C_{g-1}^{-1}B_{g-3}^{-1}A_{g-4}^{-1}. \end{aligned}$$

Hence, $F_2F_1(a_2, b_3, c_5, c_{g-1}, b_{g-3}, a_{g-4}) = (b_2, b_3, c_5, c_{g-1}, b_{g-3}, b_{g-4})$ so that $F_3 = B_2B_3C_5C_{g-1}^{-1}B_{g-3}^{-1}B_{g-4}^{-1} \in H$.

Let

$$\begin{aligned} F_4 &= F_2F_3^{-1} \\ &= A_2B_2^{-1}A_{g-4}^{-1}B_{g-4} \end{aligned}$$

$$\begin{aligned} F_5 &= RF_4R^{-1} \\ &= A_3B_3^{-1}A_{g-3}^{-1}B_{g-3} \end{aligned}$$

$$\begin{aligned} F_6 &= F_5F_3 \\ &= B_2A_3C_5C_{g-1}^{-1}A_{g-3}^{-1}B_{g-4}^{-1} \end{aligned}$$

$$\begin{aligned} F_7 &= R^{-2}F_6R^2 \\ &= B_gA_1C_3C_{g-3}^{-1}A_{g-5}^{-1}B_{g-6}^{-1} \end{aligned}$$

and

$$\begin{aligned} F_8 &= F_7^{-1} \\ &= B_{g-6}A_{g-5}C_{g-3}C_3^{-1}A_1^{-1}B_g^{-1}. \end{aligned}$$

We get $F_8F_6(b_{g-6}, a_{g-5}, c_{g-3}, c_3, a_1, b_g) = (b_{g-6}, a_{g-5}, c_{g-3}, c_3, a_1, c_{g-1})$ so that $F_9 = B_{g-6}A_{g-5}C_{g-3}C_3^{-1}A_1^{-1}C_{g-1}^{-1} \in H$.

Hence, $F_9F_8^{-1} = C_{g-1}B_g^{-1} \in H$ and then by conjugating $C_{g-1}B_g^{-1}$ with R iteratively, we get $C_iB_{i+1}^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_{10} &= F_3(C_{g-1}B_g^{-1}) \\ &= B_2B_3C_5B_g^{-1}B_{g-3}^{-1}B_{g-4}^{-1} \end{aligned}$$

and

$$\begin{aligned} F_{11} &= R^2F_{10}R^{-2} \\ &= B_4B_5C_7B_2^{-1}B_{g-1}^{-1}B_{g-2}^{-1}. \end{aligned}$$

We also have $F_{11}F_{10}(b_4, b_5, c_7, b_2, b_{g-1}, b_{g-2}) = (b_4, c_5, c_7, b_2, b_{g-1}, b_{g-2})$ so that $F_{12} = B_4C_5C_7B_2^{-1}B_{g-1}^{-1}B_{g-2}^{-1} \in H$.

Hence, $F_{12}F_{11}^{-1} = C_5B_5^{-1} \in H$ and then $C_iB_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_{13} &= F_1(B_{g-4}C_{g-4}^{-1}) \\ &= A_1B_2C_4C_{g-2}^{-1}C_{g-4}^{-1}A_{g-5}^{-1} \end{aligned}$$

and

$$\begin{aligned} F_{14} &= RF_{13}R^{-1} \\ &= A_2B_3C_5C_{g-1}^{-1}C_{g-3}^{-1}A_{g-4}^{-1}. \end{aligned}$$

Now, we get $F_{14}F_{13}(a_2, b_3, c_5, c_{g-1}, c_{g-3}, a_{g-4}) = (b_2, b_3, c_5, c_{g-1}, c_{g-3}, a_{g-4})$ so that $F_{15} = B_2B_3C_5C_{g-1}^{-1}C_{g-3}^{-1}A_{g-4}^{-1} \in H$.

Finally, $F_{15}F_{14}^{-1} = B_2A_2^{-1} \in H$ and then $B_iA_i^{-1} \in H$ for all i.

It follows from Corollary 4.0.2 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the corollary. $\square$

Lemma 4.2.12. *If $k \geq 7$ and $g \geq 2k+1$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by elements R and $A_1B_2C_4C_{g-k+4}^{-1}B_{g-k+2}^{-1}A_{g-k+1}^{-1}$.*

Proof. Let $F_1 = A_1B_2C_4C_{g-k+4}^{-1}B_{g-k+2}^{-1}A_{g-k+1}^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let

$$\begin{aligned} F_2 &= R^{k-3}F_1R^{3-k} \\ &= A_{k-2}B_{k-1}C_{k+1}C_1^{-1}B_{g-1}^{-1}A_{g-2}^{-1} \end{aligned}$$

and

$$\begin{aligned} F_3 &= F_2^{-1} \\ &= A_{g-2}B_{g-1}C_1C_{k+1}^{-1}B_{k-1}^{-1}A_{k-2}^{-1}. \end{aligned}$$

We have $F_3F_1(a_{g-2}, b_{g-1}, c_1, c_{k+1}, b_{k-1}, a_{k-2}) = (a_{g-2}, b_{g-1}, b_2, c_{k+1}, b_{k-1}, a_{k-2})$ so that $F_4 = A_{g-2}B_{g-1}B_2C_{k+1}^{-1}B_{k-1}^{-1}A_{k-2}^{-1} \in H$.

Hence, $F_4F_3^{-1} = B_2C_1^{-1} \in H$ and then by conjugating $B_2C_1^{-1}$ with R iteratively, we get $B_{i+1}C_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_5 &= F_1(B_{g-k+2}C_{g-k+1}^{-1}) \\ &= A_1B_2C_4C_{g-k+4}^{-1}C_{g-k+1}^{-1}A_{g-k+1}^{-1} \end{aligned}$$

and

$$\begin{aligned} F_6 &= RF_5R^{-1} \\ &= A_2B_3C_5C_{g-k+5}^{-1}C_{g-k+2}^{-1}A_{g-k+2}^{-1}. \end{aligned}$$

We get $F_6F_5(a_2, b_3, c_5, c_{g-k+5}, c_{g-k+2}, a_{g-k+2}) = (b_2, b_3, c_5, c_{g-k+5}, c_{g-k+2}, a_{g-k+2})$ so that $F_7 = B_2B_3C_5C_{g-k+5}^{-1}C_{g-k+2}^{-1}A_{g-k+2}^{-1} \in H$.

Hence, $F_7F_6^{-1} = B_2A_2^{-1} \in H$ and then $B_iA_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_8 &= R^{k-2}F_6R^{2-k} \\ &= A_kB_{k+1}C_{k+3}C_3^{-1}Cg^{-1}A_g^{-1} \end{aligned}$$

and

$$\begin{aligned} F_9 &= F_8^{-1} \\ &= A_g C_g C_3 C_{k-3}^{-1} B_{k+1}^{-1} A_k^{-1}. \end{aligned}$$

Then, $F_9 F_6(a_g, c_g, c_3, c_{k+3}, b_{k+1}, a_k) = (a_g, c_g, b_3, c_{k+3}, b_{k+1}, a_k)$
so that $F_{10} = A_g C_g B_3 C_{k-3}^{-1} B_{k+1}^{-1} A_k^{-1} \in H$.

Finally, $F_{10} F_9^{-1} = B_3 C_3^{-1} \in H$ and then $B_i C_i^{-1} \in H$ for all i.

It follows from Corollary 4.0.2 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the lemma. □

Corollary 4.2.13. *If $k \geq 5$ and $g \geq 2k+1$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the elements R and $A_1 B_2 C_4 C_{g-k+4}^{-1} B_{g-k+2}^{-1} A_{g-k+1}^{-1}$.*

Proof. It directly follows from Lemma 4.2.10, Lemma 4.2.11 and Lemma 4.2.12. □

4.2.2 Proof of Theorem 1.0.4

Finally, we prove Theorem 1.0.4.

Proof. If $g = 6$, let H_6 be the subgroup of $\mathrm{Mod}(\Sigma_6)$ generated by the set $\{R, RC_1 B_4 A_6 A_1^{-1} B_5^{-1} C_2^{-1}\}$. Then, $H_6 = \mathrm{Mod}(\Sigma_6)$ by Corollary 4.2.1. Hence, we are done by Lemma 4.0.3 since $R(C_1 B_4 A_6) R^{-1} = C_2 B_5 A_1$. Note that, since $R(c_1) = c_2$, $R(b_4) = b_5$ and $R(a_6) = a_1$, we have $R(C_1 B_4 A_6) R^{-1} = C_2 B_5 A_1$ which implies order of the element $R(C_1 B_4 A_6)(C_2 B_5 A_1)^{-1}$ is g.

If $g = 7$, let H_7 be the subgroup of $\mathrm{Mod}(\Sigma_7)$ generated by the set $\{R, RC_1 B_4 A_6 A_7^{-1} B_5^{-1} C_2^{-1}\}$. Then, $H_7 = \mathrm{Mod}(\Sigma_7)$ by Corollary 4.2.2. Hence, we are done by Lemma 4.0.3 since $R(C_1 B_4 A_6) R^{-1} = C_2 B_5 A_7$.

If $g = 8$, let H_8 be the subgroup of $\mathrm{Mod}(\Sigma_8)$ generated by the set $\{R, RB_1 C_4 A_7 A_8^{-1} C_5^{-1} B_2^{-1}\}$. Then, $H_8 = \mathrm{Mod}(\Sigma_8)$ by Corollary 4.2.3. Hence, we are done by Lemma 4.0.3 since $R(B_1 C_4 A_7) R^{-1} = B_2 C_5 A_8$.

If $g \geq 9$, let H_9 be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, RC_1B_4A_7A_8^{-1}B_5^{-1}C_2^{-1}\}$. Then, $H_9 = \mathrm{Mod}(\Sigma_g)$ by Corollary 4.2.4. Hence, we are done by Lemma 4.0.3 since $R(C_1B_4A_7)R^{-1} = C_2B_5A_8$. □

4.2.3 Proof of Theorem 1.0.5

Now, we can prove Theorem 1.0.5.

Proof. If $g = 10$, let H_{10} be the subgroup of $\mathrm{Mod}(\Sigma_{10})$ generated by the set $\{R, R^4A_1C_1B_3B_7^{-1}C_5^{-1}A_5^{-1}\}$. Then, $H_{10} = \mathrm{Mod}(\Sigma_{10})$ by Corollary 4.2.7. Hence, we are done by Lemma 4.0.3 since $R^4(A_1C_1B_3)R^{-4} = A_5C_5B_7$. Note that, order of R^4 is clearly 5 and hence order of the element $R^4(A_1C_1B_3)(A_5C_5B_7)^{-1}$ is also 5 by Lemma 4.0.3 since $R^4(a_1) = a_5$, $R^4(c_1) = c_5$ and $R^4(b_3) = b_7$ implies $R^4(A_1C_1B_3)R^{-4} = A_5C_5B_7$.

If $g = 9$, let H_9 be the subgroup of $\mathrm{Mod}(\Sigma_9)$ generated by the set $\{R, R^3B_1A_3C_5C_8^{-1}A_6^{-1}B_4^{-1}\}$. Then, $H_9 = \mathrm{Mod}(\Sigma_9)$ by Corollary 4.2.6. Hence, we are done by Lemma 4.0.3 since $R^3(B_1A_3C_5)R^{-3} = B_4A_6C_8$.

If $g = 8$, let H_8 be the subgroup of $\mathrm{Mod}(\Sigma_8)$ generated by the set $\{R, R^2B_1A_5C_5C_7^{-1}A_7^{-1}B_3^{-1}\}$. Then, $H_8 = \mathrm{Mod}(\Sigma_8)$ by Corollary 4.2.5. Hence, we are done by Lemma 4.0.3 since $R^2(B_1A_5C_5)R^{-2} = B_3A_7C_7$.

If $g = 7$, let H_7 be the subgroup of $\mathrm{Mod}(\Sigma_7)$ generated by the set $\{R, RC_1B_4A_6A_7^{-1}B_5^{-1}C_2^{-1}\}$. Then, $H_7 = \mathrm{Mod}(\Sigma_7)$ by Corollary 4.2.2. Hence, we are done by Lemma 4.0.3 since $R(C_1B_4A_6)R^{-1} = C_2B_5A_7$.

The remaining part of the proof is the case of $g \geq 11$. Let $k = g/g'$ so that k is the greatest divisor of g such that k is strictly less than $g/2$. Clearly, k can be any positive integer but three.

If $k = 2$, let K_2 be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, R^2A_1B_4C_8C_{10}^{-1}B_6^{-1}A_3^{-1}\}$. Then, $K_2 = \mathrm{Mod}(\Sigma_g)$ by Corollary 4.2.8. Hence, we are done by Lemma 4.0.3 since $R^2(A_1B_4C_8)R^{-2} = A_3B_6C_{10}$.

If $k = 4$, let K_4 be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set

$\{R, R^4B_1A_3C_6C_{10}^{-1}A_7^{-1}B_5^{-1}\}$. Then, $K_4 = \mathrm{Mod}(\Sigma_g)$ by Corollary 4.2.9. Hence, we are done by Lemma 4.0.3 since $R^4(B_1A_3C_6)R^{-4} = B_5A_7C_{10}$.

If $k = 1$ or $k = 5$, let K_5 be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, R^{-5}A_1B_2C_4C_{g-1}^{-1}B_{g-3}^{-1}A_{g-4}^{-1}\}$. Then, $K_5 = \mathrm{Mod}(\Sigma_g)$ by Corollary 4.2.13. Hence, we are done by Lemma 4.0.3 since $R^{-5}(A_1B_2C_4)R^5 = A_{g-4}B_{g-3}C_{g-1}$.

If $k = 6$, let K_6 be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, R^{-6}A_1B_2C_4C_{g-2}^{-1}B_{g-4}^{-1}A_{g-5}^{-1}\}$. Then, $K_6 = \mathrm{Mod}(\Sigma_g)$ by Corollary 4.2.13. Hence, we are done by Lemma 4.0.3 since $R^{-6}(A_1B_2C_4)R^6 = A_{g-5}B_{g-4}C_{g-2}$.

If $k \geq 7$, let K be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R, R^{-k}A_1B_2C_4C_{g-k+4}^{-1}B_{g-k+2}^{-1}A_{g-k+1}^{-1}\}$. Then, $K = \mathrm{Mod}(\Sigma_g)$ by Corollary 4.2.13. Hence, we are done by Lemma 4.0.3 since $R^{-k}(A_1B_2C_4)R^k = A_{g-k+1}B_{g-k+2}C_{g-k+4}$.

□

4.2.4 Proof of Theorem 1.0.6

In this section, we prove Theorem 1.0.6 which states as: for $g \geq 3k^2+4k+1$ and any positive integer k, the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by two elements of order $g/\gcd(g,k)$.

Korkmaz showed the following in the proof of Theorem 4.0.1.

Theorem 4.2.14. *If $g \geq 3$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the elements $A_iA_j^{-1}, B_iB_j^{-1}, C_iC_j^{-1}$ for all i, j.*

Sketch of the proof is as follows:
First, it is easy to check that $A_1A_2^{-1}B_1B_2^{-1}(a_1, a_3) = (b_1, a_3)$ and we also have that $B_1A_3^{-1}C_1C_2^{-1}(b_1, a_3) = (c_1, a_3)$.
Korkmaz then showed that the Dehn twist A_3 can be generated by these elements by using the lantern relation.
Hence, we have that $A_i = (A_iA_3^{-1})A_3$, $B_i = (B_iB_1^{-1})(B_1A_3^{-1})A_3$ and we also get $C_i = (C_iC_1^{-1})(C_1A_3^{-1})A_3$ are generated by given elements. This finishes the proof.

Now, we prove the next statement as a corollary to Theorem 4.2.14.

Corollary 4.2.15. *If $g \geq 3$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the elements $A_iB_i^{-1}, C_iB_i^{-1}, C_iB_{i+1}^{-1}$ for all i.*

Proof. Let us denote by H the subgroup generated by the elements $A_iB_i^{-1}, C_iB_i^{-1}, C_iB_{i+1}^{-1}$ for all i.
We have $B_iB_j^{-1} = (B_iC_i^{-1})(C_iB_{i+1}^{-1}) \cdots (B_{j-1}C_{j-1}^{-1})(C_{j-1}B_j^{-1}) \in H$ for all i, j.
Similarly, $C_iC_j^{-1} = (C_iB_i^{-1})(B_iB_j^{-1})(B_jC_j^{-1}) \in H$ for all i, j.
Finally, $A_iA_j^{-1} = (A_iB_i^{-1})(B_iB_j^{-1})(B_jA_j^{-1}) \in H$ for all i, j.

It follows from Theorem 4.2.14 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the lemma. □

Theorem 4.2.16. *If $g \geq 21$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the elements $R^2, B_1B_2A_5A_8C_{11}C_{14}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}$.*

Proof. Let $F_1 = B_1B_2A_5A_8C_{11}C_{14}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R^2, F_1\}$.

Let

$$\begin{aligned} F_2 &= R^2F_1R^{-2} \\ &= B_3B_4A_7A_{10}C_{13}C_{16}C_{18}^{-1}C_{15}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}B_5^{-1} \end{aligned}$$

and

$$\begin{aligned} F_3 &= F_2^{-1} \\ &= B_5B_6A_9A_{12}C_{15}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}. \end{aligned}$$

We have $F_3F_1(b_5, b_6, \cdots, b_3) = (a_5, b_6, \cdots, b_3)$ so that
$F_4 = A_5B_6A_9A_{12}C_{15}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1} \in H$.

Note that $\cdots$ refers to the elements remaining fixed under the given maps.

Hence, $F_4F_3^{-1} = A_5B_5^{-1} \in H$ and then by conjugating $A_5B_5^{-1}$ with R^2 iteratively, we get $A_{2i+1}B_{2i+1}^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_5 &= R^4 F_1 R^{-4} \\ &= B_5 B_6 A_9 A_{12} C_{15} C_{18} C_{20}^{-1} C_{17}^{-1} A_{14}^{-1} A_{11}^{-1} B_8^{-1} B_7^{-1} \end{aligned}$$

and

$$\begin{aligned} F_6 &= (A_7 B_7^{-1}) F_5^{-1} (B_5 A_5^{-1}) \\ &= A_7 B_8 A_{11} A_{14} C_{17} C_{20} C_{18}^{-1} C_{15}^{-1} A_{12}^{-1} A_9^{-1} B_6^{-1} A_5^{-1}. \end{aligned}$$

We get $F_6 F_1(a_7, b_8, a_{11}, \cdots, b_6, a_5) = (a_7, a_8, a_{11}, \cdots, b_6, a_5)$ so that
$F_7 = A_7 A_8 A_{11} A_{14} C_{17} C_{20} C_{18}^{-1} C_{15}^{-1} A_{12}^{-1} A_9^{-1} B_6^{-1} A_5^{-1} \in H$.

$F_7 F_6^{-1} = A_8 B_8^{-1} \in H$ and then by conjugating $A_8 B_8^{-1}$ with R^2 iteratively, we get $A_{2i} B_{2i}^{-1} \in H$ for all i.
Hence, $A_i B_i^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_8 &= (B_{12} A_{12}^{-1}) F_4 \\ &= A_5 B_6 A_9 B_{12} C_{15} C_{18} C_{16}^{-1} C_{13}^{-1} A_{10}^{-1} A_7^{-1} B_4^{-1} B_3^{-1}. \end{aligned}$$

Now, we have $F_8 F_1(\cdots, b_{12}, \cdots) = (\cdots, c_{11}, \cdots)$ so that
$F_9 = A_5 B_6 A_9 C_{11} C_{15} C_{18} C_{16}^{-1} C_{13}^{-1} A_{10}^{-1} A_7^{-1} B_4^{-1} B_3^{-1} \in H$.

Hence, $F_9 F_8^{-1} = C_{11} B_{12}^{-1} \in H$ and then by conjugating $C_{11} B_{12}^{-1}$ with R^2 iteratively, we get $C_{2i+1} B_{2i+2}^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_{10} &= (B_{11} A_{11}^{-1}) F_7 \\ &= A_7 A_8 B_{11} A_{14} C_{17} C_{20} C_{18}^{-1} C_{15}^{-1} A_{12}^{-1} A_9^{-1} B_6^{-1} A_5^{-1}. \end{aligned}$$

Similarly, we have $F_{10} F_1(\cdots, b_{11}, \cdots) = (\cdots, c_{11}, \cdots)$ so that
$F_{11} = A_7 A_8 C_{11} A_{14} C_{17} C_{20} C_{18}^{-1} C_{15}^{-1} A_{12}^{-1} A_9^{-1} B_6^{-1} A_5^{-1} \in H$.

Hence, $F_{11} F_{10}^{-1} = C_{11} B_{11}^{-1} \in H$ and then, we get $C_{2i+1} B_{2i+1}^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_{12} &= (B_{15}C_{15}^{-1})F_4 \\ &= A_5B_6A_9A_{12}B_{15}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}. \end{aligned}$$

Then, $F_{12}F_1(\cdots, b_{15}, \cdots) = (\cdots, c_{14}, \cdots)$ so that
$F_{13} = A_5B_6A_9A_{12}C_{14}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1} \in H$.

We also have $F_{13}F_{12}^{-1} = C_{14}B_{15}^{-1} \in H$ and then, we get $C_{2i}B_{2i+1}^{-1} \in H$ for all i.
Hence, $C_iB_{i+1}^{-1} \in H$ for all i.

Let

$$\begin{aligned} F_{14} &= F_7(C_{15}B_{16}^{-1}) \\ &= A_7A_8A_{11}A_{14}C_{17}C_{20}C_{18}^{-1}B_{16}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1}. \end{aligned}$$

Hence, $F_{14}F_1(\cdots, b_{16}, \cdots) = (\cdots, c_{16}, \cdots)$ so that
$F_{15} = A_7A_8A_{11}A_{14}C_{17}C_{20}C_{18}^{-1}C_{16}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1} \in H$.

Similarly, $F_{15}^{-1}F_{14} = C_{16}B_{16}^{-1} \in H$ and then, we get $C_{2i}B_{2i}^{-1} \in H$ for all i.
Finally, $C_iB_i^{-1} \in H$ for all i.

It follows from Corollary 4.2.15 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the theorem. □

Corollary 4.2.17. *If g is even and $g \geq 22$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by two elements of order $g/2$.*

Proof. Let H be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set
$\{R^2, R^2B_1B_2A_5A_8C_{11}C_{14}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}\}$.
Then, $H = \mathrm{Mod}(\Sigma_g)$ by Theorem 4.2.16. Hence, we are done by Lemma 4.0.3 since $R^2(B_1B_2A_5A_8C_{11}C_{14})R^{-2} = B_3B_4A_7A_{10}C_{13}C_{16}$. □

Generalization of Theorem 4.2.16 and Corollary 4.2.17 is as follows:

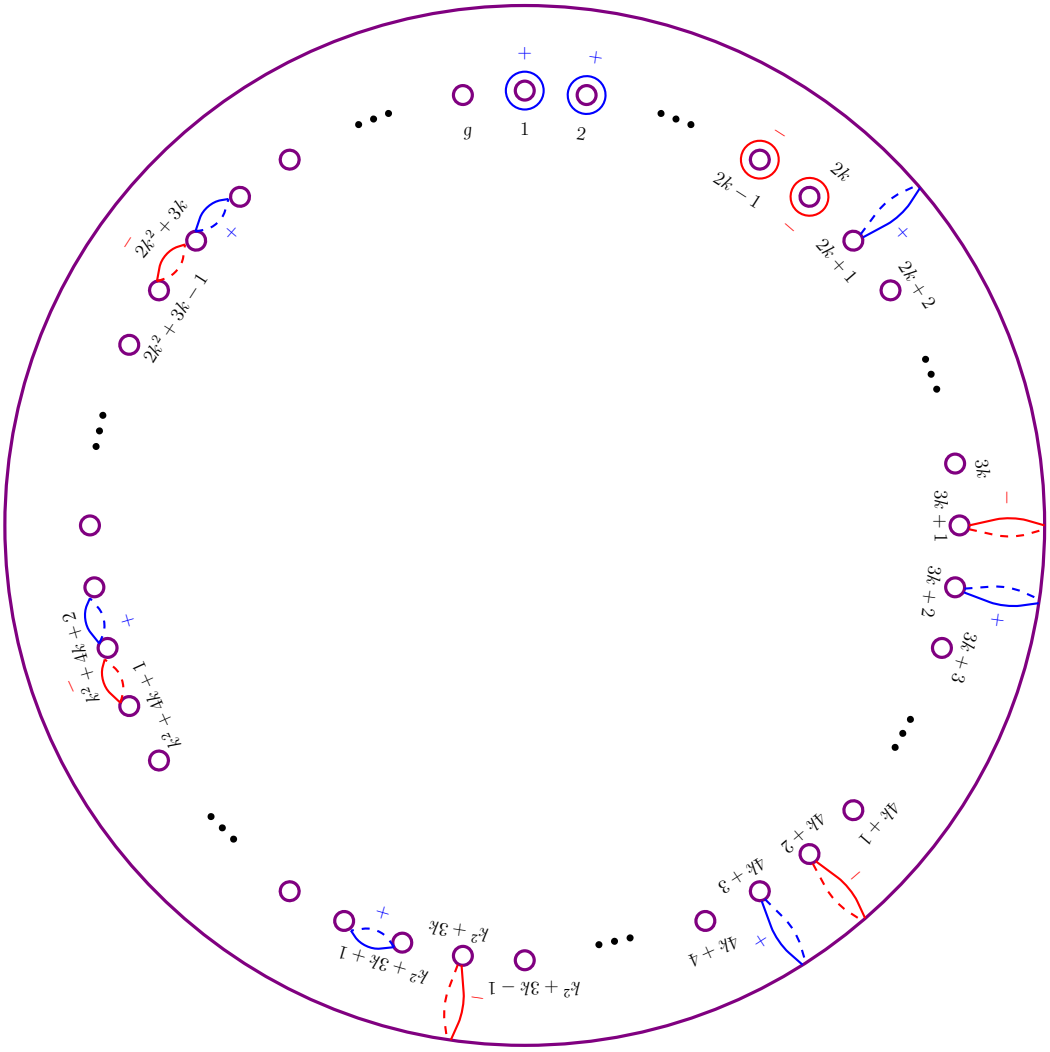


Figure 4.9: Generator for Theorem 1.0.6

Theorem 4.2.18. *For $k \geq 2$ and $g \geq 3k^2 + 4k + 1$, the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the elements $R^k and R^k F(R^k F^{-1} R^{-k})$ where $F = B_1 B_2 \cdots B_k A_{2k+1} A_{3k+2} \cdots A_{k^2+2k} C_{k^2+3k+1} C_{k^2+4k+2} \cdots C_{2k^2+3k}$.*

Proof. We define an algorithm to prove the desired result.
Let $F = B_1 B_2 \cdots B_k A_{2k+1} A_{3k+2} \cdots A_{k^2+2k} C_{k^2+3k+1} C_{k^2+4k+2} \cdots C_{2k^2+3k}$ and $F_1 = F(R^k F^{-1} R^{-k})$. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R^k, F_1\}$.

A) Use conjugation of F_1 with $R^k, R^{2k}, \cdots, R^{k^2}$ with proper multiplications to get $A_{k+1}B_{k+1}^{-1} \in H$, $A_{k+2}B_{k+2}^{-1} \in H$, $\cdots$, $A_{2k-1}B_{2k-1}^{-1} \in H$, $A_{2k}B_{2k}^{-1} \in H$, respectively.
Hence, $A_i B_i^{-1} \in H$ for all i.

B) Follow the next k steps.

1) Use conjugation of F_1 with R^{kl} for some positive integers l's with proper multiplications to get $C_{ik+1}B_{ik+1}^{-1} \in H$ and $C_{ik+1}B_{ik+2}^{-1} \in H$ for all i.

2) Use conjugation of F_1 with R^{kl} for some positive integers l's with proper multiplications to get $C_{ik+2}B_{ik+2}^{-1} \in H$ and $C_{ik+2}B_{ik+3}^{-1} \in H$ for all i.
$\cdots$

k) Use conjugation of F_1 with R^{kl} for some positive integers l's with proper multiplications to get $C_{ik}B_{ik}^{-1} \in H$ and $C_{ik}B_{ik+1}^{-1} \in H$ for all i.
Hence, $C_i B_i^{-1} \in H$ and $C_i B_{i+1}^{-1} \in H$ for all i.

It follows from Corollary 4.2.15 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof of the theorem. See Theorem 4.2.16 for an example usage of the algorithm. □

Now, we prove Theorem 1.0.6.

Proof. For $k \geq 2$ and $g \geq 3k^2 + 4k + 1$, let H be the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set $\{R^k, R^k F(R^k F^{-1} R^{-k})\}$. Then, $H = \mathrm{Mod}(\Sigma_g)$ by Theorem 4.2.18. Hence, we are done by Lemma 4.0.3 since the orders of R^k and $R^k F(R^k F^{-1} R^{-k})$ are g/d where d is the greatest common divisor of g and k. If $k = 1$, we are done by Theorem 1.0.5. □

4.3 Further Results

The mapping class group $\mathrm{Mod}(N_g)$ of closed connected nonorientable surface N_g of genus g is defined to be the group of the isotopy classes of all diffeomorphisms of N_g. Compared to orientable surfaces less is known about $\mathrm{Mod}(N_g)$. Lickorish [11], [13] showed that it is generated by Dehn twists about two-sided simple closed curves and a so-called Y-homeomorphism (or a crosscap slide). Chillingworth [2] gave a finite generating set for $\mathrm{Mod}(N_g)$ that linearly depends on g. Szepietowski [22] proved that $\mathrm{Mod}(N_g)$ is generated by three elements and by four involutions.

The twist subgroup $\mathcal{T}_g$ of $\mathrm{Mod}(N_g)$ is the group generated by Dehn twists about two-sided simple closed curves. The group $\mathcal{T}_g$ is a subgroup of index 2 in $\mathrm{Mod}(N_g)$ [13]. Chillingworth [2] showed that $\mathcal{T}_g$ can be generated by finitely many Dehn twists. Stukow [21] obtained a finite presentation for $\mathcal{T}_g$ with $(g+2)$ Dehn twist generators. Later, Omori [20] reduced the number of Dehn twist generators to $(g+1)$ for $g \geq 4$. Du [5] obtained a generating set consisting of three elements whenever $g \geq 5$ and odd. Recently, Yoshihara [25] proved that $\mathcal{T}_g$ can be generated by six involutions for $g \geq 14$ and by eight involutions if $g \geq 8$.

The extended mapping class group $\mathrm{Mod}^*(\Sigma_{g,p})$ of closed connected orientable surface $\Sigma_{g,p}$ of genus g with p punctures is defined to be the group of the isotopy classes of all diffeomorphisms of $\Sigma_{g,p}$.

The idea in Sections 4.1 and 4.2 can be used with little modifications for the extended mapping class group of orientable surfaces possibly with punctures, the mapping class group of nonorientable surfaces and twist subgroups of the mapping class group of nonorientable surfaces.

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