ALGEBRAIC STRUCTURES OF FINITE MORLEY RANK

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ABSTRACT

ALGEBRAIC STRUCTURES OF FINITE MORLEY RANK

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In the first two chapters of this thesis, the model theoretical and the axiomatic approaches to algebraic structures of finite Morley rank are explained. The last two chapters prove that if R is an infinite ring of finite Morley rank which contains no zero divisors, then R is an algebraically closed field.

Keywords: Morley Rank, Ranked Structure

ÖZ

MORLEY RANKI SONLU OLAN CEBİRSEL YAPILAR

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Tezin ilk iki bölümünde, Morley rankı sonlu olan cebirsel yapılar Modeller Kuramı açısından ve belitsel açıdan ele alınmıştır. Son iki bölümde ise Morley rankı sonlu olan, sıfır böleni içermeyen sonsuz halkaların cebirsel olarak kapalı cisimler olduğu kanıtlanmıştır.

Anahtar Sözcükler: Morley Rankı, Ranklı Yapılar

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TABLE OF CONTENTS

CHAPTER 1

SOME MODEL THEORY

1.1 Basic Definitions and Properties

1.1.1 Language and Structure

A (first order) *language* $\mathcal{L}$ is a collection of three kinds of symbols, which are relation symbols, function symbols and constant symbols. For each relation and function symbol, there corresponds a natural number which shows the arity of the relation or the function. In general, one can write

$$\mathcal{L} = \{R_i\}_{i\in I} \cup \{F_j\}_{j\in J} \cup \{c_k\}_{k\in K},$$

where I, J, K are some index sets and R_i's, F_j's, c_k's denote the relation symbols, function symbols, constant symbols respectively. In order to study groups, one uses the language $\mathcal{L} = \{\cdot,\ ^{-1}, 1\}$, where $\cdot$ is a binary function symbol, $^{-1}$ is a unary function symbol and 1 is a constant symbol. For a language $\mathcal{L}$, an $\mathcal{L}$-*structure* $\mathcal{M}$ is a pair $< M\ ,\ \mathcal{I} >$, where M is a nonempty set and $\mathcal{I}$ is an interpretation function which maps the symbols of $\mathcal{L}$ to the appropriate relations, functions and constants of M. We use the following notation: $\mathcal{I}(R) = R_{\mathcal{M}}$, $\mathcal{I}(F) = F_{\mathcal{M}}$ and $\mathcal{I}(c) = c_{\mathcal{M}}$.

The concepts of definable subset and interpretable subset are of fundamental importance in the study of structures of finite Morley rank. To give the definitions of these concepts, we need to formalize the language $\mathcal{L}$. Apart from the symbols of $\mathcal{L}$, we will use the following symbols, which are called the *primitive symbols*:

- left and right parantheses (,)
- variables v_i for $i \in \mathbb{N}$
- logical connectives $\wedge$, $\neg$
- a logical quantifier $\exists$
- a binary relation symbol, the identity $\equiv$.

Most of the concepts that we need will be defined inductively.

Notation: $\phi(m_1, \ldots, m_n)$ is used to denote $(\phi(m_1), \ldots, \phi(m_n))$, where $\phi : M \to N$ is a map.

Homomorphisms of $\mathcal{L}$-structures:

For two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, a map $\phi : M \to N$ is called a *homomorphism* iff:

1. $\phi(F_{\mathcal{M}}(\overline{m})) = F_{\mathcal{N}}(\phi(\overline{m}))$, for every function symbol F of $\mathcal{L}$ and for all $\overline{m} \in M^n$, where n is the arity of F.
2. $R_{\mathcal{M}}(\overline{m})$ iff $R_{\mathcal{N}}(\phi(\overline{m}))$, for every relation symbol R in $\mathcal{L}$ and for all $\overline{m} \in M^k$, where k is the arity of R.

3. $\phi(c_{\mathcal{M}}) = c_{\mathcal{N}}$. for every constant symbol c in $\mathcal{L}$.

Naturally. an isomorphism is defined to be a one to one and onto homomorphism. Epimorphisms, monomorphisms, endomorphisms and automorphisms are defined in the usual way.

Terms of $\mathcal{L}$:

1. All variables and constant symbols are terms.
2. $F(t_1, \ldots, t_n)$ is a term, where F is an n-ary function and $t_1, \ldots, t_n$'s are terms.
3. A sequence of symbols is not a term, unless it can be shown that it is a term by a finite number of applications of 1 and 2.

Note that terms are analogous to polynomials.

Atomic Formulas of $\mathcal{L}$:

1. $t_1 \equiv t_2$ is an atomic formula, where t_1 and t_2 are terms.
2. $R(t_1, \ldots, t_n)$ is an atomic formula, where R is an n-ary relation symbol and $t_1, \ldots, t_n$'s are terms.

Formulas of $\mathcal{L}$:

1. Every atomic formula is a formula.
2. $\neg(\alpha)$, $(\alpha \wedge \beta)$, $(\exists v_i)\alpha$ are formulas, where α and β are formulas and v_i is a variable.

3. A sequence of symbols is not a formula, unless it can be shown that it is a formula by a finite number of applications of 1 and 2.

In order to make the formulas shorter, some abbreviations will be used. The symbols $\vee$, $\Rightarrow$, $\Leftrightarrow$ and $\forall$ are abbreviations defined as follows:

$$\begin{aligned} (\alpha \vee \beta) &\quad \text{for} \quad \neg(\neg(\alpha) \wedge \neg(\beta)) \\ (\alpha \Rightarrow \beta) &\quad \text{for} \quad (\neg(\alpha) \vee \beta) \\ (\alpha \Leftrightarrow \beta) &\quad \text{for} \quad ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) \\ (\forall v_i)\alpha &\quad \text{for} \quad \neg((\exists v_i)\neg(\alpha)). \end{aligned}$$

We could have included these symbols in the language $\mathcal{L}$, but then the definitions and the proofs would be longer.

Free Variables of a Term t:

1. If t is a constant symbol, then t has no free variables.

2. If t is a variable v_i, then v_i is the only free variable of t.

3. If t is of the form $F(t_1, \ldots, t_n)$, then the set of free variables of t is the union of the sets of free variables of $t_1, \ldots, t_n$.

We use $t(v_1, \ldots, v_n)$ to denote a term whose free variables form a subset of $\{v_1, \ldots, v_n\}$.

Free Variables of a Formula α:

1. If α is an atomic formula, then all the free variables of α are the variables of α.

2. If $\alpha = \neg(\beta)$, then the set of free variables of α is the set of free variables of β.

3. If $\alpha = \beta \wedge \gamma$, then the set of free variables of α is the union of the sets of free variables of β and the free variables of γ.

4. If $\alpha = (\exists v_i)\beta$, then the free variables of α are the free variables of β except v_i.

We use $\alpha(v_1, \ldots, v_n)$ to denote a formula α whose free variables form a subset of $\{v_1, \ldots, v_n\}$.

A *sentence* is a formula which has no free variables.

The Value of a Term:

Let $\overline{m} = (m_1, \ldots, m_n) \in M^n$ and $t(\overline{v}) = t(v_1, \ldots, v_n)$ be a term. The value of the term t at $\overline{m}$, denoted by $t[\overline{m}]$, is defined inductively by:

1. If t is a variable v_i, then $t[\overline{m}] = m_i$.

2. If t is a constant symbol c, then $t[\overline{m}] = c_{\mathcal{M}}$, the interpretation of c in $\mathcal{M}$.

3. If t is of the form $F(t_1, \ldots, t_k)$, then $t[\overline{m}] = F_{\mathcal{M}}(t_1[\overline{m}], \ldots, t_k[\overline{m}])$, where $F_{\mathcal{M}}$ is the interpretation of F in $\mathcal{M}$.

$\overline{m} = (m_1, \ldots, m_n)$ *satisfies a formula* $\alpha(v_1, \ldots, v_n)$ *in* $\mathcal{M}$:

We denote it by $\mathcal{M} \models \alpha[\overline{m}]$.

1. If α is the atomic formula $t_1 \equiv t_2$, then $\mathcal{M} \models \alpha[\overline{m}]$ iff $t_1[\overline{m}] = t_2[\overline{m}]$.

2. If $\alpha = R(t_1, \ldots, t_k)$, then $\mathcal{M} \models \alpha[\overline{m}]$ iff $R_{\mathcal{M}}(t_1[\overline{m}], \ldots, t_k[\overline{m}])$.

3. If $\alpha = \neg(\beta)$, then $\mathcal{M} \models \alpha[\overline{m}]$ iff $\mathcal{M} \not\models \beta[\overline{m}]$.

4. If $\alpha = (\beta \wedge \gamma)$, then $\mathcal{M} \models \alpha[\overline{m}]$ iff $\mathcal{M} \models \beta[\overline{m}]$ and $\mathcal{M} \models \gamma[\overline{m}]$.

5. If $\alpha = (\exists v_i)\beta(\overline{v}, v_i)$, then $\mathcal{M} \models \alpha[\overline{m}]$ iff $\mathcal{M} \models \beta(\overline{m}, a)$, for some $a \in M$.

Any fact on terms (or on formulas) must be proven by induction on the complexity of the term (or of the formula). The proof of the following lemma is an example of such a proof.

Lemma 1.1 *Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures.*

(i) *If $\phi : M \to N$ is a homomorphism, $t(\overline{v})$ is a term and $\overline{m} \in M^n$, then*

$$\phi(t[\overline{m}]) = t[\phi(\overline{m})].$$

(ii) *If $\phi : M \to N$ is an isomorphism, α is a formula in $\mathcal{L}$ and $\overline{m} \in M^n$, then*

$$\mathcal{M} \models \alpha[\overline{m}] \textit{ iff } \mathcal{N} \models \alpha[\phi(\overline{m})].$$

Proof: (i) If t is a variable v_i, then $t[\overline{m}] = m_i$ and $t[\phi(\overline{m})] = \phi(m_i)$ which implies that $\phi(t[\overline{m}]) = \phi(m_i) = t[\phi(\overline{m})]$.

If t is a constant symbol c, then $t[\overline{m}] = c_{\mathcal{M}}$ and $t[\phi(\overline{m})] = c_{\mathcal{N}}$. Since ϕ is a homomorphism, $\phi(t[\overline{m}]) = \phi(c_{\mathcal{M}}) = c_{\mathcal{N}} = t[\phi(\overline{m})]$.

If t is of the form $F(t_1, \ldots, t_k)$, then $t[\overline{m}] = F_{\mathcal{M}}(t_1[\overline{m}], \ldots, t_k[\overline{m}])$ and $t[\phi(\overline{m})] = F_{\mathcal{N}}(t_1[\phi(\overline{m})], \ldots, t_k[\phi(\overline{m})])$.

$$\phi(t[\overline{m}]) = \phi(F_{\mathcal{M}}(t_1[\overline{m}], \ldots, t_k[\overline{m}])) = F_{\mathcal{N}}(\phi(t_1[\overline{m}], \ldots, t_k[\overline{m}]))$$

$= F_{\mathcal{N}}(\phi(t_1[\overline{m}]), \ldots, \phi(t_k[\overline{m}]))$, by the induction hypothesis.

$= F_{\mathcal{N}}(t_1[\phi(\overline{m})], \ldots, t_k[\phi(\overline{m})]) = t[\phi(\overline{m})]$.

Thus the induction is over and $\phi(t[\overline{m}]) = t[\phi(\overline{m})]$ for any term $t(\overline{v})$ of $\mathcal{L}$.

(ii) If α is the atomic formula $t_1 \equiv t_2$, then $\mathcal{M} \models \alpha[\overline{m}]$ iff $t_1[\overline{m}] = t_2[\overline{m}]$ iff $\phi(t_1[\overline{m}]) = \phi(t_2[\overline{m}])$ iff $t_1[\phi(\overline{m})] = t_2[\phi(\overline{m})]$, by part (i), iff $\mathcal{N} \models \alpha[\phi(\overline{m})]$.

If $\alpha = R(t_1, \ldots, t_k)$, then $\mathcal{M} \models \alpha[\overline{m}]$ iff $R_{\mathcal{M}}(t_1[\overline{m}], \ldots, t_k[\overline{m}])$ iff $R_{\mathcal{N}}(\phi(t_1[\overline{m}]), \ldots, \phi(t_k[\overline{m}]))$, since ϕ is a homomorphism,

iff $R_{\mathcal{N}}(t_1[\phi(\overline{m})], \ldots, t_k[\phi(\overline{m})])$, by part (i), iff $\mathcal{N} \models \alpha[\phi(\overline{m})]$.

If $\alpha = \neg(\beta)$, then $\mathcal{M} \models \alpha[\overline{m}]$ iff $\mathcal{M} \not\models \beta[\overline{m}]$ iff $\mathcal{N} \not\models \beta[\phi(\overline{m})]$, by induction hypothesis, iff $\mathcal{N} \models \alpha[\phi(\overline{m})]$.

If $\alpha = (\beta \wedge \gamma)$, then $\mathcal{M} \models \alpha[\overline{m}]$ iff $\mathcal{M} \models \beta[\overline{m}]$ and $\mathcal{M} \models \gamma[\overline{m}]$ iff $\mathcal{N} \models \beta[\phi(\overline{m})]$ and $\mathcal{N} \models \gamma[\phi(\overline{m})]$, by induction hypothesis, iff $\mathcal{N} \models \alpha[\phi(\overline{m})]$.

If $\alpha = (\exists v_i)\beta(\overline{v}, v_i)$, then $\mathcal{M} \models \alpha[\overline{m}]$ iff $\mathcal{M} \models \beta(\overline{m}, a)$ for some $a \in M$ iff $\mathcal{N} \models \beta(\phi(\overline{m}), \phi(a))$ for some $a \in M$ iff $\mathcal{N} \models \beta(\phi(\overline{m}), b)$ for some $b \in \phi(M) = N$, by induction hypothesis, iff $\mathcal{N} \models \alpha[\phi(\overline{m})]$.

Now the induction is over. □

1.1.2 Definable Sets

In order to use arbitrary elements of M in the formulas, it is a common trick to extend the language $\mathcal{L}$ to a new language $\mathcal{L}' = \mathcal{L} \cup \{m\}_{m \in M}$. Obviously, the interpretation of m in M will be m itself. The elements of M which are not the interpretations of constant symbols of $\mathcal{L}$ are called *parameters*.

Note that each formula $\alpha(\overline{v})$ with n free variables in $\mathcal{L}'$ can be expressed by another formula $\beta(\overline{v}, m_1 \ldots, m_k)$ with n free variables in $\mathcal{L}$, where $m_1, \ldots, m_k$ are the parameters used in $\alpha(v)$.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. Then A is said to be *definable in* $\mathcal{M}$ (or just *definable* where $\mathcal{M}$ is clear from the context) if there exists a formula $\alpha(v)$ with one free variable v in $\mathcal{L}'$ (or equivalently if there exists $\overline{m} = (m_1, \ldots, m_k) \in M^k$ and a formula $\beta(v, \overline{m})$ with one free variable v in $\mathcal{L}$) such that:

$$A = \{a \in M : \mathcal{M} \models \alpha[a]\} = \{a \in M : \mathcal{M} \models \beta[a, \overline{m}]\}.$$

More generally, a subset $A \subseteq M^n$ is said to be *definable* if there exists a formula $\alpha(\overline{v})$ with n free variables in $\mathcal{L}'$ (or equivalently if there exist $\overline{m} = (m_1, \ldots, m_k) \in M^k$ and a formula $\beta(\overline{v}, \overline{m})$ with n free variables in $\mathcal{L}$) such that:

$$A = \{\overline{a} \in M^n : \mathcal{M} \models \alpha[\overline{a}]\} = \{\overline{a} \in M^n : \mathcal{M} \models \beta[\overline{a}, \overline{m}]\}.$$

Here k parameters were used to express A, namely $m_1, \ldots, m_k$. If k is the minimum number of parameters that we use to define A, then A is called *k-definable*. In particular, A is called *0-definable* if no parameters are needed to define A.

As an example, consider $\mathcal{L} = \{+, \cdot, 0, 1\}$ and the $\mathcal{L}$-structure $\mathcal{Z}$ with the underlying set $\mathbb{Z}$ and the natural interpretation function. Then $\mathbb{N}$ is 0-definable in this structure, since

$$\mathbb{N} = \{n \in \mathbb{Z} : \mathcal{Z} \models (\exists x)(\exists y)(\exists z)(\exists w)(n = x^2 + y^2 + z^2 + w^2)\}.$$

The additive inverse of an integer is 1-definable since

$$\{-a\} = \{n \in \mathbb{Z} : \mathcal{Z} \models n + a = a + n = 0\} \quad \text{for any } a \in \mathbb{Z}.$$

An n-ary relation P defined on a definable set $A \subseteq M$ is called a *definable relation in $\mathcal{M}$* if the set

$$\{(a_1, \ldots, a_n) \in A^n : \mathcal{M} \models P(a_1, \ldots, a_n)\}$$

is definable in $\mathcal{M}$.

Let's consider the above example again. $\leq$ is a definable binary relation in this structure since

$$\{(a,b) \in \mathbb{Z}^2 : a \leq b\} = \{(a,b) \in \mathbb{Z}^2 : \mathcal{Z} \models (\exists x)(\exists y)(\exists z)(\exists w)(b - a = x^2 + y^2 + z^2 + w^2)\}.$$

A function $f : A \to B$, where $A \subseteq M^n$ and $B \subseteq M^m$ are definable subsets, is called a *definable function in $\mathcal{M}$* if its graph is definable in $\mathcal{M}$.

Lemma 1.2 *The set of definable subsets of M^n is a Boolean algebra.*

Proof: Assume that A, B are definable subsets of M^n. Then there exist formulas $\alpha(\overline{v})$ and $\beta(\overline{v})$ of n free variables defining A and B respectively.

$$A \cap B = \{\overline{m} \in M^n : \mathcal{M} \models (\alpha \wedge \beta)[\overline{m}]\},$$

$$A \cup B = \{\overline{m} \in M^n : \mathcal{M} \models (\alpha \vee \beta)[\overline{m}]\},$$

$$A \setminus B = \{\overline{m} \in M^n : \mathcal{M} \models (\alpha \wedge \neg\beta)[\overline{m}]\}$$

prove that $A \cap B, A \cup B$ and $A \setminus B$ are all definable subsets of M^n.

$M^n = \{\overline{m} \in M^n : \mathcal{M} \models \overline{m} \equiv \overline{m}\}$ implies that M^n is definable. So $M^n \setminus M^n = \phi$ and $M^n \setminus A$ are all definable. This implies that the set of definable subsets of M^n is a Boolean algebra. $\square$

Lemma 1.3 *If $A \subseteq M^n$ and $B \subseteq M^k$ are definable, then $A \times B \subseteq M^{n+k}$ is also definable.*

Proof: Let $\alpha(\overline{v})$ be a formula of n free variables defining A and $\beta(\overline{v})$ be a formula of k free variables defining B. Then

$$A \times B = \{(\overline{m}, \overline{n}) \in M^{n+k} : \mathcal{M} \models \varphi[\overline{m}, \overline{n}]\},$$

where $\varphi(\overline{m}, \overline{n}) = \alpha(\overline{m}) \wedge \beta(\overline{n})$.

Thus $A \times B$ is definable. □

Lemma 1.4 *Let $A \subseteq M^n$ and $B \subseteq M^k$ be definable sets. The projection maps $\Pi_1 : A \times B \to A$ and $\Pi_2 : A \times B \to B$ are definable.*

Proof: Let $\alpha(\overline{u})$ and $\beta(\overline{v})$ be the formulas defining A and B respectively. Then the graph of Π_1 is

$$\{(\overline{u}, \overline{v}, \overline{w}) \in M^{n+k+n} : \mathcal{M} \models \varphi(\overline{u}, \overline{v}, \overline{w})\},$$

where $\varphi(\overline{u}, \overline{v}, \overline{w}) = (\overline{u} = \overline{w}) \wedge \alpha(\overline{u}) \wedge \beta(\overline{v})$. □

Lemma 1.5 *If $A \subseteq M^{n+k}$ is definable, then the projection of A on M^n is also definable.*

Proof: Let $\alpha(\overline{v}, \overline{w})$ be the formula of $n + k$ free variables defining A, where $\overline{v} = (v_1, \ldots, v_n)$ and $\overline{w} = (w_1, \ldots, w_k)$.

The projection of A on $M^n = \{\overline{m} \in M^n : \mathcal{M} \models \varphi(\overline{m})\}$, where $\varphi(\overline{v}) = (\exists\overline{b})(\alpha(\overline{v},\overline{b}))$ is a formula of n free variables. □

Lemma 1.6 *Let $\phi : M \to M$ be an automorphism and $X \subseteq M$ be a definable set. Then $\phi(X)$ is definable.*

Proof: Let $\alpha(v) = \beta(v,\overline{a})$ be the formula which defines X. Then $X = \{m \in M : \mathcal{M} \models \beta(m,\overline{a})\}$. Note that $\mathcal{M} \models \beta(m,\overline{a})$ iff $\mathcal{M} \models \beta(\phi(m),\phi(\overline{a}))$, by Lemma 1.1 (ii).

Claim: $\phi(X) = \{m \in M : \mathcal{M} \models \beta(m,\phi(\overline{a}))\}$

To see that $\phi(X) \subseteq \{m \in M : \mathcal{M} \models \beta(m,\phi(\overline{a}))\}$, let $m \in \phi(X)$. Then there exists $x \in X$ such that $m = \phi(x)$. So, $\mathcal{M} \models \beta(x,\overline{a})$ and by the above remark $\mathcal{M} \models \beta(\phi(x),\phi(\overline{a})) = \beta(m,\phi(\overline{a}))$. This proves that $m \in \{m \in M : \mathcal{M} \models \beta(m,\phi(\overline{a}))\}$.

For the converse, let $m \in M$ and $\mathcal{M} \models \beta(m,\phi(\overline{a}))$. Since ϕ^{-1} is an automorphism, we have $\mathcal{M} \models \beta(\phi^{-1}(x),\overline{a})$. This implies that $\phi^{-1}(m) \in X$. Then $m = \phi(x)$ for some $x \in X$, and so $m \in \phi(X)$.This proves the claim. Thus $\phi(X)$ is definable. □

Now let $A \subseteq M^n$ be a definable set and $\sim$ be a definable equivalence relation on A. Then any Boolean combination of the sets of the form $A/\sim$ is called an *interpretable set in* $\mathcal{M}$. By taking $\sim$ to be the identity relation, one can see that a definable set is interpretable.

The interpretable relations and functions are defined naturally as in the definable case.

Lemma 1.7 *If A and B are interpretable sets and $f : A \rightarrow B$ is an interpretable function, then the following statements hold:*

(i) *$A \times B$ is interpretable*

(ii) *The projection maps $\Pi_1 : A \times B \rightarrow A$ and $\Pi_2 : A \times B \rightarrow B$ are interpretable*

(iii) *$\Delta(A) = \{(a, a) : a \in A\}$ is interpretable*

(iv) *For any interpretable set $C \subseteq A$, $f(C)$ is interpretable.*

(v) *For any interpretable equivalance relation E on A, the set A/E and the canonical map $A \rightarrow A/E$ are interpretable.*

Proof: To simplfy the notation, assume that $A = \frac{X}{\sim_1}$ and $B = \frac{Y}{\sim_2}$, where X and Y are definable sets and $\sim_1$ and $\sim_2$ are definable equivalance relations on X and Y respectively. Let's agree to use G_f to denote the graph of f and assume that $G_f = D/\sim_f$.

(i) By Lemma 1.3, $X \times Y$ is definable. Define a relation $\sim$ on $X \times Y$ so that $(x_1, y_1) \sim (x_2, y_2)$ iff $x_1 \sim_1 x_2$ and $y_1 \sim_2 y_2$. The relation $\sim$ is a definable equivalance relation. Now $A \times B = \frac{X \times Y}{\sim}$, and so $A \times B$ is interpretable.

(ii) The graph of Π_1 is $\{(a, b, a) : a \in A, b \in B\}$. Let $P = \{(x, y, z) : x, z \in X \wedge y \in Y \wedge x \equiv z\}$. Now P is definable. Let's define a new relation $\sim$ on P such that $(x_1, y_1, x_1) \sim (x_2, y_2, x_2)$ iff $x_1 \sim_1 x_2$ and $y_1 \sim_2 y_2$. Then the graph of Π_1 is $P/\sim$ which proves that Π_1 is an interpretable function. Similarly Π_2 is an interpretable function.

(iii) The set $L = \{(x_1, x_2) : x_1, x_2 \in X \wedge x_1 \equiv x_2\}$ is definable. Define a new relation $\sim$ on L such that $(x_1, x_1) \sim (x_2, x_2)$ iff $x_1 \sim_1 x_2$. $\Delta(A) = L/\sim$ proves that $\Delta(A)$ is interpretable.

(iv) There exists a definable subset S of X such that $C = S/\sim_1$. Let

$$T = \{y \in Y : (\exists x \in S)(x, y) \in D\}.$$

Now $f(C) = T/\sim_2$ proves that $f(C)$ is interpretable.

(v) E is an interpretable relation on A implies that there exists a definable set N and a definable equivalance relation R on N such that

$$\{(x, y) \in A \times A : E(x, y)\} = N/R.$$

If we define a relation $\sim$ on X such that for any $x_1, x_2 \in X$, $x_1 \sim x_2$ iff $x_1 \sim_1 x_2$ or $E([x_1]_1, [x_2]_1)$, then $A/E = X/\sim$.

The following argument shows that $\sim$ is definable.

$$\begin{aligned} x_1 \sim x_2 \quad &\text{iff} \quad x_1 \sim_1 x_2 \text{ or } E([x_1]_1, [x_2]_1) \\ &\text{iff} \quad x_1 \sim_1 x_2 \text{ or } ([x_1]_1, [x_2]_1) \in N/R \\ &\text{iff} \quad x_1 \sim_1 x_2 \text{ or there exists } n \in N \text{ such that } ([x_1]_1, [x_2]_1) = [n]_R. \end{aligned}$$

Equivalance class of an element with respect to a definable relation is a definable set. Since everything is definable in the last statement, $\sim$ is a definable relation on X. Thus $X/\sim = A/E$ is interpretable.

Define a relation $\sim_\Pi$ on $X \times X$ such that $(x_1, d_1) \sim_\Pi (x_2, d_2)$ iff $x_1 \sim_1 x_2$ and $E([d_1]_1, [d_2]_1)$.Then

$$\begin{aligned} G_\Pi &= \{(a, b) \in A \times A/E : E(a, [d]_1), \text{where } [d]_1 = b\} \\ &= \frac{\{(x, d) \in X \times X : E([x]_1, [d]_1)\}}{\sim_\Pi}. \end{aligned}$$

□

1.1.3 Consistency

Let $\mathcal{L}$ be a language, Σ be a set of sentences in $\mathcal{L}$ and σ be a sentence in $\mathcal{L}$. σ is said to be a *logical consequence of* Σ, in symbols $\Sigma \vdash \sigma$, iff there is a finite sequence $\sigma_1, \ldots, \sigma_n$ of sentences such that $\sigma_n = \sigma$ and each σ_i is either in Σ or a logical axiom or deduced from two earlier sentences by applying some logical rules of inference.

One can find the logical axioms and rules in any standard logic or model theory book. For example see [3].

A formula $\alpha(v_1, \ldots, v_n)$ is said to be a *logical consequence of* Σ if the sentence $(\forall v_1) \ldots (\forall v_n)\alpha(v_1, \ldots, v_n)$ is a logical consequence of Σ.

Σ is called *consistent* if no sentence of the form $\sigma \wedge \neg\sigma$ is a logical consequence of Σ. If $\Sigma \vdash (\sigma \wedge \neg\sigma)$ for some sentence σ, then Σ is called *inconsistent.*

A sentence which is satisfied in every model of $\mathcal{L}$ is called a *valid sentence.* A formula $\alpha(v_1, \ldots, v_n)$ is called a *valid formula* if the sentence

$(\forall v_1) \ldots (\forall v_n)\alpha(v_1, \ldots, v_n)$ is valid.

Theorem 1.8 (Compactness Theorem) *Let Σ be a set of sentences. Σ is consistent iff every finite subset of Σ is consistent.*

Proof: Assume that Σ is inconsistent. Then there exist a finite sequence of sentences $\sigma_1, \ldots, \sigma_n$ such that $\sigma_n = \alpha \wedge \neg\alpha$ for some sentence α, some σ_k's are logical axioms and the remaining ones belong to Σ. The remaining ones constitute a finite subset of Σ and $\alpha \wedge \neg\alpha$ is a logical consequence of it.

Thus Σ has an inconsistent finite subset. The other part is obvious.

$\square$

An $\mathcal{L}$-structure $\mathcal{M}$ is called a *model of* Σ, denoted by $\mathcal{M} \models \Sigma$, iff $\mathcal{M} \models \sigma$ for every $\sigma \in \Sigma$.

If Σ has a model, then obviously Σ is consistent. The converse is also true.

Theorem 1.9 (Extended Completeness Theorem) *Let Σ be a set of sentences. Σ has a model iff Σ is consistent.*

Proof: See [3].

Therefore the Compactness Theorem may be restated as follows:

Σ has a model iff every finite subset of Σ has a model.

A consistent set T of sentences of $\mathcal{L}$ is called a *theory.*

We use $F_n(T)$ to denote the set of all formulas in the language of T whose free variables form a subset of $\{v_1, \ldots, v_n\}$.

A formula $\alpha(v_1, \ldots, v_n)$ is said to be *consistent with* T iff

$$T \vdash (\exists v_1) \ldots (\exists v_n) \alpha(v_1, \ldots, v_n).$$

More generally, a set $\Gamma \subseteq F_n(T)$ is said to be *consistent with* T iff the conjunction of any finite number of members of Γ is consistent with T.

An *n-type* is a maximal consistent subset of $F_n(T)$.

We denote the set of all n-types of T by $S_n(T)$.

For any model $\mathcal{M}$ of T and for arbitrary elements $m_1, \ldots, m_n \in M$ the set $\Gamma(v_1, \ldots, v_n)$ consisting of all formulas $\gamma(v_1, \ldots, v_n)$ satisfied by $m_1, \ldots, m_n$ is a type. It is called the *type of* $m_1, \ldots, m_n$ *in* $\mathcal{M}$.

We use $Th(\mathcal{M}_M)$ to denote the set of all sentences in $\mathcal{L} \cup \{m\}_{m \in M}$ which are satisfied in $\mathcal{M}$.

An $\mathcal{L}$-structure $\mathcal{M}$ is called *ω-stable* iff whenever $\mathcal{N}$ is a model of $Th(\mathcal{M})$ and $|N| = \omega$, then $|S_1(Th(\mathcal{N}_N))| = \omega$.

1.2 Morley Rank and Morley Degree

Let $\mathcal{L}$ be a first order language and $\mathcal{M}$ be an $\mathcal{L}$-structure. We will define the Morley rank and the Morley degree of a formula $\phi(x)$ of one free variable in the language $\mathcal{L} \cup \{m\}_{m \in M}$, by induction.

The Morley rank, or simply the rank, of a formula is infinity or an integer greater than or equal to -1. The Morley degree, or simply the degree, of a formula is only defined if the formula is of finite Morley rank. The degree is always a positive integer.

If $\phi(x)$ is satisfied by finitely many elements of M, then $\phi(x)$ is said to be of Morley rank 0. In this case, the Morley degree of $\phi(x)$ is the number of elements in M that satisfy $\phi(x)$.

As an example, consider $\mathcal{K} =< K, +, \cdot, 0, 1 >$, where K is an algebraically closed field. Then every nonconstant polynomial $p(x) \in K[x]$ will be of rank 0 and deg($p(x)$) will be the number of distinct roots of $p(x)$.

Assume that all formulas of rank less than n have been defined. Write $S_{n+1}(x) = \{\neg\sigma(x) : rk(\sigma) \leq n\}$. Then $\phi(x)$ is said to be of rank $n+1$ iff the set of formulas $\{\phi(x)\} \cup S_{n+1}(x)$ is consistent in $\mathcal{L} \cup \{m\}_{m \in M}$ and has finitely many maximal consistent extensions. The degree of $\phi(x)$ is the number of maximal consistent extensions of $\phi(x)$ in $Th(\mathcal{M}_M)$.

Note that $S_0(x)$ is the set of all valid formulas in $Th(\mathcal{M}_M)$ and $S_1(x)$ is logically equivalent to the set $\{x \not\equiv m : m \in M\}$.

If $\phi(x)$ is inconsistent with $Th(\mathcal{M}_M)$, then we define $rk(\phi) = -1$. If $\phi(x)$ is consistent but has no rank, then we define $rk(\phi) = \infty$.

The formula $x \equiv x$ has the largest rank in a structure. $rk(M)$ is defined to be $rk(x \equiv x)$. Similarly, $deg(M)$ is $deg(x \equiv x)$. The rank and the degree of definable subsets of M are also defined. Let A be a definable subset of M. Then there exists a formula $\phi_A(x)$ such that $A = \{m \in M : \mathcal{M} \models \phi_A(m)\}$. Then $rk(A) = rk(\phi_A)$ and $deg(A) = deg(\phi_A)$.

Once more consider the structure $\mathcal{K}$. $\{x \equiv x\} \cup S_1(x)$ is consistent by compactness theorem and has no maximal consistent extensions except from itself. Thus, $rk(K) = 1$ and $deg(K) = 1$.

However, if K is not algebraically closed, then the situation will be different. Consider $\mathcal{R} =< \mathbb{R}, +, \cdot, 0, 1 >$. One can define the additive inverse of a real number in the structure $\mathcal{R}$, just like we did on page 9. Now the relation $\leq$ is definable in $\mathcal{R}$ since

$$\{(a,b) \in \mathbb{R}^2 : a \leq b\} = \{(a,b) \in \mathbb{R}^2 : (\exists x)(b - a = x \cdot x)\}.$$

For any $a, b \in \mathbb{R}$, the interval $[a, b]$ is definable, since $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. Suppose that there exists an integer $n \geq 1$ such that $rk([0,1]) = n$. Note that if $[a, b]$ and $[c, d]$ are any two closed intervals, then there exists a definable bijection $f : [a, b] \to [c, d]$. It follows that $rk([a, b]) = rk([c, d]) = n$. But the interval $[0,1]$ contains the infinitely many disjoint closed intervals $[(\frac{1}{2})^{2n}, (\frac{1}{2})^{2n+1}]$ for $n \geq 0$. The contradiction shows that $rk([0,1]) = \infty$ and hence $rk(\mathbb{R}) = \infty$.

CHAPTER 2

RANKED STRUCTURES

2.1 The Universe and the Rank Function

Throughout this section, $\mathcal{L}$ will stand for a first order language and $\mathcal{M}$ will stand for an arbitrary $\mathcal{L}$-structure.

Definition: A nonempty collection $\mathcal{U}$ of sets is called a *universe* iff for every $A, B, C \in \mathcal{U}$ the following holds:

1. $A \cap B, A \cup B, A \setminus B \in \mathcal{U}$

2. $A \times B \in \mathcal{U}$,

   the graphs of $\Pi_1 : A \times B \to A$ and $\Pi_2 : A \times B \to B$ are in $\mathcal{U}$,

   for any $C \subseteq A \times B$ $\Pi_i(C) \in \mathcal{U}$ for $i = 1, 2$.

   $\Delta(A) = \{(a, a) : a \in A\} \in \mathcal{U}$

3. $a \in A$ implies that $\{a\} \in \mathcal{U}$

4. E is an equivalance relation on A and $\{(x, y) \in A^2 : E(x, y)\} \in \mathcal{U}$ imply that $A/E \in \mathcal{U}$ and the graph of the canonical map $A \to A/E$ is in $\mathcal{U}$.

The reader probably noticed the similarity between the definition of the universe and the properties of the interpretable sets in some $\mathcal{L}$-structure

In fact, Lemma 1.7 proves that the collection of interpretable sets in $\mathcal{M}$ is a universe. We denote this universe by $\mathcal{U}_{\mathcal{L}}(\mathcal{M})$ or simply by $\mathcal{U}(\mathcal{M})$, where $\mathcal{L}$ is clear from the context.

Thus it is reasonable to call the elements of a universe *interpretable sets* and the function whose graphs lie in the universe *interpretable functions.*

Definition: Let $\mathcal{U}$ be a universe and $rk : \mathcal{U} \setminus \{\phi\} \to \mathbb{N}$ be a function. The function rk is called a *rank function* iff the following axioms are satisfied:

Axiom A: $rk(A) \geq n+1$ iff there exists a sequence of nonempty sets $\{A_i\}_{i=1}^{\infty}$ such that $A_i \subseteq A$, $A_i \in \mathcal{U}$, $A_i \cap A_j = \phi$ if $i \neq j$ and $rk(A_i) \geq n$.

Axiom B: $f : A \to B$ is an interpretable function implies that $\{b \in B : rk(f^{-1}(b)) = n\} \in \mathcal{U}$, for all $n \in \mathbb{N}$.

Axiom C: $f : A \to B$ is an interpretable function which is onto and $rk(f^{-1}(b)) = n$ for all $b \in B$ imply that $rk(A) = rk(B) + n$.

Axiom D: For any interpretable function $f : A \to B$, there exists $m \in \mathbb{Z}$ such that

$$|f^{-1}(b)| \geq m \text{ implies that } |f^{-1}(b)| = \infty, \text{ forall } b \in B.$$

Definition: A universe $\mathcal{U}$ is called a *ranked universe* if there exists a rank function on $\mathcal{U}$. $\mathcal{M}$ is called a *ranked structure* if the universe $\mathcal{U}_{\mathcal{L}}(\mathcal{M})$ consisting of interpretable subsets in $\mathcal{M}$ is a ranked universe.

$\mathcal{M}$ is a ranked group means that $\mathcal{M} = \langle G, \cdot, ^{-1}, 1, \ldots \rangle$, where $\langle G, \cdot, ^{-1}, 1 \rangle$ is a group and $\mathcal{M}$ is a ranked structure.

In [8], Poizat showed that ranked groups and ω-stable groups of finite Morley rank coincide. The proof is skipped since it requires hard model theoretic machinery. ω-stable groups of finite Morley rank will be discussed in Chapter 3.

2.2 Properties of Ranked Universes

Throughout this section, assume that $\mathcal{U}$ is a fixed ranked universe of the form $\mathcal{U}_{\mathcal{L}}(\mathcal{M})$ with the rank function $rk : \mathcal{U} \setminus \{\phi\} \to \mathbb{N}$. For convenience, we set $rk(\phi) = -1$.

For the following lemmas, assume that A and B are interpretable sets and $f : A \to B$ is an interpretable function.

Lemma 2.1 *$rk(A) \geq 1$ iff A is infinite.*

Proof: Let A be a finite set, then A does not have infinitely many nonempty pairwise disjoint subsets. Axiom A implies that $rk(A) = 0$. Now let A be infinite, then the sequence $\{a\}_{a \in A}$ fulfills the requirements of Axiom A for the case $n = 0$, thus $rk(A) \geq 1$. □

Lemma 2.2 *$A \subseteq B$ implies that $rk(A) \leq rk(B)$.*

Proof: Let $rk(A) = n$. Then Axiom A implies that A has infinitely many pairwise disjoint interpretable subsets of rank greater than or equal to $n - 1$. These sets also lie in B, so $rk(B) \geq n = rk(A)$ by Axiom A. □

Lemma 2.3 $rk(A \cup B) = max\{rk(A), rk(B)\}$.

Proof: Set $C = A \cup B$, then $rk(C) \geq max\{rk(A), rk(B)\}$ by Lemma 2.2. The proof of the converse will be done by induction on $rk(C)$. If $rk(C) = 0$, then the inequality obviously holds. Assume that $rk(C) = n$, then C has infinitely many pairwise disjoint interpretable subsets C_i of rank greater than or equal to n-1. Let $A_i = A \cap C_i$ and $B_i = B \cap C_i$. The induction hypothesis implies that for each i, either $rk(A_i) = n - 1$ or $rk(B_i) = n - 1$. Without loss of generality, assume that there are infinitely many sets A_i of rank $n - 1$. Now Axiom A implies that $rk(A) \geq n = rk(C)$. □

Now we will associate a positive integer to each element of $\mathcal{U} \setminus \{\phi\}$, which is called the degree of the set.

Definition: $deg(A)$ is defined to be 1 if for any interpretable subset $B \subseteq A$, we have either $rk(B) < rk(A)$ or $rk(A \backslash B) < rk(A)$. $deg(A)$ is defined to be d if there exist interpretable subsets $C_1, \ldots, C_d$ of A such that $A = \sqcup_{i=1}^{d} C_i$, $deg(C_i) = 1$ and $rk(C_i) = rk(A)$ for each $i = 1, \ldots, d$.

Lemma 2.4 *Degree is well-defined.*

Proof: Let A be an interpretable set and $rk(A) = n$. Assume that $A = \sqcup_{i=1}^{d} A_i = \sqcup_{j=1}^{e} B_j$, where A_i's and B_j's are all interpretable sets of rank n and degree 1. Now a one to one map will be defined from $\{1, \ldots, d\}$ into $\{1, \ldots, e\}$ to show that $d \leq e$.

For any $i \in \{1, \ldots, d\}$, we have $A_i = \sqcup_{j=1}^{e} (B_j \cap A_i)$. $rk(A_i) = n$ implies that there exists $j \in \{1, \ldots, e\}$ such that $rk(B_j \cap A_i) = n$. Since $deg(A_i) = 1$, j is unique with this property. Thus $d \leq e$.

Since everything is symmetric, $e \leq d$ which shows that $d = e$ and degree is well-defined. □

Lemma 2.5 *Let $A \cap B = \phi$ and $rk(A) = rk(B)$, then $deg(A \sqcup B) = deg(A) + deg(B)$.*

Proof: Let $rk(A) = rk(B) = n$, $deg(A) = d$ and $deg(B) = e$. Then there exist disjoint interpretable sets $A_1, \ldots, A_d$ in A and $B_1, \ldots, B_e$ in B of rank n and degree 1. Then $A \sqcup B = A_1 \sqcup \ldots A_d \sqcup B_1 \sqcup \ldots \sqcup B_e$. By Lemma 2.3, $rk(A \sqcup B) = n$ so $A \sqcup B$ can be written as a disjoint union of (d+e) interpretable sets of rank n and degree 1; i.e. $deg(A \sqcup B) = d + e = deg(A) + deg(B)$. □

Lemma 2.6 *Every nonempty interpretable set has a degree.*

Proof: Assume that A is an interpretable set of rank n and A does not have a degree. Since $deg(A) \neq 1$, there exist two disjoint interpretable sets A_1, A_2 in A such that $A = A_1 \sqcup A_2$ and $rk(A_1) = rk(A_2) = n$.

If A_1 and A_2 both have degrees, then by Lemma 2.5 A has a degree. Assume that A_1 does not have a degree and call $A_2 = B_1$. Now we can apply the same arguement to A_1 and obtain two disjoint interpretable sets A_{11} and A_{12} in $A_1 \subseteq A$ of rank n. Assume that A_{11} does not have a degree. Now set $B_2 = A_{12}$. Notice that B_1 and B_2 are of rank n and disjoint.

Continuing like this, we can obtain infinitely many disjoint interpretable sets $(B_i)_{i=1}^{\infty}$ in A each of rank n contradicting Axiom A. The contradiction implies that A has a degree. □

Lemma 2.7 *If $rk(A) > rk(B)$, then $deg(A \cup B) = deg(A)$.*

Proof: First assume that $deg(A) = 1$ and C is an interpretable subset of $A \cup B$. Then $C \cap A$ is an interpretable subset of A and either $rk(C \cap A) < rk(A)$ or $rk(A \setminus C) < rk(A)$.

Assume that $rk(C \cap A) < rk(A)$. $C = (C \cap A) \sqcup (C \cap B)$ and $rk(C) = max\{rk(C \cap A), rk(C \cap B)\}$. By assumption, $rk(C \cap A) < rk(A) = rk(A \cup B)$ and $rk(C \cap B) \leq rk(B) < rk(A) = rk(A \cup B)$. Thus $rk(C) < rk(A \cup B)$ in this case.

Now assume that $rk(A \setminus C) < rk(A)$. $((A \cup B) \setminus C) = (A \setminus C) \cup (B \setminus C)$ and $rk((A \cup B) \setminus C) = max\{rk(A \setminus C), rk(B \setminus C)\}$. By assumption $rk(A \setminus C) < rk(A) = rk(A \cup B)$ and $rk(B \setminus C) \leq rk(B) < rk(A) = rk(A \cup B)$. Thus $rk((A \cup B) \setminus C) < rk(A \cup B)$.

This proves that $deg(A \cup B) = 1 = deg(A)$.

Now let $deg(A) = d > 1$, then there exist disjoint interpretable subsets $A_1, \ldots, A_d$ of A such that $rk(A_i) = rk(A)$ and $deg(A_i) = 1$ for all $i = 1, \ldots, d$. Then $A \cup B = ((B \setminus A) \cup A_1) \sqcup A_2 \sqcup \ldots \sqcup A_d$. $rk(B \setminus A) \leq rk(B) < rk(A) = rk(A_1)$ implies that $deg((B \setminus A) \cup A_1) = 1$ by the above arguement. $rk((B \setminus A) \cup A_1) = rk(A_i) = rk(A) = rk(A \cup B)$ for all $i = 2, \ldots, d$. Thus $deg(A \cup B) = d = deg(A)$.
□

Lemma 2.8 *If $A \subseteq B$ and $rk(A) = rk(B)$, then $deg(A) \leq deg(B)$.*

Proof: Let $deg(A) = d$ then there exist interpretable subsets $A_1, \ldots, A_d$ of A

satisfying $rk(A_i) = rk(A)$, $deg(A_i) = 1$ for all $i = 1, \ldots, d$ and $A = \sqcup_{i=1}^{d} A_i$. Then $B = \sqcup_{i=1}^{d} A_i \sqcup (B \setminus A)$.

If $rk(B \setminus A) = rk(B) = rk(A)$, then Lemma 2.5 implies that $deg(B) = d + deg(B \setminus A) > d = deg(A)$.

If $rk(B \setminus A) < rk(B) = rk(A)$, then Lemma 2.7 implies that $deg(B) = deg(A)$.

Combining these two results, we get that $deg(B) \geq deg(A)$. □

Lemma 2.9 *$f : A \to B$ is an interpretable bijection implies that $f^{-1} : B \to A$ is also an interpretable bijection.*

Proof: Clearly $f : A \to B$ is a bijection implies that $f^{-1} : B \to A$ is a bijection. To see that f^{-1} is interpretable, consider the graph of f^{-1} which is

$$\{(b, f^{-1}(b)) : b \in B\} = \{(f(a), a) : a \in A\}.$$

This set is the image of $\overline{A} = \{(a, f(a)) : a \in A\}$ under the map θ defined by $\theta((a, f(a))) = (f(a), a)$. $\overline{A}$ is interpretable, since it is the graph of f. So, it suffices to show that θ is interpretable, or equivalently the set $R = \{(a, f(a), f(a), a) : a \in A\}$ is interpretable.

$S = \{(a, f(a), f(a)) : a \in A\}$ is the graph of the second projection from $\overline{A}$ onto B, so S is interpretable. Finally, R is the graph of the first projection from S onto A, which proves that R is interpretable. □

Lemma 2.10 *If there exists an interpretable bijection $f : A \to B$, then $rk(A) = rk(B)$ and $deg(A) = deg(B)$.*

Proof: Since f is one to one, $rk(f^{-1}(b)) = 0$ for all $b \in B$. Now Axiom C implies that $rk(A) = rk(B)$.

Now let $deg(A) = d$, then there exist interpretable subsets $A_1, \ldots, A_d$ in A such that $rk(A_i) = rk(A)$, $deg(A_i) = 1$ for all $i = 1, \ldots, d$ and $A = \sqcup_{i=1}^{d} A_i$. Then $B = \sqcup_{i=1}^{d} f(A_i)$ since f is a bijection. The above arguement implies that $rk(A_i) = rk(f(A_i))$, also $rk(B) = rk(A) = rk(A_i)$. Thus $deg(B) = \Sigma_{i=1}^{d} deg(f(A_i)) \geq d$.

f^{-1} is also an interpretable bijection by Lemma 2.9, so applying the same arguements to f^{-1} will prove that $deg(A) = deg(B)$. □

In particular, rank and degree are invariant under interpretable automorphisms. In fact requiring the automorphism to be interpretable is superfluous, as the following lemma shows.

Lemma 2.11 *Let $\phi : M \to M$ be an automorphism and $A \subseteq M$ be a definable subset, then $rk(A) = rk(\phi(A))$ and $deg(A) = deg(\phi(A))$.*

Note that we can not use Axiom C, since ϕ is not interpretable.

Proof: Proof will be done by induction on $rk(A)$. If $rk(A) = 0$, then A is finite and $deg(A) = |A|$. ϕ is an automorphism implies that $|A| = |\phi(A)|$ and $rk(\phi(A)) = 0$ and $deg(\phi(A)) = |\phi(A)| = |A| = deg(A)$.

Let $rk(A) = n$, then Axiom A implies that there exist a sequence $\{A_i\}_{i=1}^{\infty}$ of disjoint interpretable subsets of A of rank greater than or equal to $n-1$. The induction hypothesis implies that $rk(\phi(A_i)) \geq n-1$. ϕ is one to one implies that $\phi(A_i)$'s are disjoint and Lemma 1.6 implies that $\phi(A_i)$'s are definable. So $rk(\varphi(A)) \geq n = rk(A)$, by Axiom A. φ^{-1} is an automorphism. Thus $rk(A) = rk(\varphi(A))$.

The proof of the equality of the degrees is exactly the above proof. □

Lemma 2.12 $rk(A \times B) = rk(A) + rk(B)$.

Proof: Consider the canonical projection $\Pi_1 : A \times B \to A$. Π_1 is onto and interpretable by the second axiom of the definition of the universe. For any $a \in A$, $\Pi_1^{-1}(a) = \{a\} \times B$, so $rk(\Pi_1^{-1}(a)) = rk(\{a\} \times B) = rk(B)$ by Lemma 2.10.

Now Axiom C implies that $rk(A \times B) = rk(A) + rk(B)$. □

The next lemma is simple but will be needed in Chapter 3.

Lemma 2.13 *Assume that A is a definable set of rank n and degree 1. Let $X_1, \ldots, X_m$ be definable subsets of A of rank n also. Then $rk(X_1 \cap \ldots \cap X_m) = n$.*

Proof: It is enough to prove the statement for m=2. Assume that $rk(X_1 \cap X_2) < n$, then $rk(X_1 \setminus X_2) = n$ since $X_1 = (X_1 \cap X_2) \sqcup (X_1 \setminus X_2)$. By the same arguement, $rk(X_2 \setminus X_1) = n$. Thus there are two disjoint sets of rank n in A. This contradicts the fact that $deg(A) = 1$. □

CHAPTER 3

GROUPS OF FINITE MORLEY RANK

When we say that G is an ω-stable group of finite Morley rank, we mean that:

- G is an $\mathcal{L}$-structure, where $\mathcal{L} = \{\cdot, ^{-1}, 1\}$,
- $\mathcal{U}_{\mathcal{L}}(G)$ is a ranked universe,
- $< G, \cdot, ^{-1}, 1 >$ satisfies the following:

  $(\forall x)(\forall y)(\forall z)((x \cdot y) \cdot z = x \cdot (y \cdot z))$,

  $(\forall x)(x \cdot 1 = 1 \cdot x = x)$,

  $(\forall x)(x \cdot x^{-1} = x^{-1} \cdot x = 1)$.

Recall that, $\mathcal{U}_{\mathcal{L}}(G)$ is the collection of all interpretable subsets of G in the language $\mathcal{L}$.

However, the word "ω -stable" is usually omitted, so when one says that G is a group of finite Morley rank, one means that G is an ω -stable group of finite Morley rank.

In this chapter, G will stand for a group of finite Morley rank.

It is customary to use the term definable subset instead of the term interpretable subset. From now on. all facts will be stated in terms of definable subsets to mean an interpretable subset. Here we give some examples of definable subsets of G.

- The center of $G, Z(G) = \{g \in G : (\forall x)(xg = gx)\}$, is a definable subgroup of G.
- For $x \in G$, $C_G(x) = \{g \in G : gx = xg\}$ is a definable subgroup of G.
- For $r \in G$, the conjugacy class of r, $r^G = \{g \in G : (\exists x)(g = x^{-1}rx)\}$ is a definable subset of G.
- If H is a definable subgroup of G, then $C_G(H)$ and $N_G(H)$ are also definable subgroups of G.

3.1 The Descending Chain Condition

The purpose of this section is to prove that a group of finite Morley rank satisfies the descending chain condition on its definable subgroups; i.e. for any descending chain $(H_i)_{i \in I}$ of definable subgroups of G, there exists $k \in I$ such that $H_k = H_n$ for all $n \geq k$.

Note that an arbitrary group does not necessarily satisfy the descending chain condition on its definable subgroups. For example, for any $n \in \mathbb{Z}$, $n\mathbb{Z}$ is a definable subgroup of $\mathbb{Z}$, since $n\mathbb{Z} = \{k \in \mathbb{Z} : (\exists m)(k = nm)\}$. Now, let $n \neq \pm 1$, then

$$\mathbb{Z} \geq n\mathbb{Z} \geq n^2\mathbb{Z} \geq \cdots\cdots \geq n^k\mathbb{Z} \geq n^{k+1}\mathbb{Z} \geq \cdots\cdots$$

is a descending chain, but does not satisfy the descending chain condition.

Nor does a group of finite Morley rank satisfy the descending chain condition on its nondefinable subgroups. For example, $\mathbb{C}$ is a group of finite Morley rank and has a descending chain of subgroups which is not stationary, namely

$$\mathbb{C} \geq \mathbb{Z} \geq n\mathbb{Z} \geq n^2\mathbb{Z} \geq \cdots\cdots \geq n^k\mathbb{Z} \geq n^{k+1}\mathbb{Z} \geq \cdots\cdots, n \neq \pm 1.$$

In order to prove that a group of finite Morley rank satisfies the descending chain condition on its definable subgroups, we will need some lemmas.

When $H \leq G$ is a definable subgroup of G, it is meaningful to write $rk(G/H)$, since G/H is an interpretable set. To see this, define a relation $\sim$ on G such that $g_1 \sim g_2$ iff there exists $h \in H$ such that $g_2^{-1}g_1 = h$. Since H is definable, $\sim$ is a definable equivalance relation.

Lemma 3.1 *Let $H \leq G$ be a definable subgroup of G. Then*

$$rk(G) = rk(G/H) + rk(H).$$

Proof: Let $\varphi : G \to G/H$ be the canonical epimorphism. $rk(\varphi^{-1}(gH)) = rk(gH) = rk(H)$ for all $g \in G$. Now Axiom C implies that $rk(G) = rk(G/H) + rk(H)$. $\square$

Lemma 3.2 *Let $\phi : G \to H$ be a definable group homomorphism, then*

$$rk(G) = rk(Ker\phi) + rk(\phi(G)).$$

Proof: $\frac{G}{Ker\phi} \cong \varphi(G)$ implies that $rk(\frac{G}{Ker\phi}) = rk(\phi(G))$, by Lemma 2.10. Now Lemma 3.1 implies that $rk(\frac{G}{Ker\phi}) = rk(G) - rk(Ker\phi) = rk(\phi(G))$ which proves the result. $\square$

Lemma 3.3 *Assume that K and H are definable subgroups of G and $K \leq H$. Then, the following holds:*

$$rk(K) = rk(H) \text{ if and only if } [H : K] < \infty$$

Moreover, we have $deg(H) = [H : K]deg(K)$ in such a case.

Proof: $rk(K) = rk(H)$ iff $rk(H/K) = 0$, by Lemma 3.1, iff H/K is finite.

For the last part, write $H = \bigsqcup_{i=1}^{n} h_i K$. Since $rk(H) = rk(h_i K)$ for all $i = 1, \dots, n$; Lemma 2.5 implies that $deg(H) = \sum_{i=1}^{n} deg(h_i K) = \sum_{i=1}^{n} deg(K) = ndeg(K) = [H : K]deg(K)$. $\square$

Notice that, for any two definable subgroups $K \leq H$ of G, we also have:

$$rkK < rkH \iff [H : K] = \infty \tag{3.1}$$

and

$$K = H \iff rk(K) = rk(H) \text{ and } deg(K) = deg(H). \tag{3.2}$$

Equation 3.1 is just the contraposition of Lemma 3.3 and Equation 3.2 is an easy consequence of Lemma 3.3.

Now it is time to prove the main result of this section.

Theorem 3.4 (Macintyre) *A group G of finite Morley rank satisfies the descending chain condition on its definable subgroups.*

Proof: Let $(H_i)_{i\in I}$ be an arbitrary descending chain of definable subgroups of G. Consider the sequence $(rk(H_i))_{i\in I}$ of integers which is bounded from below by -1. Lemma 2.2 implies that $(rk(H_i))_{i\in I}$ is a decreasing sequence. Therefore, there exists $m \in I$ such that $rk(H_n) = rk(H_m)$ for all $n \geq m$. Now, consider the sequence $(deg(H_i))_{i\in I}$ of integers which is bounded from below by 1. Lemma 2.8 implies that the sequence $(deg(H_i))_{i\in I}$ is decreasing after the m-th term, so necessarily there exists $k \in I$ such that $k \geq m$ and $deg(H_n) = deg(H_k)$ for all $n \geq k$. What we proved is: There exists $k \in I$ such that for all $n \geq k, deg(H_n) = deg(H_k)$ and $rk(H_n) = rk(H_k)$. Equation 3.2 implies that there exists $k \in I$ such that $H_n = H_k$, for all $n \geq k$. □

Note that groups of finite Morley rank do not satisfy an ascending chain condition. Although the corresponding rank sequence will become stationary at some point, the corresponding degree sequence may increase without any bounds. Here is an example of such a case: Let $G = \mathbb{C}$, p be a prime number and $H_n = \{z : z^{p^n} = 1\}$. H_n's are definable subgroups of G and $H_n \leq H_{n+1}$. However, $H_n \neq H_{n+1}$; since a primitive p^{n+1}st root of unity ζ belongs to H_{n+1}, but ζ is not in H_n. Thus, the ascending chain $(H_n)_{n\in\mathbb{N}}$ never becomes stationary. The fact that for all $n \in \mathbb{N}$, $rk(H_n) = 0$ and $deg(H_n) = |H_n| = p^n$ explains why the chain does not become stationary after some point.

The following form of the previous theorem is more useful, so it is worth stating it.

Corollary 3.5 *Let H_i's be definable subgroups of G for $i \in I$. Then there exists a finite subset $J \subseteq I$ such that $\bigcap_{i\in I} H_i = \bigcap_{j\in J} H_j$. In particular, $\bigcap_{i\in I} H_i$ is also a definable subgroup of G.*

Proof: Assume that the conclusion of the corollary is false. Then there exists a sequence (i_n) in I such that $K_n = H_{i_1} \cap \ldots\ldots \cap H_{i_n}$ is a strictly decreasing chain of definable subgroups of G. This contradiction implies that the above statement is true. $\square$

Corollary 3.6 *Let $X \subseteq G$ be any subset of G, not necessarily definable. Then $C_G(X)$ is a definable subgroup of G.*

Proof: $C_G(X) = \cap_{x \in X} C_G(x)$. The previous corollary implies that there exists $x_1, \cdots, x_n \in X$ such that $C_G(X) = \cap_{i=1}^{n} C_G(x_i)$. Therefore, $C_G(X)$ can be written as a finite Boolean combination of definable subgroups of G which shows that $C_G(X)$ is definable. $\square$

One can use Theorem 3.4 or Corollary 3.5 to show that certain groups are not of finite Morley rank. Now, we will apply these facts to free abelian groups, free groups and symmetry groups of X, where X is an infinite set, to see that they are not groups of finite Morley rank.

- A free abelian group A is not of finite Morley rank.

  Let X be a basis of A, then $A \cong \sum_{x \in X} \mathbb{Z}x$. For every $n \in \mathbb{Z} \setminus \{\pm 1\}$ and $k \in \mathbb{Z}^+$, $n^k A = \{a \in A : (\exists b)(a = n^k b)\}$ is a definable subgroup of A. Now, $A \geq nA \geq n^2 A \geq \cdots\cdots \geq n^k A \geq n^{k+1} A \geq \cdots$ is a descending chain of definable subgroups of A, which is not stationary. A does not satisfy the descending chain condition which implies that A is not a group of finite Morley rank.

- A free group F is not of finite Morley rank.

Let X be a basis of F and $x \in X$. $C_G(x)$ is a definable subgroup of F. Then $C_G(x)$ is isomorphic to the infinite cyclic group, which is free abelian. However, a definable subgroup of a group of finite Morley rank is also of finite Morley rank. This proves that F is not a group of finite Morley rank.

- Sym(X) is not a group of finite Morley rank, where X is an infinite set.

  For each $x \in X$, define $S_x = \{\sigma \in Sym(X) : \sigma(x) = x\}$. S_x's are definable subgroups of Sym(X) and $\cap_{x \in X} S_x = \{id\}$. Consider an arbitrary finite subset $Y \subset X$,then obviously, $\cap_{y \in Y} S_y \neq \{id\}$. Now, Corollary 3.5 implies that Sym(X) is not a group of finite Morley rank.

3.2 The Connected Component

In this section, an important subgroup, the connected component, of a group of finite Morley rank will be introduced and some properties of it will be stated.

Throughout this section, G denotes a group of finite Morley rank.

Definition: The connected component of G, denoted by G°, is the intersection of all definable subgroups of finite index in G.

Definition: G is called connected if $G^\circ = G$.

Thus G is connected if and only if G has no proper definable subgroups of finite index.

Lemma 3.7 *Assume that G is connected and $\phi : G \to G$ is a definable group homomorphism. If $Ker\phi$ is finite then ϕ is onto.*

Proof: $Ker\phi$ is finite implies that $rk(Ker\phi) = 0$. Lemma 3.2 implies that $rk(G) - rk(Ker\phi) = rk(\phi(G))$. Thus $rk(G) = rk(\phi(G))$. Lemma 3.3 implies that $[G : \phi(G)] < \infty$. Thus $\phi(G)$ is a definable subgroup of G of finite index. G is connected implies that $G = \phi(G)$; i.e. ϕ is onto. □

Lemma 3.8 *G° is a definable subgroup of G.*

Proof: Since G satisfies the descending chain condition there exists definable subgroups $H_1, \ldots, H_n$ of G such that $[G : H_i] < \infty$ for $i = 1, \ldots, n$ and $G^{\circ} = \bigcap_{i=1}^{n} H_i$. A finite intersection of definable subgroups of G is a definable subgroup of G. □

Lemma 3.9 $[G : G^{\circ}] < \infty$

Proof: This follows from the fact that if $[G : H_1] < \infty$ and $[G : H_2] < \infty$, then $[G : H_1 \cap H_2] < \infty$. □

Lemma 3.10 $rk(G^{\circ}) = rk(G)$

Proof: Lemma 3.3 and Lemma 3.9 imply this result. □

Remarks:

1. G° has no proper definable subgroups of finite index .

2. G° is contained in any definable subgroup which is of finite index in G.

3. G° is connected.

4. A finite group is not connected unless it is the trivial group.

Lemma 3.11 *An infinite simple group G is connected.*

Proof: Assume that G is not connected. Then G has a proper definable subgroup H of finite index, say n. Consider the homomorphism $G \to Sym(g_1H, \ldots, g_nH)$ defined by $g \mapsto \sigma_g$, where $\sigma_g(g_iH) = gg_iH$. Since $\cap_{g \in G} H^g$ is the kernel of this map, $[G : \cap_{g \in G} H^g] \leq n!$. Thus $\cap_{g \in G} H^g$ is a normal subgroup of G of finite index. G is simple implies that $\cap_{g \in G} H^g = \{id\}$ or $\cap_{g \in G} H^g = G$. The first case is impossible, since G is infinite. The second case implies that $H = G$, which contradicts the assumption that H is a proper subgroup.

Thus G has no proper definable subgroups of finite index. Hence G is connected. □

Lemma 3.12 *A divisible (abelian) group G is connected.*

Proof: Let H be a proper definable subgroup of finite index in G. Then G/H is divisible and finite. Thus G/H is the trivial group, and so $G = H$. This contradiction shows that G is connected. □

Lemma 3.13 *A group G of degree 1 is connected.*

Proof: Since $rk(G) = rk(G^\circ)$ and $G^\circ \leq G$, $deg(G^\circ) \leq deg(G) = 1$. Thus $deg(G^\circ) = 1$. Equation 3.2 implies that $G = G^\circ$. Hence G is connected. □

Lemma 3.14 *If ϕ is an automorphism of G, then $\phi(G^\circ) = G^\circ$.*

Proof: $\phi(G^\circ)$ is a definable subgroup of finite index in G by Lemma 1.6. So Remark 2 implies that $G^\circ \leq \phi(G^\circ)$. ϕ^{-1} is an automorphism implies that $G^\circ \leq \phi^{-1}(G^\circ)$. So $\phi(G^\circ) \leq G^\circ$. Thus we get $G^\circ = \phi(G^\circ)$. □

Lemma 3.15 $G^\circ \trianglelefteq G$.

Proof: For any $g \in G, \phi_g(x) = g^{-1}xg$ is a definable automorphism of G. By the previous lemma, $\phi_g(G^\circ) = G^\circ$, for all $g \in G$; i.e. $g^{-1}G^\circ g = G^\circ$, for all $g \in G$. Thus G° is normal in G. □

Definition: Assume that G is acting on a definable set X. G is said to act definably on X if the map $\phi : G \times X \to X$ defined by $(g, x) \mapsto gx$ is definable.

Lemma 3.16 *If a connected group G acts definably on a finite set X, then the action is trivial.*

Proof: Define a group homomorphism $\Phi : G \to Sym(X)$ such that $\Phi(g) = \sigma_g$, where $\sigma_g(x) = gx$. Ker $\Phi = \{g \in G : (\forall x)(gx = x)\}$ is a definable subgroup of G. $G/Ker\Phi$ is isomorphic to the image $\Phi(G)$, which is a subgroup of $Sym(X)$. X is finite implies that $Sym(X)$ is finite. Since $[G : Ker\Phi] = \Phi(G) \subseteq Sym(X)$, $[G : Ker\Phi]$ is finite. G is connected, and so $Ker\Phi = G$. Thus $gx = x$, for all $g \in G$ and for all $x \in X$. □

The converse of Lemma 3.13 is also true; i.e. the degree of a connected group is 1. We need the following lemma to prove this fact.

Lemma 3.17 *Let $rk(G) = n$, $deg(G) = d$ and $X \subseteq G$ be a definable subset. Then, $rank(X) < n$ iff $\vee_{l=1}^{\infty} P_l$, where*

$$P_l : (\exists g_{11}), \ldots, (\exists g_{l(d+1)})(X \subseteq \cup_{k=1}^{l} [\cap_{i,j=1;i \neq j}^{d+1} g_{kj}^{-1} g_{ki} X^c]).$$

Proof:($\Rightarrow$) : Let $\mathcal{L}$ be the language such that G is an $\mathcal{L}$-structure and T be the complete theory of G. Define $\mathcal{L}' = \mathcal{L} \cup \{g\}_{g \in G} \cup \{a\}$ and let $S(g_1, \ldots, g_{d+1}) = \vee_{i,j=1;i \neq j}^{d+1} [g_i^{-1} g_j a \in X]$. Now define a new theory T' containing the theory T, the sentence $\{a \in X\}$ and the sentences $S(g_1, \ldots, g_{d+1})$ for each $d+1$-tuple $g_1, \ldots, g_{d+1}$ in G.

Assume that T' is consistent and G^* is a model of T'. Choose a maximal set $\{g_1, \ldots, g_r\} \subseteq G$ such that $g_i^{-1} g_j a \notin X$ if $i \neq j$. Then, necessarily $r \leq d$. Since it is the maximal set, for all $g \in G$, there exists $i \leq r$ such that $g_i^{-1} g a \in X$ or $g^{-1} g_i a \in X$, which implies that $g \in g_i X a^{-1}$ or $g \in g_i a X^{-1}$. Hence $G \subseteq \cup_{i=1}^{r} (g_i X a^{-1} \cup g_i a X^{-1})$ whose rank is equal to $rk(X) < n = rk(G)$. This contradiction implies that T' is inconsistent.

T' is inconsistent implies that for any $x \in X$ there exist $h_1, \ldots, h_{d+1} \in G$ such that $h_i^{-1} h_j x \notin X$ whenever $i \neq j$. Let's set $h_i = g_{k(x)i} = g_{ki}$ and rewrite this statement. For any $x \in X$ there exists $k \leq m$ such that $g_{ki}^{-1} g_{kj} x \notin X$ whenever $i \neq j$.

Note that the following statements are all equivalent: $g_{ki}^{-1} g_{kj} x \notin X$ if and only if $g_{ki}^{-1} g_{kj} x \in X^c$ if and only if $x \in g_{kj}^{-1} g_{ki} X^c$.

Thus we have for all $x \in X$ there exists $k \leq m$ such that $x \in g_{kj}^{-1} g_{ki} X^c$ whenever $i \neq j$. Hence for all $x \in X$ there exists $k \leq m$ such that $x \in \cap_{i,j=1;i \neq j}^{d+1} g_{kj}^{-1} g_{ki} X^c$. In other words, $X \subseteq \cup_{k=1}^{m} \cap_{i,j=1;i \neq j}^{d+1} g_{kj}^{-1} g_{ki} X^c$.

Therefore $\vee_{l=1}^{\infty}(X \subseteq \cup_{k=1}^{l} \cap_{i,j=1;i\neq j}^{d+1} g_{kj}^{-1} g_{ki} X^c)$ holds.

$(\Leftarrow)$: Since there exists $\vee_{l=1}^{\infty} P_l$, there exist l and $g_{11} \ldots, g_{l(d+1)}$ such that $X \subseteq \cup_{k=1}^{l} \cap_{i,j=1;i\neq j}^{d+1} g_{kj}^{-1} g_{ki} X^c$. For $1 \leq k \leq l$, define $X_k = \cap_{i,j=1;i\neq j}^{d+1} g_{kj}^{-1} g_{ki} X^c$. Then, $X \subseteq \cup_{k=1}^{l} X_k$, so there exists $1 \leq k_0 \leq l$ such that $rk(X) \leq rk(X_{k_0})$. For any $1 \leq k \leq l$, $\sqcup_{i=1}^{d+1} g_{ki} X_k$ is a disjoint union, because otherwise for some $i \neq j$, there exists $a \in g_{ki} X_k \cap g_{kj} X_k$. Then there exists $a_1, a_2 \in X_k$ such that $a = g_{ki} a_1 = g_{kj} a_2$ and $a_2 = g_{kj}^{-1} g_{ki} a_1 \in X_k \subseteq g_{kj}^{-1} g_{ki} X^c$. This implies that $a_1 \in X \cap X^c$, which is nonsense. Thus, in particular, $\sqcup_{i=1}^{d+1} g_{k_0 i} X_{k_0}$ is a disjoint union in G. $deg(G) = d$ implies that $rk(X_{k_0}) < n$. Since $rk(X) \leq rk(X_{k_0})$, it follows that $rk(X) < n$. □

Theorem 3.18 *G is connected iff $deg(G) = 1$.*

Proof: By Lemma 3.13, if $deg(G) = 1$, then G is connected. For the converse, assume that G is connected and $deg(G) = d$ and let $rk(G) = n$. Then, there exist definable sets $A_i \subseteq G$, for $i = 1, \ldots, d$ such that $rk(A_i) = n$, $deg(A_i) = 1$ and $G = \sqcup_{i=1}^{d} A_i$.

Now define an action of G on $\{1, \ldots, d\}$ such that $rk(gA_i \cap A_{gi}) = n$. Let's prove that this action is well defined. Fix gA_i. Then, $rk(gA_i) = rk(G) = n$ and $deg(gA_i) = 1$. $gA_i = (gA_i \cap A_1) \cup \ldots \cup (gA_i \cap A_d)$ implies that $n = rk(gA_i) = max_{1 \leq j \leq d}\{rk(gA_i \cap A_j)\}$. Thus there exists $j \in \{1, \ldots, d\}$ such that $rk(gA_i \cap A_j) = n$. If j is not unique, then gA_i can be written as a disjoint union of at least two definable sets of rank n. But this contradicts the fact that gA_i is of degree 1. This proves that the action is well defined.

To prove that the action is definable, we have to prove that the map

$(g, i) \mapsto gi$ is definable. The graph of this map is

$$\{(g, i, gi) : g \in G, i \in \{1, \ldots, d\}\} = \cup_{i,j=1}^{d}(X_{ij}^{c} \times \{i\} \times \{j\}),$$

where $X_{ij} = \{g \in G : rk(gA_i \cap A_j) < n\}$. Lemma 3.17 implies that X_{ij} is definable, so the action is definable. Lemma 3.16 implies that the action is trivial;i.e. $rk(gA_i \cap A_i) = n$, for all $g \in G$ and for all $i = 1, \ldots, d$. Now, $gA_i = \sqcup_{j=1}^{d}(gA_i \cap A_j)$, $deg(gA_i) = 1$ and $rk(gA_i \cap A_j) = n$. These imply that $rk(gA_i \cap A_j) < n$, for all $j \neq i$ and for all $g \in G$.

Define a new language $\mathcal{L}' = \mathcal{L} \cup \{g\}_{g \in G} \cup \{a\}$, where $\mathcal{L}$ is the language of groups and a new theory $T' = T \cup \{P_g\}_{g \in G}$, where T is the complete theory of G and P_g is the sentence $ga \in A_1$. (Since A_1 is definable, P_g is a first order statement.)

Now the problem is to prove that T' is consistent. First, take a finite subset Σ of T'. Then there exists $k \in \mathbb{N}$ such that $\Sigma \supseteq T \cup P_{g_1} \cup \ldots \cup P_{g_k}$. To prove that Σ is consistent, let's apply Lemma 2.13 for $A = A_1$ and $X_i = g_i^{-1}A_1 \cap A_1$, to get $rk(g_1^{-1}A_1 \cap \ldots \cap g_k^{-1}A_1 \cap A_1) = n$. Since we have $g_1^{-1}A_1 \cap \ldots \cap g_k^{-1}A_1 \cap A_1 \subseteq g_1^{-1}A_1 \cap \ldots \cap g_k^{-1}A_1 \subseteq G$, $rk(g_1^{-1}A_1 \cap \ldots \cap g_k^{-1}A_1) = n$. Thus one can choose $a \in \cap_{i=1}^{k} g_i^{-1}A_1$, which implies that $g_1 a \in A_1, \ldots, g_k a \in A_1$.

What we proved is: T' is finitely satisfiable. The compactness theorem implies that T' is satisfiable,and hence has a model say $G^\star$. $\{g\}_{g \in G} \subset \mathcal{L}$ implies that $G \leq G^\star$. Let $A_1^\star$ be the interpration of A_1 in $G^\star$. Then $ga \in A_1^\star$, for all $g \in G$, so $Ga \subseteq A_1^\star$ and we have $G \subseteq A_1^\star a^{-1}$.

If we assume that $d > 1$, then in particular, $A_2 \subseteq A_1^\star a^{-1}$. Also $A_2 \subseteq A_2^\star$. So, $A_2 \subseteq A_1^\star a^{-1} \cap A_2^\star$. $rk(A_2) = n$ and $rk(A_1^\star a^{-1} \cap A_2^\star) < n$ gives a contradiction. Therefore $d = 1$. $\square$

Corollary 3.19 $deg(G) = [G : G^\circ]$.

Proof: Let $[G : G^\circ] = d$ and $rk(G) = n$. Then $rk(G^\circ) = n$ and $deg(G^\circ) = 1$, since G° is connected. By Lemma 2.5, $G = \dot{\sqcup}_{i=1}^{d} g_i G^\circ$ implies that $deg(G) = \Sigma_{i=1}^{d} deg(g_i G^\circ) = \Sigma_{i=1}^{d} deg(G^\circ) = d = [G : G^\circ]$. □

3.3 Regular Subgroups of $GL_n(K)$

Throughout this section, K will stand for an algebraically closed field.

Definition: Let G be a group acting on a set X. The action is called *transitive*, if for all $x, y \in X$ there exists $g \in G$ such that $gx = y$. The action is called *sharply transitive* if for all $x, y \in X$, there exists unique $g \in G$ such that $gx = y$.

Sharply transitive actions are also called *regular actions*.

Definition: A subgroup $G \leq GL_n(K)$ is called *regular*, if G acts on $K^n \setminus \{0\}$ regularly.

The main result of this section is: If $G \leq GL_n(K)$ is a regular subgroup of finite Morley rank, then $n = 1$ and $G = K^*$ or $n = 2$. In fact, it was proved that in such a case $n \neq 2$. See [5].

$n = 1$ and G is a regular subgroup directly implies that $G = K^*$. From now on, it will be assumed that $n > 1$.

$V_\alpha(g)$ denotes the α-eigenspace of g and $C(g)$ stands for $C_G(g)$ and set $V = K^n$.

Lemma 3.20 *For $g, h \in G$,*

(i)$gh = hg$ *iff* g *and* h *have a common eigenvector.*

(ii)g *and* h *are conjugate iff* g *and* h *have a common eigenvalue.*

(iii)$Z(G)$ *acts as a group of scalars and* G *is not abelian.*

(iv)$C(g)$ *acts regularly on* $V_\alpha(g)$*, for each eigenvalue* α *of* g.

Proof: **(i)**($\Rightarrow$): Let α be an eigenvalue of g. $h(V_\alpha(g)) \subseteq V_\alpha(g)$, because if $v \in V_\alpha(g)$, then $gv = \alpha v$ and $g(hv) = hgv = h\alpha v = \alpha(hv)$ implies that $hv \in V_\alpha(g)$. Therefore, h acts on the subspace $V_\alpha(g)$ of K^n and h has an eigenvector in the subspace $V_\alpha(g)$.

($\Leftarrow$):Let v be the common eigenvector of g and h, then there exists $\alpha, \beta \in K$ such that $gv = \alpha v$ and $hv = \beta v$. This implies that $ghv = g(\beta v) = \beta gv = \beta\alpha v = \alpha\beta v = \alpha hv = h(\alpha v) = hgv$. Now, sharpness implies that $gh = hg$.

(ii)($\Rightarrow$):Let $g = h^k$ and let α be an eigenvalue of g. Then there exists a non-zero $v \in V_\alpha(g)$ such that $gv = \alpha v$. Now, $h(kv) = kgv = k\alpha v = \alpha(kv)$ implies that α is an eigenvalue of h.

($\Leftarrow$):Let α be the common eigenvalue of g and h, then there exists $v, w \in K^n$ such that $gv = \alpha v$ and $hw = \alpha w$. Regularity implies that there exists a unique $k \in G$ such that $kv = w$, then $k^{-1}hkv = k^{-1}(\alpha w) = \alpha v = gv$. Finally, sharpness implies that $k^{-1}hk = g$.

(iii) Let $z \in Z(G)$, z has eigenvectors, say $zv = \alpha v$. Fix an arbitrary $w \in K^n \setminus \{0\}$. By regularity, there exists $g \in G$ such that $gv = w$. Then we have

$zw = zgv = gzv = g\alpha v = \alpha gv = \alpha w$. Thus every vector in K^n lies in $V_\alpha(z)$ which shows that z acts scalarly.

G is not abelian, because G is regular and $n > 1$ implies that G can not act as a group of scalars.

(iv) For any $h \in C(g)$, $h(V_\alpha(g)) \subseteq V_\alpha(g)$. This was proven in part (i). For regularity, let $v, w \in V_\alpha(g)^*$, then there exists unique $h \in G$ satisfying $hv = w$. To see that $h \in C(g)$, consider $[g, h]v = g^{-1}h^{-1}ghv = g^{-1}h^{-1}\alpha w = v$. Now sharpness implies that $[g, h] = id$. Thus $gh = hg$ and $h \in C(g)$. □

For the following three lemmas, it will be assumed that $C(g)$ is abelian for all $g \in G \setminus Z(G)$. This assumption will allow us to prove Cherlin's theorem on division rings.

Lemma 3.21 **(i)** *For all $g \in G \setminus Z(G)$, if $V_\alpha(g) \neq 0$ then $dim(V_\alpha(g)) = 1$. In particular, $C(g) \cong K^*$.*

(ii) *G is divisible and has no subgroups of finite index.*

Proof: **(i)** Take an arbitrary $h \in C(g)$, then h has an eigenvector in $V_\alpha(g)$; say $hv = cv$. Take an arbitrary $w \in V_\alpha(g)$, then there exists $h_1 \in C(g)$ such that $h_1 v = w$. Then $hw = hh_1v = h_1hv = h_1cv = cw$. So every vector of $V_\alpha(g)$ is an element of $V_c(h)$. Thus for every $h \in C(g)$, h acts as a scalar on $V_\alpha(g)$.

If $dim(V_\alpha(g)) > 1$, then there exist two linearly independent elements $\beta_1, \beta_2 \in V_\alpha(g)$. Since $C(g)$ acts regularly on $V_\alpha(g)$, there exists $x \in C(g)$ such that $x\beta_1 = \beta_2$. However, $x \in C(g)$ implies that h acts as a scalar on $V_\alpha(g)$, thus $x\beta_1 = c\beta_1$, for some $c \in K$. This contradiction implies that $dim(V_\alpha(g)) = 1$.

(ii) For all $g \in G \setminus Z(G)$, $C(g) \cong K^*$ and K is algebraically closed imply that $C(g)$ is divisible. Fix an arbitrary $g \in G$ and $n \in \mathbb{N}$. If $g \in G \setminus Z(G)$, then $g \in C(g)$ implies that there exists $x \in C(g)$ such that $x^n = g$. If $g \in Z(G)$, then choose an arbitrary element $h \in G \setminus Z(G)$. We have $g \in C(h) \cong K^*$, so G is divisible.

If G has a subgroup H of finite index, then G/G^o is a nontrivial finite divisible group which is not possible. □

Notation: $N(g)$ denotes $N_G(C(g))$.

Lemma 3.22 *For any $g, h \in G \setminus Z(G)$, we have the following:*

(i) *$gh = hg$ iff they have the same set of eigenspaces.*

(ii) *$C(g) = C(h)$ or $C(g) \cap C(h) = Z(G)$.*

(iii) *$N(g)/C(g)$ acts regularly on the set of eigenspaces of g.*

(iv) *$C(g)$ and $C(h)$ are conjugate.*

Proof: **(i)** Let α be an eigenvector of g. $gh = hg$ implies that $h(V_\alpha(g)) \subseteq V_\alpha(g)$. This fact and $dim(V_\alpha(g)) = 1$ prove that $V_\alpha(g)$ is an eigenspace of h. The converse is trivial by Lemma 3.20(i).

(ii) First let's prove that to commute is an equivalance relation on $G \setminus Z(G)$. It is trivially reflexive and symetric. To see the transitivity, let $g \in C(h)$ and $h \in C(k)$, then $g, k \in C(h)$. $C(h)$ is abelian implies that $gk = kg$.

Therefore, the $C(g) \setminus Z(G)$'s partition $G \setminus Z(G)$, that is either $C(g) =$

$C(h)$ or $C(g) \cap C(h)$ is the empty set in $G \setminus Z(G)$. Since, $Z(G)$ is a subset of each centralizer, second case implies that $C(g) \cap C(h) = Z(G)$.

(iii) Let α and β be eigenvalues of g. Consider some arbitrary vectors in $v \in V_\alpha(g)^*$, $w \in V_\beta(g)^*$ and some $k \in N(g)$. kv is an eigenvector of g, since $g(kv) = (gk)v = kk^{-1}gkv$ and $k^{-1}gk \in C(g)$ implies that $k(k^{-1}gk)v = k(cv) = c(kv)$, for some $c \in K$.

G is regular implies that there exists $k \in G$ such that $kv = w$. To prove that the action is transitive, it is enough to prove that $k \in N(g)$. $k^{-1}gk(v) = k^{-1}gw = k^{-1}\beta w = \beta v$ implies that v is an eigenvector of g and g^k. Thus g and g^k commute and $g \in C(g) \cap C(g^k)$. If $C(g) \cap C(g^k) = Z(G)$, then $g \in Z(G)$, which is not the case. Now part (ii) implies that $C(g) = C(g^k)$.

$k^{-1}C(g)k = C(g^k) = C(g)$ implies that $k \in N(g)$. Since the dimensions of the eigenspaces are 1, $kV_\alpha(g) = V_\beta(g)$.

If $kV_\alpha(g) = V_\alpha(g)$, then k has a common eigenvector with g and $k \in C(g)$. This proves that $N(g)/C(g)$ acts regularly on the set of eigenspaces of g.

(iv) Let α and β be eigenvalues of g and h, respectively. Then there exist $v, w \in K^n$ such that $gv = \alpha v$ and $hw = \beta w$. Regularity implies that there exists $k \in G$ such that $kv = w$. Thus $k^{-1}hkv = k^{-1}\beta v = \beta v$, that is g and $k^{-1}hk$ have a common eigenvalue. So g and $k^{-1}hk$ commute. Since all centralizers are abelian, we have $C(g) = C(k^{-1}hk)$.

The map $\phi : G \to G$ which is defined by $\phi(g) = k^{-1}gk$ is an automorphism. Hence $\phi(C(h)) = C(\phi(h)) = C(k^{-1}hk)$. On the otherhand $\phi(C(h)) =$

$k^{-1}C(h)k$. Combining these observations, we get that $C(g) = C(k^{-1}hk) = k^{-1}C(h)k$ as desired. $\square$

Before starting the last lemma, let's recall all the assumptions once more. G is a regular subgroup of $GL_n(K)$, K is an algebraically closed field, $n > 1$ and $C(g)$ is abelian for all $g \in G \setminus Z(G)$. $C(g)$ and $N(g)$ stand for $C_G(g)$ and $N_G(C(g))$ respectively.

Lemma 3.23 **(i)** *n=2 and char* $K \neq 2$

(ii) *Every* $g \in G \setminus Z(G)$ *has two distinct eigenvalues and* $[N(g) : C(g)] = 2$.

Proof: **(i)** Lemma 3.22(iii), for any $g \in G \setminus Z(G)$, $N(g)/C(g)$ acts regularly on the set of eigenspaces of g. The action is regular implies that $|\frac{N(g)}{C(g)}|$ = number of eigenspaces of g= number of eigenvalues of g. Let's call this number m. By Lemma 3.22(iv), for any $g, h \in G \setminus Z(G)$, there exists $y \in G$ such that $C(g) = y^{-1}C(h)y$. Moreover we have $y^{-1}\frac{N(h)}{C(h)}y = \frac{N(g)}{C(g)}$. Since $N(h)/C(h)$ and $N(g)/C(g)$ are conjugate, they have the same cardinality. Therefore for any $g, h \in G \setminus Z(G)$, g and h have the same number of eigenvalues.

Now suppose that $m = 1$ and char $K = q$, then there exists $r, e \in \mathbb{N}$ such that $n = q^r e$ and $q \not| e$. For any h in the commutator subgroup G', det$h = 1$. Note that the constant term of the characteristic polynomial of h is $(-1)^n$det$h = (-1)^n$. Let a be the unique eigenvalue of h, then $(-1)^n = (-1)^n a^n$, so $a^n = 1$. char $K = q$ implies that $a^e = 1$. Now $h^e v = a^e v = v$ and sharpness implies that $h^e = 1$.

char $K = q$ implies that the polynomial $x^e - 1$ does not have any multiple roots. Thus the minimal polynomial of h does not have any multiple roots which implies that h is diagonalizable. h has one eigenvalue implies that h is similar to a scalar matrix. Thus h is a scalar matrix. Therefore there is a bijection between G' and the set of e-th roots of unity. Thus G' is finite. Elements of G' are scalar matrices implies that $G' \leq Z(G)$.

Fix an arbitrary $x \in G$ and consider the map $\theta : G \to G'$, defined by $\theta(y) = x^{-1}y^{-1}xy$. θ is a homomorphism, since $\theta(y)\theta(y_1) = x^{-1}y^{-1}xyx^{-1}y_1^{-1}xy_1 = x^{-1}y_1^{-1}x(x^{-1}y^{-1}xy)y_1 = x^{-1}(yy_1)^{-1}xyy_1 = \theta(yy_1)$. G is divisible implies that its homomorphic image $[x, G]$ is divisible. G' is finite implies that $[x, G]$ is the identity group for all $x \in G$. Thus $G = Z(G)$. This contradicts with Lemma 3.20(iii).

Therefore $m > 1$ and $|\frac{N(g)}{C(g)}| = m > 1$. Let p be a prime number which divides m, then there exists $h \in N(g) \setminus C(g)$ such that $h^p \in C(g)$. $h \in C(h)$ but $h \notin C(g)$ implies that $C(g) \neq C(h)$, so $C(g) \cap C(h) = Z(G)$ by Lemma 3.22(ii). Thus $h^p \in Z(G)$, that is h^p is a scalar.

h has m distinct eigenvalues, say $\alpha_1, \ldots, \alpha_m$. For each $i = 1, \ldots, m$, α_i^p is an eigenvalue of h^p and h^p has only one eigenvalue. Hence $\alpha_i^p = \alpha_j^p$ for all $i, j = 1, \ldots, m$. If char $K = p$, then $\alpha_i = \alpha_j$, which is a contradiction. Thus char $K \neq p$.

For any $x \in C(g)$, $h^{-1}xh \in C(g)$, since $h \in N(g)$. Thus $\sigma : C(g) \to C(g)$, $\sigma(x) = h^{-1}xh$ is an automorphism and $\sigma^p(x) = h^{-p}xh^p = x$ implies that $\sigma^p = id$.

Now let U_p be the set of p-elements of $C(g) \cong K^*$. Then $U_p \cong \mathbb{Z}(p^\infty)$. To see that $\sigma(U_p) \subseteq U_p$, let $x \in U_p$, then for some $r \in \mathbb{N}$, $x^{p^r} = 1$ and $\sigma(x)^{p^r} = \sigma(x^{p^r}) = h^{-1}x^{p^r}h = 1$. This proves that $\sigma(x) \in U_p$.

Now suppose that $p > 2$. Then $Aut(\mathbb{Z}(p^\infty)) \cong \mathbb{Z}_p^*$ has no elements of order p. For the proof see [1]. Thus $\sigma = id$ on U_p. By definition, $U_p \subseteq C(g)$ and $\sigma = id$ implies that $h^{-1}xh = x$ for all $x \in U_p$. So $U_p \subseteq C(h)$. Thus $U_p \subseteq C(g) \cap C(h) = Z(G)$.

Claim: $Z(G)$ is p-divisible.

Let $a \in Z(G) \leq C(g)$. By Lemma 3.21(i), $C(g)$ is p-divisible, so there exists $x \in C(g)$ such that $x^p = a$ and there exists $y \in C(g)$ such that $y^p = x$. Now our aim is to show that $y^p \in Z(G)$. Define $z = \sigma(y)y^{-1}$. Then $z^{p^2} = \sigma(y^{p^2})y^{-p^2}$, since $\sigma(y), y^{-1} \in C(g) \cong K^*$ which is abelian. Now $z^{p^2} = \sigma(y^{p^2}y^{-p^2}) = \sigma(x^p)x^{-p} = x^p x^{-p} = 1$, since σ is identity on $Z(G)$. Thus $z \in U_p \leq Z(G)$ and $\sigma(z) = z$. $\sigma(y) = zy$ implies that $\sigma^2(y) = \sigma(z)\sigma(y) = zzy = z^2y$. Thus $\sigma^p(y) = z^py$. But at the same time $\sigma^p(y) = y$, so $z^p = 1$ which implies that $z \in U_p \leq Z(G)$. So we have $\sigma(y^p) = \sigma(y)^p = (zy)^p = z^py^p = y^p$. This implies that $y^p \in C(h)$. Also $y \in C(g)$ implies that $y^p \in C(g)$. Therefore $y^p \in C(g) \cap C(h) = Z(G)$ which proves that $Z(G)$ is p-divisible.

Since $h^p \in Z(G)$, h^p acts like a scalar, that is there exists $\lambda \in K$ such that $h^pv = \lambda v$. $Z(G)$ is p-divisible implies that there exists $z \in Z(G)$ such that $z^pv = \lambda v$. By replacing h with $z^{-1}h$, one can assume that $h^p = 1$.

Let $v \neq 0$ be an eigenvector of g, so for some $c \in K$, $gv = cv$. Let $w = v + hv + \ldots + h^{p-1}v$, then $h(w) = hv + h^2v + \ldots + h^pv = hv + \ldots + h^{p-1}v + v = w$.

Claim: $w \neq 0$.

Lemma 3.22(iii) implies that $N(g)/C(g)$ acts regularly on the set of eigenspaces of g. So $hv \notin < v >$, because of sharpness. Moreover, all h^iv's belong to different eigenspaces, for $i = 0, \ldots, p-1$. A fact from linear algebra implies that sum of the vectors from different eigenspaces are linearly independent, so their sum can not be zero. Thus $w \neq 0$.

Now h fixes a nonzero vector. But then sharpness implies a contradiction. Thus $p = 2$. $p = 2$ implies that char $K \neq 2$ and $h^2 \in Z(G)$. Hence h^2 acts like a scalar, say λ. Then we have $h^2 - \lambda I = 0$, which shows that the minimal polynomial of h divides the polynomial $x^2 - \lambda$. Therefore h has at most two distinct eigenvalues; i.e. $m \leq 2$. Since $m > 1$ was assumed, necessarily $m = 2$. h has two distinct eigenvalues and the degree of the minimal polynomial of h is less than or equal to 2 imply that h is diagonalizable. The sum of the dimension of eigenspaces of h is equal to the dimension of V, which is n. Lemma 3.21(i) implies that the dimension of each eigenspace is 1, so we can conclude that $n = 2$.
$\square$

Here is the exact statement of the fact that will be used to prove Cherlin's theorem on division rings.

Theorem 3.24 *Let G be a regular subgroup of $GL_n(K)$ of finite Morley rank, where K is an algebraically closed field. Suppose that $C_G(g)$ is abelian for all $g \in G \setminus Z(G)$. Then one of the following possibilities holds.*

(i) $n = 1$ *and* $G = K^*$.

(ii) $n = 2$ *and* $[N_G(C_G(g)) : C_G(g)] = 2$.

CHAPTER 4

FIELDS AND RINGS OF FINITE MORLEY RANK

4.1 Fields of Finite Morley Rank

4.1.1 Galois Theory

Throughout this section, the letters E, F, K, L, L' will stand for arbitrary fields.

Definition: An extension $F : K$ is called *algebraic*, if every element $a \in F$ is a root of a polynomial $f(x) \in K[x]$.

Lemma 4.1 *If $F : K$ is a finite extension, then it is algebraic.*

Proof: Assume that $[F : K] = n$, then for any $a \in F$, the set $\{1, a, a^2, \ldots, a^n\}$ is linearly dependent over K; i.e. there exists $c_0, \ldots, c_n \in K$ such that $c_0 1 + c_1 a + \ldots + c_n a^n = 0$. Thus a is the root of the polynomial $f(x) = c_0 + c_1 x + \ldots + c_n x^n \in K[x]$, which implies that a is algebraic over K, for any $a \in F$. □

Definition: An irreducible polynomial $f(x) \in K[x]$ is called *separable*, if the roots of f are pairwise different. An arbitrary polynomial $p(x) \in K[x]$ is called *separable*, if its irreducible factors are separable.

Lemma 4.2 *Let $f(x) \in K[x]$ and $a \in K$ be a root of $f(x)$. Then a is a multiple root of $f(x)$ iff $f'(a) = 0$.*

Proof: ($\Rightarrow$): If a is a multiple root of $f(x)$, then there exists $h(x) \in K[x]$ such that $f(x) = (x-a)^2h(x)$ and $f'(x) = 2(x-a)h(x)+(x-a)^2h'(x)$. Then $f'(a) = 0$.

($\Leftarrow$):a is a root of $f(x)$ implies that there exists $g(x) \in K[x]$ such that $f(x) = (x-a)g(x)$. So $f'(x) = g(x) + (x-a)g'(x)$ and $f'(a) = 0$ implies that $g(a) = 0$. Thus there exists $h(x) \in K[x]$ such that $g(x) = (x-a)h(x)$ and $f(x) = (x-a)^2h(x)$ which proves that a is a multiple root of $f(x)$. □

Lemma 4.3 *Let $f(x) \in K[x]$ be an irreducible polynomial. Then the following are equivalent:*

(i) *$f(x)$ is separable*

(ii) *$gcd(f(x), f'(x)) = 1$ in $K[x]$*

(iii) $f'(x) \neq 0$.

Proof: **(i)$\Rightarrow$(ii):** Assume that $gcd(f(x), f'(x)) = h(x) \neq 1$. Then $deg\ h \geq 1$. Then $h(x)$ has a root x_0 in some extension field L of K. There exists $f_1(x), g_1(x) \in K[x]$ such that $f(x) = h(x)f_1(x)$ and $f'(x) = h(x)g_1(x)$. Now, we have $f(x_0) = f'(x_0) = 0$. Lemma 4.2 implies that x_0 is not a simple root of $f(x)$. Thus, $f(x)$ is not separable.

(ii)⇒(iii): If $f'(x) = 0$, then $gcd(f(x), f'(x)) = f(x) \neq 1$.

(iii)⇒(i): $f(x)$ is irreducible, $f'(x) \neq 0$ and $deg\ f > deg\ f'$, so $gcd(f(x), f'(x)) = 1$. Now there exists $g(x), h(x) \in K[x]$ such that $1 = f(x)g(x) + f'(x)h(x)$. Let x_0 be an arbitrary root of $f(x)$ in some extension field of K. Then we have $1 = f'(x_0)h(x_0)$, and so $f'(x_0) \neq 0$. Lemma 4.2 implies that x_0 is a simple root of $f(x)$. Hence $f(x)$ is separable. □

Definition: K is called a *perfect field*, if either char $K = 0$ or char $K = p$ and $K = K^p$.

Theorem 4.4 *Let K be a perfect field and $p(x) \in K[x]$ be an irreducible polynomial. Then $p(x)$ is separable.*

Proof: First consider the case when char K=0. Let $p(x) \in K[x]$ be an irreducible polynomial. If $deg\ p \leq 1$, then there is nothing to prove. If $deg\ p > 1$, then $p'(x) \neq 0$. Lemma 4.3 implies that $p(x)$ is separable.

Now consider the case when char K=p. Since K is perfect, in this case we have $K = K^p$. Let $p(x) = \Sigma_{i=0}^{n} a_i x^i$ be an irreducible polynomial over K and assume that $p(x)$ is not separable. Then Lemma 4.3 implies that $p'(x) = 0$; i.e. $\Sigma_{i=1}^{n} i a_i x^{i-1} = 0$ and so $i a_i = 0$ for all $i = 1, \dots, n$. Thus $a_i = 0$ or $p|i$ for all $i = 1, \dots, n$. Therefore $p(x) = \Sigma_{i=0}^{k} a_{pi} x^{pi}$. Since $K = K^p$, for each a_{pi}, there exists $b_i \in K$ such that $b_i^p = a_{pi}$. Thus, $p(x) = \Sigma_{i=0}^{k} b_i^p x^{pi} = (\Sigma_{i=0}^{k} b_i x^i)^p$. This contradicts the fact that $p(x)$ is irreducible. So necessarily $p(x)$ is separable. □

Definition: Let $p(x) \in K[x]$ be a nonconstant polynomial. A field extension L of K is called *a splitting field over K of $p(x)$* if p can be written as a product of linear factors in $L[x]$ and L is minimal with this property. In other words, $L = K(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are the roots of p.

Lemma 4.5 *Let $f(x) \in K[x]$ and deg $f = n \geq 1$. Then there exists a splitting field F of $f(x)$ satisfying $[F : K] \leq n!$.*

Proof: See Theorem V.3.2 in [6].

Lemma 4.6 *Let $f(x) \in K[x]$, σ be an automorphism of K and L be a splitting field over K of the polynomial $f(x)$. Then σ can be extended to an automorphism of L.*

Proof: See Theorem V.3.8 in [6].

Notations: $Aut_K L = \{\sigma \in Aut(L) : \sigma(k) = k$, for all $k \in K\}$, where $K \subseteq L$ and $G^L = \{x \in L : g(x) = x$, for all $g \in G\}$, where $G \subseteq Aut_K L$.

Definition: Let L be a finite extension of K. L is called a *Galois extension* of K if $(Aut_K L)^L = K$.

Theorem 4.7 *If L is a splitting field over K of a separable polynomial $f(x) \in K[x]$, then $L : K$ is a Galois extension.*

Proof: Lemma 4.5 directly implies that the extension is finite. To show that the extension is Galois, it is enough to prove that $(Aut_K L)^L \subseteq K$.

The proof will be done by induction on r=number of roots of $f(x)$ not lying in K. $r = 0$ implies that all roots lie in K. So $L = K$ and $[L : K]$ is Galois.

Now let $r > 0$, then for some $a \in L \setminus K$. $f(a) = 0$. Let $g(x)$ be the minimal polynomial of a over K. Then, $f(x) = g(x)h(x)$ for some $h(x) \in K[x]$. $f(x)$ is still separable as a polynomial over $K(a)[x]$ and L is the splitting field of $f(x)$ over $K(a)$. By induction hypothesis, $[L : K(a)]$ is Galois, so $(Aut_{K(a)}L)^L = K(a)$. $Aut_{K(a)}L \subset Aut_K L$ implies that $(Aut_K L)^L \subset (Aut_{K(a)}L)^L = K(a)$.

Thus, any $y \in (Aut_K L)^L$ can be written as $y = k_0 + k_1 a + \ldots + k_{s-1}a^{s-1}$, where $s = deg\ g$. $f(x)$ is separable implies that $g(x)$ is separable. So $g(x)$ has s distinct roots, say $a = a_1, \ldots, a_s$. The maps $\phi_i : K(a) \to K(a_i)$ defined by $\phi_i(k) = k$ for all $k \in K$ and $\phi_i(a) = a_i$ are field isomorphisms for all $i = 1, \ldots, s$.

Lemma 4.6 implies that each ϕ_i can be extended to an automorphism Φ_i of L, so $\Phi_i \in Aut_K L$. Thus, we have $k_0 + k_1 a_i + \ldots + k_{s-1}a_i^{s-1} = \Phi_i(y) = y = k_0 + k_1 a + \ldots + k_{s-1}a^{s-1}$. Define $p(x) = k_{s-1}x^{s-1} + \ldots + k_1 x + k_0 - y$, then $p(a_i) = 0$ for all $i = 1, \ldots, s$. $deg\ p \leq s - 1$ and $p(x)$ has s distinct roots implies that $p(x) = 0$. Thus, $y = k_0 \in K$. $\square$

Definition: Let S be a nonempty set of homomorphisms from L into L'. S is called *linearly independent* if for any $k_1, \ldots, k_n \in L'$ and $s_1, \ldots, s_n \in S$, we have:

$k_1 s_1(x) + \ldots\ldots + k_n s_n(x) = 0$ for all $x \in L$ implies that $k_i = 0$ for all $i = 1, \ldots, n$.

Lemma 4.8 *Let S be an arbitrary subset of homomorphisms from L into L', then S is linearly independent.*

Proof: If S is not linearly independent, then there exist $a_1, \ldots, a_n \in L'$ and $\sigma_1, \ldots, \sigma_n \in S$ such that

$$a_1\sigma_1(u) + \ldots\ldots + a_n\sigma_n(u) = 0, \text{ forall } u \in L. \qquad (4.1)$$

Among all such relations, choose the one with n minimal. Clearly, $n > 1$. $\sigma_1 \neq \sigma_2$ implies that there exists $v \in L$ such that $\sigma_1(v) \neq \sigma_2(v)$. Now, we have

$$a_1\sigma_1(u)\sigma_1(v) + \ldots\ldots + . a_n\sigma_n(u)\sigma_n(v) = 0. \tag{4.2}$$

Multiply Equation 4.1 with $\sigma_1(v)$ to get

$$a_1\sigma_1(u)\sigma_1(v) + \ldots\ldots + a_n\sigma_n(u)\sigma_1(v) = 0. \tag{4.3}$$

The difference of Equation 4.2 and Equation 4.3 gives

$$a_2[\sigma_2(v) - \sigma_1(v)]\sigma_2(u) + \ldots\ldots + a_n[\sigma_n(v) - \sigma_1(v)]\sigma_n(u) = 0$$

Now $a_2 \neq 0$ and $\sigma_2(v) \neq \sigma_1(v)$. This contradicts with the minimality of n.

Therefore S is linearly independent. □

Lemma 4.9 *Let L and L' be finite extensions of K. There are at most $[L : K]$ distinct field homomorphisms defined from L into L' fixing K. In particular, $|Aut_K L| \leq [L : K]$.*

Proof: Denote the set of K-linear maps from L into L' by $Hom_K(L, L')$. In fact, $Hom_K(L, L')$ is a vector space over L'. Let $\sigma_1, \ldots, \sigma_n$ be distinct field homomorphisms from L to L'. Lemma 4.8 implies that the set $\{\sigma_1, \ldots, \sigma_n\}$ is linearly independent, so $n \leq dim_{L'}(Hom_K(L, L'))$.

Now our aim is to prove that $[L : K] = dim_{L'}(Hom_K(L, L'))$. Let $\{a_1, \ldots, a_m\}$ be a K-basis for L and define $f_i \in Hom_K(L, L')$ such that

$$f_i(a_j) = \begin{cases} 1_{L'} & \text{if i=j} \\ 0_{L'} & \text{if i}\neq\text{j.} \end{cases}$$

If $\Sigma_{i=1}^{m} b_i f_i = 0$ for some $b_i \in L'$, then $b_j = \Sigma_{i=1}^{m} b_i f_i(a_j) = 0$, so $b_j = 0$ for all $j = 1, \ldots, m$. Hence $\{f_1, \ldots, f_m\}$ is linearly independent.

If $f \in Hom_K(L, L')$, then $f = \Sigma_{i=1}^{m} f(a_i) f_i$. Therefore $\{f_1, \ldots, f_m\}$ is an L'-basis for $Hom_K(L, L')$, which proves that $[L : K] = dim_{L'}(Hom_K(L, L'))$. □

Lemma 4.10 *Let G be a finite subgroup of $Aut(L)$ and $K = G^L$. Then,*

(i) $[L : K] = |G|$

(ii) $Aut_K L = G$.

Proof: **(i)** Lemma 4.9 implies that $[L : K] \geq |Aut_K L| \geq |G|$. Thus it is enough to prove that $[L : K] \leq |G|$. Let $|G| = n$ and $G = \{\sigma_1, \ldots, \sigma_n\}$. Now our aim is to show that any n+1 elements in L are linearly dependent over K. That will imply $[L : K] < n + 1$.

Let $a_0, \ldots, a_n \in L$ and consider the system of equations:

$$\sigma_i^{-1}(a_0)x_0 + \ldots\ldots + \sigma_i^{-1}(a_n)x_n = 0, \text{ for } i = 1, \ldots, n.$$

There are n equations in n+1 variables, so there exists a nontrivial solution of this system, say $b_0, \ldots, b_n$. But then, for any $c \in L$, $cb_0, \ldots, cb_n$ is also a solution for the system. Assume that $b_0 \neq 0$. Apply σ_i to the equation:

$$\sigma_i^{-1}(a_0)cb_0 + \ldots\ldots + \sigma_i^{-1}(a_n)cb_n = 0$$

to get :

$$\Sigma_{k=0}^{n} a_k \sigma_i(cb_k) = 0 \text{ for all } i = 1, \ldots, n.$$

When we take the sum over all equations, we will get:

$$\Sigma_{k=0}^{n} a_k \Sigma_{i=1}^{n} \sigma_i(cb_k) = 0.$$

$\Sigma_{i=1}^{n} \sigma_i(cb_k) \in G^L = K$ for all $c \in L$ implies that the map $\Sigma_{i=1}^{n} \sigma_i$ is defined from L into K. Since, σ_i's are linearly independent, this map is nonzero. Thus, there exists $c \in L$ such that $\Sigma_{i=1}^{n} \sigma_i(cb_0) \neq 0$. Therefore, $\{a_0, \dots, a_n\}$ is a linearly dependent set.

(ii) Trivially, $G \subseteq Aut_K L$. Part (i) implies that $|G| = [L : K] \geq |Aut_K L|$. Thus, $G = Aut_K L$. □

Lemma 4.11 *If $L : K$ is a finite extension then the following are equivalent:*

(i) *$L : K$ is a Galois extension*

(ii) $|Aut_K L| = [L : K]$.

Proof:(i)⇒(ii): $Aut_K L$ is a finite subgroup of $Aut(L)$ and by our assumption $(Aut_K L)^L = K$. Thus Lemma 4.10(i) can be applied for $G = Aut_K L$ to get $[L : K] = |Aut_K L|$.

(ii)⇒(i): Let $G = Aut_K L$. By Lemma 4.10 and the assumption, we have $[L : G^L] = |G| = [L : K]$. $K \subseteq G^L$ implies that $[L : G^L][G^L : K] = [L : K]$ and $[G^L : K] = 1$, so $G^L = K$; i.e. $(Aut_K L)^L = K$ implies that $L : K$ is a Galois extension. □

Theorem 4.12 *If $L : K$ is a Galois extension and E is an intermediate field, then $L : E$ is also a Galois extension.*

Proof: In order to show that $L : E$ is Galois, it is enough to prove that $|Aut_E L| \geq [L : E]$. First let's see that

$$|\{\sigma|_E : \sigma \in Aut_K L\}| = \frac{|Aut_K L|}{|Aut_E L|} \tag{4.4}$$

Note that $\sigma \in Aut_K L$ implies that $\sigma|_E$ is a field homomorphism from E to L and its restriction to K is the identity map.

For $\sigma, \tau \in Aut_K L$, $\sigma|_E = \tau|_E$ iff $\tau^{-1}\sigma|_E = id|_E$ iff $\tau^{-1}\sigma \in Aut_E L$ iff $\tau Aut_E L = \sigma Aut_E L$. Thus Equation 4.4 is proven.

Lemma 4.9 implies that $|\{\sigma|_E : \sigma \in Aut_K L\}| \leq |\{\sigma : E \to L : \sigma$ is an embedding and $\sigma|_K = id\}| \leq [E : K]$. When we combine this result with Equation 4.4 we get: $[E : K] \geq |\{\sigma|_E : \sigma \in Aut_K L\}| = \frac{|Aut_K L|}{|Aut_E L|}$, so $|Aut_E L| \geq \frac{|Aut_K L|}{[E:K]} = \frac{[L:K]}{[E:K]} = [L : E]$.

Thus $L : E$ is Galois. □

Theorem 4.13 *Let $L : K$ be a Galois extension. Then the map defined from the set of subgroups of $Aut_K L$ into the intermediate fields of L and K that sends U to U^L is a bijection. The inverse of this map takes E to $Aut_E L$.*

Proof: It is enough to prove that $Aut_{U^L} L = U$, for any $U \leq Aut_K L$ and $(Aut_E L)^L = E$, for any intermediate field E.

For the first one; since U is finite, we can apply Lemma 4.10(ii) for $G = U$ and get $Aut_{U^L} L = U$.

For the second one; $E \subseteq (Aut_E L)^L$ clearly holds. By Theorem 4.12, $L : E$ is Galois. Therefore $[L : E] = |Aut_E L| = [L : (Aut_E L)^L]$ by Lemma 4.10(i). Thus $E = (Aut_E L)^L$. □

Definition: An extension L of K is said to be a *cyclic extension of degree n*, if $L : K$ is a Galois extension and $Aut_K L$ is a finite cyclic group of order n.

Theorem 4.14 *Let char K=p, where p is a prime number and F be a cyclic extension of K of degree p. Then there exists $\alpha \in F$ such that $F = K(\alpha)$ and $\alpha^p - \alpha - a = 0$ for some $a \in K$.*

Proof: Let σ be the generator of the cyclic group $Aut_K F$. Now we will see that there exists $\alpha \in F$ such that $\sigma(\alpha) - \alpha = 1_K$. Let's agree to use $T_i(x)$ as an abbreviation for $x + \sigma(x) + \ldots + \sigma^i(x)$ for any $x \in F$ and for all $i \in \{0, 1, \ldots, p-1\}$. Lemma 4.8 implies that there exists $z \in F$ such that $T_{p-1}(z) \neq 0$. Now let $w = -(T_{p-1}(z))^{-1} z$ and $\alpha = \Sigma_{i=0}^{p-2}(i+1)\sigma^i(w)$. Then $T_{p-1}(w) = -1_K$ and $\sigma(\alpha) - \alpha = \Sigma_{i=0}^{p-2}(i+1)\sigma^{i+1}(w) - \Sigma_{i=0}^{p-2}(i+1)\sigma^i(w)$

$$= \Sigma_{i=1}^{p-1}(i)\sigma^i(w) - \Sigma_{i=0}^{p-2}(i+1)\sigma^i(w)$$

$$= -1.\sigma^0(w) - \Sigma_{i=1}^{p-2}\sigma^i(w) + (p-1)\sigma^{p-1}(w)$$

$$= -\Sigma_{i=0}^{p-1}\sigma^i(w) = -T_{p-1}(w) = 1_K.$$

$\sigma(\alpha) - \alpha = 1$ implies that $\sigma(\alpha) = \alpha + 1 \neq \alpha$, so $\alpha \notin K$.

$[F : K] = p$ implies that there are no intermediate fields, so necessarily $F = K(\alpha)$. In order to show that $\alpha^p - \alpha \in K$, it is enough to prove that σ fixes $(\alpha^p - \alpha)$. $\sigma(\alpha^p - \alpha) = \sigma(\alpha^p) - \sigma(\alpha) = (\sigma(\alpha))^p - \sigma(\alpha) = (\alpha+1)^p - (\alpha+1) = \alpha^p - \alpha$.

Thus there exists $a \in K$ such that $\alpha^p - \alpha = a$. Hence $\alpha^p - \alpha - a = 0$ for some $a \in K$. □

Theorem 4.15 *Assume that K contains a primitive n-th root of unity ζ and F is a cyclic extension of K of degree n. Then there exists $\alpha \in F$ such that $F = K(\alpha)$ and $\alpha^n - a = 0$, for some $a \in K$.*

Proof: Let σ be the generator of the cyclic group $Aut_K F$. Now we will see that there exists $\alpha \in F$ such that $\zeta = \alpha^{-1}\sigma(\alpha)$. Since $\{\sigma^i\}_{i=0}^{n-1}$'s are distinct automorphisms of F, by Lemma 4.8, there exists $y \in F$ such that $\Sigma_{i=0}^{n-1}\zeta^{i+1}\sigma^i(y)$ is nonzero. Call this element α^{-1}.

$$\sigma(\alpha^{-1}) = \Sigma_{i=0}^{n-1}\zeta^{i+1}\sigma^{i+1}(y) = \Sigma_{i=1}^{n}\zeta^i\sigma^i(y)$$

$$= \Sigma_{i=1}^{n-1}\zeta^i\sigma^i(y) + \zeta^n\sigma^n(y) = \Sigma_{i=1}^{n-1}\zeta^i\sigma^i(y) + y$$

$$= y + \Sigma_{i=1}^{n-1}\zeta^i\sigma^i(y) = \Sigma_{i=0}^{n-1}\zeta^i\sigma^i(y) = \zeta^{-1}\Sigma_{i=0}^{n-1}\zeta^{i+1}\sigma^i(y) = \zeta^{-1}\alpha^{-1}.$$

Thus $\sigma(\alpha) = \zeta\alpha$ which implies that $\sigma(\alpha^n) = \sigma(\alpha)^n = \zeta^n\alpha^n = \alpha^n$. So $\alpha^n \in K$; i.e., $\alpha^n = a$ for some $a \in K$ and $\alpha^n - a = 0$ for some $a \in K$. □

4.1.2 Macintyre's Theorem

Lemma 4.16 *An infinite division ring of finite Morley rank, considered as an additive group, is connected.*

Proof: Let D be a division ring of finite Morley rank. Consider D as an additive group and let D° be the connected component of D. $I = \cap_{k\in D} kD^\circ$ is a left ideal of D, so $I = D$ or $I = \{0\}$. By descending chain condition, we can express this left ideal as $\cap_{k\in D} kD^\circ = \cap_{i=1}^{n} k_i D^\circ$. Since $[D : D^\circ]$ is finite, $[D : \cap_{i=1}^{n} k_i D^\circ]$ is also finite. D is infinite and $[D : I]$ is finite imply that $I \neq \{0\}$. So necessarily $I = D$; i.e. $\cap_{k\in D} kD^\circ = D$.

Therefore $kD^\circ = D$ for all $k \in D$. In particular we have $D^\circ = D$ by choosing $k = 1$. Thus D is connected. $\square$

Lemma 4.17 *If K is an infinite field of finite Morley rank, then K^+ and K^* are connected groups.*

Proof: K^+ is connected by Lemma 4.16. Thus $deg(K) = 1$ by Theorem 3.18. Now Lemma 2.7 implies that $deg(K^*) = 1$. Applying Theorem 3.18 once more, we get that K^* is connected. $\square$

Theorem 4.18 (Macintyre) *An infinite field of finite Morley rank is an algebraically closed field.*

Proof: Let L be an infinite field of finite Morley rank. Let's first see that L is perfect. Assume that char $L = q$ and consider the homomorphism $\phi : L^* \to L^*$ of multiplicative groups defined by $a \mapsto a^q$. $x^q - 1 = 0$ has at most q solutions in L^* and so $Ker\phi$ is finite. Lemma 4.17 and Lemma 3.7 imply that ϕ is onto, so $L^q = L$.

Now let's assume that L is not algebraically closed and try to get a contradiction. $L[x]$ must contain an irreducible polynomial $p(x)$ of degree $n > 1$. Let L' be the splitting field of $p(x)$ over L, then $[L' : L] \leq n!$. Now Theorem 4.4 implies that $p(x)$ is separable and Theorem 4.7 implies that L' is a Galois extension of L, which is different from L.

Now let E be a finite field extension of L, then one can view E as a structure defined in L as follows:

Let $[E : L] = d < \infty$, then E can be considered as a vector space of dimension d over L. Fix a basis for E. Now E and L^d are isomorphic as additive groups. The multiplication rule of the basis elements determines the multiplication rule of the whole extension field. The multiplication rule for the basis elements of L^d can be defined so that the resulting structure is isomorphic to E.

Since the multiplication rule for the basis elements of L^d can be expressed by finite number of statements containing parameters only from the field L, L^d can be viewed as a structure defined in L, so is E.

Thus we have $rk(E) \leq [E : L]rk(L)$ and so E is of finite Morley rank. Hence any finite extension of a field of finite Morley rank is also of finite Morley rank.

We have thus shown that if L is an infinite field of finite Morley rank which is not algebraically closed, then there exists a Galois extension L' of L such that L' also has finite Morley rank.

Now consider all pairs (F, K) satisfying: $F : K$ is a (finite) Galois extension and F is an infinite field of finite Morley rank. Among all such pairs, choose one with minimal degree and call it (F, K). $[F : K]$ contains no intermediate fields. Otherwise we will get a contradiction with the minimality assumption by Theorem 4.12. Theorem 4.13 implies that $Aut_K F$ has no nontrivial subgroups. Thus $Aut_K F$ is cyclic of prime order, say p.

We have two cases: Either char $K = p$ or char $K \neq p$.

If char $K = p$, then Theorem 4.14 implies that there exists $\alpha \in F$ such that $F = K(\alpha)$ and $\alpha^p - \alpha - a = 0$ for some $a \in K$. thus $Irr(\alpha, K) | x^p - x - a$. In fact, $Irr(\alpha, K) = x^p - x - a$, since $p = [F : K] = [K(\alpha) : K] = deg\ Irr(\alpha, K)$.

Claim: $x^p - x - a = 0$ has a solution in K.

To prove the claim, consider the map $\theta : K \to K$ defined by $k \mapsto k^p - k$. θ is a homomorphism of the additive groups and $Ker\theta$ is finite. Thus θ is onto, by Lemma 4.17 and Lemma 3.7. In particular, there exists $c \in K$ such that $\theta(c) = a$, so $c^p - c - a = 0$. This proves the claim.

Therefore $Irr(\alpha, K)$ is reducible in K. Thus we reached a contradiction in this case.

If char$K \neq p$, then $gcd(p, char\ K) = 1$. In order to apply Theorem 4.15, we have to show that K contains a primitive p-th root of unity. Let L be the splitting field of $g(x) = x^p - 1$ over K. Then $L = K(u)$ for some primitive p-th root of unity u. $x^p - 1 = (x - 1)g_0(x)$, where $g_0(x) = x^{p-1} + \ldots + x + 1$. $Irr(u, K) | g_0(x)$ implies that $[L : K] = deg(Irr(u, K)) \leq p - 1$. The minimality assumption forces that $L = K$. Thus K contains all p-th roots of unity. Now Theorem 4.15 implies that there exists $\alpha \in F$ such that $F = K(\alpha)$ and $\alpha^p - a = 0$. Then $Irr(\alpha, K) = x^p - a$ for some $a \in K$. Define a homomorphism $\varphi : K^* \to K^*$ such that $\varphi(k) = k^p$. $Ker\varphi$ is finite and so φ is onto. Thus $Irr(\alpha, K)$ is reducible over K, which is again a contradiction.

Since we got contradictions in both cases, K must be algebraically closed. $\square$

4.2 Rings of Finite Morley Rank

When we say that R is a ring, we mean that it is associative and distributive, but does not necessarily contain a multiplicative identity.

Lemma 4.19 *If R is a ring without any zero divisors of finite Morley rank, then R is a division ring.*

Proof: To show that R is a division ring, it is enough to prove that R contains a multiplicative identity and every nonzero element has a multiplicative inverse.

Let a be a nonzero element of R satisfying $aR \neq R$, then a^nR is a definable additive subgroup of R for all $n \geq 1$ and $a^nR \neq a^{n+1}R$. To see the last part, assume that $a^nR = a^{n+1}R$, for some $n \geq 1$. Then, for all $r \in R$, there exists $r' \in R$ such that $a^nr = a^{n+1}r'$, so $a^n(r - ar') = 0$, which implies that $a^n = 0$ or $r = ar'$. Then $a = 0$ or $r = ar'$. The contradiction implies that $a^nR \neq a^{n+1}R$, for all $n \geq 1$. However we have obtained a descending chain of definable subgroups which is not stationary. This contradicts the descending chain condition. Thus for all $a \in R^*$, $aR = R$ and by symmetry for all $a \in R^*$, $Ra = R$.

Fix an arbitrary element $a \in R$, then there exists $b \in R$ such that $ab = a$. To see that b is the unique element with this property, let $ab' = a$,for some $b' \in R$. But then $ab' = ab = a$ which implies that $ab' - ab = a(b' - b) = 0$, so $b' = b$. Let $r \in R^*$ be an arbitrary element, then there exists $s \in R$ such that $r = sa$ which implies that $rb = sab = sa = r$, so $rb = r$ for all $r \in R$. Similarly, there exists $b' \in R$ such that $b'r = r$,for all $r \in R$. Finally, $b = b'b = b'$, which shows that R has a multiplicative identity.

Since an arbitrary element $r \in R$ satisfies $rR = R = Rr$, there exist $r_1, r_2 \in R$ such that $rr_1 = 1 = r_2r$ which implies that $r_1 = r_2rr_1 = r_2$, so $r_1 = r_2$. Therefore all nonzero elements have an inverse in R.

Thus R is a division ring. ▭

Lemma 4.20 *Let D be an infinite division ring of finite Morley rank.*

(i) *If K is an infinite subfield of D, then D is a finite dimensional vector space over K.*

(ii) *If $Z(D)$ is an infinite algebraically closed field, then $Z(D) = D$.*

Proof:(i) Let $m > rk(D)$ and assume that D has m linearly K- independent elements $z_1, \ldots, z_m$. There is a bijection between

$$D \supseteq Kz_1 + \ldots\ldots + Kz_m \leftrightarrow Kz_1 \times \ldots\ldots \times Kz_m.$$

Thus we have

$$m > rk(D) \geq rk(Kz_1 + \ldots + Kz_m) = rk(Kz_1 \times \ldots \times Kz_m)$$

$$= \Sigma_{i=1}^{m} rk(Kz_i) = \Sigma_{i=1}^{m} rk(K) \geq m.$$

This is a contradiction, and so D has at most $rk(D)$ linearly K- independent elements.

Thus $dim_K D \leq rk(D) < \infty$.

(ii) Let L be the smallest field containing $Z(D)$ and some element $a \in D \setminus Z(D)$. Part (i) implies that $[D : Z(D)] < \infty$, and so $[L : Z(D)] < \infty$. Thus $L : Z(D)$

is an algebraic extension by Lemma 4.1. Hence a is a root of a polynomial $p(x) \in Z(D)[x]$.

Therefore $a \in Z(D)$, since $Z(D)$ is algebraically closed by assumption.

□

Lemma 4.21 (Reineke) *Let G be a group of finite Morley rank such that $G = x^G \cup \{1\}$ for some element $x \in G$. Then $|G| = 1$ or 2.*

Proof: In order to see that G contains a torsion element, let's assume that G is torsion free and get a contradiction.

There exists $h \in G$ such that $g^h = g^{-1}$, so we get $g^{h^2} = h^{-2}gh^2 = h^{-1}g^{-1}h = g$ which proves that $g \in C(h^2) \setminus C(h)$. Thus $C(h) \neq C(h^2)$.

Note that $G = x^G \cup \{1\}$ implies that for any $a, b \in G \setminus \{1\}$, there exists $c \in G$ such that $a = b^c$. In particular, there exists $a \in G$ such that $h^a = h^2$.

Since $g \in C(h^2) \setminus C(h)$, we obtain that $g^a \in C(h^2)^a \setminus C(h)^a = C(h^4) \setminus C(h^2)$.

Note that $(h^2)^a = h^4$, so $C(h^2) \neq C(h^4)$. By similar arguments, one can obtain an infinite ascending chain of definable subgroups of G;

$$C(h) < C(h^2) < C(h^4) < \cdots.$$

By taking the centralizer of the elements of the chain, a non-stationary descending chain of definable subgroups of G is obtained. This contradiction shows that G is not torsion free.

Now, there exists an element $g \in G$ of order some prime number p. Since all nonidentity elements of G are conjugate, we see that each $y \in G \setminus \{1\}$ has order p.

If p=2, then G is abelian so $x^G = \{x\}$. Thus $G = \mathbb{Z}_2$.

If p>2, then choose $g \in G$ such that $x^{-1} = g^{-1}xg$ then $x^{(-1)^n} = x^{(g^n)}$, for all n. In particular, for n=p we have $x^{-1} = x^{g^p} = x$, so $p = 2$. This is a contradiction. □

Lemma 4.22 *Let $\langle K, +, \cdot, 0, 1\rangle$ be an infinite field and G be a group acting on K as field automorphisms such that the structure $\langle K, +, \cdot, 0, 1, G\rangle$ is of finite Morley rank. Then the action of G on K is trivial.*

Proof: Since G acts on K as field automorphisms, we have a map $\phi : G \to Aut(K)$. Let $H = \frac{G}{Ker\phi}$. Then the action of H on K is faithful.

Claim 1: The subfield $K_g = \{k \in K : gk = k\}$ is finite for every $g \in H^*$.

To prove the claim, assume that K_g is infinite for some $g \in H^*$. Theorem 4.18 implies that K_g is an algebraically closed field. Thus K is a finite extension of K_g. Lemma 4.1 implies that the extension is algebraic and since K_g is algebraically closed, we have $K = K_g$ which implies that $g = 1$. Thus K_g must be finite for every $g \in H^*$.

Corollary V.5.8 in [6] implies that finite fields have unique extensions of any degree. In particular, K_g has a unique quadratic extension L. Since K is algebraically closed, $L \subseteq K$.

Claim 2: $K_{g^2} = L$.

Since L is a quadratic extension of K, $L = K(a)$ where a is a root of a quadratic polynomial $p(x) = x^2 + cx + d \in K_g[x]$. Let $\bar{a}$ be the other root of $p(x)$. Then $gp(a) = (ga)^2 + c(ga) + d = p(ga)$. On the other hand $gp(a) = g0 = 0$ implies that $p(ga) = 0$, thus $ga = a$ or $ga = \bar{a}$. Either case implies that $g^2 a = a$, and similarly $g^2\bar{a} = \bar{a}$. Therefore $L \leq K_{g^2}$.

Now consider an arbitrary element $x \in K_{g^2}$. Then $a = x + gx$ and $b = x \cdot gx$ are in K_g and $x^2 - ax + b = 0$. Thus x lies in the quadratic extension of K_g, namely in L. Therefore, $K_{g^2} \leq L$ which proves that $L = K_{g^2}$.

We can apply the same argument to the fields $K_{g^{2^n}}$ and conclude that each $K_{g^{2^n}}$ is finite and their cardinalities increase without any bounds. Now consider the interpretable map $\varphi : H \times K \to H \times K$ defined by $(h, k) \mapsto (h, k - hk)$. Then $\varphi^{-1}(g^{2^n}, 0) = \{(g^{2^n}, k) : k \in K_{g^{2^n}}\}$, which implies that $|\varphi^{-1}(g^{2^n}, 0)| = |K_{g^{2^n}}|$. This contradicts with Axiom D, so $H = 1$ and $G = Ker\phi$. Thus the action is trivial. □

Theorem 4.23 (Cherlin) *Infinite rings of finite Morley rank containing no zero divisors are algebraically closed fields.*

Proof: Let D be such a ring. Then Lemma 4.19 and Lemma 4.16 imply that D is a connected division ring. Let D be a counterexample of smallest Morley rank. Assume that $Z(D)$ is infinite. Then Theorem 4.18 implies that $Z(D)$ is an algebraically closed field, and so $Z(D) = D$ by Lemma 4.20(ii). Hence $Z(D)$ is finite.

Now assume that $C_D(r)$ is finite for some $r \in D \setminus Z(D)$. Since one can write

$$D^* = \cup_{x \in D^*} x^{D^*} = \cup_{x \notin Z(D)} x^{D^*} \cup Z(D)^* \tag{4.5}$$

and the map defined from $\frac{D^*}{C_{D^*}(r)} \to r^{D^*}$ defined by $dC_{D^*}(r) \mapsto r^d$ is a bijection, we have $rk(D^*) = rk(r^{D^*})$.

Applying Lemma 2.5 and Lemma 2.7 to the Equation 4.5, we get

$$1 = deg(D^*) = \Sigma_{x \notin Z(D)} deg(x^{D^*}) = \Sigma_{x \notin Z(D)} 1.$$

Therefore, $D^* = Z(D)^* \cup r^{D^*}$, which implies that

$$\frac{D^*}{Z(D)^*} = \{1\} \cup (rZ(D)^*)^{D^*/Z(D)^*}.$$

Lemma 4.21 implies that $|D^*/Z(D)^*| = 1$ or 2. But D^* is infinite and $Z(D)^*$ is finite, which is a contradiction.

Therefore for all $r \in D \setminus Z(D)$, $C_D(r)$ is an infinite proper subring. The minimality assumption implies that $C_D(r)$ is an algebraically closed field for all $r \in D \setminus Z(D)$.

Fix an arbitrary element $r \in D \setminus Z(D)$ and let $K = C_D(r)$. Let K act on D by right multiplication and consider D as a right vector space over K. Lemma 4.20 implies that $dim_K(D) = n < \infty$.

Now let's see that D^* is a regular subgroup of $GL_n(K)$. Define a map $\phi : D^* \to GL_n(K)$ such that $\phi(d)(x) = dx$, for all $d \in D^*$ and for all $x \in D$. For all $d \in D^*$ and for all $x, y \in D$, we have

$$\phi(d)(x + y) = d(x + y) = dx + dy = \phi(d)(x) + \phi(d)(x)$$

and for all $d \in D^*$, for all $x \in D$, and for all $k \in K$, we have

$$\phi(d)(xk) = dxk = \phi(d)(x)k.$$

These two statements imply that $\phi(d)$ is a linear transformation for all $d \in D^*$. For regularity, let $x, y \in D^*$. Then $\phi(yx^{-1})(x) = y$.

D^* satisfies the condition: For all $r \in D^* \setminus Z(D)^*$, $C_{D^*}(r)$ is abelian. Therefore we can apply Theorem 3.24 and conclude that $n = 1$ or $n = 2$. $n = 1$ implies that $K = D$, which is not the case. Thus $n = 2$, and we can apply Theorem 3.24(ii) for $C_G(g) = C_{D^*}(r) = K^*$ to conclude that there exists $x \in D^* \setminus K^*$ satisfying that $x \in N_{D^*}(K^*)$ and $x^2 \in K^*$.

Note that $N_{D^*}(K^*)$ acts on K definably as a group of field automorphisms. By Lemma 4.22 this action is trivial, that is $x^{-1}kx = k$ for all $k \in K$. Thus in particular, $x^{-1}rx = r$, since $r \in K$ which implies that $x \in C_D(r) = K$. This is a contradiction, since $x \in D^* \setminus K^*$. □

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