March 4, 2016

Schedule: Tuesday 13^{40} 1hr
Friday 12^{40} 2hr
Sunday 13^{40} 2hs (every other week)

Time Independent Perturbation Theory

$$\Rightarrow E_n^{(1)} = {}^{(0)}\langle \psi_n | H' | \psi_n \rangle^{(0)}$$

$$\Rightarrow |\psi_n\rangle^{(1)} = \sum_{n \neq k} \frac{\langle \psi_k^{(0)} | H' | \psi_n \rangle^{(0)}}{E_n^{(0)} - E_k^{(0)}} |\psi_k\rangle^{(0)}$$

$$\left(\delta \langle \psi_n | \right) |\psi_n\rangle^{(0)} = 0 \Leftarrow$$

} non-degenerate perturbation theory

If $\exists$ degeneracy $\{|\psi_n\rangle^{(0)}\}$ degenerate.

$$(H')_{nm} = {}^{(0)}\langle \psi_n | H' | \psi_m \rangle^{(0)} \}$$

Diagonalize H'

Example two bosons in an infinite potential well.

$$H' = \lambda \delta(x_1 - x_2)$$

$$H^0 = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1, x_2)$$

$$V(x) = \begin{cases} 0 & -\frac{L}{2} < x_{1,2} < \frac{L}{2} \\ \infty & \text{otherwise} \end{cases}$$

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left[\psi_1(x_1)\psi_2(x_2) + \psi_1(x_2)\psi_2(x_1) \right]$$

Identical Particles

$|\lambda| = 1$

$$\Rightarrow |\psi(x_1,x_2)| = |\psi(x_2,x_1)| \Leftarrow$$

$$\psi(x_2,x_1) = \lambda\,\psi(x_1,x_2) = \lambda(\lambda\psi(x_2,x_1))$$

$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

[other values of λ look: parastatistics]

$$\psi(x_2,x_1) = \pm\,\psi(x_1,x_2)$$

$\lambda = +1$: bosons : particles with integer spin

$\lambda = -1$: fermions : particles with half integer spin

$$H^\circ\left[\psi_1(x_1)\psi_2(x_2)\right] = E\left[\psi_1(x_1)\psi_2(x_2)\right] \qquad -\tfrac{L}{2} \le x_1, x_2 \le \tfrac{L}{2}$$

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2}\right)\psi_1\psi_2 = E\,\psi_1\psi_2$$

$$\left[\frac{-\hbar^2}{2m}\psi_1''\psi_2 - \frac{\hbar^2}{2m}\psi_1\psi_2'' = E\,\psi_1\psi_2\right]\frac{1}{\psi_1\psi_2}$$

$$\underbrace{\left(\frac{-\hbar^2}{2m}\frac{\psi_1''}{\psi_1}\right)}_{\text{depends only on } x_1} + \underbrace{\left(\frac{-\hbar^2}{2m}\frac{\psi_2''}{\psi_2}\right)}_{\text{depends only on } x_2} = \underbrace{E}_{\text{constant}}$$

E_1 $\qquad$ E_2

$$\frac{-\hbar^2}{2m}\psi_1'' = E_1\psi_1 \quad ; \quad \frac{-\hbar^2}{2m}\psi_2'' = E_2\psi_2 \; ; \; E = E_1 + E_2$$

$$E_1 = \frac{\hbar^2 k_1^2}{2m} \qquad E_2 = \frac{\hbar^2 k_2^2}{2m}$$

$$\psi_+'' = -k_+^2 \psi_+ \Rightarrow \psi_+ = A e^{ik_+ x} + B e^{-ik_+ x}$$

$$\psi_+\left(x=-\frac{L}{2}\right) = \psi_+\left(x=\frac{L}{2}\right) = 0$$

$$A e^{ik_+\frac{L}{2}} + B e^{-ik_+\frac{L}{2}} = 0$$

$$A e^{-ik_+\frac{L}{2}} + B e^{ik_+\frac{L}{2}} = 0$$

$$A = \mp B$$

$A = B \quad 2A\cos k_+\frac{L}{2} = 0 \Rightarrow k_+\frac{L}{2} = \left(n+\frac{1}{2}\right)\pi$

$$(k_+)_n = \frac{2}{L}\left(n+\frac{1}{2}\right)\pi$$

$$n = 0, 1, 2, \ldots$$

$$\psi_+ = A\cos(k_{+n} x)$$

$$1 = \int_{-L/2}^{L/2} dx\,|\psi(x)|^2 = \int_{-L/2}^{L/2} dx\,|A|^2 \cos^2(k_{+n}x) = |A|^2\frac{L}{2}$$

$$|A| = \sqrt{\frac{2}{L}} \Rightarrow A = \sqrt{\frac{2}{L}}$$

$$\psi_+^{(n)} = \sqrt{\frac{2}{L}}\cos(k_n^+ x) \quad |n>$$

$A = -B$

$$\psi = A\sin(k_- x)$$

$$A\sin\left(k_-\frac{L}{2}\right) = A\sin\left(k_-\left(-\frac{L}{2}\right)\right) = 0$$

$$k_-\frac{L}{2} = n\pi \qquad n = 0, 1, \ldots$$

$$\boxed{k_- = \frac{2\pi}{L}n}$$

$$\psi_{k_-}^{(n)}(x) = \sqrt{\frac{2}{L}}\sin(k^- x) = |n>$$

Parity

$P: x \to -x$

$[H^{(0)}, P] = 0$

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$$

$$P\,\frac{f(x) \mp f(-x)}{2} = \mp\,\frac{f(x) \mp f(-x)}{2}$$

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_n(x_1)\psi_m(x_2) + \psi_n(x_2)\psi_m(x_1)\right] \quad (\text{if } n \neq m)$$

$$= \psi_n(x_1)\psi_n(x_2) \quad (\text{if } n = m)$$

b) Perturbation corrections.

$$E = \frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m}$$

$$= \frac{\hbar^2}{2m}\,\frac{2\pi}{L}\left[\left(n+\left(\tfrac{1}{2}\right)\right)^2 + \left(m+\left(\tfrac{1}{2}\right)\right)^2\right]$$

$$a^2 + b^2 = c^2 + d^2 \Leftarrow$$

One particle energy eigenvalues

$$\left(\frac{\hbar^2}{2m}\right)\left(n+\left(\tfrac{1}{2}\right)\right)^2$$

$\frac{1}{4}$, $\frac{9}{4}$, $\frac{25}{4}$, ...

$\left(\frac{\hbar^2}{2m}\right)$ 2 particle energies

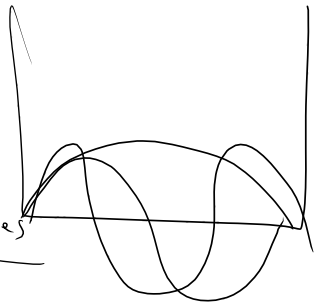
Ground $\frac{1}{2}$

1st excited $\frac{5}{4}$

2nd excited 2

3rd excited $\frac{5}{2}$

...



$$V = \lambda \delta(x_1 - x_2)$$

$$\psi_0 = \frac{2}{L} \cos\left(\frac{\pi}{L} x_1\right) \cos\left(\frac{\pi}{L} x_2\right)$$

$$E_0^{(1)} = \langle \psi_0 | V | \psi_0 \rangle = \int dx_1 dx_2 \, \psi_0^*(x_1, x_2) V \psi_0(x_1, x_2)$$

$$= \lambda \int_{L/2} dx_1 \left| \psi_0^*(x_1, x_1) \right|^2$$

$$E_0^{(1)} = \frac{4\lambda}{L^2} \int_{-L/2} dx_1 \cos\left(\frac{\pi}{L} x_1\right)^4 < \frac{4\lambda}{L}$$

$$E_0 = \frac{2\hbar^2}{2m} \left(\frac{2}{L} \cdot \frac{1}{2}\right)^2 = \frac{\hbar^2 \pi^2}{m L^2}$$

$$\frac{E_0^{(1)}}{E_0} < \frac{4\lambda}{4} \cdot \frac{m L^2}{\hbar^2 \pi^2} = \frac{4 m L \lambda}{\hbar^2 \pi^2}$$

Hydrogen Atom

$$\left[\frac{-\hbar^2}{2m} \nabla^2 - \frac{e^2}{4\pi\varepsilon_0} \frac{1}{r} \right] \psi = E \psi$$

$$\left[\frac{\hbar^2}{2m} \right] = J m^2 \qquad \left[\frac{e^2}{4\pi\varepsilon_0} \right] = J m$$

$$\left[\frac{\hbar^2}{2m} \right] \left[\frac{4\pi\varepsilon_0}{e^2} \right] = m$$

$$\boxed{\frac{\hbar^2}{m} \frac{4\pi\varepsilon_0}{e^2} = a_B}$$

$$[\hbar c] = J s \frac{m}{s} = J m = \left[\frac{e^2}{4\pi\varepsilon_0} \right]$$

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \simeq \frac{1}{137}$$

$$\langle r^n \rangle \propto a_B^n$$

$\frac{\hbar}{mc}$: compton wavelength

$$a_B = \frac{\hbar}{\alpha m c}$$

$$E_n = -\frac{E_0}{n^2} = -\frac{13.6\ eV}{n^2}$$

$E_0 = +\frac{V}{2}$ classically

$$= +\frac{\alpha \hbar c}{r}\frac{1}{2} = +\frac{\hbar c \alpha}{2a_B} = \frac{\hbar c \alpha}{2\frac{\hbar}{\alpha m c}} = \frac{mc^2\alpha^2}{2}$$

$$E_0 = \frac{m}{3\hbar^2}\left(\frac{e}{4\pi\varepsilon_0}\right)^2 = \frac{m^2c^2\alpha^2}{3}$$

$$E = \frac{p^2}{2m} + V(\vec{r})$$

$\frac{p^2}{2m}$: non-relativistic KE

relativistic KE $= \sqrt{p^2c^2 + m^2c^4} - mc^2$

non-relativistic limit $\frac{p}{mc} \ll 1$

$$\sqrt{p^2c^2 + m^2c^4} = mc^2\left(1 + \left(\frac{p}{mc}\right)^2\right)^{1/2}$$

$$= mc^2\left[1 + \frac{1}{2}\left(\frac{p}{mc}\right)^2 + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{p}{mc}\right)^4 \frac{1}{2} + \cdots\right]$$

$$KE = \frac{1}{2}mc^2\left(\frac{p}{mc}\right)^2 + mc^2\left(-\frac{1}{8}\right)\left(\frac{p}{mc}\right)^4$$

$$= \frac{p^2}{2m} - \frac{p^4}{8m^3c^2}$$

$$H = H^0 - \underbrace{\frac{\hat{p}^4}{8m^3c^2}}_{H'} \qquad H' = \frac{-\hat{p}^4}{8m^3c^2} = -\frac{(\hat{p}^2)^2}{8m^3c^2}$$

$H^0\psi^0_{nlm} = E^0_n\psi^0_{nlm}$: degenerate system.

$$[H', L^2] = 0$$

$$0 = \langle n'l'm'|[H', L^2]|nlm\rangle = \langle n'l'm'|H'\hbar^2 l(l+1)|nlm\rangle - \langle n'l'm'|\hbar^2 l'(l'+1)H'|nlm\rangle$$

$$\{[l'(l'+1)] - [l(l+1)]\}\langle n'l'm'|H'|nlm\rangle = 0$$

if $l \neq l'$ $\langle n'l'm'|H'|nlm\rangle = 0$

$$[H', L_z] = 0$$

$$\Rightarrow (m'-m)\langle n'l'm'|H'|nlm\rangle = 0$$

$$\Rightarrow \langle n'l'm'|H'|nlm\rangle = 0 \quad \text{if} \quad m \neq m'$$

In general.

Assume $|n\rangle$ and $|m\rangle$ are degenerate states. Assume $\exists$ an operator $\mathcal{O}$, st $[H', \mathcal{O}] = 0$ if $\mathcal{O}|n\rangle = \mu_n|n\rangle$ and $\mathcal{O}|m\rangle = \mu_m|m\rangle$

Then $\langle n|H'|m\rangle = 0$ if $M_m \neq M_n$

$$E^{(1)}_{nlm} = \langle nlm|H'|nlm\rangle$$

$$= \langle nlm| -\frac{\hat{p}^4}{8m^3c^2}|nlm\rangle$$

$$= -\frac{1}{8m^3c^2}\langle nlm|\hat{p}^4|nlm\rangle$$

$$\hat{p}^2|nlm\rangle = 2m(E-V)|nlm\rangle$$

$$\langle nlm|\hat{p}^4|nlm\rangle = \langle nlm|[2m(E-V)]^2|nlm\rangle$$

$$\langle nlm|H'|nlm\rangle = -\frac{1}{2\cancel{8}m^{\cancel{3}}c^2}\cancel{4}m^2\left(E^2 - 2E\langle V\rangle + \langle V^2\rangle\right)$$

$$= -\frac{1}{2mc^2}\left(E^2 - 2E\left(-\frac{e^2}{4\pi\varepsilon_0}\right)\left\langle\frac{1}{r}\right\rangle + \left(-\frac{e^2}{4\pi\varepsilon_0}\right)\left\langle\frac{1}{r^2}\right\rangle\right)$$

$$\left\langle\frac{1}{r^2}\right\rangle = \frac{1}{\left(l+\frac{1}{2}\right)a_B^2}$$

$$\left\langle\frac{1}{r}\right\rangle = \frac{1}{n^2 a_B}$$

$$E^{(1)}_{nlm} \propto \frac{\left(E^{(0)}_{nlm}\right)^2}{mc^2} \Rightarrow \frac{E^{(1)}_{nlm}}{E_{nlm}} \simeq \frac{E^{(0)}_{nlm}}{mc^2} \simeq \frac{10\,eV}{5\times10^5\,eV}$$

$$\frac{E^{(1)}_{nlm}}{E_{nlm}} \simeq \mathcal{O}(10^{-5})$$

Runge-Lenz vector?