# CHARACTERIZATION OF QUASIDIAGONAL ISOMORPHISMS BETWEEN SOME TYPES OF KÖTHE SPACES

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## ABSTRACT

## CHARACTERIZATION OF QUASIDIAGONAL ISOMORPHISMS BETWEEN SOME TYPES OF KÖTHE SPACES



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This thesis consists of two basic parts. The first part contains a survey on linear topological invariants in the class of Köthe spaces. In the second part, we present a characterization of quasidiagonal isomorphisms between spaces of the types $\mathcal{D}_\infty(\kappa,\gamma,\text{a})$, $\mathcal{H}_\infty(\kappa,\gamma,\text{a})$, and $\mathcal{D}_0(\kappa,\gamma,\text{a})$. These concrete Köthe spaces are investigated by functional analysists led by V.P. Zahariuta.



Keywords: Nuclear, Schwartz Köthe Spaces, Linear Topological Invariants, Quasidiagonal Isomorphisms.

# ÖZ

## BAZI TİP KÖTHE UZAYLARI ARASINDAKİ QUASİDİAGONAL İZOMORFİZMALARIN KARAKTERİZASYONU

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Bu tez iki ana bölümden oluşmaktadır. İlk bölüm lineer topolojik değişmezler konusunda genel bilgiler icermektedir. İkinci bölümde ise $\mathcal{D}_\infty(\kappa,\gamma,\mathrm{a})$, $\mathcal{H}_\infty(\kappa,\gamma,\mathrm{a})$ ve $\mathcal{D}_0(\kappa,\gamma,\mathrm{a})$ tip uzaylar arasındaki quasidiagonal izomorfizmaların karakterizasyonu sunuyoruz. Bu somut Köthe uzayları detaylı olarak Prof. Dr. V.P. Zahariuta tarafından yönlendirilmekte olan fonksiyonel analiz grubu tarafından incelenmektedir.

Anahtar Sözcükler: Nükleer, Schwartz Köthe Uzayları, Lineer Topolojik Değişmezler, Quasidiagonal İzomorfizmalar.

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# CHAPTER 1
# DEFINITIONS AND PRELIMINARY RESULTS

## 1.1 LOCALLY CONVEX SPACES

For undefined concepts and general terminology we refer the reader to [11], [15] or [19] and on invariants to [24].

**Definition 1.1** An ordered pair, $(X, \mathcal{T})$, where $X$ is a vector space over $\mathbb{K}$ , (which will be taken as $\mathbb{C}$ or $\mathbb{R}$) , and $\mathcal{T}$ is a Hausdorff topology on $X$, will be called a locally convex topological vector space (l.c.s.), if the mapping $(x, y) \mapsto x + y$ of $X \times X$ into $X$ and the mapping $(\lambda, x) \mapsto \lambda x$ of $\mathbb{K} \times X$ into $X$ are both continuous and moreover, for every point there exists a neighborhood basis $\mathfrak{U}$ consisting of convex sets.

In a l.c.s. addition is not only a continuous mapping but also a homeomorphism. Therefore it is no loss of generality to deal only with neighborhoods of the origin , and to take the neighborhoods of the origin to be absolutely convex sets , that is, if $A \in \mathfrak{U}$, and $x, y \in A$, $|\alpha| + |\beta| \leq 1$, then $\alpha x + \beta y \in A$. Hence from now on we will denote a l.c.s. by $(X, \mathfrak{U})$.

**Definition 1.2** A l.c.s. $(X,\mathfrak{U})$ will be called a Frechet space, in case $\mathfrak{U}$ can be chosen to contain atmost countably infinitely many sets, say $\{U_n\}_{n=1}^{\infty}$, and $(X,d)$ is a complete metric space where

$$d(x,y) \triangleq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_{U_n}(x-y)}{1+p_{U_n}(x-y)} \text{ and } p_{U_n} \text{ is the Minkowski functional of } U_n.$$

## 1.2 NUCLEAR AND SCHWARTZ SPACES

**Definition 1.3** Let $(X,|.|)$, and $(Y,\|.\|)$ be two normed linear spaces. A linear and continuous mapping $T: X \to Y$ will be called <u>nuclear</u>, in case there exist $(a_n)_{n=1}^{\infty} \subset X'$, and $(b_n)_{n=1}^{\infty} \subset Y$ such that
$Tx = \sum_{n=1}^{\infty} a_n(x)b_n$ and $\sum_{n=1}^{\infty} \|a_n\|\|b_n\| < \infty$ where $\|a_n\|$ is the operator norm of $a_n$. $T$ is called <u>compact</u>, if $\overline{T(U)} \subset Y$ is compact where $U \triangleq \{x \in X : |x| \leq 1\}$.

**Notation:** Let $(X,\mathfrak{U})$ be a l.c.s.. If $U$ is in $\mathfrak{U}$, then $\tau^U : X \to X/\ker p_U$ is the quotient map $\tau^U(x) \triangleq [x]_U \triangleq \{y \in X : x-y \in \ker p_u\}$. Now if $U,V$ in $\mathfrak{U}$ satisfy $V \preceq U$ , that is, $V \subset \lambda U$ for some $\lambda > 0$, then we will denote the continuous mapping $[x]_V \mapsto [x]_U$ of $E/\ker p_V$ into $E/\ker p_U$ by $\tau_V^U$.

**Definition 1.4** A l.c.s. $(X,\mathfrak{U})$ is called <u>nuclear</u> (respectively, <u>Schwartz</u>), if for every $U \in \mathfrak{U}$ there exists $V \in \mathfrak{U}$ such that $V \preceq U$ and $\tau_V^U$ is a nuclear (respectively, compact) mapping.

## 1.3 BASES IN A L.C.S.

**Definition 1.5** Let $(X, \mathfrak{U})$ be a l.c.s., and let $(b_n)_{n=1}^{\infty}$ be a sequence in $X$. Then $(b_n)_{n=1}^{\infty}$ is called a basis, if for every $x \in X$ there exists a uniquely determined sequence $(\lambda_n)_{n=1}^{\infty}$ of scalars such that $x = \sum_{n=1}^{\infty} \lambda_n b_n$.
Now let $(b_n)_{n=1}^{\infty}$ be a basis in $X$. Then $(b_n)_{n=1}^{\infty}$ is called
**(1)** a Schauder basis, if for each $n$ the mapping $x \mapsto f_n(x) \triangleq \lambda_n$ is continuous.
**(2)** an equicontinuous basis, if $\{f_n\}_{n=1}^{\infty}$ is equicontinuous.
**(3)** an unconditional basis, in case $\sum_{n=1}^{\infty} \lambda_n b_n$ converges unconditionally in $X$.
**(4)** an absolute basis, in case

$$\forall U \in \mathfrak{U}\ \exists V \in \mathfrak{U} \text{ such that } \sum_{n=1}^{\infty} |\lambda_n| p_U(b_n) \leq p_V(x)\ \forall x \in X.$$

**Theorem 1.6 (Dynin-Mitiagin)** In a nuclear Frechet space any basis is absolute.

## 1.4 SOME SPECIAL OPERATORS BETWEEN L.C.S.

**Definition 1.7** Let $X$, $Y$ be two l.c.s. with unconditional bases respectively $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$. An operator $T : X \to Y$
**(i)** is called quasi-diagonal (q.d.) (with respect to these bases), if $Tx_n = \gamma_n y_{\sigma(n)}$ where $\sigma : \mathbb{N} \to \mathbb{N}$ is a permutation, and $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence of scalars.
Now a q.d. operator $T$ will be called
**(ii)** permutable, if $\gamma_n = 1$ for each $n \in \mathbb{N}$
**(iii)** diagonal, if $\sigma(n) = n$ for all $n$ in $\mathbb{N}$.

If, moreover, the mapping $T$ is an isomorphism of the type respectively (i),(ii),(iii), then the bases $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ are called respectively quasi-equivalent, permutably equivalent, diagonally equivalent.

Now if $T$ is a q.d. isomorphism (q.d.i), and $\gamma_n = 1$, $\sigma(n) = n$ for all $n$ in $\mathbb{N}$, then the bases are called equivalent. Moreover, we will say a l.c.s. $X$ has the quasi equivalence property (q.e.p.), if every unconditional basis is quasi-equivalent to the natural coordinate vector basis, $\{e_n\}_{n=1}^{\infty}$ where $e_n = (0, ..., 1, 0, ...)$, 1 is in the $i-$th place and 0 elsewhere.

Isomorphisms of types respectively (i),(ii),(iii) will be denoted respectively by $X \overset{q.d.i}{\cong} Y$, $X \overset{p}{\cong} Y$, $X \overset{d}{\cong} Y$.

## 1.5 KÖTHE SEQUENCE SPACES

**Definition 1.8** An infinite matrix $[a_{pn}]$ with non-negative entries will be called a Köthe matrix, if the following conditions are fulfilled:

**(i)** for every $n$ there exists $p$ such that $a_{pn} > 0$;

**(ii)** for each $p$ and for each $n$, $a_{p,n} \leq a_{p+1,n}$.

And the sequence space

$$\mathcal{K}([a_{pn}]) \triangleq \{(\gamma_n)_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}} : \ \|(\gamma_n)_{n=1}^{\infty}\|_p \triangleq \sum_{n=1}^{\infty} |\gamma_n| a_{pn} < \infty \text{ for each } p \in \mathbb{N}\}$$

will be called a Köthe sequence space.

**Fact 1.9** $(\mathcal{K}([a_{pn}]), \{|.|\}_{p=1}^{\infty})$ and $(\mathcal{K}([b_{pn}]), \{\|.\|\}_{p=1}^{\infty})$ are equal as sets if and only if $I : \mathcal{K}([a_{pn}]) \to \mathcal{K}([b_{pn}])$, defined by $Ie_i \triangleq e_i$ for all $i \in \mathbb{N}$, is isomorphism.

**Proof:**($\Leftarrow$) It follows from the fact that $|e_n|_p = a_{pn}$ and $\|e_n\|_p = b_{pn}$.

($\Rightarrow$): Let us assume the contrary. Without loss of generality, suppose that there is some $p_0$ such that for all $q$, $C > 0$ there is some $n$ $a_{p_0,n} > Cb_{qn}$. Letting $C = q^2$, $q \in \mathbb{N}$, we inductively define a subsequence $(n_q)$ of $\mathbb{N}$ satisfying $a_{p_0,n_q} > q^2 b_{q,n_q}$. Now define the following sequence $(x_n)_{n=1}^{\infty}$

$$x_n \triangleq \begin{cases} \frac{1}{a_{p_0,n_q}} & \text{if } n = n_q \text{ for some } q \\ 0 & \text{otherwise} \end{cases}$$

Now let $q$ be given. Then

$$\sum_{n=1}^{\infty} |x_n| b_{qn} = \sum_{p=1}^{\infty} \frac{b_{q,n_p}}{a_{p_0,n_p}} = \sum_{p=1}^{q} \frac{b_{q,n_p}}{a_{p_0,n_p}} + \sum_{p=q+1}^{\infty} \frac{b_{q,n_p}}{a_{p_0,n_p}} \leq \sum_{p=1}^{q} \frac{b_{q,n_p}}{a_{p_0,n_p}} + \sum_{p=q+1}^{\infty} \frac{b_{p,n_p}}{a_{p_0,n_p}}$$

$$< \sum_{p=1}^{q} \frac{b_{q,n_p}}{a_{p_0,n_p}} + \sum_{p=q+1}^{\infty} \frac{1}{p^2} < \infty.$$

However, for $q = p_0$, $\sum_{n=1}^{\infty} |x_n| a_{qn} = \infty$.

Let $(X, \{U_n\}_{n=1}^{\infty})$ be a nuclear Frechet space with a basis $(b_n)_{n=1}^{\infty}$. Then $X$ is isomorphic to $\mathcal{K}([a_{pn}])$, where $a_{kn} \triangleq p_{U_k}(b_n)$.

The following theorem is called **Grothendick-Pietsch Criterion** for Köthe spaces.

**Theorem 1.10** A Köthe space $\mathcal{K}([a_{pn}])$ is nuclear (respectively, Schwartz) if and only if for every $p$ in $\mathbb{N}$ there exists $q$ in $\mathbb{N}$ such that $\sum_{n=1}^{\infty} \frac{a_{pn}}{a_{qn}} < \infty$ (respectively, $\frac{a_{pn}}{a_{qn}} \to 0$ as $n \to \infty$).

## 1.6 SOME SPECIAL KÖTHE SPACES

**(1)** $\Lambda_\alpha(a) \triangleq \mathcal{K}([a_{pn}])$ where $\alpha$ is in $(-\infty,\infty)$, $a_{pn} \triangleq e^{r_p a_n}$, $r_p$ increases to $\alpha$, and $a_n$ is an increasing sequence of non-negative reals, is called a finite type power series space. For example, the space of all analytic functions in the open unit disc with the topology of uniform convergence on compact sets, is isomorphic to $\Lambda_0(n)$.

**(2)** $\Lambda_\infty(a) \triangleq \mathcal{K}([a_{pn}])$ where $a_{pn} \triangleq e^{r_p a_n}$, $r_p$ increases to $\infty$, and $a_n$ is an increasing sequence of non-negative reals, is called an infinite type power series space. For example, the set of all entire functions is isomorphic to $\Lambda_\infty(n)$.

**(3)** $E(\lambda,a) \triangleq \mathcal{K}([a_{pn}])$ where $a_{pn} \triangleq e^{(\frac{-1}{p}+\lambda_n p)a_n}$, $0 < \lambda_n \leq 1$, $a_n \geq 1$. For instance, if $\lambda_n \to 0$, then $E(\lambda,a)$ is isomorphic to $\Lambda_0(a)$, if $\liminf_{n\to\infty} \lambda_n > 0$, then $E(\lambda,a)$ is isomorphic to $\Lambda_\infty(a)$. For more details, see [24], in which isomorphic classification of such spaces can also be found.

**(4)** Let $\mathcal{K}([a_{pn}])$, $\mathcal{K}([b_{pn}])$ be two Köthe spaces. Then by the fact that absolute convergence implies unconditional convergence, $\mathcal{K}([a_{pn}]) \times \mathcal{K}([b_{pn}])$ is isomorphic to $\mathcal{K}([c_{pn}])$ where

$$c_{pn} \triangleq \begin{cases} a_{pk} & \text{if n=2k} \\ b_{pk} & \text{if n=2k-1} \end{cases} \quad , k \in \mathbb{N}$$

Now $\mathcal{K}([a_{pn}])\hat{\otimes}_\pi\mathcal{K}([b_{pn}])$ and $\mathcal{K}([d_{(i,j),p}])$ where $d_{(i,j),p} \triangleq a_{pi}b_{pj}$, are isomorphic by the following mapping $e_i \otimes e_j \mapsto e_{\sigma(i,j)}$, where $\sigma : \mathbb{N}^2 \to \mathbb{N}$ is a bijection.

The following two types of Köthe spaces are generalised from the particular counter-examples given by M.M.Dragilev (see [7] and the remark following Theorem 2.7 in section 2.1).

**(5)** $\mathcal{D}_0(\kappa,\gamma,a) \triangleq \mathcal{K}([a_{pn}])$ where $a_{pn} \triangleq e^{[\frac{-1}{p}+\gamma_n\chi(p-\kappa_n)]a_n}$, $p \in I\!N$, $1 \le a_n$ is increasing to infinity, $\gamma_n \ge 0$, $\kappa : I\!N \to I\!N$ is a function, $\chi \triangleq \chi_{[0,\infty)}$, the characteristic function of $[0,\infty)$. (See section 2.5.E) P.B. Djakov and V.P. Zahariuta examined the case $\gamma_n = 1$ for each $n \in I\!N$ and gave an isomorphic characterization of such spaces. Now if $\sup_{n\in I\!N} \kappa_n < \infty$, then $\mathcal{D}_0(\kappa,\gamma,a) \overset{d}{\cong} \Lambda_0(a)$. If $\lim_{n\to\infty} \kappa_n = \infty$, then $\mathcal{D}_0(\kappa,\gamma,a) \cong \Lambda_0(a)$ . Hence in general it is assumed that $\kappa$ has infinitely many finite limit points. Now let $\nu$ be a subsequence in $I\!N$. If $\lim_{n\in\nu} \kappa_n = \infty$, then $\mathcal{K}([a_{pn}]_{p\in I\!N, n\in\nu}) \cong \Lambda_0(b)$ and if $sup_{n\in\nu}\kappa_n < \infty$, then $\mathcal{K}([a_{pn}]_{p\in I\!N,n\in\nu}) \overset{d}{\cong} \Lambda_0(b)$ where $b$ is the sequence $\{a_n\}_{n\in\nu}$. More precisely, each basic subspace has a finite type power series subspace.

**(6)** $\mathcal{D}_\infty(\kappa,\gamma,a) \triangleq \mathcal{K}([a_{pn}])$ where $a_{pn} \triangleq e^{[p+\gamma_n\chi(p-\kappa_n)]a_n}$, $p \in I\!N$, $1 \le a_n$ is increasing to infinity, $\gamma_n \ge 1$, $\kappa : I\!N \to I\!N$ is a function, $\chi \triangleq \chi_{[0,\infty)}$. By a similar argument as in (5), we can say that each basic subspace of $\mathcal{D}_\infty(\kappa,\gamma,a)$ has a basic infinite type power series subspace.

**(7)** $\mathcal{H}_\infty(\kappa,\gamma,a) \triangleq \mathcal{K}([a_{pn}])$ where $a_{pn} \triangleq e^{[p+\gamma_n \min(p,\kappa_n)]a_n}$, where $\kappa,\gamma,a$ are as in (6). In this case, again all basic subspaces have basic infinite type power series subspaces. (See [13])

The definitions below are introduced by M.M. Dragilev [6].

**Definition 1.11** A Köthe space $\mathcal{K}([a_{pn}])$ belongs to the class

**(i)** $d_0$ and is called <u>regular</u>, if $\frac{a_{p,n+1}}{a_{p,n}} \leq \frac{a_{p+1,n+1}}{a_{p+1,n}}$ for all $p$, $n$ in $I\!N$;

**(ii)** $d_1$, if $\exists p\, \forall q\, \exists r$ such that $sup_{n \in I\!N} \frac{a_{qn}^2}{a_{p,n}a_{r,n}} < \infty$;

**(iii)** $d_2$, if $\forall p\, \exists q\, \forall r$ such that $sup_{n \in I\!N} \frac{a_{p,n}a_{r,n}}{a_{qn}^2} < \infty$;

**(iv)** $D_1$, if it is in $d_0$ and $d_1$;

**(v)** $D_2$, if it is in $d_0$ and $d_2$;

**(vi)** $D$, if it is in $D_1$ or $D_2$.

**Definition 1.12** Let $\mathcal{K}([a_{pn}])$ be a Köthe space. Then

**(i)** it will be called <u>stable</u>, if $\mathcal{K}([a_{pn}]) \times \mathcal{K}([a_{pn}]) \overset{q.d.i.}{\cong} \mathcal{K}([a_{pn}])$.

**(ii)** it will be called <u>weakly stable</u>, if $\mathcal{K}([a_{pn}]) \times I\!K \cong \mathcal{K}([a_{pn}])$.

**(iii)** it will be called <u>unstable</u>, if $\exists s\, \forall p\, \exists q\, \forall r\, \lim_{n\to\infty} \frac{a_{p,n}a_{r,n}}{a_{q,n}a_{s,n}} = 0$.

**Definition 1.13** Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ be sequences of positive real numbers. Then we will say that the sequences are <u>weakly equivalent</u> and denote by $(a_n)_{n=1}^{\infty} \asymp (b_n)_{n=1}^{\infty}$, if there exists $C > 0$ such that for every $n$ $\frac{a_n}{C} \leq b_n \leq Ca_n$.

# CHAPTER 2
# LINEAR TOPOLOGICAL INVARIANTS

## 2.1 THE CLASSICAL INVARIANTS $\Gamma$, $\Gamma'$

**Definition 2.1** Let $\mathfrak{L}$ be a subclass of the all l.c.s.'s and let $\tau$ be a set valued function on $\mathfrak{L}$ . Then we will say that $\tau$ is a linear topological invariant (l.t.i.), if $X$, $Y$ are both in $\mathfrak{L}$ and are linear isomorphic , then $\tau(X) = \tau(Y)$.

**Definition 2.2** Let $(X, \mathfrak{U})$ be a (l.c.s.), and let $U$, $V$ be in $\mathfrak{U}$ such that $V \preceq U$. Then

$$d_n(V, U) \triangleq \inf\{\alpha > 0 : \ V \subset L + \alpha U \text{ for some } L \in \mathcal{F}_n\}$$

where $\mathcal{F}_n$ is the class of at most $n-$ dimensional subspaces of $X$, will be called the $n-$th Kolmogorov diameter of $V$ with respect to $U$.

**Definition 2.3** Let $(X, \mathfrak{U})$ be a l.c.s., and let $U$, $V$ be in $\mathfrak{U}$ such that $V \preceq U$. Then we call the sequence space

$$\Gamma(X) \triangleq \{(\gamma_n) \in I\!K^{I\!N} : \ \forall\, U \in \mathfrak{U}\ \exists\, V \in \mathfrak{U} \text{ such that } V \preceq U,\ \gamma_n d_n(V, U) \to 0\}$$

the diametral dimension of $X$ and we call the following space

$$\Gamma'(X) \triangleq \{(\gamma_n) \in I\!K^{I\!N} : \ \exists\, U \in \mathfrak{U}\, \forall V \in \mathfrak{U} \text{ such that } V \preceq U,\ \frac{\gamma_n}{d_n(V, U)} \to 0\}$$

the dual diametral dimension of $X$.

Since obviously $d_n(T(V),T(U)) \leq d_n(V,U)$ if $T$ is a linear map, then

**Fact 2.4** Both $\Gamma$ and $\Gamma'$ are l.t.i.'s.

Because of the following lemma (see [21]) these invariants are very useful in investigating Köthe spaces with regular matrices.

**Lemma 2.5** Let $K([a_{pn}])$ be a regular Köthe space. Then for every $n \in I\!N$

$$d_n(U_p,U_q) = \frac{a_{qn}}{a_{pn}} \quad \text{where } U_k \triangleq \{(x_n) \in I\!K^{I\!N} : \ \|(x_n)\|_k \triangleq \sum_{n=1}^{\infty} |x_n| a_{kn} \leq 1\}.$$

By the following theorem, we can say that both of the invariants $\Gamma, \Gamma'$ are complete in the class of all Schwartz power series spaces; that is, if $X, Y$ are Schwartz p.s.s., and if either $\Gamma(X) = \Gamma(Y)$ or $\Gamma'(X) = \Gamma'(Y)$, then $X$ and $Y$ are isomorphic. To prove the theorem we will observe the following:

**Fact 2.6** Let $X \triangleq \Lambda_0(a)$and let $Y \triangleq \Lambda_\infty(b)$ be Schwartz p.s.s., that is, $a_n, b_n \nearrow \infty$. Then

**(i)** $\Gamma(X) = \bigcap_{p \in I\!N}\{(\gamma_n)_{n=1}^{\infty} \in I\!K^{I\!N} : \ \gamma_n e^{\frac{-a_n}{p}} \to 0\} \triangleq A$;

**(ii)** $\Gamma'(X) = \bigcup_{p \in I\!N}\{(\gamma_n)_{n=1}^{\infty} \in I\!K^{I\!N} : \ \gamma_n e^{\frac{1}{p}a_n} \to 0\} \triangleq B$;

**(iii)** $\Gamma(Y) = \bigcup_{p \in I\!N}\{(\gamma_n)_{n=1}^{\infty} \in I\!K^{I\!N} : \ \frac{\gamma_n}{e^{pa_n}} \to 0\} \triangleq C$

**(iv)** $\Gamma'(Y) = \bigcap_{p \in I\!N}\{(\gamma_n)_{n=1}^{\infty} \in I\!K^{I\!N} : \ \gamma_n e^{pa_n} \to 0\} \triangleq D$.

**Proof:** We will prove (iii) and (iv). The rest can be handled similarly.

**(iii):** Let $(\gamma_n)_{n=1}^{\infty}$ be in $\Gamma(Y)$. Then for each $p$ there exists $q_0$ such that $\gamma_n e^{(p-q)b_n} \to 0$. Thus, in particular, if we let $p \triangleq 1$, then $q_0$ exists such that $\gamma_n e^{(1-q_0)b_n} \to 0$. This implies $\Gamma(Y) \subset C$.

Conversely, let $(\gamma_n)_{n=1}^{\infty}$ belong to $C$. Then there exists $q_0$ such that $\frac{\gamma_n}{e^{q_0 b_n}} \to 0$. Now If $p$ is given, then choose $q \triangleq q_0 + p$. So we have $\frac{\gamma_n}{e^{(q-p)b_n}} = \frac{\gamma_n}{e^{q_0 b_n}} \to 0$, which implies that $C \subset \Gamma(Y)$.

(iv): Let $(\gamma_n)_{n=1}^{\infty}$ be in $\Gamma'(Y)$. But then there is $p_0$ such that for all $q$ $\gamma_n e^{(q-p_0)a_n} \to 0$. Hence for $q \triangleq p + q_0$, we have $\gamma_n e^{(q-p_0)a_n} = \gamma_n e^{pa_n} \to 0$. Conversely, let $(\gamma_n)_{n=1}^{\infty}$ be in $D$. Let $p_0 \triangleq 1$ and $q$ be given. Then $\gamma_n e^{(q-p_0)a_n} = \gamma_n e^{(q-1)a_n} \to 0$.

**Remark:** In the case of nuclear p.s.s., since $\ell_\infty$ and $\ell_1$ norms are equivalent, one has $\Gamma(X) = X$ and $\Gamma'(Y) = Y$.

The following theorem is due to B.S. Mitiagin.

**Theorem 2.7** Let $X = \Lambda_\alpha(a)$, $Y = \Lambda_\alpha(b)$, where $a_i, b_i \to \infty$, and $\alpha = 0, \infty$. Then the following conditions are equivalent:

(i) $X \cong Y$;

(ii) $\Gamma(X) = \Gamma(Y)$;

(iii) $\Gamma'(X) = \Gamma'(Y)$;

(iv) $(a_i)_{i=1}^{\infty} \asymp (b_i)_{i=1}^{\infty}$;

(v) $X \overset{qdi}{\cong} Y$.

**Proof:** Because the proof for $\alpha = \infty$ is similar to that for $\alpha = 0$, we will give a proof of the case $\alpha = 0$.

**(i)⇒(ii), (iii):** This follows from the Fact 2.4.

**(ii)⇒ (iv):** Let $k$ be in $I\!N$ and let $M_k \triangleq \{n \in I\!N : \ e^{-b_n} > (k-1)e^{\frac{-a_n}{k}}\}$. Then $M_1 = I\!N, M_{k+1} \subset M_k$ for all $k$ in $I\!N$.

*Claim: The sets $M_k$'s are finite from some $k_0$ on.*

*Proof of the claim:* Assume that all $M_k$'s are infinite. Now let $(m_k)_{k=1}^{\infty}$ be a subsequence of natural numbers such that $m_k \in M_k$ for all $k$. Define the sequence $(\gamma_n)_{n=1}^{\infty}$ by

$$\gamma_n \triangleq \begin{cases} \frac{k-1}{e^{-b_n}} & \text{if } n = m_k \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

Let $l$ be given. So $\gamma_{m_k} e^{-a_{m_k}} = \frac{(m_k - 1)}{e^{-b_{m_k}}} e^{\frac{-a_{m_k}}{l}} < e^{\frac{a_{m_k}}{k}} e^{\frac{-a_{m_k}}{l}} = e^{(\frac{1}{k} - \frac{1}{l}) a_{m_k}} \to 0$ as $k \to \infty$. Thus $\gamma_n e^{\frac{-a_n}{l}} \to 0$ as $n \to \infty$. However, $\gamma_{m_k} e^{-b_{m_k}} = m_k - 1 \not\to 0$, which is a contradiction. Hence $M_k$'s are finite from some $k_0$ on. It implies that $\sup_{n \in N} \frac{b_n}{a_n} < \infty$. The result follows if we change the roles of the sequences.

**(iii)⇒(iv):** Assume (iii) holds. For $p = 2$, the sequence $(e^{-b_n})_{n=1}^{\infty}$ is in set $\tilde{\Gamma}(Y)$ and hence $e^{-b_n} e^{\frac{a_n}{p_0}} \to 0$ for some $p_0$. It implies that from some $n_0$ on we have $e^{\frac{a_n}{p_0}} \leq e^{b_n}$ or $\frac{a_n}{p_0} \leq b_n$. Now if we let $M \triangleq \max(p_0, \{\frac{a_n}{b_n} : n < n_0\})$, we get for all $n$, $\frac{1}{M} \leq \frac{b_n}{a_n}$. Changing the roles of $a$ and $b$, we get the result.

**(iv)⇒(v):** If (iv) is true, then there is some $C > 0$ such that $\frac{a_n}{C} \leq b_n \leq C a_n$. Because for each $p$ we have $e^{\frac{-C a_n}{p}} \leq e^{\frac{-b_n}{p}} \leq e^{\frac{-a_n}{pC}}$, the mapping $I : X \to Y$, defined by $I e_i \triangleq e_i$ for every $i$, is a quasi-diagonal isomorphism.

**(v)⇒(i):** Clear.

In [6], M.M. Dragilev proved the following theorem stating the completeness of the invariant $\Gamma$ in the class $D$, which is the union of regular $d_1$ and $d_2$ classes. Later, he gave an interesting example showing that the $\Gamma$ invariant is not sufficient to examine spaces in $d_0$ (see [7]).

**Theorem 2.8** Let $X = K([a_{pn}])$, and $Y = K([b_{pn}])$ be two $D-$spaces. Then $X \cong Y$ if and only if they belong to the same $D_i-$class $(i = 1, 2)$ and $\Gamma(X) = \Gamma(Y)$.

L. Crone, W.B. Robinson ([2]) and V.P. Kondakov ([14]) showed that the invariant $\Gamma'$ is complete in the class of all regular Köthe spaces. In 1975, P.B. Djakov gave a considerably simpler proof of this result. ([3])

**Theorem 2.9** Let $X = \mathcal{K}([a_{pn}])$ and $Y = \mathcal{K}([b_{pn}])$ be regular nuclear spaces. Then the following statements are equivalent:

($i$) $X \cong Y$;

($ii$) $\Gamma'(X) = \Gamma'(Y)$;

($iii$) $X \overset{qdi}{\cong} Y$.

Hence in each NFS with a regular basis all bases are q.e.. However the insufficiency of the invariant $\Gamma'$ is observed by M.M. Dragilev, B.S. Mitiagin, S. Rolewicz (see [20]) in the following example:

The space $\Lambda_0(n) \times \Lambda_\infty(n)$ has no regular basis whereas $\Lambda_0(n)$ has a regular basis. Therefore $\Lambda_0(n) \times \Lambda_\infty(n)$ is not isomorphic to $\Lambda_0(n)$ , but they have the same diametral and dual diametral dimensions. By this example, we can conclude that $\Gamma$, $\Gamma'$ are too rough to examine spaces without regular basis. Hence it is necessary to consider more general invariant(s) which take the irregularity of bases into account.

## 2.2 AN INVARIANT ON CARTESIAN PRODUCTS OF L.C.S.

**Definition 2.10** Let $X$, $Y$ be two l.c.s.. We will denote $X \approx Y$, and say $X$ is nearly isomorphic to $Y$, in case there is a linear continuous mapping satisfying the following conditions

**(i)** $T(X)$ is closed in $Y$, and $T : X \to T(X)$ is an open mapping,

**(ii)** $\dim\ker T < \infty$, **(iii)** $\dim Y/T(X) < \infty$.

Hence we can regard $T$ as a "nearly $1-1$" and "nearly onto" map.

**Definition 2.11** Let $(X, \mathfrak{U})$ be a l.c.s.. Then we will say that the space $X$ belongs to the class $\mathcal{M}$, and call it a Montel space, if every bounded set is relatively compact, i.e. its closure is compact and $X$ is barreled.

**Definition 2.12** Let $X, Y$ be two l.c.s.. We will write $(X, Y) \in \mathfrak{K}$, if for every linear continuous $T : X \to Y$, there exists a neigborhood $U$ of the origin in $X$ such that $\overline{T(U)}$ is compact in $Y$.

**Definition 2.13** Let $X$ be a l.c.s. and $s \in \mathbb{Z}$. If $s > 0$, then $X^{(s)}$ will denote an arbitrary subspace, say $L$, of $X$ such that $X \cong L \times \mathbb{K}^s$. If $s < 0$, it will be an arbitrary l.c.s., say L, such that $L \cong X \times \mathbb{K}^{-s}$

The proof of the following lemma can be found in [22].

**Douady-Zahariuta's Lemma 2.14** Let $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$ be cartesian products of l.c.s.. Assume $(X_1, Y_2) \in \mathfrak{K}$, $(Y_1, X_2) \in \mathfrak{K}$. Then $X \approx Y$ if and only if $X_i \approx Y_i$ $(i = 1, 2)$.

By the help of above lemma, V.P. Zahariuta defined the following invariant (see [22]) on the class of all cartesian products of the form $X \times Y$ where $(X, Y) \in \mathfrak{K}$.

**Definition 2.15** Let $\mathfrak{X}$, $\mathfrak{Y}$ be classes of l.c.s. such that $(X, Y) \in \mathfrak{K}$ whenever $X$ is in $\mathfrak{X}$ and $Y$ is in $\mathfrak{Y}$. Then

$$\tilde{\Gamma}(X \times Y) \triangleq \{(\Gamma(X^{(s)}), \Gamma(Y^{(-s)})\}_{s \in \mathbb{Z}}$$

is a linear topological invariant on the class $\mathfrak{X} \times \mathfrak{Y} \triangleq \{X \times Y : \ X \in \mathfrak{X} \text{ and } Y \in \mathfrak{Y}\}$.

To prove the following theorem we will need the observations below:

**Fact 2.16** Let $X$ and $Y$ be l.c.s.. Then
**(i)** $Y \cong X^{(s)}$ if and only if $X \cong Y^{(-s)}$
**(ii)** $(X^{(s)})^{(d)} \cong X^{(s+d)}$
**(iii)** If $X \triangleq \mathcal{K}([a_{pn}])$ belongs to class $D$ and $s$ is a nonnegative integer, then both $X \times I\!K^s$ and $X$ belong to the same $d_i$ class.
**Proof:(i)** It is enough to prove when $X$ is isomorphic to $Y^{(s)}$ and $s$ is positive. Now if this is the case, then there exists a subspace of $X$ with dimension $s$ and $X \cong Y \times I\!K^s$. But then $Y$ will be isomorphic to $X^{(-s)}$.
**(ii)** Again it suffices to prove the statement for nonnegative $s, d$. Now Let $Y$ be isomorphic to $X^{(s)}$ and let $Z$ be isomorphic to $Y^{(d)}$. Then we have $X \cong Y \times I\!K^s$ and $Y \cong X \times I\!K^d$. And thus we get $X \cong Y \times I\!K^s \cong Z \times I\!K^{s+d}$.
**(iii)** Let $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{s}$ be the natural bases of $X$ and $I\!K^s$, respectively. Then $\{(0, f_1), ..., (0, f_s), (e_1, 0), (e_2, 0), ...\}$ is an absolute basis for $X \times I\!K^s$ with the following norm system $\|(x, y)\|_p \triangleq |x|_p + \|y\|$ where $\|.\|$ is any norm in $I\!K^s$. Then the Köthe matrix, except the first $s$ rows, will be of the same type as that of $X$.

**Proposition 2.17**(see [22]) Let $X \triangleq \mathcal{K}([a_{pn}])$ belong to the class $d_2$, let $Y \triangleq \mathcal{K}([b_{pn}])$ belong to the classes $\mathcal{M}$ and $d_1$. Then all the linear continuous maps from $X$ into $Y$ are compact.

**Theorem 2.18** If $X, Y \in \mathfrak{D} \triangleq d_2 \times d_1 \bigcap \mathcal{M}$, then $X \cong Y$ if and only if $\check{\Gamma}(X) = \check{\Gamma}(Y)$.
**Proof:** If $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$ both belong to class $\mathfrak{D}$ and are isomorphic, then by the above proposition and Douady-Zahariuta's Lemma 2.14, there is some $s$ in $\mathbb{Z}$ such that $Y_1 \cong {X_1}^{(s)}$ and $Y_2 \cong {X_2}^{(-s)}$. Since $X_1 \cong {Y_1}^{(-s)}$,

$X_2 \cong {Y_2}^{(s)}$, for $d \in \mathbb{Z}$, we get ${X_1}^{(d)} \cong {Y_1}^{(d-s)}$ and ${X_2}^{(-d)} \cong {Y_2}^{(s-d)}$. This implies that $\tilde{\Gamma}(X) \subset \tilde{\Gamma}(Y)$. Changing the roles of the spaces, we conclude that $\tilde{\Gamma}(X)$ is an invariant in the class $\mathfrak{D}$.

Now conversely, let us assume that $X$ is not isomorphic to $Y$. But then for all integer $s$ we have either $Y_1 \ncong {X_1}^{(s)}$ or $Y_2 \ncong {X_2}^{(-s)}$. Thus, by the help of the Theorem 2.8 and above fact, we can conclude that $\tilde{\Gamma}(X)$ is not equal to $\tilde{\Gamma}(X)$.

Now let $\Lambda_\alpha(a)$ be a p.s.s. and let $s \in \mathbb{N}$. If we let $b \triangleq (a_{k+s})_{k=1}^{\infty}$, then $\Lambda_\alpha(b)$ is a subspace of $\Lambda_\alpha(a)$ with $\Lambda_\alpha(a) \cong \Lambda_\alpha(b) \times \mathbb{K}^s$. Hence $(\Lambda_\alpha(a))^{(s)}$ can be taken as $\Lambda_\alpha(b)$. Thus the previous theorem has the following corollary.

**Theorem 2.19** Let $X = \Lambda_0(a^1) \times \Lambda_\infty(a^2)$, $Y = \Lambda_0(b^1) \times \Lambda_\infty(b^2)$ where $a^i$, $b^i$ are increasing to infinity $(i = 1, 2)$. Then $X \cong Y$ if and only if there exists $s \in \mathbb{Z}$ such that $(a^1_{k+s})_{k=1}^{\infty} \asymp (b^1_k)_{k=1}^{\infty}$ and $(a^2_{k-s})_{k=1}^{\infty} \asymp (b^2_k)_{k=1}^{\infty}$.

## 2.3 THE INVARINATS M, m OF MITIAGIN

**Definition 2.20** Let $(a_i)_{i=1}^{\infty}$ be a positive sequence, and $t > \tau$ be positive numbers. Then

$$m_a(t) \triangleq \#\{i \in \mathbb{N} : \; a_i \le t\}, \; M_a(\tau, t] \triangleq \#\{i \in \mathbb{N} : \; \tau < a_i \le t\}$$

where $\#\, A$ stands for the cardinality of the set $A$.

**Definition 2.21** Let $(a_i)_{i=1}^{\infty}$, $(b_i)_{i=1}^{\infty}$ be two positive sequences. We write $M_a \approx M_b$, in case there exists $C > 0$ such that

$$M_a(\tau, t] \le M_b(\frac{\tau}{C}, tC] \text{ and } M_b(\tau, t] \le M_a(\frac{\tau}{C}, tC], \text{ whenever } 0 < \tau < t < \infty.$$

And we write that $m_a \approx m_b$, in case there exists $C > 0$ such that

$$m_a(t) \leq m_b(tC), \text{ and } m_b(t) \leq m_a(tC), \text{ for each } 0 < t < \infty.$$

The proofs of the following results can be found in [17], [24]; here we only state the results. Firstly, in the general case, the invariant $M$ is complete in the class of all p.s.s..

**Theorem 2.22** Let $(a_i)_{i=1}^{\infty}$, $(b_{i=1})_{i=1}^{\infty}$ be two positive sequences. Then the following conditions are equivalent: $(i)$ $\Lambda_\alpha(a) \cong \Lambda_\alpha(b)$ $(\alpha = 0, \infty)$;
$(ii)$ $M_a \approx M_b$;
$(iii)$ $\Lambda_\alpha(a) \overset{p}{\cong} \Lambda_\alpha(b)$ $(\alpha = 0, \infty)$.

Now, in the particular case of Schwartz p.s.s., that is, spaces of the form $\Lambda_\alpha(a)$ where $\alpha = 0, \infty$ and $a_n \nearrow \infty$, all the invariants $m$, $M$, $\Gamma$, $\Gamma'$ are complete.

**Theorem 2.23** Let $(a_i)_{i=1}^{\infty}$, $(b_{i=1})_{i=1}^{\infty}$ be two positive sequences increasing to $\infty$. Then the following statements are equivalent:
**(i)** $\Lambda_\alpha(a) \cong \Lambda_\alpha(b)$ $(\alpha = 0, \infty)$;
**(ii)** $m_a \approx m_b$;
**(iii)** $M_a \approx M_b$;
**(iv)** $\Gamma(\Lambda_\alpha(a)) = \Gamma(\Lambda_\alpha(b))$ $(\alpha = 0, \infty)$;
**(v)** $\Gamma'(\Lambda_\alpha(a)) = \Gamma'(\Lambda_\alpha(b))$ $(\alpha = 0, \infty)$;
**(vi)** $\Lambda_\alpha(a) \overset{qdi}{\cong} \Lambda_\alpha(b)$ $(\alpha = 0, \infty)$.

## 2.4 The $\beta$ INVARIANT OF ZAHARIUTA

For the proofs that we do not give and a more complete list of references, see [24].

**Definition 2.24** Let $(X, \mathfrak{U})$ be a l.c.s., and $A,\ B \subset X$ be absolutely convex subsets of $X$. Then

$$\beta(A, B) \triangleq \sup_{L \in \mathcal{F}} \{\dim L : \ A \cap L \subset B\}$$

where $\mathcal{F}$ is the set of all finite dimensional subspaces of $X$.

**Remark:** The invariant $\beta$ is closely connected to the so called Bernstein diameters, $b_i(V, U)$ (see[8]). Namely, $\beta(V, U)) = \sup_{L \in \mathcal{F}} \sup\{\alpha > 0 : \ \frac{1}{b_i(V,U)} \leq 1\}$.

The following, which is a direct consequence of the definition of an invariant, has practical use in estimates.

**Lemma 2.25** Let $X$ be a l.c.s. and let $V_1 \subset V_2$, $U_2 \subset U_1$ be absolutely convex subsets of $X$. Then $\beta(V_1, U_1) \leq \beta(V_2, U_2)$.

**Definition 2.26** Let $\mathcal{K}([a_{pn}])$ be a Köthe space and let $U_p$, $U_q$ be two absolutely convex neighborhoods of the origin. Then

$$U_p^{1-\alpha} U_q^{\alpha} \triangleq \{(x_n)_{n=1}^{\infty} : \ \sum_{n=1}^{\infty} |x_n| a_{pn}^{1-\alpha} a_{qn}^{\alpha} \leq 1\},$$

where $\alpha \in [0, 1]$, will be called an interpolative neighborhood of the origin.

**Interpolation Lemma 2.27** Let $a^0, b^0, a^1, b^1$ be positive sequences, and assume that $\ell_1(a^0) \xrightarrow{T} \ell_1(b^0)$ and $\ell_1(a^1) \xrightarrow{T} \ell_1(b^1)$ are linear and continuous. Then $T$ can be regarded as a linear continuous mapping from $\ell_1(a^\alpha)$ into $\ell_1(b^\alpha)$
where $a_i^\alpha \triangleq ({a^0}_i)^{1-\alpha}({a^1}_i)^\alpha$ and $b_i^\alpha \triangleq ({b^0}_i)^{1-\alpha}({b^1}_i)^\alpha$ and $\alpha \in [0,1]$. Moreover, $\|T\|_{\ell_1(a^\alpha)\to\ell_1(b^\alpha)} \leq \max(\|T\|_0, \|T\|_1)$.

**Lemma 2.28** Let $(a_i)_{i=1}^\infty, (b_i)_{i=1}^\infty$ be two positive sequences. Then
**(i)** $\beta(B^e(b), B^e(a)) = \#\{i \in I\!N : \frac{b_i}{a_i} \leq 1\}$
where $B^e(b) \triangleq \{(x_n) \in I\!K^{I\!N} : \sum_{n=1}^\infty |x_n| b_n \leq 1\}$.
**(ii)** $B^e(a \vee b) \subset B^e(a) \cap B^e(b) \subset 2B^e(a \vee b)$ where $a \vee b \triangleq (\max\{a_i, b_i\})_{i=1}^\infty$.
**(iii)** $\overline{conv}(B^e(a) \cup B^e(b)) = B^e(a \wedge b)$ where $a \wedge b \triangleq (\min\{a_i, b_i\})_{i=1}^\infty$, $\overline{conv}(A)$ is the closed absolute convex hull of the set $A$.

**Proof:**
(i) Let $X \triangleq \ell_1(a) \cap \ell_1(b)$, $A \triangleq \{i : \frac{b_i}{a_i} \leq 1\}$ and let $Y$ be the subspace of $X$ generated by $\{e_i : i \in A\}$. Now let $L$ be any finite dimensional subspace of $X$ such that $L \cap B^e(a) \subset B^e(b)$ and let $P$ be the canonical projection from $X$ onto $Y$, defined by $Px \triangleq \sum_{i\in A} x_i e_i$. To conclude $\beta(B^e(a), B^e(b)) \leq \#\{i : \frac{b_i}{a_i} \leq 1\}$, it suffices to show that the restiriction of the projection on $L$ is $1-1$. Now let us assume the contrary. Then there is some non-zero element, say $z$, in the kernel of $P$. Thus $(I-P)z = z$ and $w \triangleq \frac{z}{\|z\|_{\ell_1(a)}} \in B^e(a)$. Hence we have

$$\|w\|_{\ell_1(b)} = \|(I-P)w\|_{\ell_1(b)} = \frac{1}{\|z\|_{\ell_1(a)}} \sum_{i\notin A} |z_i| b_i > \frac{1}{\|z\|_{\ell_1(a)}} \sum_{i\notin A} |z_i| a_i$$

$$= \frac{1}{\|z\|_{\ell_1(a)}} \|(I-P)z\|_{\ell_1(a)} = \frac{\|z\|_{\ell_1(a)}}{\|z\|_{\ell_1(a)}} = 1.$$

Consequently, $\|z\|_{\ell_1(b)} > \|z\|_{\ell_1(a)} \geq |z\|_{\ell_1(b)}$, which is a contradiction.
To prove the converse, let $(x_n)_{n=1}^\infty$ be in $Y$. Then

$$\|x\|_{\ell_1(b)} = \sum_{n=1}^\infty |x_n|\ b_n \leq \sum_{n=1}^\infty |x_n|\ a_n = \|x\|_{\ell_1(a)}.$$

It implies that $Y \cap B^e(a) \subset B^e(b)$. Therefore

$$\beta(B^e(a), B^e(b)) \geq \#\{i : \frac{b_i}{a_i} \leq 1\}.$$

**(ii)** This follows from the definition of $B^e(b)$.

**(iii)** Clearly, $B^e(a) \cup B^e(b) \subset B^e(a \vee b)$. Since $B^e(a \vee b)$ is absolute convex and contains $B^e(a) \cup B^e(b)$, $\overline{conv}(B^e(a) \cup B^e(b))$ lies entirely in $B^e(a \vee b)$.
Conversely, if $x \triangleq \{x_n\}_{n=1}^{\infty}$ is in $B^e(a \vee b)$, then let us define
$A \triangleq \{n \in I\!N : a_n \leq b_n\}$ and for each $m$, let $x^m$ be the sequence $(x_1, x_2, ..., x_m, 0, ...)$. Then since $\|x - x^m\|_{\ell_1} \to 0$, it will be sufficient if we prove that $x^m$'s are in $\overline{conv}(B^e(a) \cup B^e(b))$. Now define $\alpha \triangleq \sum_{n \in A} |x_n|\ a_n$ and $\beta \triangleq \sum_{n \notin A} |x_n|\ b_n$. It implies that

$$1 \geq \sum_{n \in I\!N} |x_n|\ (a \vee b)_n = \sum_{n \in A} |x_n|\ (a \vee b)_n + \sum_{n \notin A} |x_n|\ (a \vee b)_n$$

$$= \sum_{n \in A} |x_n|\ a_n + \sum_{n \notin A} |x_n|\ b_n = \alpha + \beta.$$

It is no loss of generality to assume that both $\alpha$ and $\beta$ are nonzero. Because for otherwise $x$ would be in either $B^e(a)$ or in $B^e(b)$. Now since

$$x^m = \alpha \sum_{n \in A} \frac{x_i}{\alpha} e_i + \beta \sum_{i \notin A} \frac{x_i}{\beta} e_i$$

$$\text{and } 1 = \|\sum_{i \in A} \frac{x_i}{\alpha} e_i\|_{\ell_1(a)} = \|\sum_{i \notin A} \frac{x_i}{\alpha} e_i\|_{\ell_1(b)},$$

$x^m$'s are in the absolute convex hull of $B^e(a) \cup B^e(b)$.

The below is a direct consequence of Lemma 2.25 and Lemma 2.28.

**Lemma 2.29** Let $a^1, a^2, a^3, a^4$ be positive sequences. Then
(i) $\#\{i \in I\!N : \frac{\max(a_i^3, a_i^2)}{\min(a_i^2, a_i^1)} \le 1\} = \#\{i \in I\!N : \frac{a_i^3}{a_i^2} \le 1,\ \frac{a_i^2}{a_i^1} \le 1\}$
$\le \beta(\overline{B^e(a^3) \cap B^e(a^2)}, \overline{conv}(B^e(a^1) \cup B^e(a^2)))$.
(ii) $\beta(B^e(a^2) \bigcap B^e(a^3), \overline{conv}(B^e(a^1) \cup B^e(a^4))) \le \#\{i \in I\!N : \frac{\frac{1}{2}\max(a_i^3, a_i^2)}{\min(a_i^4, a_i^1)} \le 1\}$
$\le \#\{i \in I\!N : \frac{a_i^3}{a_i^4} \le 2,\ \frac{a_i^2}{a_i^1} \le 2\}$.

Using the above properties of $\beta$, one can use it as a two parameter ( by using more complicated interpolative neighborhoods, also as a multi-parameter) invariant as follows: If $(E, \mathfrak{U})$ is isomorphic to $(Y, \mathfrak{V})$, then

$$\forall U_1\ \exists V_1\ \forall V_2 \subset V_1\ \exists U_2, C > 0 \text{ such that}$$
$$\forall t, \tau > 0 \text{ it follows that } \beta(tV_2, \tau V_1) \le \beta(CtU_2, \frac{\tau}{C}U_1)$$

and also the converse inequality with an appropriate symmetry.

By the help of the invariants $\beta$ and $M$, M. Yurdakul, V.P.Zahariuta gave an alternative proof of the Theorem 2.19

## 2.5 SOME RECENT RESULTS

The following results are obtained by using more developed forms of the above invariant $\beta$, basically using more complicated neighborhoods formed by taking intersection, convex hull of interpolative neighborhoods of the origin.

**(A)** P.B.Djakov, M.Yurdakul, V.P.Zahariuta gave a complete classification of cartesian products of spaces of the form $\Lambda_0(a) \times \Lambda_\infty(b)$ where $a_i, b_i$ are arbitrary positive sequences (not necessarily increasing to infinity). (see [4])

**Theorem 2.30** Let $X = \Lambda_0(a) \times \Lambda_\infty(b)$, $Y = \Lambda_0(\tilde{a}) \times \Lambda_\infty(\tilde{b})$. Then the following conditions are equivalent:

**(i)** $X \cong Y$;

**(ii)** $X \overset{qdi}{\cong} Y$;

**(iii)** either $X,\ Y$ are both Schwartz spaces and there exists $s \in \mathbb{Z}$, permutations $\sigma,\ \gamma : \mathbb{N} \to \mathbb{N}$ such that $(\tilde{a}_k)_{k=1}^\infty \asymp (a_{\sigma(k)+s})_{k=1}^\infty$, $(\tilde{b}_k)_{k=1}^\infty \asymp (a_{\gamma(k)-s})_{k=1}^\infty$,
or $X,\ Y$ are non-Schwartz spaces and $M_a \approx M_{\tilde{a}}$, $M_b \approx M_{\tilde{b}}$.

**(B)** With the help of the invariant $\beta$, A.Goncharov and V.P.Zahariuta gave some necessary conditions of isomorphism of spaces of the form $\Lambda_0'(a)\hat{\otimes}_\pi\Lambda_0(b)$, $\Lambda_0'(a)\hat{\otimes}_\pi\Lambda_\infty(b)$, $\Lambda_\infty'(a)\hat{\otimes}_\pi\Lambda_0(b)$, $\Lambda_\infty'(a)\hat{\otimes}_\pi\Lambda_\infty(b)$ where $a$ and $b$ are nuclear exponent sequences, that is, the corresponding p.s.s's are nuclear. (see [8])

**(C)** A.Goncharov, T.Terzioğlu, V.P.Zahariuta gave complete isomorphic classification of spaces of the form $s\hat{\otimes}_\pi\Lambda_\infty'(a)$ (see [9]). Later, in [10], they proved similar results for the spaces of the form $\Lambda_\infty(a)\hat{\otimes}_\pi\Lambda_\infty'(b)$ where at least one of the sequences $a,\ b$ increasing to infinity is stable.

**(D)**In [13], M.Kocatepe and V.P.Zahariuta investigated an invariant in the class of Köthe spaces of the type $\mathcal{H}_\infty(\kappa, \gamma, a)$ and gave some examples of this type that show their invariant is stronger than some of the old invariants. Namely, the examples they gave have the same $D_\varphi$ whereas they can be distinguished by this new invariant.

**(E)** P.B.Djakov, V.P.Zahariuta gave a complete isomorphic classification of stable spaces of the form $\mathcal{K}([a_{pn}])$ where $a_{pn} = e^{[-\frac{1}{p}+\chi(p-\kappa_n)]a_n}$. (see [5]) In fact, they showed that if two such spaces are isomorphic then there is a quasidagonal isomorphism between them. We state the result below:

**Theorem 2.31** Let $X \triangleq \mathcal{D}_0(\kappa, \gamma, a)$ and $Y \triangleq \mathcal{D}_0(\tilde{\kappa}, \hat{\gamma}, \dot{a})$, where $\gamma_n \triangleq \tilde{\gamma}_n \triangleq 1$ for each $n$ in $I\!N$, be stable Köthe spaces. Then

$$X \cong Y \text{ if and only if } X \overset{q.d.i.}{\cong} Y.$$

# CHAPTER 3
# QUASIDIAGONAL MAPS BETWEEN SOME TYPES KÖTHE SPACES

The techniques we use in this chapter were introduced by P.B.Djakov and V.P.Zahariuta in [5].

In this chapter we will classify the quasidiagonal mappings between spaces of the types $\mathcal{D}_\infty(\kappa, \gamma, a)$, $\mathcal{D}_0(\kappa, \gamma, a)$, and $\mathcal{H}_\infty(\kappa, \gamma, a)$, for definitions see section 1.6. Moreover, we will characterize such mappings between spaces one of the type $\mathcal{D}_\infty(\kappa, \gamma, a)$ and the other of the type $\mathcal{H}_\infty(\tilde{\kappa}, \tilde{\gamma}, \tilde{a})$. Finally, we will conclude this chapter by a counterexample which shows that the class of spaces of the form $\mathcal{D}_\infty(\kappa, \gamma, a)$ is not identical (in topological sense) with that of spaces of the form $\mathcal{H}_\infty(\tilde{\kappa}, \tilde{\gamma}, \tilde{a})$.

Since the proofs of the results are similar, we will give only one of them completely, and state the others.

Now let $\kappa$, $\tilde{\kappa}$ be functions from $I\!N$ to $I\!N$, $\pi$ is a permutation of $I\!N$, and let $m$ be in $I\!N$. Then we define the following sets

$I_1 \triangleq \{\nu = (i_n) : \ \kappa_{i_n}, \ \tilde{\kappa}_{\pi(i_n)} \to \infty\}$;
$I_2 \triangleq \{\nu = (i_n) : \ \kappa_{i_n} \to \infty, \ (\tilde{\kappa}_{\pi(i_n)})_{n=1}^{\infty}$ is bounded $\}$;
$I_3 \triangleq \{\nu = (i_n) : \ (\kappa_{i_n})_{n=1}^{\infty}$is bounded $, \tilde{\kappa}_{\pi(i_n)} \to \infty\}$;
$I_4 \triangleq \{\nu = (i_n) : \ (\kappa_{i_n})_{n=1}^{\infty}, \ (\tilde{\kappa}_{\pi(i_n)})_{n=1}^{\infty}$are bounded $\}$.
$I\!N_m \triangleq \{i \in I\!N : \ \kappa_i = m\}$.

**Theorem 3.1** Let $T : \mathcal{D}_{\infty}(\kappa, \gamma,\text{a}) \to \mathcal{D}_{\infty}(\tilde{\kappa}, \tilde{\gamma}, \tilde{a})$ be a quasidiagonal operator defined by $Te_i = e^{t_i} e_{\pi(i)}$. Then $T$ is an isomorphism if and only if
**(i)** $(a_i)_{i=1}^{\infty} \asymp (\tilde{a}_{\pi(i)})_{i=1}^{\infty}$;
**(ii)** For all $\nu \in I_1$ there exists $D$ such that $\left|\frac{t_{i_n}}{a_{i_n}}\right| \leq D$ ;
For all $\nu \in I_2$ there exists $D > 0$ such that $\left|\frac{t_{i_n} + \tilde{\gamma}_{\pi(i_n)} \tilde{a}_{\pi(i_n)}}{a_{i_n}}\right| \leq D$;
For all $\nu \in I_3$ there exists $D > 0$ such that $\left|\frac{t_{i_n} - \gamma_{i_n} a_{i_n}}{a_{i_n}}\right| \leq D$ ;
For all $\nu \in I_4$ there exists $D > 0$ such that $\left|\frac{t_{i_n} + \tilde{\gamma}_{\pi(i_n)} \tilde{a}_{\pi(i_n)} - \gamma_{i_n} a_{i_n}}{a_{i_n}}\right| \leq D$.

**Proof: (necessity)** Let $T$ be an isomorphism in the given form. Then by the continuity of $T$ and $T^{-1}$ we get

$$\forall p_1 \ \ \exists \tilde{p_1} \ \ \forall \tilde{p_2} > \tilde{p_1} \ \ \exists p_2 > p_1 \ \ \exists M > 0 \text{ such that}$$

$$|e_i|_{p_1} \leq e^{M/2} ||Te_i||_{\tilde{p}_1}, \ ||Te_i||_{\tilde{p}_2} \leq e^{M/2} |e_i|_{p_2} \text{ for each } i \in I\!N. \text{ Hence}$$

$$\frac{||Te_i||_{\tilde{p}_2}}{||Te_i||_{\tilde{p}_1}} \leq e^M \frac{|e_i|_{p_2}}{|e_i|_{p_1}} \text{ for each } i \in I\!N \qquad \text{(A)}$$

where $\{|.|\}_{k=1}^{\infty}$ respectively $\{||.||\}_{k=1}^{\infty}$ denote the fundamental norms in $\mathcal{D}_{\infty}(\kappa, \gamma,\text{a})$ respectively $\mathcal{D}_{\infty}$ $(\tilde{\kappa}, \tilde{\gamma}, \tilde{a})$.

Similarly,

$$\forall \tilde{q}_1 \ \exists q_1 \ \forall q_2 > q_1 \ \exists \tilde{q}_2 > \tilde{q}_1 \ \exists \tilde{M} > 0 \text{ such that}$$

$\|Te_i\|_{\tilde{q}_1} \leq e^{\tilde{M}/2}|e_i|_{q_1}$, $|e_i|_{q_2} \leq e^{\tilde{M}/2}\|Te_i\|_{\tilde{q}_2}$ for each $i \in I\!N$ . Hence

$$\frac{|e_i|_{q_2}}{|e_i|_{q_1}} \leq e^{\tilde{M}} \frac{\|Te_i\|_{\tilde{q}_2}}{\|Te_i\|_{\tilde{q}_1}} \text{ for each } i \in I\!N \qquad \text{(B)}$$

Now (A) gives

$$\frac{e^{t_i} e^{[(\tilde{p_2}+\tilde{\gamma}_{\pi(i)}\chi(\tilde{p_2}-\tilde{\kappa}_{\pi(i)}))\tilde{a}_{\pi(i)}]}}{e^{t_i} e^{[(\tilde{p_1}+\tilde{\gamma}_{\pi(i)}\chi(\tilde{p_1}-\tilde{\kappa}_{\pi(i)}))\tilde{a}_{\pi(i)}]}} \leq e^M \frac{e^{[(p_2+\gamma_i\chi(p_2-\kappa_i))a_i]}}{e^{[(p_1+\gamma_i\chi(p_1-\kappa_i))a_i]}} \text{ for each } i \in I\!N.$$

Taking the logarithm of both sides, we have

$$[\tilde{p_2} - \tilde{p_1} + \tilde{\gamma}_{\pi(i)}\{\chi(\tilde{p_2} - \tilde{\kappa}_{\pi(i)}) - \chi(\tilde{p_1} - \tilde{\kappa}_{\pi(i)})\}]\tilde{a}_{\pi(i)}$$
$$\leq M + [p_2 - p_1 + \gamma_i\{\chi(p_2 - \kappa_i) - \chi(p_1 - \kappa_i)\}]a_i \text{ for each } i \in I\!N.$$

**(i)** Let $(i_n)$ be in $I_1$. Let $p_2 > p_1 \triangleq 1$, $\tilde{p_2} \triangleq \tilde{p_1} + 1 > \tilde{p}_1$. Since $\kappa_{i_n}$, $\tilde{\kappa}_{\pi(i_n)} \to \infty$ , there exists $n_0 \in I\!N$, such that for all $n \geq n_0$ $\kappa_{i_n} > p_2$, $\tilde{\kappa}_{\pi(i)} > \tilde{p_2}$. So (A) becomes

$$[\tilde{p_2} - \tilde{p_1} + \tilde{\gamma}_{\pi(i_n)}\{0 - 0\}]\tilde{a}_{\pi(i_n)} \leq M + [p_2 - p_1 + \gamma_{i_n}\{0 - 0\}]a_{i_n} \text{ for each } n \geq n_0.$$

But since $a_{i_n} \to \infty$, we have $\limsup_{n\to\infty} \frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} \leq \frac{p_2-p_1}{\tilde{p_2}-\tilde{p_1}} < \infty$. Likewise, using the inequality (B) with $\tilde{q}_2 > \tilde{q}_1 \triangleq 1$, $q_2 \triangleq q_1 + 1 > q_1$, we get $(\tilde{a}_{\pi(i_n)})_{n=1}^{\infty} \asymp (a_{i_n})_{n=1}^{\infty}$.

**(ii)** Let $(i_n)$ be in $I_2$. Let $p_2 > p_1 \triangleq 1$, $\tilde{p_2} \triangleq \tilde{p_1} + 1 > \sup_{n\in I\!N} \tilde{\kappa}_{\pi(i_n)}$. Because $\kappa_{i_n} \to \infty$, there exists $n_0 \in I\!N$ such that for every $n \geq n_0$, $\kappa_{i_n} > p_2$. Thus (A) becomes

$$[\tilde{p_2} - \tilde{p_1} + \tilde{\gamma}_{\pi(i_n)}\{1 - 1\}]\tilde{a}_{\pi(i_n)} \leq M + [p_2 - p_1 + \gamma_{i_n}\{0 - 0\}]a_{i_n} \text{ for each } n \geq n_0.$$

But since $a_{i_n} \to \infty$, we have $\limsup_{n\to\infty} \frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} \leq \frac{p_2-p_1}{\tilde{p_2}-\tilde{p_1}} < \infty$. Similarly, using the inequality (B) with $\tilde{q}_2 > \tilde{q}_1 \triangleq 1 + \sup_{n\in I\!N} \tilde{\kappa}_{\pi(i_n)}$, $q_2 \triangleq q_1 + 1 > q_1$, we get

$(\tilde{a}_{\pi(i_n)})_{n=1}^{\infty} \asymp (a_{i_n})_{n=1}^{\infty}$.

**(iii)** Let $(i_n)$ be in $I_3$. Let $p_2 > p_1 \triangleq 1 + \sup_{n \in \mathbb{N}} \kappa_{i_n}$, $\tilde{p_2} \triangleq \tilde{p_1} + 1 > \tilde{p}_1$. Since $\tilde{\kappa}_{\pi(i_n)} \to \infty$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ $\tilde{\kappa}_{\pi(i_n)} > p_2$. Hence (A) becomes

$$[\tilde{p_2} - \tilde{p_1} + \tilde{\gamma}_{\pi(i_n)}\{0 - 0\}]\tilde{a}_{\pi(i_n)} \leq M + [p_2 - p_1 + \gamma_{i_n}\{1 - 1\}]a_{i_n} \text{ for all } n \geq n_0.$$

Now as $a_{i_n} \to \infty$, we obtain $\limsup_{n \to \infty} \frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} \leq \frac{p_2 - p_1}{\tilde{p_2} - \tilde{p_1}} < \infty$. By the same token, using (B) with $\tilde{q}_2 > \tilde{q}_1 \triangleq 1$, $q_2 \triangleq q_1 + 1 > 1 + \sup_{n \in \mathbb{N}} \kappa_{i_n}$, we get $(\tilde{a}_{\pi(i_n)})_{n=1}^{\infty} \asymp (a_{i_n})_{n=1}^{\infty}$.

**(iv)**

Let $(i_n)$ be in $I_4$. Let $p_2 > p_1 \triangleq 1 + \sup_{n \in \mathbb{N}} \kappa_{i_n}$, $\tilde{p_2} \triangleq \tilde{p_1} + 1 > \sup_{n \in \mathbb{N}} \tilde{\kappa}_{\pi(i_n)} + 1$. Therefore (A) becomes

$$[\tilde{p_2} - \tilde{p_1} + \tilde{\gamma}_{\pi(i_n)}\{1 - 1\}]\tilde{a}_{\pi(i_n)} \leq M + [p_2 - p_1 + \gamma_{i_n}\{1 - 1\}]a_{i_n} \quad \text{for all } n \in \mathbb{N}.$$

But since $a_{i_n} \to \infty$, we have $\limsup_{n \to \infty} \frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} \leq \frac{p_2 - p_1}{\tilde{p_2} - \tilde{p_1}} < \infty$. Similarly, using the inequality (B) with $\tilde{q}_2 > \tilde{q}_1 \triangleq 1 + \sup_{n \in \mathbb{N}} \tilde{\kappa}_{\pi(i_n)}$, $q_2 \triangleq q_1 + 1 > 1 + \sup_{i \in \mathbb{N}} \kappa_{i_n}$, we get $(\tilde{a}_{\pi(i_n)})_{n=1}^{\infty} \asymp (a_{i_n})_{n=1}^{\infty}$.

**Claim:** *There exists $C > 0$ such that $\frac{1}{C} \leq \frac{\tilde{a}_{\pi(i)}}{a_i} \leq C$ for all $i \in \mathbb{N}$*

Let us assume the contrary, i.e. for each $C > 0$ there exists $i \in \mathbb{N}$ such that either $\frac{\tilde{a}_{\pi(i)}}{a_i} > C$ or $\frac{\tilde{a}_{\pi(i)}}{a_i} < \frac{1}{C}$. Letting $C = n \in \mathbb{N}$, we get a sequence $(i_n)$ such that for every $n$ in $\mathbb{N}$ either $\frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} > n$ or $\frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} < \frac{1}{n}$. Passing to a subsequence of $(i_n)$, say $(i_{n_m})$, we either get $\frac{\tilde{a}_{\pi(i_{n_m})}}{a_{i_{n_m}}} \to \infty$ or $\frac{\tilde{a}_{\pi(i_{n_m})}}{a_{i_{n_m}}} \to 0$ . So without loss of generality say $\frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} \to \infty$. Now if $\kappa_{i_n} \to \infty$, then $\tilde{\kappa}_{\pi(i_n)} \not\to \infty$. For otherwise, we would have a sequence $(i_n) \in I_1$ whereas $(a_{i_n}) \not\asymp (\tilde{a}_{\pi(i_n)})$, which is a contradiction. Hence there is a subsequence of $(i_n)$, say $(i_{n_m})$, such that $(\tilde{\kappa}_{\pi(i_{n_m})})$ is bounded. But in this case, we would have a sequence $(i_{n_m}) \in I_2$ such that $(a_{i_{n_m}}) \not\asymp (\tilde{a}_{\pi(i_{n_m})})$, which is a contradiction. Thus $\kappa_{i_n} \not\to \infty$. But then there is a subsequence $(i_{n_m})$ such that $(\kappa_{i_{n_m}})$ is bounded. Now if $\tilde{\kappa}_{\pi(i_{n_m})} \to \infty$, then we would have a sequence $(i_{n_m}) \in I_3$ such that $(a_{i_{n_m}}) \not\asymp (\tilde{a}_{\pi(i_{n_m})})$, which is a contradiction. So $\tilde{\kappa}_{\pi(i_{n_m})} \not\to \infty$. But then there is a subsequence of $(i_{n_m})$ say, $(i_{n_{m_l}})$, such that $\tilde{\kappa}_{\pi(i_{n_{m_l}})}$ is bounded. As a result, we would have a sequence $(i_{n_{m_l}}) \in I_4$ but $(a_{i_{n_{m_l}}}) \not\asymp (\tilde{a}_{\pi(i_{n_{m_l}})})$, which is a contradiction. Therefore there exists $C > 0$ such that $\frac{1}{C} \leq \frac{\tilde{a}_{\pi(i)}}{a_i} \leq C$ for all $i \in I\!N$.

Now since $T$ is continuous for each $\tilde{p} \in I\!N$ there exists $p \in I\!N$, $M > 0$ such that $||Tx||_{\tilde{p}} \leq e^M |x|_p$ for all $x \in \mathcal{D}_\infty(\kappa,\gamma,\text{a})$ and since $T^{-1}$ is continuous for this $p$ there exists $\tilde{\tilde{p}}$, $\tilde{M} > 0$ such that $|x|_p \leq e^{\tilde{M}} ||Tx||_{\tilde{\tilde{p}}}$ for every $x \in \mathcal{D}_\infty(\kappa,\gamma,\text{a})$. Putting $x = e_i$ and taking the logaritm of all sides, we have

$$t_i + [\tilde{p} + \tilde{\gamma}_{\pi(i)}\chi(\tilde{p} - \tilde{\kappa}_{\pi(i)})]\tilde{a}_{\pi(i)} \leq M + [p + \gamma_i\chi(p - \kappa_i)]a_i;$$

$$-t_i + [p + \gamma_i\chi(p - \kappa_i)]a_i \leq \tilde{M} + [\tilde{\tilde{p}} + \tilde{\gamma}_{\pi(i)}\chi(\tilde{\tilde{p}} - \tilde{\kappa}_{\pi(i)})]\tilde{a}_{\pi(i)} \text{ for all } i \in I\!N.$$

Now

(i) Let $(i_n)$ be in $I_1$. As $\kappa_{i_n}, \tilde{\kappa}_{\pi(i_n)} \to \infty$ there exists $n_0 \in I\!N$ such that for each $n \geq n_0$ $\kappa_{i_n} > p$, $\tilde{\kappa}_{\pi(i_n)} > \tilde{\tilde{p}}$. Hence we obtain

$$-\frac{\tilde{M}}{a_{i_n}} - \tilde{\tilde{p}}\frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} + p \leq \frac{t_{i_n}}{a_{i_n}} \leq \frac{M}{a_{i_n}} - \tilde{p}\frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} + p \qquad \text{for all } n \geq n_0.$$

Now since $a_{i_n} \to \infty$ there exists $n_1 \geq n_0$ such that for all $n \geq n_1$ $\frac{1}{a_{i_n}} < \tilde{\tilde{p}}C$. Thus

$$-2\tilde{\tilde{p}}\tilde{M}C \leq \frac{t_{i_n}}{a_{i_n}} \leq 2\tilde{p}MC \qquad n \geq n_1.$$

Now let

$$D \triangleq \max(2\tilde{\hat{p}}\tilde{M}C,\ 2\tilde{\hat{p}}MC,\ \{\left|\frac{t_{i_n}}{a_{i_n}}\right| :\ n < n_1\}).$$

Therefore

$$\left|\frac{t_{i_n}}{a_{i_n}}\right| \leq D \quad n \in I\!N.$$

**(ii)** Let $(i_n)$ be in $I_2$. Let $\tilde{p} \triangleq 1 + \sup_{n\in I\!N} \tilde{\kappa}_{\pi(i_n)}$. Since $\kappa_{i_n} \to \infty$ there exists $n_0 \in I\!N$ such that for all $n \geq n_0$, $\kappa_{i_n} > p$. Thus we have

$$-\frac{\tilde{M}}{a_{i_n}} - \tilde{\tilde{p}}\frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} + p \leq \frac{t_{i_n} + \tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)}}{a_{i_n}} \leq \frac{M}{a_{i_n}} - \tilde{p}\frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} + p \qquad \text{for all } n \geq n_0.$$

Now because $a_{i_n} \to \infty$ there exists $n_1 \geq n_0$ such that for all $n \geq n_1$ $\frac{1}{a_{i_n}} < C$. Thus

$$(-\tilde{\hat{p}} - \tilde{M})C \leq \frac{t_{i_n} + \tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)}}{a_{i_n}} \leq Mp + p \qquad n \geq n_1.$$

Letting

$$D \triangleq \max(p + Mp,\ (\tilde{\hat{p}} + \tilde{M})C,\ \{\left|\frac{t_{i_n} + \tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)}}{a_{i_n}}\right| :\ n < n_1\}),$$

we have

$$\left|\frac{t_{i_n} + \tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)}}{a_{i_n}}\right| \leq D \qquad n \in I\!N.$$

**(iii)** Let $(i_n)$ lie $I_3$. Let $\tilde{p} \triangleq 1$ and $p > \sup_{n\in I\!N}(\kappa_{i_n})$. As $\tilde{\kappa}_{\pi(i_n)} \to \infty$ there exists $n_0 \in I\!N$ such that for each $n \geq n_0$ $\tilde{\kappa}_{\pi(i_n)} > \tilde{\tilde{p}}$. Therefore

$$-\frac{\tilde{M}}{a_{i_n}} - \tilde{\tilde{p}}\frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} + p \leq \frac{t_{i_n} - \gamma_{i_n}a_{i_n}}{a_{i_n}} \leq \frac{M}{a_{i_n}} - \tilde{p}\frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} + p \text{ for all } n \geq n_0.$$

Now as $a_{i_n} \to \infty$ there exists $n_1 \geq n_0$ such that for all $n \geq n_1$ $\frac{1}{a_{i_n}} < C$. Thus

$$-\tilde{\tilde{p}}C - p\tilde{M} \leq \frac{t_{i_n} - \gamma_{i_n}a_{i_n}}{a_{i_n}} \leq Mp + p \qquad \text{for all } n \geq n_1.$$

Let

$$D \triangleq \max(p + Mp,\ \tilde{p}C + p\tilde{M},\ \left\{\left|\frac{t_{i_n} - \gamma_{i_n} a_{i_n}}{a_{i_n}}\right| :\ n < n_1\right\}).$$

Hence

$$\left|\frac{t_{i_n} - \gamma_{i_n} a_{i_n}}{a_{i_n}}\right| \leq D \quad \text{for all } n \in I\!N.$$

**(iv)** Let $(i_n)$ be in $I_4$. Let $\check{p} \triangleq 1 + \sup_{n\in I\!N} \tilde{\kappa}_{\pi(i_n)}, p > \sup_{n \in I\!N} \kappa_{i_n}, \tilde{p} > \sup_{n\in I\!N} \tilde{\kappa}_{\pi(i_n)}$. As a result

$$-\frac{\tilde{\tilde{M}}}{a_{i_n}} - \tilde{\tilde{p}}\frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} + p \leq \frac{t_{i_n} - \tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)} - \gamma_{i_n} a_{i_n}}{a_{i_n}} \leq \frac{M}{a_{i_n}} - \check{p}\frac{\tilde{a}_{\pi(i_n)}}{a_{i_n}} + p \text{ for all } n \in I\!N.$$

Because $a_{i_n} \to \infty$ there exists $n_0$ such that for each $n \geq n_0$ $\frac{1}{a_{i_n}} < 1$. Therefore

$$-\check{p}C - \tilde{M} \leq \frac{t_{i_n} + \tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)} - \gamma_{i_n} a_{i_n}}{a_{i_n}} \leq M + p \quad \mathit{for\ all} \qquad n \geq n_0.$$

Now let

$$D \triangleq \max(p + M,\ \tilde{p}C + \tilde{M},\ \{\left|\frac{t_{i_n} + \tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)} - \gamma_{i_n} a_{i_n}}{a_{i_n}}\right| :\ n < n_0\}).$$

Hence

$$\left|\frac{t_{i_n} + \tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)} - \gamma_{i_n} a_{i_n}}{a_{i_n}}\right| \leq D \quad \mathit{for\ all}\ n \in I\!N$$

**(Sufficiency):** Let $\tilde{p}$ be given.

CLAIM 1: *There exist* $p_1 > \tilde{p},\ D_1 > 0$ *such that for all* $i$ *such that*

$\kappa_i > p_1,\ \tilde{\kappa}_{\pi(i)} > p_1 \quad \left|\frac{t_i}{a_i}\right| \leq D_1.$

Suppose the contrary,

for each $p_1 > \tilde{p}$, $D_1 > 0$ there exists $i \in I\!N$ such that $\kappa_i > p_1$ $\tilde{\kappa}_{\pi(i)} > p_1$ but $\left|\frac{t_i}{a_i}\right| > D_1$. Letting $p_1 = D_1 = n \in I\!N$ , we get a sequence $(i_n)$ in $I_1$ but $\left|\frac{t_{i_n}}{a_{i_n}}\right| \to \infty$, which is a contradiction.

CLAIM 2: *There exist* $p_2 > p_1,\ D_2 > D_1$ *such that for all* $i \in I\!N$ *such that*

$\kappa_i > p_2,\ \tilde{\kappa}_{\pi(i)} \leq p_1 \quad \left|\frac{t_i+\tilde{\gamma}_{\pi(i)}\tilde{a}_{\pi(i)}}{a_i}\right| \leq D_2.$

For otherwise,

for all $p_2 > p_1$, $D_2 > D_1$ there exists $i \in I\!N$ such that $\kappa_i > p_2$, $\tilde{\kappa}_{\pi(i)} \leq p_1$ but $\left|\frac{t_i+\tilde{\gamma}_{\pi(i)}\tilde{a}_{\pi(i)}}{a_i}\right| > D_2$. If we let $p_2 = D_2 = n \in I\!N$, there will be a sequence $(i_n)$ in $I_2$ with $\left|\frac{t_{i_n}+\tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)}}{a_{i_n}}\right| \to \infty$, which is a contradiction.

CLAIM 3: *There exist* $p_3 > p_2,\ D_3 > D_2$ *such that for every* $i \in I\!N$ *such that*

$\kappa_i \leq p_2,\ \tilde{\kappa}_{\pi(i)} > p_3 \quad \left|\frac{t_i-\gamma_i a_i}{a_i}\right| \leq D_3.$

If otherwise,

for all $p_3 > p_2$, $D_3 > D_2$ there exists $i \in I\!N$ $\kappa_i \leq p_2$, $\tilde{\kappa}_{\pi(i)} > p_3$ but $\left|\frac{t_i-\gamma_i a_i}{a_i}\right| > D_3$. Letting $p_3 = D_3 = n \in I\!N$, we have a sequence $(i_n) \in I_3$ but $\left|\frac{t_{i_n}-\gamma_{i_n} a_{i_n}}{a_{i_n}}\right| \to \infty$, which is a contradiction.

CLAIM 4: *There exist* $D_4 > D_3$ *such that for all* $i \in I\!N$ *such that*

$\kappa_i \leq p_2,\ \tilde{\kappa}_{\pi(i)} \leq p_3 \quad \left|\frac{t_i+\tilde{\gamma}_{\pi(i)}\tilde{a}_{\pi(i)}-\gamma_i a_i}{a_i}\right| \leq D_4.$

Assume the contrary,

for all $D_4 > D_3$, there exists $i \in I\!N$ $\kappa_i \leq p_2$, $\tilde{\kappa}_{\pi(i)} \leq p_3$ $\left|\frac{t_i+\tilde{\gamma}_{\pi(i)}\tilde{a}_{\pi(i)}-\gamma_i a_i}{a_i}\right| > D_4$. Now let $D_4 = n \in I\!N$. Then there is a sequence $(i_n) \in I_4$ such that

$\left|\frac{t_{i_n}+\tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)}-\gamma_{i_n}a_{i_n}}{a_{i_n}}\right| \to \infty$, which is a contradiction.

Now let $D \triangleq D_4,\ p > \max(p_2, \tilde{p}C + D)$.

**(i)** For all $i \in I\!N$ such that $\kappa_i > p_2,\ \tilde{\kappa}_{\pi(i)} > p_1$

$t_i + \tilde{p}\tilde{a}_{\pi(i)} + \tilde{\gamma}_{\pi(i)}\ \chi(\tilde{p} - \tilde{\kappa}_{\pi(i)})\ \tilde{a}_{\pi(i)} = t_i + \tilde{p}\tilde{a}_{\pi(i)} \leq Da_i + \tilde{p}Ca_i \leq pa_i$
$\leq pa_i + \gamma_i\chi(p - \kappa_i)a_i$.

**(ii)** For each $i \in I\!N$ such that $\kappa_i > p_2,\ \tilde{\kappa}_{\pi(i)} \leq p_1$

$t_i + \tilde{p}\tilde{a}_{\pi(i)} + \tilde{\gamma}_{\pi(i)}\ \chi(\tilde{p} - \tilde{\kappa}_{\pi(i)})\ \tilde{a}_{\pi(i)} \leq t_i + \tilde{p}\tilde{a}_{\pi(i)} + \tilde{\gamma}_{\pi(i)}\tilde{a}_{\pi(i)} \leq Da_i + \tilde{p}Ca_i$
$\leq pa_i = pa_i + \gamma_i\chi(p - \kappa_i)a_i$.

**(iii)** For every $i \in I\!N$ such that $\kappa_i \leq p_2,\ \tilde{\kappa}_{\pi(i)} > p_3$

$t_i + \tilde{p}\tilde{a}_{\pi(i)} + \tilde{\gamma}_{\pi(i)}\ \chi(\tilde{p} - \tilde{\kappa}_{\pi(i)})\ \tilde{a}_{\pi(i)} = t_i + \tilde{p}\tilde{a}_{\pi(i)} \leq Da_i + \gamma_i a_i + \tilde{p}\tilde{a}_{\pi(i)}$
$\leq (\tilde{p}C + D)a_i + \gamma_i a_i \leq pa_i + \gamma_i\chi(p - \kappa_i)a_i$.

**(iv)** For all $i \in I\!N$ such that $\kappa_i \leq p_2,\ \tilde{\kappa}_{\pi(i)} \leq p_3$

$t_i + \tilde{p}\tilde{a}_{\pi(i)} + \tilde{\gamma}_{\pi(i)}\ \chi(\tilde{p} - \tilde{\kappa}_{\pi(i)})\ \tilde{a}_{\pi(i)} \leq t_i + \tilde{p}\tilde{a}_{\pi(i)} + \tilde{\gamma}_{\pi(i)}\tilde{a}_{\pi(i)} \leq (D + \tilde{p}C)a_i + \gamma_i a_i$
$\leq pa_i + \gamma_i\chi(p - \kappa_i)a_i$.

Hence $||Te_i||_{\tilde{p}} \leq |e_i|_p$ for each $i \in I\!N$. Now if $x \in \mathcal{D}_\infty(\kappa, \gamma, \mathrm{a})$, $x = \sum_{i \in I\!N} x_i e_i$, then with $T_N x \triangleq \sum_{i=1}^{N} x_i e_i$ we get
$||T_N x||_{\tilde{p}} \leq \sum_{i=1}^{N} |x_i|||Te_i||_{\tilde{p}} \leq \sum_{i=1}^{N} |x_i||e_i|_p \leq |x|_p$ for each $N \in I\!N$.
Hence $\{T_N\}_{N=1}^{\infty}$ is a sequence of continuous linear mappings, which converges pointwise to $Tx \triangleq \sum_{i=1}^{\infty} x_i Te_i$. Hence by Banach-Steinhaus Theorem $T$ is continuous.

Now let $\tilde{\tilde{p}} > \max(p_3, (\tilde{p}+D)C)$.

**(i)** For all $i \in I\!N$ such that $\kappa_i > p_2$ , $\check{\kappa}_{\pi(i)} > p_1$ we have
$-t_i + \check{p}a_i + \gamma_i\ \chi(\tilde{p}-\kappa_i)\ a_i = -t_i + \check{p}a_i \le Da_i + \check{p}a_i \le (D+\tilde{p})C\check{a}_{\pi(i)}$
$\le \tilde{\tilde{p}}\check{a}_{\pi(i)} + \check{\gamma}_{\pi(i)}\chi(\tilde{\tilde{p}} - \check{\kappa}_{\pi(i)})\check{a}_{\pi(i)}$.

**(ii)** For each $i \in I\!N$ such that $\kappa_i > p_2$, $\check{\kappa}_{\pi(i)} \le p_1$ we get
$-t_i + \check{p}a_i + \gamma_i\ \chi(\tilde{p}-\kappa_i)\ a_i = -t_i + \check{p}a_i \le pa_i + \check{\gamma}_{\pi(i)}\check{a}_{\pi(i)} + \check{p}a_i \le (D+\tilde{p})C\check{a}_{\pi(i)} +$
$\check{\gamma}_{\pi(i)}\check{a}_{\pi(i)} \le \tilde{\tilde{p}}\check{a}_{\pi(i)} + \check{\gamma}_{\pi(i)}\chi(\tilde{\tilde{p}} - \check{\kappa}_{\pi(i)})\check{a}_{\pi(i)}$.

**(iii)** For every $i \in I\!N$ such that $\kappa_i \le p_2$, $\check{\kappa}_{\pi(i)} > p_3$ it follows that
$-t_i + \check{p}a_i + \gamma_i\ \chi(\tilde{p}-\kappa_i)\ a_i \le -t_i + \check{p}a_i + \gamma_i a_i \le Da_i + \check{p}a_i \le (D+\tilde{p})C\check{a}_{\pi(i)}$
$\le \tilde{\tilde{p}}\check{a}_{\pi(i)} \le \tilde{\tilde{p}}\check{a}_{\pi(i)} + \check{\gamma}_{\pi(i)}\chi(\tilde{\tilde{p}} - \check{\kappa}_{\pi(i)})\check{a}_{\pi(i)}$.

**(iv)** For all $i \in I\!N$ such that $\kappa_i \le p_2$, $\check{\kappa}_{\pi(i)} \le p_3$ we have
$-t_i + \check{p}a_i + \gamma_i\ \chi(\tilde{p}-\kappa_i)\ a_i \le -t_i + \check{p}a_i + \gamma_i a_i \le Da_i + \check{\gamma}_{\pi(i)}\check{a}_{\pi(i)} + \check{p}a_i$
$\le (D+\tilde{p})C\check{a}_{\pi(i)} + \check{\gamma}_{\pi(i)}\check{a}_{\pi(i)} \le \tilde{\tilde{p}}\check{a}_{\pi(i)} + \check{\gamma}_{\pi(i)}\chi(\tilde{\tilde{p}} - \check{\kappa}_{\pi(i)})\check{a}_{\pi(i)}$

Therefore $|e_i|_{\tilde{p}} \le ||Te_i||_{\tilde{\tilde{p}}}$ *for all* $i \in I\!N$, if $x \in \mathcal{D}_\infty(\check{\kappa}, \check{\gamma}, \check{a})$ , $x = \sum_{i\in I\!N} x_i e_i$, then with $T_N^{-1}x \triangleq \sum_{i=1}^N x_i e_i$, then it follows that
$|T_N^{-1}x|_{\tilde{p}} \le \sum_{i=1}^N |x_i||T^{-1}e_i|_{\tilde{p}} \le \sum_{i=1}^N |x_i|\ ||e_i||_{\tilde{\tilde{p}}} \le ||x||_{\tilde{\tilde{p}}}$
Thus $\{T_N^{-1}\}_{N=1}^\infty$ is a sequence of linear mappings, which converges pointwise to $T^{-1}x \triangleq \sum_{i=1}^\infty x_i T^{-1}e_i$. Thus by Banach-Steinhaus Theorem $T^{-1}$ is continuous. Hence $T$ is an isomorphism.

**Corollary 3.2** The following conditions are equivalent:

(i) $\mathcal{D}_\infty(\kappa,\gamma,\text{a}) \in d_1$;

(ii) $\exists m_0 \ \ \forall m \geq m_0 \ \ \sup_{i\in I\!N_m} \gamma_i < \infty$;

(iii) $\mathcal{D}_\infty(\kappa,\gamma,\text{a}) \overset{qdi}{\cong} \Lambda_\infty(a)$;

(iv) $\mathcal{D}_\infty(\kappa,\gamma.\text{a}) \cong \Lambda_\infty(a)$.

**Proof: (i)⇒(ii):**

Since $\mathcal{D}_\infty(\kappa,\gamma,\text{a}) \in d_1$, there exists $p_0$, for each $q$, there exists $r$ such that $\sup_{i\in I\!N}(\frac{a_{iq}^2}{a_{ip_0}a_{ir}}) \triangleq e^M < \infty$ for some $M > 0$.

Or $\ e^{[2q+2\gamma_i\chi(q-\kappa_i)]a_i} \leq e^M e^{[p_0+r+\gamma_i\chi(p_0-\kappa_i)+\gamma_i\chi(r-\kappa_i)]a_i}$ for each $i \in I\!N$.

Or $\ [2q + 2\gamma_i\chi(q-\kappa_i)]a_i \leq M + [p_0 + r + \gamma_i\chi(p_0-\kappa_i) + \gamma_i\chi(r-\kappa_i)]a_i$ for every $i \in I\!N$.

Since $\ a_i \to \infty$ there exists $B > 0$ such that $\sup_{i\in I\!N} \frac{M}{a_i} \leq B$. Thus $[2q + 2\gamma_i\chi(q-\kappa_i)] \leq B + [p_0 + r + \gamma_i\chi(p_0-\kappa_i) + \gamma_i\chi(r-\kappa_i)]$ for all $i \in I\!N$

Let $m_0 \triangleq p_0 + 1$ and let $m \geq m_0$. But then there exists $r$ such that $[2m + 2\gamma_i\chi(m-\kappa_i)] \leq B + [p_0 + r + \gamma_i\chi(p_0-\kappa_i) + \gamma_i\chi(r-\kappa_i)]$ for every $i \in I\!N$.

If $I\!N_m \neq \phi$, then $2m + 2\gamma_i \leq B + [p_0 + r + \gamma_i 0 + \gamma_i\chi(r-\kappa_i)] \leq B + p_0 + r + \gamma_i$.

Therefore $\gamma_i \leq B + p_0 + r - m$ for each $i \in I\!N$. Hence $\sup_{i\in I\!N_m}(\gamma_i) < \infty$.

**(ii) ⇒(iii):**

The map $\ I : \mathcal{D}_\infty(\tilde{\kappa},\tilde{\gamma},\text{a}) \to \Lambda_\infty(\text{a})$, defined by $Ie_i = e_i$ is an isomorphism, where $\tilde{\kappa}_i = i, \tilde{\gamma}_i = 1$. Thus it suffices to show that $\mathcal{D}_\infty(\tilde{\kappa},\tilde{\gamma},\tilde{a}) \overset{qdi}{\cong} \mathcal{D}_\infty(\kappa,\gamma,\text{a})$. Now let $J_1 = \{i \in I\!N : \kappa_i < m_0\}$, $J_2 = I\!N \setminus J_1$.

Define $T : \mathcal{D}_\infty(\tilde{\kappa},\tilde{\gamma},\tilde{a}) \to \mathcal{D}_\infty(\kappa,\gamma,\text{a})$ by $Te_i = e^{t_i}e_i$, where

$$t_i \triangleq \begin{cases} \gamma_i a_i & \text{if } i \in J_1 \\ a_i & \text{otherwise} \end{cases}$$

Now let $\nu = (i_n) \subset I\!N$ be such that $\kappa_{i_n} \to \infty$. Let $\nu = \nu_1 \cup \nu_2$ where $\nu_i = \nu \cap J_i \ \ i = 1,2$. Since $\kappa_{i_n} \to \infty$ , $\nu_1$ is finite. Now let $D_\nu \triangleq \max_{i\in\nu_1}\gamma_i$. Thus for each $i \in \nu$, $\left|\frac{t_i}{a_i}\right| = \begin{cases} \gamma_i & \text{if } i \in \nu_1 \\ 1 & \text{if } i \in \nu_2 \end{cases} \leq D_\nu$.

Now let $\nu = (i_n) \subset I\!N$ be such that $(\kappa_{i_n})$ is bounded. Then there exists $m \in I\!N$ such that $\kappa_{i_n} \leq m$. Now let $\nu = \nu_1 \cup \nu_2$; $\nu_i = \nu \cap J_i \ \ i = 1,2$. If $i \in \nu_1$, $\left|\frac{t_i - \gamma_i a_i}{a_i}\right| = 0$;

if $i \in \nu_2$,

$$\left|\frac{t_i - \gamma_i a_i}{a_i}\right| = \gamma_i - 1 \leq \sup\{\gamma_j - 1 : j \in \cup_{s=m_0}^{m} \mathbb{N}_s\} \triangleq D \text{ for some } D.$$

Hence $\left|\frac{t_i-\gamma_i a_i}{a_i}\right| \leq D$ for all $i \in \nu$. So by the previous theorem
$\mathcal{D}_\infty(\kappa, \gamma.\mathrm{a}) \overset{qdi}{\cong} \mathcal{D}_\infty(\tilde{\kappa}, \tilde{\gamma}, \tilde{a})$.

**(iii) $\Rightarrow$ (iv):** Clear.

**(iv) $\Rightarrow$ (i):** Follows from the fact that $d_1$ is a linear topological invariant.

**Theorem 3.3** Let $T : \mathcal{D}_0(\kappa,\gamma,\mathrm{a}) \to \mathcal{D}_0(\tilde{\kappa}, \tilde{\gamma}, \tilde{a})$ be a quasi-diagonal operator defined by $Te_i = e^{t_i} e_{\pi(i)}$. Then $T$ is an isomorphism if and only if

**(i)** $(a_i)_{i=1}^{\infty} \asymp (\tilde{a}_{\pi(i)})_{i=1}^{\infty}$;

**(ii)** For each $\nu \in I_1$, $\lim_{n\to\infty} \left|\frac{t_{i_n}}{a_{i_n}}\right| = 0$ ;
For each $\nu \in I_2$, $\lim_{n\to\infty} \left|\frac{t_{i_n}+\tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)}}{a_{i_n}}\right| = 0$;
For each $\nu \in I_3$, $\lim_{n\to\infty} \left|\frac{t_{i_n}-\gamma_{i_n} a_{i_n}}{a_{i_n}}\right| = 0$ ;
For each $\nu \in I_4$, $\lim_{n\to\infty} \left|\frac{t_{i_n}+\tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)}-\gamma_{i_n} a_{i_n}}{a_{i_n}}\right| = 0$.

**Proof:** The proof of this theorem is similar to that of Theorem 3.1, so we omit it.

**Corollary 3.4** The following statements are equivalent:

**(i)** $\mathcal{D}_0(\kappa,\gamma,\mathrm{a}) \in d_2$;

**(ii)** There exists a sequence $(q_j)_{j=1}^{\infty}$ strictly increasing to $\infty$ such that $\forall j\ \forall r > q_j\ \exists i_r$ such that $\sup\{\gamma_i :\ i \in \mathbb{N}_r, i \geq i_r\} \leq \frac{1}{j}$;

**(iii)** $\mathcal{D}_0(\kappa,\gamma,\mathrm{a}) \overset{qdi}{\cong} \Lambda_0(\mathrm{a})$;

**(iv)** $\mathcal{D}_0(\kappa,\gamma,\mathrm{a}) \cong \Lambda_0(\mathrm{a})$.

**Proof: (i)⇒(ii):**

Since $\mathcal{D}_0(\kappa,\gamma,\text{a}) \in d_2$, for all $p$ there exists $q$ such that for all $r$

$\sup_{i\in\mathbb{N}}(\frac{a_{ip}a_{ir}}{a_{iq}^2}) \triangleq e^M < \infty$ for some $M > 0$.

Or $\quad e^{[\frac{-1}{p}\frac{-1}{r}+\gamma_i(\chi(p-\kappa_i)+\chi(r-\kappa_i))]a_i} \leq e^M e^{[\frac{-2}{q}+2\gamma_i\chi(q-\kappa_i)]a_i}$ for each $i \in \mathbb{N}$.

Or $\;\frac{-1}{p} - \frac{1}{r} + \gamma_i(\chi(p-\kappa_i) + \chi(r-\kappa_i)) \leq \frac{M}{a_i} - \frac{2}{q} + 2\gamma_i\chi(q-\kappa_i)$ for each $i \in \mathbb{N}$.

Let $r > q$, $i \in \mathbb{N}_r$, then $\frac{-1}{p}-\frac{1}{r}+\gamma_i\{0+1\} \leq \frac{M}{a_i}-\frac{2}{q}+0$ and hence $\gamma_i \leq \frac{M}{a_i}-\frac{2}{q}+\frac{1}{p}+\frac{1}{r}$.

Because $a_i \to \infty$ there exists $i_r$ such that $\frac{M}{a_i} \leq \frac{2}{q} - \frac{1}{r}$ holds for all $i \geq i_r$, $i \in \mathbb{N}_r$.

Hence $\gamma_i < \frac{1}{p}$ for each $i \in \mathbb{N}_r$, $i \geq i_r$. Thus for all $j$ there is $q_j$ such that for all $r > q_j$ there exists $i_r$ such that for all $i \geq i_r$ with $i \in \mathbb{N}_r \quad \gamma_i < \frac{1}{j}$.

**(ii) ⇒(iii):**

Since $\mathcal{D}_0(\tilde{\kappa},\tilde{\gamma},\tilde{a})$, where $\tilde{\kappa}_i \triangleq i$ and $\tilde{\gamma}_i \triangleq \frac{1}{2}$, is isomorphic to $\Lambda_0(\tilde{a})$, it is sufficient to show that $\mathcal{D}_0(\kappa,\gamma,a) \overset{q.d.i}{\cong} \mathcal{D}_0(\tilde{\kappa},\tilde{\gamma},\tilde{a})$. Let us define the following sets

$$J_1 \triangleq \bigcup_{j\geq 1}\ \bigcup_{r\in(q_j,q_{j+1}]} \{i:\ \kappa_i = r,\ \ i \geq i_r\},$$

$$J_2 \triangleq \bigcup_{j\geq 1}\ \bigcup_{r\in(q_j,q_{j+1}]} \{i:\ \kappa_i = r,\ \ i < i_r\},$$

$$J_3 \triangleq \{i:\ \kappa_i < q_1\}.$$

Let $T : \mathcal{D}_0(\kappa,\gamma,a) \to \mathcal{D}_0(\tilde{\kappa},\tilde{\gamma},\tilde{a})$ be defined by $Te_i = e^{t_i}e_i$

where $t_i \triangleq \begin{cases} \gamma_i a_i & \text{if } i \in J_1 \cup J_3 \\ 0 & \text{if } i \in J_2 \end{cases}$

Now let $\nu = (i_n) \subset \mathbb{N}$ be such that $\kappa_{i_n} \to \infty$. Let $\nu = \nu_1 \cup \nu_2 \cup \nu_3$ where $\nu_i = \nu \cap J_i \quad i = 1,2,3$. Since $\kappa_{i_n} \to \infty$ , $\nu_3$ is finite. Now

$$\lim_{i\in\nu_1} \frac{t_i}{a_i} = \lim_{i\in\nu_1} \frac{\gamma_i a_i}{a_i} \leq \lim_{j\to\infty} \frac{1}{j}.$$

Now let $\nu = (i_n) \subset \mathbb{N}$ be such that $(\kappa_{i_n})$ is bounded. But then there exists $j_0 \in \mathbb{N}$ such that for all $i \in \nu$ $\kappa_i \leq q_{j_0}$. Now let $\nu = \nu_1 \cup \nu_2 \cup \nu_3$ ; $\nu_i = \nu \cap J_i \quad i = 1,2,3$. $\nu_2$ is composed of at most finitely many indices. Thus we have

$$\lim_{i\in\nu_1} \frac{t_i - \gamma_i a_i}{a_i} = \lim_{i\in\nu_3} \frac{t_i - \gamma_i a_i}{a_i} = 0.$$

Therefore $\mathcal{D}_0(\kappa,\gamma,a)$ is quasidiagonally isomorphic to $\mathcal{D}_0(\tilde{\kappa},\tilde{\gamma},\tilde{a})$. Consequently, we get the result.

**(iii) $\Rightarrow$ (iv):** Clear.

**(iv) $\Rightarrow$ (i):** It is clear from the fact that $d_2$ is a linear topological invariant.

**Theorem 3.5** Let $T:\mathcal{H}_\infty(\kappa,\gamma,\text{a})\to\mathcal{H}_\infty\ (\tilde{\kappa},\tilde{\gamma},\tilde{a})$ be defined by $Te_i=e^{t_i}e_{\pi(i)}$

Then $T$ is an isomorphism if and only if

**(i)** For all $\nu\triangleq(i_n)\in I_1$ one has $(\tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)})\asymp(\gamma_{i_n}a_{i_n})$;

For all $\nu\triangleq(i_n)\in I_2$ one has $(\tilde{a}_{\pi(i_n)})\asymp(\gamma_{i_n}a_{i_n})$ ;

For all $\nu\triangleq(i_n)\in I_3$ one has $(\tilde{\gamma}_{\pi(i_n)}\tilde{a}_{\pi(i_n)})\asymp(a_{i_n})$;

For all $\nu\triangleq(i_n)\in I_4$ one has $(\tilde{a}_{\pi(i_n)})\asymp(a_{i_n})$.

**(ii)** For every $\nu\in I_1$ there exists $D>0$ such that $\left|\frac{t_{i_n}}{\gamma_{i_n}a_{i_n}}\right|\leq D$ ;

For every $\nu\in I_2$ there exists $D>0$ such that $\left|\frac{t_{i_n}+\tilde{\gamma}_{\pi(i_n)}\tilde{\kappa}_{\pi(i_n)}\tilde{a}_{\pi(i_n)}}{\gamma_{i_n}a_{i_n}}\right|\leq D$ ;

For every $\nu\in I_3$ there exists $D>0$ such that $\left|\frac{t_{i_n}-\gamma_{i_n}\kappa_{i_n}a_{i_n}}{a_{i_n}}\right|\leq D$ ;

For every $\nu\in I_4$ there exists $D>0$ such that $\left|\frac{t_{i_n}+\tilde{\gamma}_{\pi(i_n)}\tilde{\kappa}_{\pi(i_n)}\tilde{a}_{\pi(i_n)}-\gamma_{i_n}\kappa_{i_n}a_{i_n}}{a_{i_n}}\right|\leq D$.

**Proof:** It can be proved similarly as Theorem 3.1.

**Corollary 3.6** The following statements are equivalent:

**(i)** $\mathcal{H}_\infty(\kappa,\gamma,\text{a})\ \in d_1$;

**(ii)** There exists $m_0$ such that for all $m\geq m_0$, $\sup\{\gamma_i:\ i\in\mathbb{N}_m\}<\infty$;

**(iii)** $\mathcal{H}_\infty(\kappa,\gamma,\text{a})\overset{qdi}{\cong}\Lambda_\infty(\tilde{a})$ for some $\tilde{a}$;

**(iv)** $\mathcal{H}_\infty(\kappa,\gamma,\text{a})\cong\Lambda_\infty(\tilde{a})$ for some $\tilde{a}$.

**Proof:(i)$\Rightarrow$(ii):** Since $\mathcal{H}_\infty(\kappa,\gamma,\text{a})\in d_1$ there exists $p_0$ such that for each $q$ there exists $r$ such that $\sup_{i\in\mathbb{N}}(\frac{a_{iq}^2}{a_{ip_0}a_{ir}})\triangleq M<\infty$.

Or for each $i\in\mathbb{N}$

$$e^{[2q+2\gamma_i\min(q,\kappa_i)]a_i}\leq e^M e^{[p_0+r+\gamma_i\min(p_0,\kappa_i)+\gamma_i\min(r,\kappa_i)]a_i}.$$

Or for all $i$ in $I\!N$

$$[2q + 2\gamma_i \min(q, \kappa_i)]a_i \leq M + [p_0 + r + \gamma_i \min(p_0, \kappa_i) + \gamma_i \min(r, \kappa_i)]a_i.$$

Since $a_i \to \infty$ there exists $B > 0$ such that $\sup_{i \in I\!N}(\frac{M}{a_i}) \leq B$. So for every $i \in I\!N$

$$[2q + 2\gamma_i \min(q, \kappa_i)] \leq B + [p_0 + r + \gamma_i \min(p_0, \kappa_i) + \gamma_i \min(r, \kappa_i)]$$

Now let $m_0 = p_0 + 1$ and let $m \geq m_0$. But then there exists $r$ such that for all $i \in I\!N$

$$[2m + 2\gamma_i \min(m, \kappa_i)] \leq B + [p_0 + r + \gamma_i \min(p_0, \kappa_i) + \gamma_i \min(r, \kappa_i)].$$

Without loss of generality assume $I\!N_m \neq \phi$. Then $2m + 2\gamma_i m \leq B + [p_0 + r + \gamma_i p_0 + \gamma_i m]$. Thus $\gamma_i(m - p_0) \leq B + p_0 + r - 2m$. Hence $\sup_{i \in I\!N_m} \gamma_i \leq \frac{B + p_0 + r - 2m}{m - p_0} < \infty$.
**(ii)⇒ (iii):** Since the map $I : \mathcal{H}_\infty(\hat{\kappa}, \hat{\gamma}, \hat{a}) \to \Lambda_\infty(\hat{a})$, defined by $Ie_i = e_i$, is an isomorphism, where $\hat{\kappa}_i = i$, $\hat{\gamma}_i = 1$, one has $\mathcal{H}_\infty(\hat{\kappa}, \hat{\gamma}, \hat{a}) \overset{qdi}{\cong} \Lambda_\infty(\hat{a})$. Hence it suffices to show that $\mathcal{H}_\infty(\tilde{\kappa}, \tilde{\gamma}, \tilde{a}) \overset{qdi}{\cong} \mathcal{H}_\infty(\kappa, \gamma, \text{a})$. Now let $J_1 = \{i : \kappa_i \leq m_0\}$, $J_2 = I\!N \setminus J_1$.

$$\tilde{a}_i \triangleq \begin{cases} a_i & \text{if } i \in J_1 \\ \gamma_i a_i & \text{otherwise} \end{cases}$$

Clearly $\tilde{a}_i \to \infty$. Let $T : \mathcal{H}_\infty(\kappa, \gamma, \text{a}) \to \mathcal{H}_\infty(\tilde{\kappa}, \tilde{\gamma}, \tilde{a})$, be defined by $Te_i = e^{t_i} e_i$, where

$$t_i \triangleq \begin{cases} \gamma_i \kappa_i a_i & \text{if } i \in J_1 \\ \gamma_i a_i & \text{otherwise} \end{cases}$$

Now let $\nu = (i_n) \in I_1$ $\nu = \nu_1 \cup \nu_2$, where $\nu_i = \nu \cap J_i$ $i = 1, 2$. Because $\kappa_{i_n} \to \infty$, $\nu_1$ is at most finite. Then

$$\frac{\tilde{\gamma}_i\tilde{a}_i}{\gamma_i a_i} = \frac{\tilde{a}_i}{\gamma_i a_i} = \begin{cases} \frac{1}{\gamma_i} & \text{if } i \in \nu_1 \\ 1 & \text{otherwise} \end{cases} . \text{ Hence ,}$$

$\min\{1, \{\frac{1}{\gamma_i} : i \in \nu_1\}\} \le \frac{\tilde{\gamma}_i\tilde{a}_i}{\gamma_i a_i} \le 1 \Rightarrow (\tilde{\gamma}_i\tilde{a}_i)_{i=1}^{\infty} \asymp (\gamma_i a_i)_{i=1}^{\infty}$. Now

$$\left|\frac{t_i}{\gamma_i a_i}\right| = \begin{cases} \frac{\gamma_i \kappa_i a_i}{\gamma_i a_i} & \text{if } i \in \nu_1 \\ 1 & \text{otherwise} \end{cases} \le D \triangleq \max_{i \in \nu_1} \kappa_i .$$

Now let $\nu \in I_3$. Hence there is some $m \ge m_0$ such that for all $i \in \nu$ $\kappa_i \le m$ $\nu = \nu_1 \cup \nu_2$ , where $\nu_i = \nu \cap J_i$ $i = 1, 2$. Then

$$\frac{\tilde{\gamma}_i\tilde{a}_i}{a_i} = \frac{\tilde{a}_i}{a_i} = \begin{cases} 1 & \text{if } i \in \nu_1 \\ \gamma_i & \text{otherwise} \end{cases}$$

By assumption, $\sup_{i\in\nu_2} \gamma_i < \infty$. So $1 \le \frac{\tilde{\gamma}_i\tilde{a}_i}{a_i} \le \sup_{i\in\nu_2} \gamma_i \Rightarrow (\tilde{\gamma}_i\tilde{a}_i)_{i=1}^{\infty} \asymp (a_i)_{i=1}^{\infty}$. And

$$\left|\frac{t_i - \gamma_i\kappa_i a_i}{a_i}\right| = \begin{cases} 0 & \text{if } i \in \nu_1 \\ \gamma_i(\kappa_i - 1) & \text{otherwise} \end{cases} \le \sup_{i\in\nu_2} \gamma_i(\kappa_i - 1) < \infty .$$

**(iii)⇒ (iv):** Clear.

**(iv)⇒ (i):** It follows from of the fact that $d_1$ is a linear topological invariant.

**Theorem 3.7** Let $T : \mathcal{H}_\infty(\kappa, \gamma, \mathrm{a}) \to \mathcal{D}_\infty\ (\tilde{\kappa}, \tilde{\gamma}, \tilde{a})$ be defined by $Te_i = e^{t_i} e_{\pi(i)}$. Then $T$ is an isomorphism if and only if

**(i)** For each $\nu \in I_1 \cup I_2$, $(\tilde{a}_{\pi(i_n)}) \asymp (\gamma_{i_n} a_{i_n})$;

For each $\nu \in I_3 \cup I_4$, $(\tilde{a}_{\pi(i_n)}) \asymp (a_{i_n})$;

**(ii)** For each $\nu \in I_1$ there exists $D > 0$ such that $\left|\frac{t_{i_n}}{\gamma_{i_n} a_{i_n}}\right| \leq D$;

For each $\nu \in I_2$ there exists $D > 0$ such that $\left|\frac{t_{i_n} + \tilde{\gamma}_{\pi(i_n)} \tilde{a}_{\pi(i_n)}}{\gamma_{i_n} a_{i_n}}\right| \leq D$ ;

For each $\nu \in I_3$ there exists $D > 0$ such that $\left|\frac{t_{i_n} - \gamma_{i_n} \kappa_{i_n} a_{i_n}}{a_{i_n}}\right| \leq D$ ;

For each $\nu \in I_4$ there exists $D > 0$ such that $\left|\frac{t_{i_n} + \tilde{\gamma}_{\pi(i_n)} \tilde{a}_{\pi(i_n)} - \gamma_{i_n} \kappa_{i_n} a_{i_n}}{a_{i_n}}\right| \leq D$ .

**Proof:** This can be proved by using similar arguments introduced in the proof of Theorem 3.1.

**Corollary 3.8** If $\mathcal{H}_\infty(\kappa, \gamma, \mathrm{a}) \overset{qdi}{\cong} \mathcal{D}_\infty(\tilde{\kappa}, \tilde{\gamma}, \tilde{a})$, then both spaces belong to class $d_1$

**Proof:** Let $T : \mathcal{H}_\infty(\kappa, \gamma, \mathrm{a}) \to \mathcal{D}_\infty(\tilde{a}, \tilde{\gamma}, \tilde{a})$, defined by $Te_i = e^{t_i} e_{\pi(i)}$, be the isomorphism.

But then $(\tilde{a}_{\pi(i_n)}) \asymp (\gamma_{i_n} a_{i_n})$ for all $(i_n) \in I_1 \cup I_2$, $(\tilde{a}_{\pi(i_n)}) \asymp (a_{i_n})$ for every $(i_n) \in I_3 \cup I_4$.

Now let $\tilde{p} \triangleq 1$. Then by a similar argument introduced in the proof of Theorem 3.1, there exist $p_3 > p_2 > p_1 > 1$, $D_4 > D_3 > D_2 > D_1 \geq 1$ such that

for all $\kappa_i > p_2$, $\tilde{\kappa}_{\pi(i)} > p_1$, $\frac{1}{D_1} \leq \frac{\tilde{a}_{\pi(i)}}{\gamma_i a_i} \leq D_1$ ; $\quad (a)$

for all $\kappa_i > p_2$, $\tilde{\kappa}_{\pi(i)} \leq p_1$, $\frac{1}{D_2} \leq \frac{\tilde{a}_{\pi(i)}}{\gamma_i a_i} \leq D_2$; $\quad (b)$

for all $\kappa_i \leq p_2$, $\tilde{\kappa}_{\pi(i)} > p_3$, $\frac{1}{D_3} \leq \frac{\tilde{a}_{\pi(i)}}{a_i} \leq D_3$ ; $\quad (c)$

for all $\kappa_i \leq p_2$, $\tilde{\kappa}_{\pi(i)} \leq p_3$, $\frac{1}{D_4} \leq \frac{\tilde{a}_{\pi(i)}}{a_i} \leq D_4$ ; $\quad (d)$.

Now let $p_0 \triangleq p_2$.

But then there exist $\tilde{p}_3 > \tilde{p}_2 > \tilde{p}_1 > p_0$, $\tilde{D}_4 > \tilde{D}_3 > \tilde{D}_2 > \tilde{D}_1 > D_4$ such that

For every $\kappa_i > \tilde{p}_2$, $\tilde{\kappa}_{\pi(i)} > \tilde{p}_1$, $\frac{1}{\tilde{D}_1} \leq \frac{\tilde{a}_{\pi(i)}}{\tilde{\gamma}_i a_i} \leq \tilde{D}_1$; $\quad (a')$

for every $\kappa_i > \tilde{p}_2, \tilde{\kappa}_{\pi(i)} \leq \tilde{p}_1$, $\frac{1}{\tilde{D}_2} \leq \frac{\tilde{a}_{\pi(i)}}{\tilde{\gamma}_i a_i} \leq \tilde{D}_2$; $\quad (b')$

for all $\kappa_i \leq \tilde{p}_2$, $\tilde{\kappa}_{\pi(i)} > \tilde{p}_3$, $\frac{1}{\tilde{D}_3} \leq \frac{\tilde{a}_{\pi(i)}}{a_i} \leq \tilde{D}_3$; $\quad (c')$

for every $\kappa_i \leq \tilde{p}_2$, $\tilde{\kappa}_{\pi(i)} \leq \tilde{p}_3$, $\frac{1}{\tilde{D}_4} \leq \frac{\tilde{a}_{\pi(i)}}{a_i} \leq \tilde{D}_4$ $\quad (d')$.

Thus for all $i \in I\!N$ such that $p_0 < \kappa_i \leq \tilde{p}_2$, by $(a)$, $(b)$, we have $\frac{1}{D_2} \leq \frac{\tilde{a}_{\pi(i)}}{\gamma_i a_i} \leq D_1$ and by $(c')$, $(d')$, we have $\frac{1}{\tilde{D}_4} \leq \frac{\tilde{a}_{\pi(i)}}{a_i} \leq \tilde{D}_4$. Hence if we let $m_0 \triangleq p_0$ and $m > m_0$ be given, then $\{\gamma_i\}_{i \in I\!N_m}$ is bounded. This implies that $\mathcal{H}_\infty(\kappa, \gamma, \mathrm{a})$ is isomorphic to $\Lambda(\tilde{\tilde{a}})$ for some $\tilde{\tilde{a}}$. Therefore $\mathcal{H}_\infty(\kappa, \gamma, \mathrm{a})$, $\mathcal{D}_\infty$ $(\tilde{\kappa}, \tilde{\gamma}, \tilde{a})$ belong to class $d_1$.

The following example will show that there exist $\mathcal{D}_\infty(\kappa, \gamma, \mathrm{a})$-type spaces which are not isomorphic to any $\mathcal{H}_\infty(\tilde{\kappa}, \tilde{\gamma}, \tilde{a})$-type spaces, also that there exist $\mathcal{H}_\infty(\kappa, \gamma, \mathrm{a})$-type spaces which are not isomorphic to any $\mathcal{D}_\infty(\tilde{\kappa}, \tilde{\gamma}, \tilde{a})$-type spaces. That is to say both classes of spaces are not identical to each other.

**Counterexample:** Let $\gamma_n = n, (\kappa_n) = (1, 1, 2, 1, 2, 3, \ldots)$, $a_n = 2^{2^n}$

Then $\mathcal{D}_\infty(\kappa, \gamma, \mathrm{a})$ can not be in $d_1$, since $\gamma_n \to \infty$. (see Corollary 3.2$(ii)$)

$\mathcal{D}_\infty(\kappa, \gamma, \mathrm{a})$ *is unstable* :

Let $s = 1$, for all $p \in I\!N$, let $q = p + 2$, for all $r \in I\!N$. Because $\chi$ is nondecreasing function and $q > p$, $\chi(p - \kappa_{n+1}) - \chi(q - \kappa_{n+1}) \leq 0$. Now observing that $\frac{\gamma_n a_n}{a_{n+1}} \to 0$, we can choose some $n_0$ such that $\frac{\gamma_n a_n}{a_{n+1}} < \frac{1}{2r}$ for all $n \geq n_0$. Hence we have

$$\frac{a_{p,n+1}}{a_{q,n+1}} \frac{a_{r,n}}{a_{s,n}} = e^{[p-q+\gamma_{n+1}\{\chi(p-\kappa_{n+1})-\chi(q-\kappa_{n+1})\}]a_{n+1}} e^{[r-s+\gamma_n\{\chi(r-\kappa_n)-\chi(s-\kappa_n)\}]a_n}$$

$$\leq e^{[p-q]} e^{[r-s+\gamma_n]\frac{a_n}{a_{n+1}} a_{n+1}} \leq e^{[p-q+2(r-s)\frac{\gamma_n a_n}{a_{n+1}}]a_{n+1}} \leq e^{[p-q+1]a_{n+1}} \leq e^{-a_{n+1}} \to 0.$$

Similarly, $\mathcal{H}_\infty(\kappa, \gamma, \mathrm{a})$ is not in $d_1$ (see Corollary 3.6$(ii)$) and is unstable.

Hence both $\mathcal{D}_\infty(\kappa, \gamma, \mathrm{a})$ and $\mathcal{H}_\infty(\kappa, \gamma, \mathrm{a})$ satisfy Bessaga's conjecture and thus they have q.e.p. ([18]). Observe the following fact

**Fact 3.9** Having quasiequivalence property is a l.t.i. in the class of Köthe spaces.

**Proof of the fact:** Let $X \triangleq \mathcal{K}([a_{pn}])$ have the q.e.p. and let

$U : Y \triangleq \mathcal{K}([b_{pn}]) \to X$ be an isomorphism. If $\{x_n\}$ is any unconditional basis in $Y$, then $\{Ue_n\}, \{Ux_n\}$ are two unconditional bases in $X$. Thus there exist a permutation $\pi : I\!N \to I\!N$, a sequence of scalars $(\alpha_n)$ such that the map

$S : X \to X$, defined by $S(Ue_n) \triangleq \alpha_n Ux_{\pi(n)}$ is an isomorphism. Consequently $Y \xrightarrow{U} X \xrightarrow{S} X \xrightarrow{U^{-1}} Y$ is an isomorphism.

Let us go back to the example. If $\mathcal{D}_\infty(\kappa,\gamma,\mathrm{a})$ is isomorphic to $\mathcal{H}_\infty(\tilde{\kappa},\tilde{\gamma},\dot{a})$ for some $\tilde{\kappa},\tilde{\gamma},\dot{a}$, then by the above fact $\mathcal{H}_\infty(\tilde{\kappa},\tilde{\gamma},\dot{a})$ would have the q.e.p..
Let $T : \mathcal{H}_\infty(\tilde{\kappa},\tilde{\gamma},\dot{a}) \to \mathcal{D}_\infty(\kappa,\gamma,a)$ be the isomorphism. Then $\{Te_n\}$ is a basis in $\mathcal{D}_\infty(\kappa,\gamma,\mathrm{a})$ , again by q.e.p. there exist $\sigma : I\!N \to I\!N$, a sequence of scalars $(t_n)$ such that the map $R : \mathcal{D}_\infty(\kappa,\gamma,\mathrm{a}) \to \mathcal{D}_\infty(\kappa,\gamma,\mathrm{a})$, defined by $R(Te_n) = t_n e_{\sigma(n)}$ is an isomorphism. Hence $\mathcal{H}_\infty\ (\tilde{\kappa},\tilde{\gamma},\dot{a}) \xrightarrow{T} \mathcal{D}_\infty(\kappa,\gamma,\mathrm{a}) \xrightarrow{R} \mathcal{D}_\infty(\kappa,\gamma,\mathrm{a})$ is an isomorphism. Therefore $\mathcal{H}_\infty(\tilde{\kappa},\tilde{\gamma},\dot{a}) \overset{qdi}{\cong} \mathcal{D}_\infty(\kappa,\gamma,\mathrm{a})$. It follows from Corollary 3.7 that $\mathcal{D}_\infty(\kappa,\gamma,\mathrm{a})$ is in $d_1$, which is a contradiction.
Therefore $\mathcal{D}_\infty(\kappa,\gamma,\mathrm{a})$ can not be isomorphic to $\mathcal{H}_\infty(\dot{\kappa},\dot{\gamma},\dot{a})$, for any $\dot{\gamma},\dot{\kappa},\dot{a}$. Similarly, $\mathcal{H}_\infty(\kappa,\gamma,\mathrm{a}) \not\cong \mathcal{D}_\infty(\dot{\kappa},\tilde{\gamma},\dot{a})$. for any $\dot{\gamma},\dot{\kappa},\dot{a}$.

# CHAPTER 4
# CONCLUSIONS AND SUGGESTIONS

In this study, we examined the quasidiagonal isomorphisms between $\mathcal{D}_\infty(\kappa,\gamma,a)$, $\mathcal{H}_\infty(\kappa,\gamma,a)$, $\mathcal{D}_0(\kappa,\gamma,a)$ types spaces.

As consequences, we proved that the only spaces either of the first two types belonging to the class $d_1$ are those which are isomorphic (or equivalently quasidiagonally isomorphic) to an infinite type power series space. Furthermore, it was shown that the only spaces of type $\mathcal{D}_0(\kappa,\gamma,a)$ belonging to the class $d_2$ are the finite type power series spaces.

Each basic subspace of a space of the type $\mathcal{D}_0(\kappa,\gamma,a)$ has a basic finite type power series subspace whereas basic subspaces those of the other types have basic infinite type power series subspaces. Hence we can say that the class of spaces of the type $\mathcal{D}_0(\kappa,\gamma,a)$ is completely different (in topological sense) from those of the spaces of the other types. Therefore we discussed whether or not the other two classes are identical (in topological sense).

By giving an example, we were able to show that these classes are not identical. On the other hand, it was shown that a space which is quasidiagonally isomorphic to a space in each of these two classes is a power series space of infinite type.

We could not prove that a stable space either of these three types has q.e.p.. In this case, it can be concluded that the intersection of the class of spaces of the type $\mathcal{D}_\infty(\kappa,\gamma,\mathrm{a})$ and the class of spaces of the type $\mathcal{H}_\infty(\kappa,\gamma,\mathrm{a})$ is the class of infinite type power series spaces.

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