WKB Appoximation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\psi = E\psi$$

$$\psi = e^{\frac{i}{\hbar}S(x)}$$

$$\vec{\nabla}\psi = \frac{i}{\hbar}(\vec{\nabla}S)\psi$$

$$\nabla^2\psi = \left[\frac{i}{\hbar}(\nabla^2 S) + \left(\frac{i}{\hbar}\right)^2(\vec{\nabla}S)^2\right]\psi$$

$$\boxed{\frac{(\vec{\nabla}S)^2}{2m} + V(r) - \left(\frac{i\hbar}{2m}\nabla^2 S\right) = E}$$

Classical limit: $\hbar \to 0$

$$\boxed{\frac{1}{2m}(\vec{\nabla}S)^2 + V(r) = E} \Longleftarrow \text{Hamilton-Jacobi Eqn.}$$

S: action

$$S[q] = \int_{t_i}^{t_f} dt\, L(\dot{q}(t), q(t), t)$$

$$\left.\begin{aligned} q(t_i) &= q_i \\ q(t_f) &= q_f \end{aligned}\right\} \text{fixed} \qquad \delta S = 0$$

$$S[q_f] = \int dt\, L(\dot{q}(t), q(t), t)$$

$$P_f \equiv p(t_f) = \left(\frac{\partial S}{\partial q_f}\right) \Longleftrightarrow p \Rightarrow \frac{\partial}{\partial q}$$

$$\frac{P_f^2}{2m} + V(r) = E \Rightarrow \boxed{\frac{(\nabla S)^2}{2m} + V(r) = E}$$

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + \dots$$

1D case

$$\hbar^0 : \quad \left(\frac{dS}{dx}\right)^2 \frac{1}{2m} + V(x) = E$$

$$\frac{dS}{dx} = \mp\sqrt{2m(E-V)} = \mp p(x)$$

$$\hbar^{(1)} \quad = \frac{1}{2m}\left(\frac{\partial S_0}{\partial x} + \hbar \frac{\partial S_1}{\partial x}\right)^2 + V(r) - i\frac{\hbar}{2m}\left(\frac{d^2 S_0}{dx^2} + \hbar \frac{d^2 S_1}{dx^2}\right) = E$$

$$\cancel{\frac{1}{2m}} \cdot 2\left(\frac{dS_0}{dx}\right)\left(\frac{dS_1}{dx}\right) - \cancel{\frac{i}{2m}} \frac{d^2 S_0}{dx^2} = 0$$

$$\frac{dS_1}{dx} = -\frac{1}{2} \frac{\frac{d}{dx}\left(\frac{dS_0}{dx}\right)}{\left(\frac{dS_0}{dx}\right)}$$

$$= -\frac{1}{2}\frac{d}{dx}\ln\left(\frac{dS_0}{dx}\right)$$

$$S_1 = -\frac{1}{2}\ln\left(\frac{dS_0}{dx}\right) + \text{const}$$

$$\frac{dS_0}{dx} = \mp p(x) = \mp\sqrt{2m(E-V)}$$

$$S(x) = \mp\int_{x_0}^{x} p(x')\,dx' + \hbar\left(-\frac{1}{2}\ln\left(\frac{dS_0}{dx}\right) + \text{const}\right)$$

$$\boxed{\psi = \exp\left\{\frac{i}{\hbar}S(x)\right\} = e^{\pm i\int p(x)dx}\,\frac{N}{\sqrt{p(x)}}} \Leftarrow$$

$$|\psi|^2 \propto \frac{1}{p(x)}$$

$$\Rightarrow \frac{d^2 S_0}{dx^2} \ll \left(\frac{dS_0}{dx}\right)\left(\frac{dS_1}{dx}\right)$$

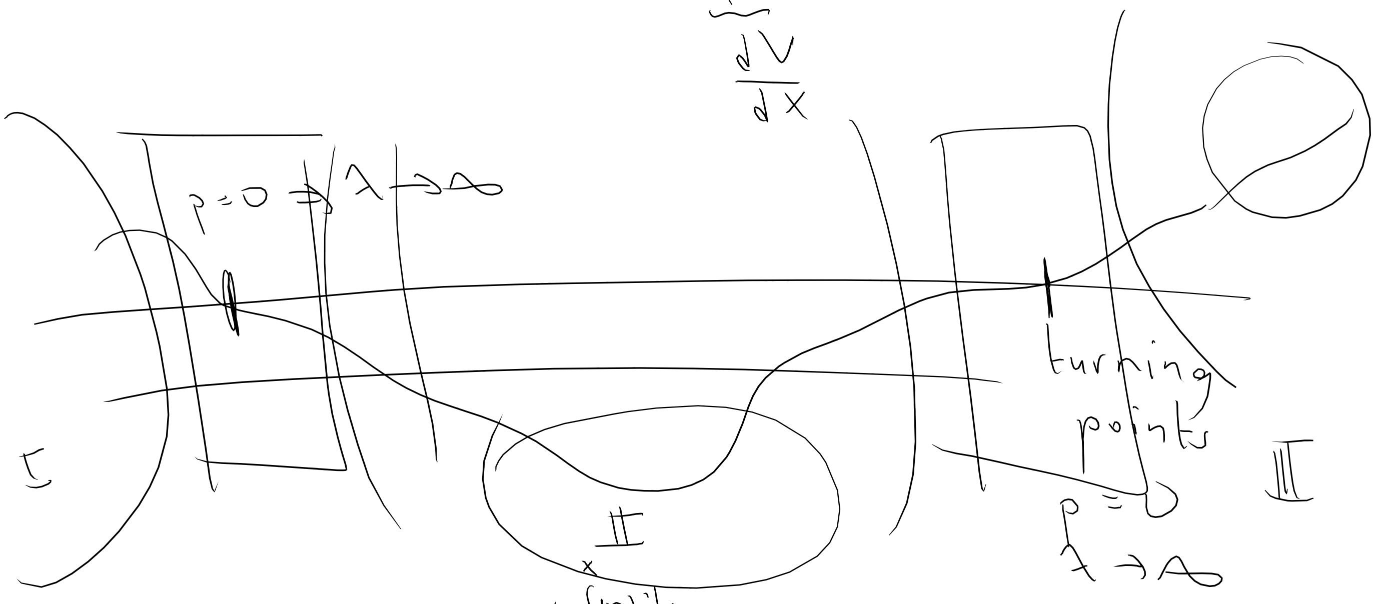
$$\underbrace{\left(\frac{dp}{dx}\right)} \ll p\,\frac{dS_1}{dx}$$

$$p = \frac{h}{\lambda}$$

$$\left(\frac{d\lambda}{dx}\right)\frac{1}{\lambda^{\cancel{2}}} \ll \frac{\cancel{+}}{\cancel{\lambda}}\ \frac{dS_1}{dx}$$

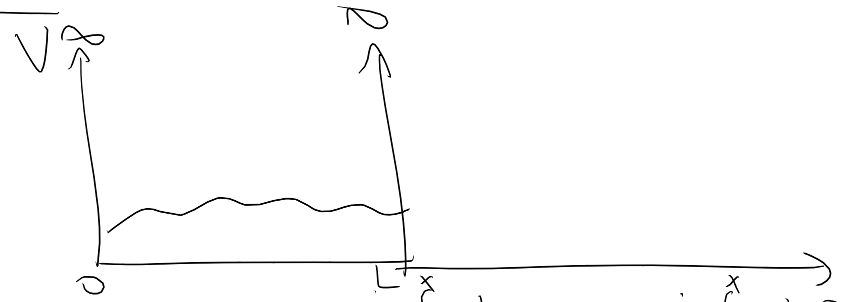
$$\frac{d\lambda}{dx}\frac{1}{\lambda} \ll \frac{dS_1}{dx} \propto \frac{d}{dx}\ln(p) = \frac{1}{p}\underbrace{\frac{dp}{dx}}$$

$$\left(\frac{d\lambda}{dx}\right)\frac{1}{\lambda} \ll \frac{1}{p}\ \underbrace{\frac{dp}{dt}}_{\underbrace{F}_{\frac{dV}{dx}}}\ \frac{1}{v}$$



$$\psi(x) = \frac{1}{\sqrt{|p|}}\begin{cases} C e^{+\int_{x_0}^{x} |p|\,dx} & \text{region I} \\ D e^{\frac{i}{\hbar}\int_{x}^{} p\,dx} + E e^{-\frac{i}{\hbar}\int p\,dx} & \text{region II} \\ F e^{-\int_{x_0'}^{x} |p|\,dx} & \text{region III} \end{cases}$$

Example



$$\psi(x) = \frac{1}{\sqrt{p}}\left[D e^{\frac{i}{\hbar}\int_0^x p\,dx} + E e^{-\frac{i}{\hbar}\int_0^x p\,dx}\right]$$

$$\psi(x=0) = 0 \Rightarrow D = -E$$

$$\psi(x) = \frac{1}{\sqrt{p}}\, 2iD \sin\left[\frac{1}{\hbar}\int_0^x p\,dx\right]$$

$$\psi(x=L) = 0 \Rightarrow \sin\left[\frac{1}{\hbar}\int_0^L p\,dx\right] = 0$$

$$\boxed{\int_0^L p\,dx = n\pi\hbar}$$ Bohr's quantization condition

$$\frac{d^2 S_0}{dx^2} \ll 2\left(\frac{dS_0}{dx}\right)\left(\frac{dS_1}{dx}\right)$$

$$\frac{dp}{dx} \ll p\,\frac{dS_1}{dx} = p\,\frac{d}{dx}(\ln p) = p\,\frac{1}{p}\frac{dp}{dx}$$

$$h^{(2)}: \frac{1}{2m}\left(S_0' + \hbar S_1' + \hbar^2 S_2'\right)^2 + V(r) - \frac{i\hbar}{2m}\left(S_0'' + \hbar S_1'' + \hbar^2 S_2''\right) = E$$

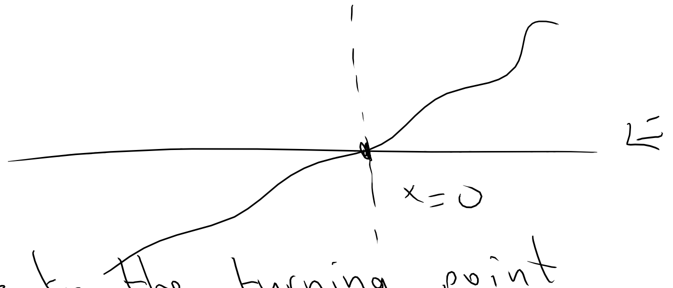
$$\frac{1}{2m}\left[2S_0' S_2' + (S_1')^2\right] - \frac{i S_1}{2m} = 0$$

$$S_2' = \left[-i S_1 - (S_1')^2\right]\frac{1}{S_0}$$

$$S_2' = \left[+\frac{i}{2}\ln p - \left(-\frac{1}{2}\frac{p'}{p}\right)^2\right]\frac{1}{S_0} \ll 1$$

$$\left(\frac{p'}{p}\right) \sim \frac{\lambda'}{\lambda}$$

Connection Formulas



Close to the turning point

$$V(x) = V(0) + V'(0)x$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (V_0 + V_0' x)\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \underbrace{(V_0 + V_0' x - E)}\psi = 0$$

$$y \equiv V_0 + V_0' x - E$$

$$\frac{d}{dx} = V_0' \frac{d}{dy}$$

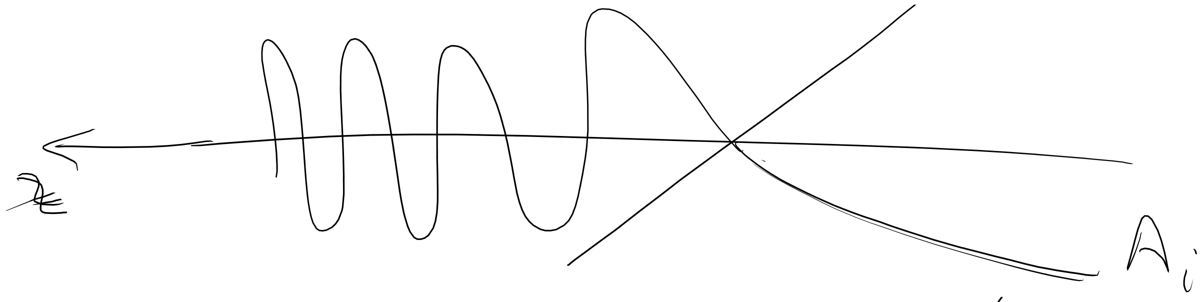
$$-\frac{\hbar^2}{2m}(V_0')^2 \frac{d^2\psi}{dy^2} + y\psi = 0$$

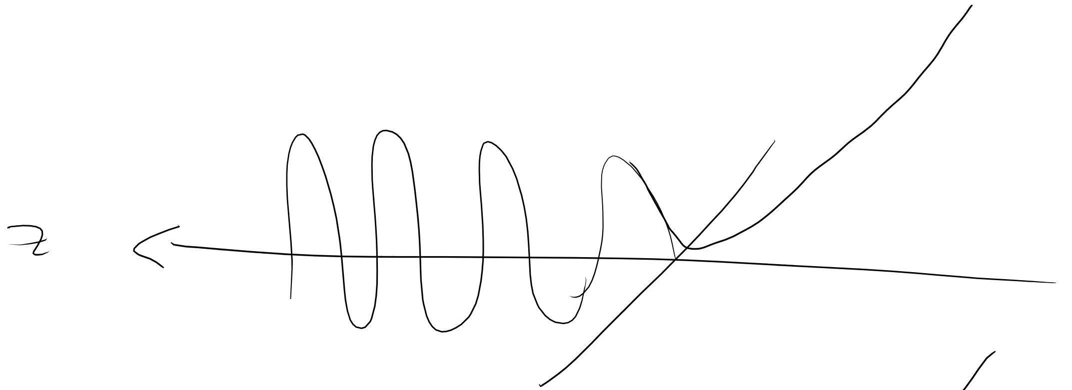
$$y = \alpha z$$

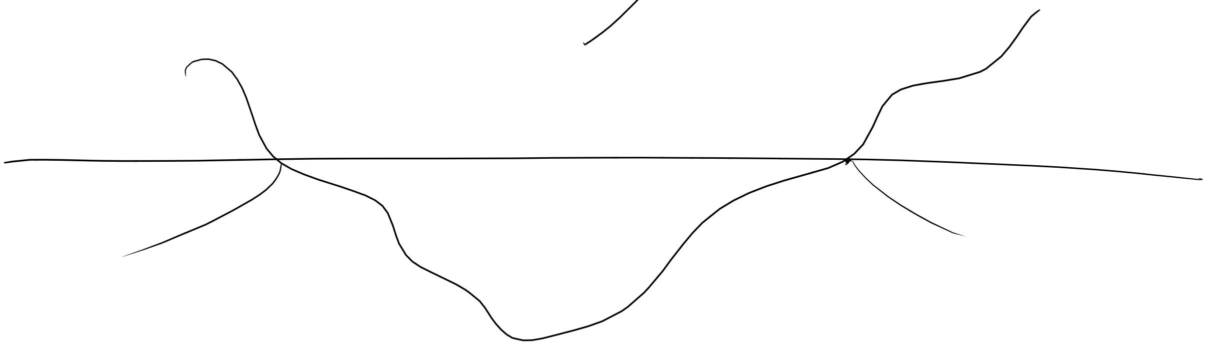
$$-\frac{\hbar^2}{2m}(V_0')^2 \frac{1}{\alpha^3}\frac{d^2\psi}{dz^2} + \cancel{\alpha} z\psi = 0$$

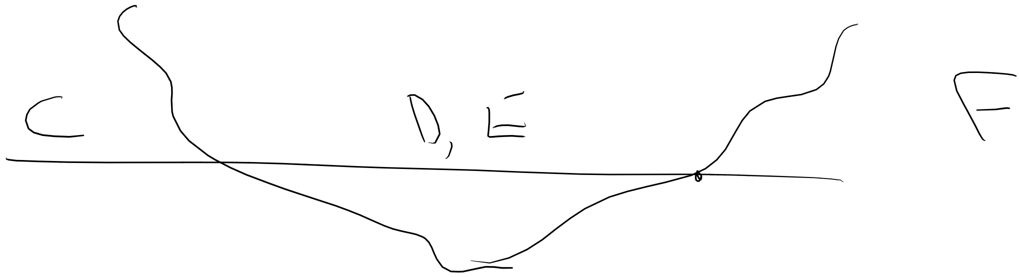
$$\alpha^3 = -\frac{\hbar^2}{2m}(V_0')^2$$

$$\boxed{\frac{d^2\psi}{dz^2} + z\psi = 0} \qquad \psi = a\,A_i(z) + b\,B_i(z)$$









$$\int p\,dx = \int \sqrt{2m(E - V(x))}\,dx$$

$$\simeq \int \sqrt{2m(E - V_0 - V_0' x)}\,dx$$

$$= \left[2m(E - V_0 - V_0' x)\right]^{3/2} \frac{2}{3} \frac{1}{(-2mV_0')}$$

$$V_0 = E$$

$$\int p\,dx \;\propto\; (-V_0' x)^{3/2} \propto (-x)^{3/2}$$

on the left

$$\psi = D e^{\frac{i}{\hbar}(-x)^{3/2}(\,)} + E e^{-\frac{i}{\hbar}(-x)^{3/2}(\,)}$$

on the right

$$\psi = F e^{-(\,)(x)^{3/2}}$$

$$A_i(z) \underset{z\to\infty}{=} (\,) e^{-(\,)x^{3/2}}$$

$$B_i(z) = (\,) e^{+(\,)x^{3/2}}$$

match the two soln. on the right of the turning point $\Rightarrow b = 0$. $a \propto$

$$\boxed{a = \sqrt{\frac{4\pi}{\alpha\hbar}}\; F}$$

$$A_i(z) \simeq \frac{1}{\sqrt{\pi}(-z)^{1/4}} \sin\left(\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{\pi}(-z)^{1/2}} \frac{1}{2i}\left[e^{i\left(\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right)} + e^{-i\left(\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right)}\right]$$

in the allowed region by matching at the turning point on the right

$$\psi(x) \doteq \frac{D}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int_x^{x_2} p(x')\,dx' + \frac{\pi}{4}\right]$$

by matching at the other turning point

$$\psi(x) = \frac{D'}{\sqrt{p(x)}} \sin\left[-\frac{1}{\hbar}\int_{x_1}^{x} p(x')\,dx' - \frac{\pi}{4}\right]$$

$$\frac{1}{\hbar}\int_x^{x_2} p(x')\,dx' + \frac{\pi}{4} = -\frac{1}{\hbar}\int_{x_1}^{x} p(x')\,dx' - \frac{\pi}{4} + n\pi$$

$$\boxed{\frac{1}{\hbar}\int_{x_1}^{x_2} p(x')\,dx' = n\pi + \frac{\pi}{2}}$$

$\Longleftarrow$ energy eigenvalues!