1^{st} Midterm April 3^{rd}, 2016 ; 13^{40}

$$\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-\frac{i}{\hbar}E_n t}$$

Two state system

eigenstates of H_0: $\{|a\rangle, |b\rangle\}$

$$H = H_0 + H'(t)$$

$$|\Psi(t)\rangle = c_a(t) e^{-\frac{i}{\hbar}E_a t} |a\rangle + c_b(t)|b\rangle e^{-\frac{i}{\hbar}E_b t}$$

if $H' = 0$, $c_a(t), c_b(t)$ independent of time.

$$H|\Psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle$$

$$c_a e^{-\frac{i}{\hbar}E_a t}(\cancel{H_0} + H')|a\rangle + c_b e^{-\frac{i}{\hbar}E_b t}(\cancel{H_0} + H')|b\rangle$$

$$= i\hbar\Big[\dot{c}_a e^{-\frac{i}{\hbar}E_a t}|a\rangle - \cancel{\frac{i}{\hbar}E_a c_a e^{-\frac{i}{\hbar}E_a t}|a\rangle}$$
$$+ \dot{c}_b e^{-\frac{i}{\hbar}E_b t}|b\rangle - \cancel{\frac{i}{\hbar}E_b c_b e^{-\frac{i}{\hbar}E_b t}|b\rangle}\Big]$$

$$c_a e^{-\frac{i}{\hbar}E_a t} H'|a\rangle + c_b e^{-\frac{i}{\hbar}E_b t} H'|b\rangle$$

$$= i\hbar \dot{c}_a e^{-\frac{i}{\hbar}E_a t}|a\rangle + i\hbar \dot{c}_b e^{-\frac{i}{\hbar}E_b t}|b\rangle$$

$$i\hbar \dot{c}_a e^{-\frac{i}{\hbar}E_a t} = c_a e^{-\frac{i}{\hbar}E_a t}\langle a|H'|a\rangle$$
$$+ c_b e^{-\frac{i}{\hbar}E_b t}\langle a|H'|b\rangle$$

$$i\hbar \dot{c}_b e^{-\frac{i}{\hbar}E_b t} = c_a e^{-\frac{i}{\hbar}E_a t}\langle b|H'|a\rangle + c_b e^{-\frac{i}{\hbar}E_b t}\langle b|H'|b\rangle$$

$$\dot{c}_a = -\frac{i}{\hbar} c_a \langle a| e^{+\frac{i}{\hbar}H_0 t} H' e^{-\frac{i}{\hbar}H_0 t}|a\rangle - \frac{i}{\hbar} c_b \langle a| e^{\frac{i}{\hbar}H_0 t} H' e^{-\frac{i}{\hbar}H_0 t}|b\rangle$$

$$\dot{c}_b = -\frac{i}{\hbar} c_a \langle b| e^{\frac{i}{\hbar}H_0 t} H' e^{-\frac{i}{\hbar}H_0 t}|a\rangle - \frac{i}{\hbar} c_b \langle b| e^{\frac{i}{\hbar}H_0 t} H' e^{-\frac{i}{\hbar}H_0 t}|b\rangle$$

$$\frac{d}{dt}\begin{pmatrix} c_a \\ c_b \end{pmatrix} = -\frac{i}{\hbar} \underbrace{\left(e^{\frac{i}{\hbar}H_0 t} H' e^{-\frac{i}{\hbar}H_0 t}\right)_{2\times 2}}_{H'^I(t)} \begin{pmatrix} c_a \\ c_b \end{pmatrix}$$

$$\int_0^t dt \boxed{\frac{d}{dt} C = \int_0^t -\frac{i}{\hbar} H'^I(t)\, C\, dt}$$

$$C(t=0) = C_0$$

$$C(t) = C_0 - \frac{i}{\hbar}\int_0^t dt\, H'^I(t')\quad C(t')$$

$$C(t) = C_0 - \frac{i}{\hbar}\int_0^t dt_1 H'^I(t_1)\left(C_0 - \frac{i}{\hbar}\int_0^{t_1} dt_2 H'^I(t_2) C(t_2)\right)$$

$$= C_0 - \frac{i}{\hbar}\int_0^t dt_1 H'^I(t_1) C_0 + \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 H'^I(t_1) H'^I(t_2)\, C(t_2)$$

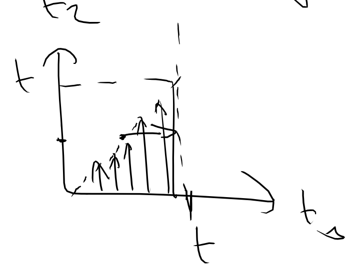
$$C(t) = C_0 + \left(\frac{-i}{\hbar}\right) \int_0^t dt_1 \, H'^I(t_1)\, C_0$$

$$\Rightarrow + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \, H'^I(t_1)\, H'^I(t_2)\, C_0$$

$$+ \left(\frac{-i}{\hbar}\right)^3 \int_0^t dt_1 \int dt_2 \int dt_3 \, H'^I(t_1) H'^I(t_2) H'^I(t_3)\, C_0$$

$$+ \ldots$$

$$= \left[\sum \left(\frac{-i}{\hbar}\right)^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \, H'^I(t_1) \ldots H'^I(t_n) \right] C_0$$



$$\int_0^t dt_1 \int_0^{t_1} dt_2 \, H'^I(t_1)\, H'^I(t_2)$$

$$= \int_0^t dt_2 \int_{t_2}^t dt_1 \, H'^I(t_1)\, H'^I(t_2)$$

$$= \int_0^t dt_1 \int_{t_1}^t dt_2 \, H'^I(t_2)\, H'^I(t_1)$$

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \, H'^I(t_1)\, H'^I(t_2) =$$

$$\frac{1}{2!} \int_0^t dt_1 \int_0^t dt_2 \underbrace{\left[H'^I(t_1)\, H'^I(t_2)\, \Theta(t_1 - t_2) + H'^I(t_2)\, H'^I(t_1)\, \Theta(t_2 - t_1) \right]}_{\mathcal{T}\{H'^I(t_1)\, H'^I(t_2)\}}$$

$\mathcal{T}\{H'^I(t_1)\, H'^I(t_2)\}$: time ordered product

$$\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \, H'^I(t_1) \ldots H'^I(t_n) = \frac{1}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n \, \mathcal{T}\left(H'^I(t_1) \ldots H'^I(t_n)\right)$$

$\mathcal{T}$: is linear. $\qquad$ a, b : any operator

$$\mathcal{T}(a+b) = \mathcal{T}(a) + \mathcal{T}(b)$$

$$\int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n\, H'^I(t_1) \cdots H'^I(t_n) = \frac{1}{n!}\, \mathcal{T}\left(\int_0^t dt'\, H'^I(t')\right)^n$$

$$C(t) = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!}\, \mathcal{T}\left(\int_0^t dt'\, H'^I(t')\right)^n C_0$$

$$\boxed{C(t) = T\exp\left\{\frac{-i}{\hbar}\int_0^t dt'\, H'^I(t')\right\} C_0}$$

$$T\exp\left\{\frac{-i}{\hbar}\int_0^t dt'\, H'^I(t')\right\}$$