Midterm November 6, Wednesday

start at 13^{40}

end at 17^{00}

prepare one page of cheat sheet to be handed in with your solutions.

$f(x)$

df

$x(t)$

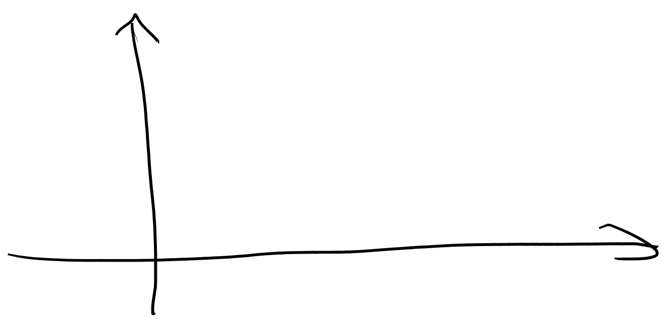
$v = \frac{dx}{dt}$

$dx = ?$

$dx = x(t+dt) - x(t)$

dQ

$Q(P, V, T, \text{path})$



$f(Q)$

$$\Delta U = \Delta Q - \Delta W$$

Microcanonical ensemble (fixed E)

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N} = \frac{2}{3}\frac{Nk}{E}$$

fixed

$$E = \frac{3}{2}NkT$$

canonical ensemble
(T fixed, E can have any value)

$$\langle E \rangle = A + TS$$

$$A = -kT\ln Q$$

$$\text{average } S = -\left(\frac{\partial A}{\partial T}\right)$$

$$\langle E \rangle = \frac{3}{2}NkT$$

$$\frac{\Delta E}{E} \sim \frac{1}{\sqrt{N}}$$

$$E \quad = \quad \langle E \rangle$$

$$\Rightarrow Q_N = \frac{Q_1^N}{N!}\left[1 +\right.$$

Q_1: partition function of a single particle

$$\frac{d\rho}{dt} = 0 \qquad \frac{\partial \rho}{\partial t} = 0$$

$$\|$$

$$\frac{\partial \rho}{\partial t} + \{\rho, H\} = 0 \qquad \boxed{\{\rho, H\} = 0}$$

ρ for SHO

$$\{\rho, H\} = 0 \Rightarrow \rho(H)$$

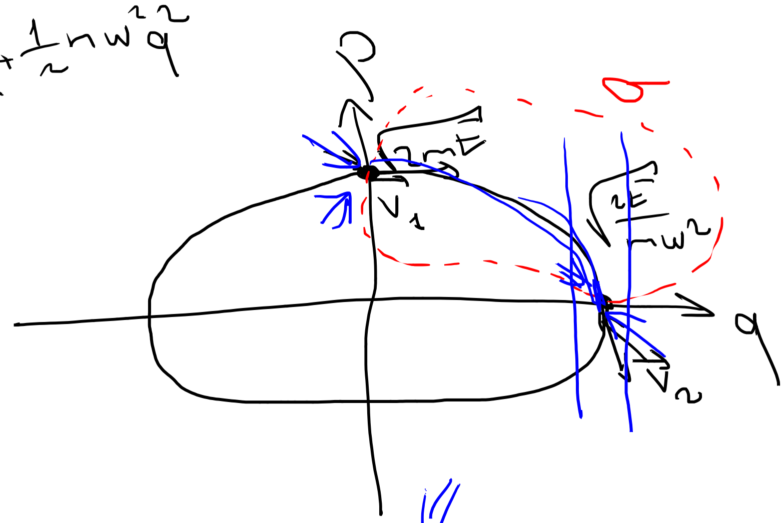
Microcanonical ensemble E is fixed

$$\rho(p,q) = 0 \quad \text{if} \quad H(p,q) \neq E$$

$$\rho = \text{const} \qquad \text{otherwise}$$

$$\boxed{\rho(p,q) = \rho_0 \, \delta\left[E - \left(\frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2\right)\right]}$$

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$



$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q \end{cases}$$

$$\frac{\partial \rho}{\partial t} + \nabla_{p,q}(\vec{j}) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial p}(\rho \dot{p}) + \frac{\partial}{\partial q}(\rho \dot{q})$$

$$\Rightarrow \vec{j} = \rho \vec{v} = \rho \overbrace{(\dot{p}, \dot{q})}^{\vec{v}}$$

$$(p_0, q_0)$$

$$H = \frac{1}{2m} p_0^2 x_p^2 + \frac{1}{2} m\omega^2 q_0^2 x_q^2$$

$$= \underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial p}\dot{p} + \frac{\partial \rho}{\partial q}\dot{q}}_{\frac{d\rho}{dt}} + \rho \underbrace{\left(\frac{\partial \dot{p}}{\partial p} + \frac{\partial \dot{q}}{\partial q}\right)}_{=0}$$

because the system is Hamiltonian

$$\vec{v}_1 = (\dot{p}, \dot{q})_{(p,q)=(\sqrt{2mE},0)} = \left(-m\omega^2 q, \frac{p}{m}\right)_{(p,q)=(\sqrt{2mE},0)} = \left(0, \sqrt{\frac{2E}{m}}\right)$$

$$\vec{v}_2 = \left(-m\omega^2 q, \frac{p}{m}\right)_{(p,q)=\left(0,\sqrt{\frac{2E}{m\omega^2}}\right)} = \left(-\sqrt{2m\omega^2 E}, 0\right)$$

$$|\vec{v}_1| = \sqrt{2mE} \neq \omega\sqrt{2mE} = |\vec{v}_2| \Rightarrow \frac{|\vec{v}_1|}{|\vec{v}_2|} = \frac{1}{\omega}$$

$$\rho v_1 = \rho_0 \delta\left(E - \left(\frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2\right)\right)\sqrt{2mE}\,\Big|_{q=0}$$

$$\Rightarrow \boxed{\rho v_1 = \rho_0 \sqrt{2mE}\,\delta\left(E - \frac{p^2}{2m}\right)_{p\to\sqrt{2mE}}} = \rho_0 \sqrt{2mE}(2m)\,\delta(2mE - p^2)$$

$$\rho v_2 = \rho_0 \delta(E - H)\sqrt{2mE}\,\omega\,\Big|_{p=0}$$

$$\Rightarrow \boxed{\rho v_2 = \rho_0 \sqrt{2mE}\,\omega\,\delta\left(E - \frac{1}{2}m\omega^2 q^2\right)} = \rho_0 \sqrt{2m\omega^2 E}\,\frac{2}{m\omega^2}\,\delta\left(\frac{2E}{m\omega^2} - q^2\right)$$

$$\frac{\rho v_2}{\rho v_1} = \omega$$

$$\rho v_2 = \rho_0 2\sqrt{\frac{2E}{m\omega^2}}\,\delta\left(q^2 - \frac{2E}{m\omega^2}\right)_{(p=0)}\Big|_{q=\ldots}$$

$$\rho v_1 = \rho_0 \sqrt{2mE}(2m)\,\delta\left(p^2 - 2mE\right)_{(q=0)}\Big|_{p=\ldots}$$

$$\rho v_1 \neq \rho v_2$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0$$

$$0 = \int_\sigma \vec{\nabla}(\rho \vec{v})\, dp\, dq = \int_{\text{boundary}} \rho \vec{v} \cdot d\vec{\omega} \simeq \rho V_2 - \rho V_1$$

$$\rho_2 V_2 - \rho_1 V_1 = 0$$

$$\Rightarrow \boxed{\underline{dp\, dq} = \underline{d\ell \cdot d\mathcal{H}}} \Leftarrow \text{Canonical transformation}$$

$$\rho(p,q) = \begin{cases} 0 & \text{if } H(p,q) \neq E \\ \neq 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \rho \neq \rho_0\, \delta(\underbrace{E - H(p,q)})$$

$$(p,q) \to (\ell, \mathcal{H}) \qquad \dot{\ell} \overset{?}{=} 1$$

$$dp\, dq = d\ell\, d\mathcal{H} \qquad \dot{\mathcal{H}} \overset{?}{=} 0$$

$$\rho(\ell, \mathcal{H}) = \underbrace{\tilde{\rho}(\ell)}\,\delta(\mathcal{H} - E)$$

$$\ln\left(e^{-\beta E} g(E)\right)\Big|_{E=U} = \ln\left(e^{-\beta \cancel{H}} g(\cancel{H})\right)$$

$$+ \frac{\partial \ln\left(e^{-\beta E} g(E)\right)}{\partial E}\Big|_{E=U} (E-U) + \frac{\partial^2 \ln\left(e^{-\beta E} g(E)\right)}{\partial E^2}\Big|_{E=U} \frac{(E-U)^2}{2} = 0$$

$$+ \ldots$$

$$= \left[-\beta U + \overbrace{\ln g(U)}\right] + \frac{\partial^2}{\partial E^2} \ln\left(e^{-\beta E} g(E)\right) \frac{(E-U)^2}{2!}$$

$$+ \ldots$$

$$= \left[-\beta U + \frac{S(U)}{k}\right] + \frac{\partial^2}{\partial E^2} \ln\left(e^{-\beta E} g(E)\right)\Big|_{E=U} \frac{(E-U)^3}{2!}$$

$$S = k\ln\Omega(E)$$

$\Omega(E)$ = # of states of energy E

$\Sigma(E)$ = # of states having energy $< E$

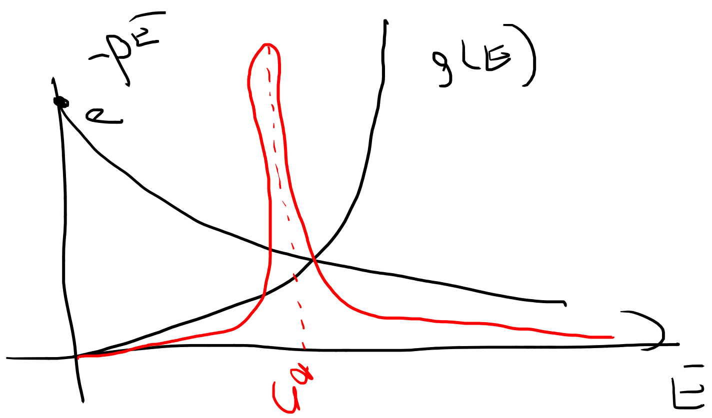
$$\sum_{E-\frac{\Delta}{2}}^{E+\frac{\Delta}{2}}\Omega(E) = \Sigma\left(E+\frac{\Delta}{2}\right) - \Sigma\left(E-\frac{\Delta}{2}\right)$$

$$\simeq \frac{\partial\Sigma}{\partial E}\Delta$$

$$\Rightarrow \Omega(E) \simeq \underbrace{\frac{\partial\Sigma}{\partial E}}_{g(E)}\Delta$$

$$\ln\Omega(E) = \underbrace{\ln\left(\boxed{\frac{\partial\Sigma}{\partial E}}E\right)}_{\sim 10^{23}} + \underbrace{\ln\frac{\Delta}{E}}_{\sim 23} \simeq \ln g(E)$$

$$e^{-\beta E} g(E) = P(E)$$: probability density for energy



U^* = most probable energy

U = average energy

$$\frac{U^* - U}{U} \longrightarrow 0$$

$$\frac{\partial}{\partial E}\left(e^{-\beta E} g(E)\right)\Big|_{E=U^*=U} = 0$$