

maximum

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial^2 f}{\partial x^2} < 0$$

$$f(x, y)$$

if (x_0, y_0) is a maximum

$$\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = \left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} = 0$$

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} < 0 & \frac{\partial^2 f}{\partial y^2} < 0 \\ \frac{\partial^2 f}{\partial x \partial y} \end{cases}$$

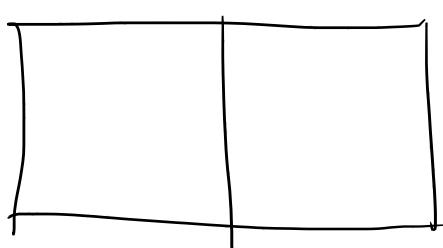
$$\frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y \partial x}$$

$$\Delta f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \rightarrow 0$$

i) $\boxed{\Delta S \geq 0}$ $\Delta A \leq 0$

ii) $\sigma^2 S$? in equilibrium



$$\left(\frac{\partial S}{\partial E_1} \right)_V = 0$$

$$\frac{\partial S_1}{\partial E_1} = \frac{\partial S_2}{\partial E_2}$$

$$\frac{\partial^2 S}{\partial E_1^2} < 0 \rightarrow \downarrow \downarrow 0$$

$$\Delta A = -S \Delta T - P \Delta V$$

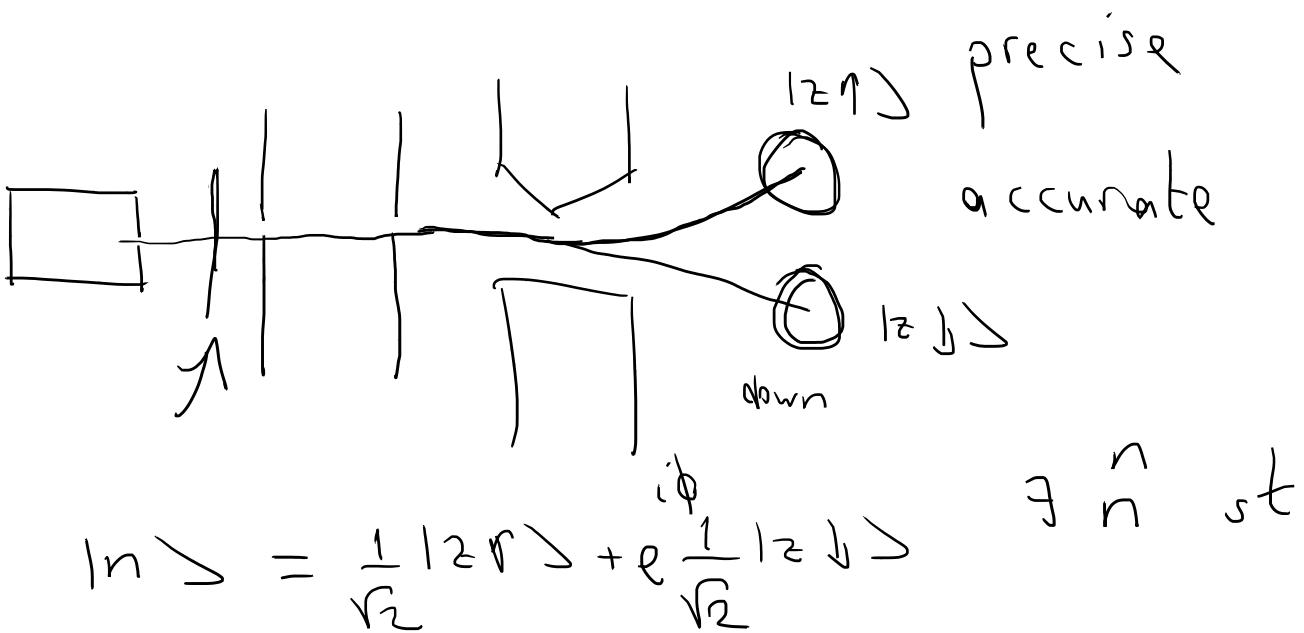
$$\begin{cases} \Delta T = 0 \\ \Delta V = 0 \end{cases} \quad \boxed{\Delta A = 0}$$

$$A = E - T S$$

$$Q = \sum e^{-\beta E}$$

$$Q = \sum e^{-\beta(E - \mu_n)}$$

$$\lim_{N \rightarrow \infty} \tan^{-1}(xN) \frac{2}{\pi} = \text{sign}(x)$$



$$|n\rangle = \frac{1}{\sqrt{2}} |z\rangle + e^{i\phi} \frac{1}{\sqrt{2}} |z\rangle$$

$$\hat{n} \cdot \vec{S} |n\rangle = \frac{n}{2} |n\rangle$$

$$P(n=+) = \frac{1}{2} \quad P(n=-) = \frac{1}{2}$$

$$|\bar{\Psi}_1\rangle, |\bar{\Psi}_2\rangle, |\bar{\Psi}_3\rangle, \dots$$

$$\langle O \rangle = p_1 \langle \Psi_1 | O | \Psi_1 \rangle + p_2 \langle \Psi_2 | O | \Psi_2 \rangle + \dots$$

$$\langle O \rangle = \underbrace{\sum p_i \langle \Psi_i | O | \Psi_i \rangle}_{\begin{array}{l} \text{QM average} \\ \text{statistical average} \end{array}} \neq (a|\Psi_1\rangle + b|\Psi_2\rangle + \dots)$$

mixed state

$\underbrace{a|\Psi_1\rangle + b|\Psi_2\rangle + \dots}_{\text{pure state}}$

$$\begin{aligned} \langle O \rangle &= \sum p_i \langle \Psi_i | O | \Psi_i \rangle & 1 = \sum \lambda_n \delta_{nn} \\ &= \sum p_i \sum_{nm} \langle \Psi_i | n \rangle \langle n | O | m \rangle \langle m | \Psi_i \rangle & \text{O}_{nn} \\ &= \sum_{nn} \left(\sum p_i \langle \Psi_i | n \rangle \langle m | \Psi_i \rangle \right) \langle n | O | m \rangle & \text{property of the measured quantity} \\ &= \sum_{nn} \Omega_{nn} = \sum_{m=0}^{\infty} \Omega_{mm} = \text{Tr}(\rho O) & \text{property of the measured quantity} \end{aligned}$$

$$\hat{\rho} = \sum p_i |\Psi_i\rangle \langle \Psi_i|$$

Sch. Picture

$|\Psi\rangle$ time dep.

\hat{O} time indep

$\hat{\rho}$ time dependent

Hil. Picture

$|J\rangle$ time indep.

$\hat{\rho}$ time dep.

$$\Psi_{AB} = \frac{1}{\sqrt{2}} (| \uparrow \downarrow \rangle_{AB} + | \downarrow \uparrow \rangle_{AB})$$

$$\rho_A = \frac{1}{2} |\uparrow\rangle \langle \uparrow| + \frac{1}{2} |\downarrow\rangle \langle \downarrow|$$

$$\begin{aligned}
 \langle O_A \rangle &= \text{Tr}_{AB}(\rho_{AB} O_A) \\
 &= \text{Tr}_A(\text{Tr}_B \rho_{AB} O_A) \\
 &= \text{Tr}_A \left[\underbrace{\text{Tr}_B \rho_{AB}}_{\rho_A} O_A \right]
 \end{aligned}$$

" $\hat{\rho}$ commutes with \hat{N} "

$$\begin{aligned}
 \hat{\rho}(t) &= \sum_i p_i |2_i(t)\rangle \langle 2_i(t)| = \sum_i p_i |2_i\rangle \langle 2_i| e^{\frac{i}{\hbar} H t} \\
 \langle O \rangle(t) &= \text{Tr} \rho(t) O = \text{Tr} \rho \stackrel{H}{\circ} O(t) \\
 |2\rangle &= e^{-\frac{i}{\hbar} H t} |2_0\rangle \\
 \langle 2| &= \langle 2_0 | e^{+\frac{i}{\hbar} H t} \\
 |u\rangle \langle 2| &= e^{-\frac{i}{\hbar} H t} |2_0\rangle \langle 2_0 | e^{\frac{i}{\hbar} H t}
 \end{aligned}$$

Ex

$$|z+\rangle$$

$$|x+\rangle$$

$$\rho = \frac{1}{2} |z+\rangle \langle z+| + \frac{1}{2} |x+\rangle \langle x+|$$

$$|x+\rangle = \frac{1}{\sqrt{2}} (|z+\rangle + |z-\rangle)$$

$$\rho = \frac{1}{2} |z+\rangle \langle z+| + \frac{1}{4} \left(|z+\rangle \langle z-| + |z-\rangle \langle z+| \right) \underbrace{(|z+\rangle \langle z-|)}$$

$$= |z+\rangle \langle z+| \left(\frac{1}{2} + \frac{1}{4} \right)$$

$$+ |z-\rangle \langle z-| \left(\frac{1}{4} \right)$$

$$+ (|z+\rangle \langle z-| + |z-\rangle \langle z+|) \left(\frac{1}{4} \right)$$

$$\rho = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \quad \text{Tr } \rho = 1$$

$$\rho^2 = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$\rho^2 = \begin{pmatrix} \frac{10}{16} & \frac{4}{16} \\ \frac{4}{16} & \frac{2}{16} \end{pmatrix} \quad \text{Tr } \rho^2 = \frac{3}{4} < 1$$

\Rightarrow mixed state.

$$\langle S_z \rangle = \text{Tr} (S_z g)$$

$$= \text{Tr} \left[\frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \right]$$

$$= \frac{\hbar}{8} \text{Tr} \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \frac{\hbar}{4}$$

$$\langle S_x \rangle = \text{Tr} (S_x g)$$

$$= \text{Tr} \left[\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \right]$$

$$= \frac{\hbar}{8} \text{Tr} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{\hbar}{4}$$

$$\hat{\frac{\partial \hat{g}}{\partial t}} = () [\hat{H}, \hat{g}] = 0 \Rightarrow \hat{g}(\hat{H}, \hat{N}, \cancel{\hat{x}}, \cancel{\hat{y}})$$

$\hat{S}_x^2 e^{-\beta(\hat{H} - \mu \hat{N})}$

 $g(\hat{H}, \hat{N})$

$$S^{\text{eq}} = \sum S_n \ln \langle n \rangle_{\text{eq}}$$

$$H(n) = E_n \ln \langle n \rangle$$

$$S^{\text{eq}} = \sum e^{-\beta E_n} \ln \langle n \rangle_{\text{eq}}$$

$\langle \vec{r}_1 \vec{r}_2 | \rho | \vec{r}_1 \vec{r}_2 \rangle$ = probability density that the particles are in \vec{r}_1 and \vec{r}_2

$$S_{nn} = \sum p_i \langle q_i | n \rangle \langle n | q_i \rangle$$

$$S_{nn} = \sum p_i \underbrace{\langle q_i | n \rangle}_{\rho_n(\vec{r}_1, \vec{r}_2)}^2 = \text{probability that the system is in state } |n\rangle$$

$$\langle \vec{r}_1 \vec{r}_2 | \rho | \vec{r}_1 \vec{r}_2 \rangle = \frac{1}{Q} \sum e^{-\beta \left(\frac{P_1^2}{2n} + \frac{P_2^2}{2n} \right)}$$

$$= \frac{1}{Q} \frac{1}{\rho_1 \rho_2} e^{-\beta \left(\frac{P_1^2}{2n} + \frac{P_2^2}{2n} \right)} \langle \vec{r}_1 \vec{r}_2 | \vec{p}_1 \vec{p}_2 \rangle$$

$$\langle \vec{r}_1 \vec{r}_2 | \vec{p}_1 \vec{p}_2 \rangle = \frac{1}{N} (e^{i \vec{p}_1 \cdot \vec{r}_1} e^{i \vec{p}_2 \cdot \vec{r}_2} + e^{i \vec{p}_2 \cdot \vec{r}_1} e^{i \vec{p}_1 \cdot \vec{r}_2})$$