$$\cancel{dq} = \qquad dV \quad N d\mu$$

$$\sum e^{-\beta(\varepsilon - \mu N)} \equiv e^{-\beta \phi}$$

$$\phi = E - TS - \mu N$$

$$q = \ln e^{-\beta G} = \boxed{-\beta \phi \equiv q}$$

$$\Rightarrow d\phi = -S\,dT - P\,dV - N\,d\mu$$

$$\phi(T, \mu, V) = V f(T, \mu) = -P(\mu, T) V$$

$$q(T, \mu, V) = V g = +\frac{P(\mu, T)}{kT} V$$

$$\left(\frac{\partial \phi}{\partial V}\right)_{T,\mu} = -P = f(T, \mu)$$

$$G = E - TS + PV$$

$$dG = -S\,dT + V\,dP - \mu\,dN$$

$$G(T, P, N) = -\mu N = E - TS + PV$$

$$PV = TS - \mu N - E$$

$$q = \frac{PV}{kT} = \frac{S}{k} - \frac{\mu N}{kT} - \frac{E}{kT}$$

$$\boxed{q = N(T, V, \mu)} \Leftarrow$$

$$\left(\frac{\partial q}{\partial V}\right)_{\mu,T} = \left(\frac{\partial N}{\partial V}\right)_{\mu,T}$$

$$\left(\frac{\partial q}{\partial N}\right)_{T,V} = 1$$

(μ, N)

(p, V)

(T, S)

$$\bar{N} = N(T, V, \mu) \Rightarrow \mu(N, T, V) = \mu\left(\frac{N}{V}, T\right)$$

$$\left(\frac{\partial x}{\partial y}\right)_u \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial y}{\partial u}\right)_x = -1$$

$[\hat{\rho}, \hat{H}] = 0$ in equilibrium

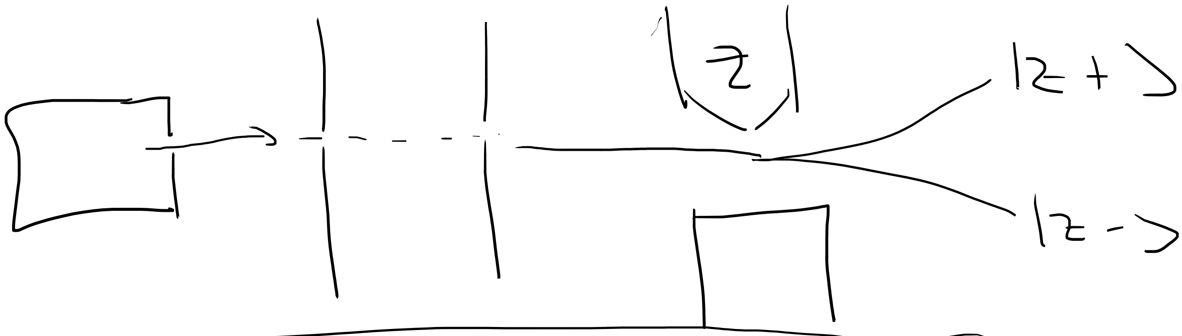
$\Rightarrow \hat{\rho}(\hat{H})$

$$\psi(\vec{r}, t) = \sum a_i(t) \psi_i(\vec{r})$$

$$H \psi_i(\vec{r}) = E_i \psi_i(\vec{r})$$

$|a_i(t)|^2$: probability of observation

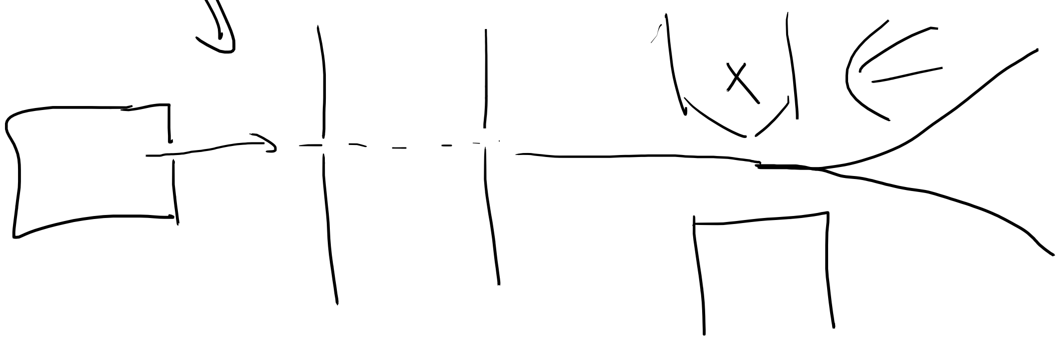
$$\rho_{nm}(t) = a_n(t) a_m^*(t)$$



$$\boxed{|n\rangle = \frac{1}{\sqrt{2}}|z+\rangle + e^{i\phi}\frac{1}{\sqrt{2}}|z-\rangle}$$

assume $\phi = 0$

$$|n\rangle = \frac{1}{\sqrt{2}}|z+\rangle + \frac{1}{\sqrt{2}}|z-\rangle = |x+\rangle$$



$$\rho = \frac{1}{2}|z+\rangle\langle z+| + \frac{1}{2}|z-\rangle\langle z-|$$

$$|z+\rangle = \frac{1}{\sqrt{2}}\left(|x+\rangle + |x-\rangle\right)$$

$$|z-\rangle = \frac{1}{\sqrt{2}}\left(|x+\rangle - |x-\rangle\right)$$

$$\rho = \frac{1}{2}|x+\rangle\langle x+| + \frac{1}{2}|x-\rangle\langle x-|$$

$$\langle O \rangle = \langle \psi | O | \psi \rangle$$

$$= \sum_{ij} \langle \psi_i | O | \psi_j \rangle a_i^* a_j$$

$$\sum |n\rangle\langle n| = 1 \quad = \sum_n \sum_{ij} \langle n|\psi_j\rangle\langle \psi_i | O | n \rangle a_i^* a_j$$

$$= \sum_n \left(\sum_{ij} |\psi_j\rangle\langle \psi_i | O \, a_i^* a_j \right)_{nn}$$

$$= T_r \left(\sum_{ij} |\psi_j\rangle\langle \psi_i | a_i^* a_j \right) O$$

$$\langle O \rangle_\psi = T_r \left(\underbrace{|\psi\rangle\langle \psi|}\, O \right)$$

$$p_n \quad |\psi_n\rangle, \quad p_{n'}, |\psi_{n'}\rangle \quad \ldots$$

$$\langle\langle O \rangle\rangle = \sum_n p_n \langle O \rangle_{\psi_n}$$

$$= \sum_n p_n T_r \, |\psi_n\rangle\langle \psi_n | O$$

$$= T_r \left[\underbrace{\left(\sum_n p_n |\psi_n\rangle\langle \psi_n| \right)}_{\rho} O \right]$$

$|x+\rangle \quad \frac{1}{4}$

$\left\{|z+\rangle \quad \frac{3}{4}\right\}$

$\langle S_z \rangle = ?$

$$\boxed{\rho = \frac{1}{4}|x+\rangle\langle x+| + \frac{3}{4}|z+\rangle\langle z+|}$$

$\{|z+\rangle, |z-\rangle\}$

$|x+\rangle = \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle)$

$|x-\rangle = \frac{1}{\sqrt{2}}(|z+\rangle - |z-\rangle)$

$$\rho = \frac{1}{8}(|z+\rangle + |z-\rangle)(\langle z+| + \langle z-|) + \frac{3}{4}|z+\rangle\langle z+|$$

$$= |z+\rangle\langle z+|\left(\frac{7}{8}\right) + |z-\rangle\langle z-|\left(\frac{1}{8}\right)$$

$$+ \frac{1}{8}(|z+\rangle\langle z-| + |z-\rangle\langle z+|)$$

$$\rho = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix} \qquad \mathrm{Tr}\,\rho = 1$$

$$\langle S_z \rangle = \mathrm{Tr}\,\rho S_z = \mathrm{Tr}\,\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{8}\begin{pmatrix} 7 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{\hbar}{16}\mathrm{Tr}\begin{pmatrix} 7 & \\ & -1 \end{pmatrix} = \frac{\hbar}{16}\cdot 6 = \frac{3}{8}\hbar$$

$$\langle S_z \rangle = \frac{\hbar}{2}\left(\frac{3}{4}\right) + 0\left(\frac{1}{4}\right)$$

In equilibrium ρ is diagonal in energy basis $[\hat{\rho}, \hat{H}] = 0$

$\langle \vec{r}_1', \vec{r}_2' | \rho | \vec{r}_1, \vec{r}_2 \rangle = ?$ for free particles.

For the simple harmonic oscillator

$\langle r' | \hat{\rho} | r \rangle = ?$

$$\boxed{\langle r' | e^{-\beta \hat{H}} | r \rangle = ?}$$

$\checkmark$ $H = \frac{1}{2m} p^2 + \frac{1}{2} m w^2 x^2$

$\checkmark$ $H \psi_n(r) = \hbar w \left(n + \frac{1}{2}\right) \psi_n(r)$

$\checkmark$ $H |n\rangle = \hbar w \left(n + \frac{1}{2}\right) |n\rangle$

$|n\rangle \qquad \psi_n(r)$

$\checkmark$ $|n\rangle = \int dr \psi_n(r) |r\rangle$

$\psi_n(r) = \langle r | n \rangle$

$|n\rangle = \left(\int dr |r\rangle \underbrace{\langle r | n \rangle}_{\psi_n(r)} \right) \Leftarrow$

$\langle r' | e^{-\beta H} | r \rangle =$

$$\mathbb{1} = \int dr \, |r\rangle\langle r|$$

$$\mathbb{1} = \sum_n |n\rangle\langle n|$$

$$\langle r'| e^{-\beta H} \int dr'' \, |r''\rangle \underbrace{\langle r''|r\rangle}_{\delta(r-r'')} = \langle r'| e^{-\beta H} |r\rangle$$

$$\sum_n \langle r'| \underbrace{e^{-\beta H} |n\rangle} \langle n|r\rangle$$

$$= \sum_n e^{-\beta E_n} \underbrace{\langle r'|n\rangle}_{\psi_n(r')} \underbrace{\langle n|r\rangle}_{\psi_n^*(r)}$$

$$\boxed{= \sum_{n=0}^{\infty} e^{-\beta E_n} \psi_n(r') \psi_n^*(r) = \langle r'| e^{-\beta H} |r\rangle}$$

$$H|n\rangle = \hbar\omega \, n|n\rangle$$

$$e^{-\beta E_n} \xrightarrow[T \to 0]{} \begin{cases} 0 & n = 1, \ldots \\ 1 & \end{cases}$$

$$\boxed{\langle r'| e^{-\beta H} |r\rangle \xrightarrow[T \to 0]{} \psi_0(r') \psi_0^*(r)}$$

$$\langle r| e^{-\beta H} |r\rangle \xrightarrow[T \to 0]{} |\psi_0(r)|^2$$

$$\langle r | e^{-\beta H} | r \rangle = e^{-\beta E_0} |\psi_0(r)|^2$$

$$T \ll T_0 = \frac{\hbar \omega}{k}$$

$\boxed{\text{Lattice Field Theory}}$

$$\frac{p^2}{2m} = \frac{3}{2} kT \Rightarrow p = \sqrt{3mkT} \Leftarrow$$

$$\Rightarrow \lambda = \frac{h}{p} = \frac{h}{\sqrt{3mkT}}$$

$$\frac{h}{\sqrt{3mkT}} \simeq \lambda_T$$

N, V $\qquad \left(\frac{V}{N}\right)^{1/3} \sim \lambda_T$

Ideal gas law might be valid if $\left(\frac{V}{N}\right)^{1/3} T^{1/2} \gg 1$

$\frac{V}{N} T^{3/2}$ is large

$$\langle \vec{r}_1 \vec{r}_2 | e^{-\beta \hat{H}} | \vec{r}_1 \vec{r}_2 \rangle$$

$$\hat{H} = \frac{\hat{\vec{p}}_1^{\,2}}{2m} + \frac{\hat{\vec{p}}_2^{\,2}}{2m}$$

$$\psi_{\vec{p}_1 \vec{p}_2}(\vec{r}_1 \vec{r}_2) = \left(e^{i\vec{p}_1 \cdot \vec{r}_1 + i\vec{p}_2 \cdot \vec{r}_2}\right) \frac{1}{\sqrt{V}} = \langle \vec{r}_1 \vec{r}_2 | \vec{p}_1 \vec{p}_2 \rangle$$

$$\psi_{\vec{p}_1 \vec{p}_2}(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2V}} \left[e^{i\vec{p}_1 \cdot \vec{r}_1 + i\vec{p}_2 \cdot \vec{r}_2} \mp e^{+i\vec{p}_1 \cdot \vec{r}_2 + i\vec{p}_2 \cdot \vec{r}_1} \right]$$

$\boxed{(\text{anyons})}$

$$\langle \vec{r}_1 \vec{r}_2 | e^{-\beta \hat{H}} | \vec{r}_1 \vec{r}_2 \rangle$$

$$= \sum_{p_1 p_2} \langle \vec{r}_1 \vec{r}_2 | e^{-\beta \hat{H}} | \vec{p}_1 \vec{p}_2 \rangle \overbrace{\langle \vec{p}_1 \vec{p}_2 | \vec{r}_1 \vec{r}_2 \rangle}$$

$$= \sum_{p_1 p_2} e^{-\beta H(p_1, p_2)} \frac{1}{V} e^{i\vec{p}_1 \cdot \vec{r}_1 + \cancel{i\vec{p}_2 \vec{r}_2}} \cancel{e^{-i(\vec{p}_1 \vec{r}_1 + i\vec{p}_2 \vec{r}_2)}}$$

$$\langle \vec{r}_1 \vec{r}_2 | e^{-\beta \hat{H}} | \vec{r}_1 \vec{r}_2 \rangle = \sum_{p_1 p_2} e^{-\beta \frac{p_1^2}{2m}} e^{-\beta \frac{p_2^2}{2m}}$$

$$\langle \vec{r}_1 \vec{r}_2 | e^{-\beta \hat{H}} | \vec{r}_1 \vec{r}_2 \rangle = \left(\sum_{p} \frac{e^{-\beta \frac{p^2}{2m}}}{V} \right)^2$$

$$\rho(\vec{r}_1,\vec{r}_2) = \rho(\vec{r}_1)\,\rho(\vec{r}_2) = \frac{\left(\sum_p e^{-\beta\frac{p^2}{2m}}\right)^2 \frac{1}{V}}{Z}$$

$$\int d^3r_1\, d^3r_2\, \rho(\vec{r}_1,\vec{r}_2) = 1$$

$$\rho(\vec{r}_1,\vec{r}_2) = \frac{1}{V^2}$$

$$\rho_{nm} = \sum_i p_i\, a_n^i\, a_m^{*i}$$

$$\left(p_i, \underbrace{\sum a_n^i \psi_n}\right)$$

$$\boxed{\rho_{nn} = \sum_i \underline{p_i}\,|a_n^i|^2}$$

$$\boxed{\left\langle e^{i(\theta_n-\theta_m)}\right\rangle = 0}$$

assumption