Final : Take Home

Date : January 4^{th}, 2020

exam will be available at 10:00 am

duration until January 6, 10:00 am

Make-up

$$q = \frac{pV}{kT} = -\overbrace{\sum_{\substack{\text{single}\\ \text{particle}\\ \text{states}}} \ln\left(1 - z e^{-\beta\varepsilon}\right)}^{o}$$

$$z \equiv e^{\beta\mu}$$

$$U = -\left(\frac{\partial q}{\partial \beta}\right)_{V,z} \qquad \varepsilon e^{-\beta(\varepsilon-\mu)} = -\frac{\partial}{\partial\beta}\left(\frac{e^{-\beta\varepsilon}}{z}\right)$$

$z = 1$ for photons & phonons

$\Rightarrow C_V$

$$\bar{N} = z\left(\frac{\partial}{\partial z} q\right)_{V,T}$$

$\vdots$

Einstein model

$$\sum_{\substack{\text{single}\\ \text{particle}\\ \text{states}}} f(\omega_s, \beta, z) \longrightarrow 3N f(\omega_E, \beta, z)$$

$$U = \sum_{\substack{\text{single}\\ \text{particle}\\ \text{states}}} \hbar\omega_s \frac{1}{e^{\hbar\beta\omega_s} - 1} = \frac{\hbar\omega_E}{e^{\hbar\beta\omega_E} - 1} 3N$$

1 km

$$\hbar\beta\omega_E \ll 1 \Rightarrow T \gg \frac{\hbar\omega_E}{k} = \Theta_E$$

$$\hbar\beta\omega_E \gg 1 \Rightarrow T \ll \frac{\hbar\omega_E}{k} = \Theta_E$$

$\Downarrow$

$$U = 3N\frac{\hbar\omega_E}{e^{\beta\hbar\omega_E}-1} \simeq 3N\frac{\cancel{\hbar\omega_E}}{(\cancel{1}+\beta\cancel{\hbar\omega_E})-\cancel{1}} = 3NkT$$

$$U = (3N+3N)\frac{kT}{2} = 3NkT \Rightarrow C_V = 3Nk$$

Low temperature limit

$$\beta\hbar\omega_E \gg 1$$

$$U = 3N\hbar\omega e^{-\beta\hbar\omega_E}$$

$$C_V = \frac{\partial U}{\partial T} = 3N\hbar\omega_E\left(\frac{\hbar\omega_E}{kT^2}\right)e^{-\beta\hbar\omega_E}$$

$$C_V = 3Nk\left(\frac{T_E}{T}\right)^2 e^{-\frac{T_E}{T}}$$

Debye Model

$$\sum_{\substack{\text{single particle states}}} = V\frac{1}{2\pi^2}\int\frac{d\omega}{c^3}\,\omega^2$$

phonons $\rightarrow$ single particle states

$$k = \frac{\omega}{c}$$

$$\int\frac{d^3q}{(2\pi)^3} = \frac{V}{(2\pi)^3}4\pi\int\frac{k^2dk}{c^3} = \frac{V}{2\pi^2}\int\frac{d\omega\,\omega^2}{c^3}\cdot 3$$

$$\sum_{\substack{\text{single}\\\text{particle}\\\text{states}}} \qquad \frac{V}{2\pi^2}\int_0^{\omega_D} d\omega\, \omega^2 \left(\frac{1}{c_L^3}+\frac{2}{c_T^3}\right)$$

$\omega_D \Leftarrow$ Debye Frequency

$$3N = \sum_{\substack{\text{single}\\\text{particle}\\\text{states}}} 1 = \frac{V}{2\pi^2}\int_0^{\omega_D} d\omega\, \frac{\omega^2}{c^3}; \quad \frac{1}{c^3} = \frac{1}{3}\left[\frac{1}{c_L^3}+\frac{2}{c_T^3}\right]$$

$$3N = \frac{V}{6\pi^2}\frac{\omega_D^3}{c^3} \Rightarrow \omega_D = \left(18\pi^2 \frac{V}{N} c^3\right)^{1/3}$$

$$U = \sum_{\substack{\text{single}\\\text{particle}\\\text{states}}} \frac{\hbar\omega_s}{e^{\beta\hbar\omega_s}-1} = \frac{V}{2\pi^2}\int_0^{\omega_D} d\omega\, \frac{\omega^2}{e^{\beta\hbar\omega}-1}$$

$$\frac{\hbar\omega_D}{k} \equiv \Theta_D \quad \text{Debye temperature}$$

<u>Large temperature limit</u>

$$T \gg \Theta_D \qquad \hbar\omega_D\beta \ll 1$$

$$U = \frac{V}{2\pi^2}\int_0^{\omega_D} \frac{d\omega}{c^3}\, \omega^2 \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$$

$$\simeq \frac{V}{2\pi^2}\int_0^{\omega_D} \frac{d\omega}{c^3} \frac{\omega^2\, \cancel{\hbar\omega}}{(\cancel{1}+\beta\cancel{\hbar\omega})-\cancel{1}}$$

$$\simeq \frac{V}{2\pi^2}\frac{1}{\beta}\int_0^{\omega_D} \frac{d\omega}{c^3}\, \omega^2$$

$$U \simeq \frac{V}{2\pi^2}(kT)\int_0^{\omega_D}\frac{d\omega}{c^3}\,\omega^2$$

$$= kT\left[\underbrace{\frac{V}{2\pi^2}\int_0^{\omega_D}\frac{d\omega}{c^3}\,\omega^2}_{\equiv\, 3N}\right]$$

$$U = 3NkT$$

Low temperature limit:

$$U = \frac{V}{2\pi^2}\int_0^{\omega_D}\frac{d\omega\,\omega^2}{c^3}\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$$

$$\beta\hbar\omega = x \Rightarrow d\omega = \frac{x}{\beta\hbar}$$

$$U = \frac{V}{2\pi^2 c^3}\int_0^{x_0}\frac{dx\,x^2}{(\beta\hbar c)^3}\frac{\left(\frac{x}{\beta}\right)}{e^x-1}$$

$$U = \frac{V}{2\pi^2 c^3}\frac{1}{\beta^4}\frac{1}{(\hbar c)^3}\int_0^{x_0}dx\frac{x^3}{e^x-1}$$

$$x_0 \gg 1$$

$$U \simeq \left[\frac{Vk^4}{2\pi^2\hbar^3c^6}\int_0^{\infty}dx\frac{x^3}{e^x-1}\right]T^4$$

$$C_V \propto T^3$$

$$\sum_{\text{states}} e^{-\beta \varepsilon}$$

$$\frac{1}{e^{\beta(\varepsilon-\mu)}-1} \xrightarrow[\mu\to 0]{} \frac{1}{e^{\beta\varepsilon}-1} \xrightarrow[\varepsilon\to 0]{} \infty$$

$$\lambda_T \sim \left(\frac{V}{N}\right)^{1/3}$$

$$\sum_{\text{microstates}}$$

He^4 : boson

He^3 : fermion

$\psi(\text{spin}\,1, \text{spin}\,2)$

$$p \propto e^{-\beta\varepsilon}$$

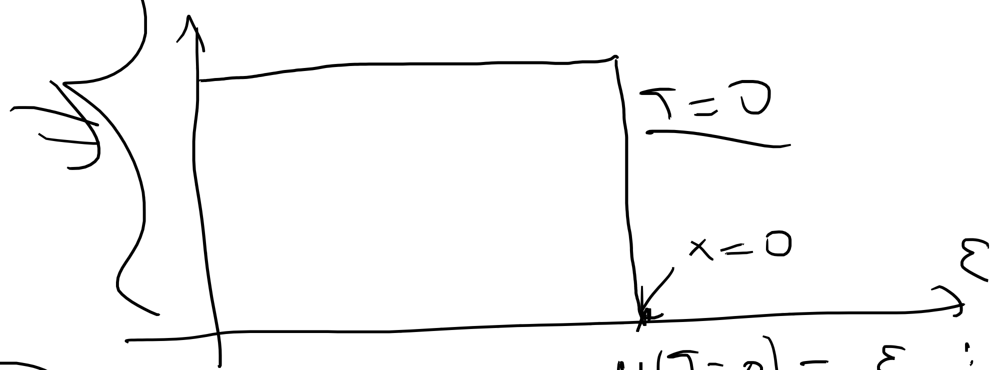
Fermions

$$q = \frac{PV}{kT} = + \sum_{\substack{\text{single}\\\text{particle}\\\text{states}}} \ln\left(1 + e^{-\beta\varepsilon} z\right)$$

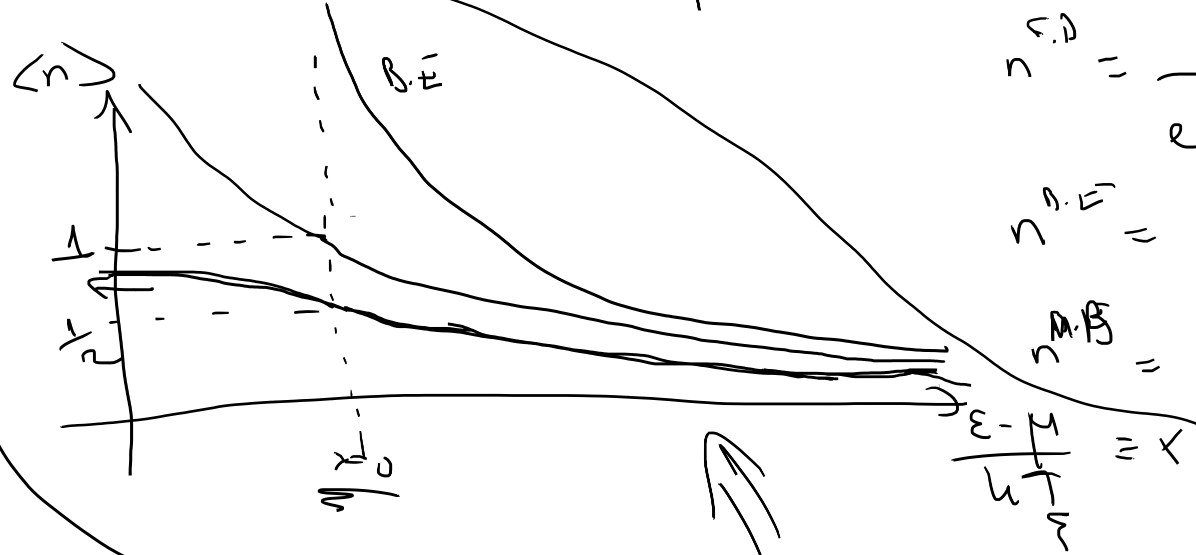
$$N = \sum_{\substack{\text{single}\\\text{particle}\\\text{states}}} \frac{1}{e^{\beta\varepsilon} z^{-1} + 1} \equiv \sum_{\substack{\text{single}\\\text{particle}\\\text{states}}} n^{F.D}(\varepsilon)$$

$$U = \sum_{\substack{\text{single}\\\text{particle}\\\text{states}}} \frac{\varepsilon}{e^{\beta\varepsilon} z^{-1} + 1}$$

$$n_{F.D}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} \xrightarrow[\beta\to\infty]{} \begin{cases} 0 & \varepsilon > \mu \\ 1 & \varepsilon < \mu \end{cases}$$



$\mu(T=0) \equiv \varepsilon_F$: fermi energy



$$n^{F.D} = \frac{1}{e^{x} + 1}$$

$$n^{B.E} = \frac{1}{e^{x} - 1}$$

$$n^{M.B} = e^{-x}$$

$$n = \frac{1}{e^{\beta\xi - 1}_{z} + a} \qquad a = \mp 1, 0$$

$e^{\beta\mu} = z \gtrsim 1$ QM effects can show themselves

$e^{\beta\mu} = z \to 0$ QM effects disappear

$$dU = TdS - PdV + \mu \underset{1}{\overset{\shortparallel}{dN}}$$

$$N = \sum_{\substack{\text{single}\\\text{particle}\\\text{states}}} \underbrace{\frac{1}{e^{\beta(\varepsilon-\mu)} + 1}}_{\Theta(\varepsilon_f - \varepsilon)}$$

at $T = 0$

$$N = \sum_{\substack{\text{single}\\\text{particle}\\\text{states}}} \Theta(\varepsilon_f - \varepsilon)$$

$$N = g\int \frac{d^3p\, d^3q}{(2\pi\hbar)^3}\, \Theta(\varepsilon_f - \varepsilon)$$

$$= \frac{gV}{(2\pi\hbar)^3} 4\pi \int dp\, p^2\, \Theta(\varepsilon_f - \varepsilon)$$

$$\varepsilon = \frac{p^2}{2m} \Rightarrow p = \sqrt{2m\varepsilon}$$

$$dp = \frac{\sqrt{2m}}{2} \frac{d\varepsilon}{\sqrt{\varepsilon}}$$

$$N = \frac{gV}{(2\pi\hbar)^3} 4\pi \left(\frac{\sqrt{2m}}{2}\right)^3 \int_0^{\varepsilon_f} d\varepsilon\, \sqrt{\varepsilon}$$

$$\boxed{N = \frac{gV}{(2\pi\hbar)^3} 2\pi (2m)^{3/2} \varepsilon_f^{3/2} \frac{2}{3}} \Rightarrow \text{solve for } \varepsilon_f$$

ε_F (in metals) $\sim eV \sim 10^4 K - 10^5 K$

$$U(T=0) = \sum_{\substack{\text{single}\\ \text{particle}\\ \text{states}}} \Theta(\varepsilon_F - \varepsilon)\, \varepsilon$$

$$= \underbrace{\frac{gV}{(2\pi\hbar)^3} 4\pi \left(\frac{\sqrt{2m}}{2}\right)^3}\int_0^{\varepsilon} d\varepsilon \sqrt{\varepsilon}\, \varepsilon$$

$$\frac{U(T=0)}{N} = \frac{\int_0^{\varepsilon_F} d\varepsilon \sqrt{\varepsilon}\, \varepsilon}{\int_0^{\varepsilon_F} d\varepsilon \sqrt{\varepsilon}} = \frac{\frac{2}{5}\varepsilon_F^{5/2}}{\frac{2}{3}\varepsilon_F^{3/2}} = \frac{3}{5}\varepsilon_F$$

$$\boxed{U(T=0) = \frac{3}{5} N \varepsilon_F}$$

$$C_V = ? \quad \frac{\partial U}{\partial T}$$

$$q = \frac{PV}{kT} = \sum_{\substack{\text{single}\\ \text{particle}\\ \text{states}}} \ln\left(1 + e^{-\beta(\varepsilon-\mu)}\right)$$

$$= \frac{gV}{(2\pi\hbar)^3} 4\pi \left(\frac{\sqrt{2m}}{2}\right)^3 \int_0^{\infty} d\varepsilon \underbrace{\sqrt{\varepsilon}}_{= \frac{2}{3}\frac{d}{d\varepsilon}\varepsilon^{3/2}} \ln\left(1 + e^{-\beta(\varepsilon-\mu)}\right)$$

$$= \frac{2}{3} \frac{gV}{(2\pi\hbar)^3} 4\pi \left(\frac{\sqrt{2m}}{2}\right)^3 \int_0^{\infty} d\varepsilon\, \varepsilon^{3/2} \left(-\frac{\partial}{\partial \varepsilon}\right) \ln(\ldots)$$

$=$

$$\frac{\partial}{\partial \varepsilon} \ln\left\{1+e^{-\beta(\varepsilon-\mu)}\right\} = \frac{-\beta e^{-\beta(\varepsilon-\mu)}}{1+e^{-\beta(\varepsilon-\mu)}} = \frac{-\beta}{e^{\beta(\varepsilon-\mu)}+1}$$

$$\frac{PV}{kT} = \frac{2}{3}\frac{1}{kT} \underbrace{\frac{gV}{(2\pi\hbar)^3}\frac{4\pi(\sqrt{2m})^3}{2}\int_0^\infty d\varepsilon\,\sqrt{\varepsilon}\,\frac{\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}}_{U}$$

$$\boxed{PV = \frac{2}{3}U}$$

at $T=0$

$$PV = \frac{2}{\cancel{3}}\frac{\cancel{3}}{5}N\varepsilon_F = \frac{2}{5}N\varepsilon_F$$

$$\varepsilon_F = \frac{p_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m} = \frac{1}{2}m v_F^2$$