Hyperfine Interaction

$$H^{h.f.} = A\vec{S}_e\cdot\vec{S}_p\,\delta^3(\vec{r}) + \frac{B}{r^3}\left[(\vec{S}_e\cdot\hat{r})(\vec{S}_p\cdot\hat{r}) - \frac{1}{3}\vec{S}_e\cdot\vec{S}_p\right]$$

scalar interaction

tensor interaction

$l = 0$

if $l \neq 0$ $\quad \psi \propto r^l$

$$\delta E = \langle\psi| H^{h.f} |\psi\rangle$$

$$|\psi\rangle = |nlm\rangle \otimes |S_e\rangle \otimes |S_p\rangle$$

for $l=0$

$$H' = A\,\vec{S}_e\cdot\vec{S}_p\,\delta^3(\vec{r})$$

$$= \frac{A}{2}\left[(\vec{S}_e+\vec{S}_p)^2 - \vec{S}_e^2 - \vec{S}_p^2\right]$$

$$= \frac{A}{2}\left[\vec{S}_H^2 - \hbar^2\,\frac{1}{2}\left(\frac{1}{2}+1\right) - \hbar^2\,\frac{1}{2}\left(\frac{1}{2}+1\right)\right]$$

$$H' = \frac{\langle A\rangle}{2}\left[\vec{S}_H^2 - \hbar^2\frac{3}{2}\right]\delta^3(\vec{r})$$

$$\left(s=\frac{1}{2}\right)\oplus\left(s=\frac{1}{2}\right) = \underbrace{(S_H=0)}_{\text{singlet}} \oplus \underbrace{(S_H=1)}_{\text{triplet}}$$

$$\left\{\,|S_H=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_e\downarrow_p\rangle - |\downarrow_e\uparrow_p\rangle\right)\right.$$

$$\text{deg.}\begin{cases} |S_H=1,\ S_z=1\rangle = |\uparrow\uparrow\rangle \\ |S_H=1,\ S_z=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ |S_H=1,\ S_z=-1\rangle = |\downarrow\downarrow\rangle \end{cases}$$

$$\langle H' \rangle^{S_H=0} = -\hbar^2 \frac{3}{4} A |\psi(\vec{r}=0)|^2$$

$$\langle H' \rangle^{S_H=1} = \langle A \rangle \frac{\hbar^2}{4} |\psi(\vec{r}=0)|^2$$

$$T = \frac{B}{r^3} \left[(\vec{S}_e \cdot \hat{r})(\vec{S}_p \cdot \hat{r}) - \frac{1}{3} \vec{S}_e \cdot \vec{S}_p \right]$$

$$(\vec{S}_e \cdot \hat{r})(\vec{S}_p \cdot \hat{r}) = \frac{1}{2} \left[\left((\vec{S}_e \cdot \hat{r}) + (\vec{S}_p \cdot \hat{r}) \right)^2 - (\vec{S}_e \cdot \hat{r})^2 - (\vec{S}_p \cdot \hat{r})^2 \right]$$

$$= \frac{1}{2} \left[(\vec{S}_H \cdot \hat{r})^2 - \frac{\hbar^2}{2} \right]$$

$$(\vec{S}_e \cdot \vec{S}_p) = \frac{1}{2} \left[\vec{S}_H^2 - \frac{3\hbar^2}{2} \right]$$

$$H'_B = \frac{B}{r^3} \left\{ \frac{1}{2} \left[(\vec{S}_H \cdot \hat{r})^2 - \frac{1}{3} \vec{S}_H^2 \right] + \frac{1}{2} \left(\cancel{-\frac{\hbar^2}{2} + \frac{1}{3} \frac{3\hbar^2}{2}} \right) \right\}$$

$$H'_B = \frac{B}{r^3} \frac{1}{2} \left[(\vec{S}_H \cdot \hat{r})^2 - \frac{1}{3} \vec{S}_H^2 \right]$$

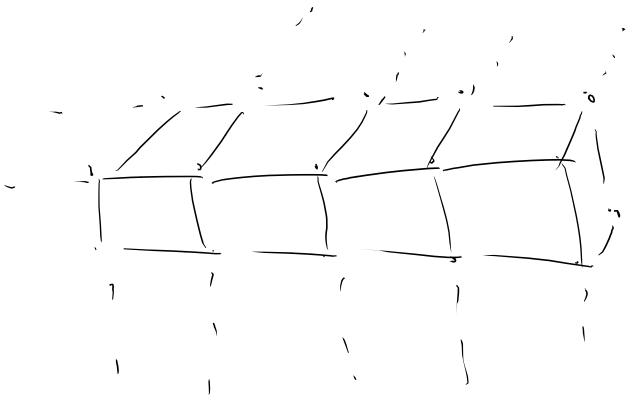
$$= \frac{B}{r^3} \frac{1}{2} \sum_{ij=x,y,z} S_H^i S_H^j \left(\hat{r}^i \hat{r}^j - \frac{1}{3} \delta^{ij} \right)$$

$$= \frac{B}{2r^3} \left(S_H^i S_H^j - \frac{1}{3} \delta^{ij} \vec{S}_H^2 \right) \left(\frac{x^i x^j}{r^2} - \frac{1}{3} \delta^{ij} \right)$$

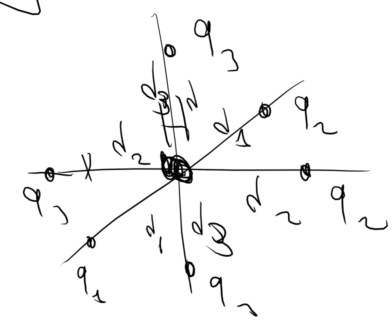
$$\langle \psi' | H'_B | \psi \rangle = \frac{B}{2} \langle \psi'^{space} | \frac{1}{r^3} \left(\frac{x^i x^j}{r^2} - \frac{1}{3} \delta^{ij} \right) | \psi^{space} \rangle$$

$$\langle \psi'^{spin} | \left(S_H^i S_H^j - \frac{1}{3} \delta^{ij} \vec{S}_H^2 \right) | \psi^{spin} \rangle$$

Pr 6.34



model



$r \ll d_i$

$\vec{r} \to -\vec{r}$

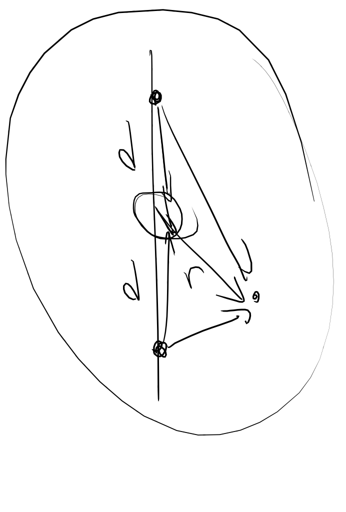
a)

$$H' = V_0 + 2(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) - (\beta_1 + \beta_2 + \beta_3) r^2$$

$$\beta_i = -\frac{e q_i}{4\pi\varepsilon_0 d_i^3} \; ; \; V_0 = 2(\beta_1 d_1^2 + \beta_2 d_2^2 + \beta_3 d_3^2)$$

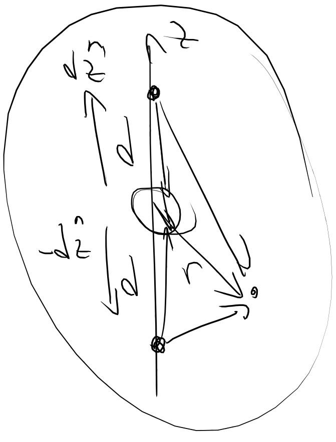
$$\vec{P} = \langle \psi | q\vec{r} | \psi \rangle = 0$$

$$H' = \sum \frac{-e q_i}{4\pi\varepsilon_0} \frac{1}{|\vec{r} - \vec{r}_i|}$$



$r \ll d$

$r \ll \vec{r}_i$



$$V = \frac{-eq}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{r}+d\hat{z}|}+\frac{1}{|\vec{r}-d\hat{z}|}\right)$$

$$|\vec{r}+d\hat{z}| = \sqrt{(\vec{r}+d\hat{z})^2}$$

$$= \sqrt{r^2+d^2+2rd\cos\theta}$$

$$\frac{1}{|\vec{r}\mp d\hat{z}|} = \left(r^2+d^2\mp 2rd\cos\theta\right)^{-1/2}$$

$$= d^{-1}\left(1+\left(\frac{r}{d}\right)^2\mp 2\frac{r}{d}\cos\theta\right)^{-1/2}$$

$$= \frac{1}{d}\left[1\pm\frac{r}{d}\cos\theta+\mathcal{O}\left(\left(\frac{r}{d}\right)^2\right)\right]$$

$$\boxed{V_0 = -\frac{eq}{4\pi\varepsilon_0}\frac{1}{d}2}$$

r^2 terms:

$$\frac{1}{|\vec{r}\mp d\hat{z}|} = \frac{1}{d}\left\{1+\left(-\frac{1}{2}\right)\left[\left(\frac{r}{d}\right)^2\mp 2\frac{r}{d}\cos\theta\right] + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left[\left(\frac{r}{d}\right)^2\mp 2\frac{r}{d}\cos\theta\right]^2\frac{1}{2} + \mathcal{O}\left(\left(\frac{r}{d}\right)^3\right)\right\}$$

$$= \frac{1}{d}\left\{1-\frac{1}{2}\left(\frac{r}{d}\right)^2 \pm \frac{r}{d}\cos\theta + \frac{3}{4}\,4\left(\frac{r}{d}\right)^2\cos^2\theta\,\frac{1}{2}+\mathcal{O}\left(\frac{r}{d}\right)^3\right\}$$

$$\frac{1}{|\vec{r}+d\hat{z}|}+\frac{1}{|\vec{r}-d\hat{z}|}=\frac{2}{d}\left\{1-\frac{1}{2}\left(\frac{r}{d}\right)^2+\frac{3}{2}\left(\frac{r}{d}\right)^2\cos^2\theta\right\}$$

$$=\frac{2}{d}\left\{1-\frac{1}{2}\left(\frac{r}{d}\right)^2\left(1-3\cos^2\theta\right)\right\}$$

$$=\frac{2}{d}\left\{1-\frac{1}{2}\left(\frac{r}{d}\right)^2\left[1-3(\hat{r}\cdot\hat{z})(\hat{r}\cdot\hat{z})\right]\right\}$$

$$=\frac{2}{d}\left\{1-\frac{1}{2}\left(\frac{r}{d}\right)^2\left[1-3\frac{z^2}{r^2}\right]\right\}$$

$$V=\underbrace{\frac{-eq}{4\pi\varepsilon_0}\frac{2}{d}}\left\{1-\frac{1}{2}\left(\frac{r}{d}\right)^2\left[1-3\frac{z^2}{r^2}\right]\right\}$$

$$=2\beta d^2\left\{1-\frac{1}{2}\frac{r^2}{d^2}\left[1-\frac{3z^2}{r^2}\right]\right\}$$

$$H'=2\sum\beta_i d^2-\sum\beta_i r^2+3\sum_i\beta_i x_i^2$$

b) Correction to the Ground state energy.

$$\delta E_G=\langle\psi_G|H'|\psi_G\rangle=V_0+3\sum_i\langle\psi_G|\beta_i x_i^2|\psi_G\rangle-\left(\sum\beta_i\right)\langle\psi_G|r^2|\psi_G\rangle$$

$$\langle\psi_G|x^2|\psi_G\rangle=\langle\psi_G|y^2|\psi_G\rangle$$
$$=\langle\psi_G|z^2|\psi_G\rangle$$
$$=\frac{1}{3}\langle\psi_G|r^2|\psi_G\rangle$$

$$\langle\psi_G|H'|\psi_G\rangle=V_0+\cancel{3}\cancel{\sum_i\beta_i\frac{1}{3}\langle\psi_G|r^2|\psi_G\rangle}-\cancel{\sum_i\beta_i\langle\psi_G|r^2|\psi_G\rangle}$$

$$\boxed{\delta E_G=V_0}$$

c) Correction to the first excited state.

$|n=2, l=0\rangle$: One
states
$|n=2, l=1\rangle$: three

$$\langle n l' m' | r^2 | n l m \rangle = \int d^3r \, R^*_{nl'}(r) R_{nl}(r) r^2 Y^*_{l'm'}(\Omega) Y_{lm}(\Omega)$$

$$= \left(\int_0^\infty dr \, r^4 R_{nl'} R_{nl}\right) \underbrace{\int d\Omega \, Y^*_{l'm'} Y_{lm}}_{\delta_{mm'}\delta_{ll'}}$$

$$\overset{0}{\overset{\shortparallel}{\langle n l' m' | [r^2, L^2] | n l m \rangle}} =$$

$$-\hbar^2 \left[l'(l'+1) - l(l+1)\right] \langle n l' m' | r^2 | n l m \rangle$$

$$\langle n l' m' | [r^2, L_z] | n l m \rangle = \hbar (m - m') \langle n l' m' | r^2 | n l m \rangle$$
$$\overset{\shortparallel}{0}$$

$\beta_x = \beta_y = \beta_z$

$H' = V_0$ already diagonal

$\beta_x = \beta_i \neq \beta_z$

$$H' = V_0 + 3\beta_x \left(x^2 + y^2 + z^2 - z^2\right) + 3\beta_z z^2$$

$$- (2\beta_x + \beta_z) \left(x^2 + y^2 + z^2\right)$$

$$= V_0 + (\beta_x - \beta_z) r^2 + (3\beta_z - 3\beta_x) z^2$$

$$H' = V_0 + (\beta_x - \beta_z) r^2 + 3(\beta_z - \beta_x) z^2$$

$$\langle n l' m' | [L_z, z^2] | n l m \rangle = \hbar (m' - m) \langle n l' m' | z^2 | n l m \rangle$$
$$\| \quad 0$$

$$\langle n l' m' | z^2 | n l m \rangle = 0 \quad \text{if} \quad m \neq m'$$

$\underline{n=2}$ only non-zero off diagonal matrix element is

$$\langle 2 1 0 | z^2 | 2 0 0 \rangle = \delta$$

$$H' = \begin{pmatrix} \varepsilon & 0 & \delta & 0 \\ 0 & \varepsilon' & 0 & 0 \\ \delta & 0 & \varepsilon'' & 0 \\ 0 & 0 & 0 & \varepsilon''' \end{pmatrix} \begin{array}{l} l=0 \quad m=0 \\ l=1 \quad m=1 \\ l=1 \quad m=0 \\ l=1 \quad m=-1 \end{array}$$

"good basis"

$|l=1; m=1\rangle$, $|l=1; m=-1\rangle$

$c|l=0; m=0\rangle + s|l=1; m=0\rangle$

$-s|l=0; m=0\rangle + c|l=1; m=0\rangle$