$H(p,q;\lambda)$

$\lambda = \lambda(t)$

$\psi_n(x,t)$

$\lambda(0) = \lambda(T)$

Berry Phase

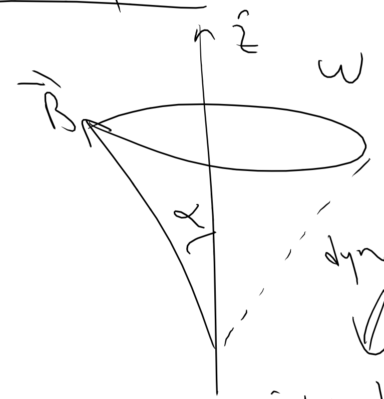
$$\rightarrow \psi_n(x,T) = \psi_n(x,0)\, e^{-\frac{i}{\hbar}\int_0^T E_n(t')dt'}\, e^{-i\gamma_n(T)}$$

$$\gamma_n(t) = i\int_{\vec{\lambda}(0)}^{\lambda(t)} \langle\psi_n|\nabla_\lambda|\psi_n\rangle\cdot d\vec{\lambda}$$

$$\gamma_n(T) = i\oint \langle\psi_n|\vec{\nabla}_\lambda|\psi_n\rangle\cdot d\vec{\lambda}$$

$$= i\oint \vec{\nabla}_\lambda \times \langle\psi_n|\vec{\nabla}_\lambda|\psi_n\rangle\cdot d\vec{a}$$

Example $\quad T = \frac{2\pi}{\omega}$



$\frac{\omega}{\omega_1} \rightarrow 0$

$E_\mp = \mp\frac{\hbar\omega_1}{2}$

dynamical phase.

geometrical phase

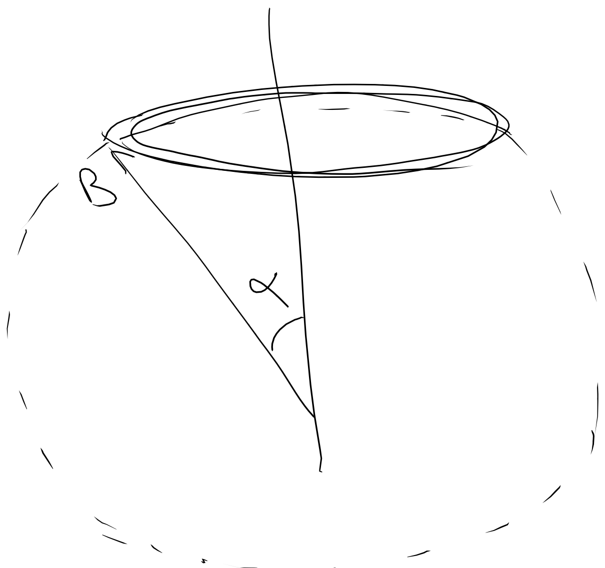
$$\chi(t) \simeq e^{i\omega_1 t/2}\, \underbrace{e^{i(\omega\cos\alpha)t/2}\, e^{-i\omega t/2}}_{\text{geometrical phase}}\, \chi_+(t)$$

$$\Rightarrow +i\left[\left(\frac{\omega}{\omega_1}\right)\sin\alpha\sin\left(\frac{\omega_1 t}{2}\right)\right]e^{-i\omega t/2}\,\chi_-(t)$$

$$\gamma_n(t) = (\cos\alpha - 1)\frac{\omega t}{2}$$

$$\gamma_n(T) = (\cos\alpha - 1)\frac{\cancel{\omega}}{\cancel{2}}\frac{\cancel{2}\pi}{\cancel{\omega}}$$

$$\boxed{\gamma_n(T) = \pi(\cos\alpha - 1)}$$



$$B^2\int d\Omega = B^2\int_0^{2\pi} d\phi \int_0^{\alpha} d\theta \sin\theta$$

$$= 2\pi B^2(-\cos\alpha + 1)$$

$$\Omega = 2\pi(1 - \cos\alpha)$$

$$\boxed{\gamma_n(T) = -\frac{\Omega}{2}}$$

$(\theta = \alpha)$

$$\chi_+ = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$\langle\chi_+|\nabla|\chi_+\rangle$$

$$\nabla\chi_+ = \frac{\partial\chi_+}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\chi_+}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\chi_+}{\partial\phi}\hat{\phi}$$

$$r \equiv B$$

$$\nabla\times\langle\chi_+|\nabla\chi_+\rangle = \overbrace{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}}\left[\sin\theta\left(i\underbrace{\frac{\sin^2(\theta/2)}{r\sin\theta}}\right)\right]\hat{r}$$

$$= \frac{i}{2r^2}\hat{r}$$

$$\langle \chi_+ | \nabla \chi_+ \rangle = i \frac{\sin^2(\theta/2)}{r\sin\theta}\hat{\phi}$$

$$\gamma_n = i \int \nabla \times \langle \chi_+ | \nabla \chi_+ \rangle \cdot d\vec{a}$$

$$= -\frac{1}{2}\int \underbrace{\frac{1}{r^2}\hat{r}\cdot d\vec{a}}_{d\Omega}$$

$$\boxed{\gamma_n = -\frac{\Omega}{2}} \Leftarrow$$

Aharonov-Bohm Effect

$$\psi(\vec{r}) \propto e^{i\lambda\phi}\,\delta(r-b)$$

$(\vec{A}, \phi)$

$$H \neq \frac{-\hbar^2}{2m}\nabla^2 + q\phi$$

$$H = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - q\vec{A}\right)^2 + q\varphi$$

$\vec{E}, \vec{B}$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\left.\begin{array}{l} \vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda(\vec{r},t) \\ \phi \to \phi' = \phi - \dfrac{\partial \Lambda}{\partial t} \end{array}\right\}$$

$$\vec{E} \to \vec{E}' = \vec{E}$$

$$\vec{B} \to \vec{B}' = \vec{B}$$

$$H = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - q\vec{A}\right)^2 + q\varphi \Leftarrow$$

$$\psi \to \psi' = e^{i\Lambda(\vec{r},t)}\psi \Rightarrow \psi = e^{-i\Lambda}\psi'$$

$$H(\vec{A},\varphi)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$-i\hbar\vec{\nabla}\psi = \left[(-\hbar\nabla\Lambda)\psi' - i\hbar\nabla\psi'\right]e^{-i\Lambda}$$

$$= e^{-i\Lambda}\left(-i\hbar\vec{\nabla} - \hbar\vec{\nabla}\Lambda\right)$$

$$\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{q}{\cancel{\hbar}}\vec{A}\right)^2\psi$$

$$= e^{-i\Lambda}\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \hbar\vec{\nabla}\Lambda - \frac{q}{h}\vec{A}\right)^2\psi$$

$$-q\,A' = -q\vec{A} - \hbar\vec{\nabla}\Lambda$$

$$H(A,\phi)\psi = e^{-i\Lambda} H(\underbrace{\vec{A} + \frac{\hbar}{q}\vec{\nabla}\Lambda}_{\vec{A}'}, \phi)\psi'$$

$$i\hbar \frac{\partial \psi}{\partial t} = i\hbar \frac{\partial}{\partial t}\left(e^{-i\Lambda}\psi'\right)$$

$$= i\hbar\left[-i\frac{\partial \Lambda}{\partial t}\psi' + \frac{\partial \psi'}{\partial t}\right]e^{-i\Lambda}$$

$$= \left(\hbar\frac{\partial \Lambda}{\partial t}\right)\psi' e^{-i\Lambda} + i\hbar\frac{\partial \psi'}{\partial t}e^{-i\Lambda}$$

$$\cancel{e^{-i\Lambda}} H(A',\phi)\psi' = \left(i\hbar\frac{\partial \psi'}{\partial t}\right)\cancel{e^{-i\Lambda}} + \left(\hbar\frac{\partial \Lambda}{\partial t}\psi'\right)\cancel{e^{-i\Lambda}}$$

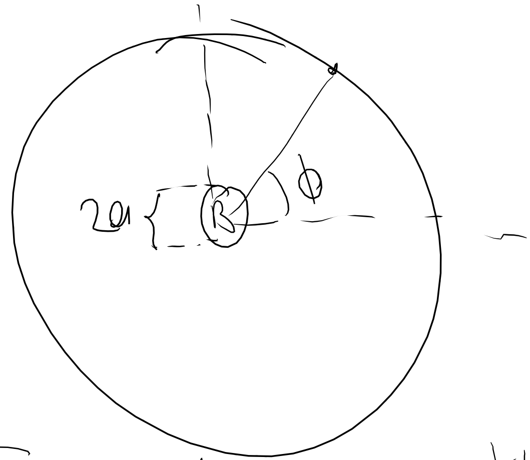
$$\underbrace{\boxed{H(A',\phi) - \hbar\frac{\partial \Lambda}{\partial t}}}_{H(A',\phi')}\psi' = i\hbar\frac{\partial \psi'}{\partial t}$$

$$H = \cdots + q\phi$$

$$\phi' = \phi - \frac{\hbar}{q}\frac{\partial \Lambda}{\partial t}$$

$$A' = \vec{A} + \frac{\hbar}{q}\vec{\nabla}\Lambda$$

$$\psi = \psi(\phi)$$



$$\vec{A} = \frac{\Phi}{2\pi r}\hat{\phi} \qquad (r > a)$$

$$\phi = 0 \qquad \vec{\nabla}\cdot\vec{A} = 0 \Longleftarrow \text{Gauge (fixing condition)}$$

$$\vec{\nabla}\times\vec{A} = \vec{B} = 0 \quad (r > a) \qquad \underbrace{\vec{\nabla}\cdot\vec{A} - \frac{\partial\phi}{\partial t} = 0}_{\text{Lorenz gauge}}$$

$$\Phi = B\pi a^2$$

$$H = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - q\vec{A}\right)^2$$

$$= \frac{1}{2m}\left[-\hbar^2\nabla^2 + 2i\hbar q\vec{A}\cdot\vec{\nabla} + \frac{q^2\Phi^2}{(2\pi)^2 b^2}\right]$$

$$= \frac{1}{2m}\left[-\frac{\hbar^2}{b^2}\frac{d^2}{d\phi^2} + \left(\frac{q\phi}{2\pi b}\right)^2 + \frac{i\hbar q\phi}{\pi b^2}\frac{d}{d\phi}\right]$$

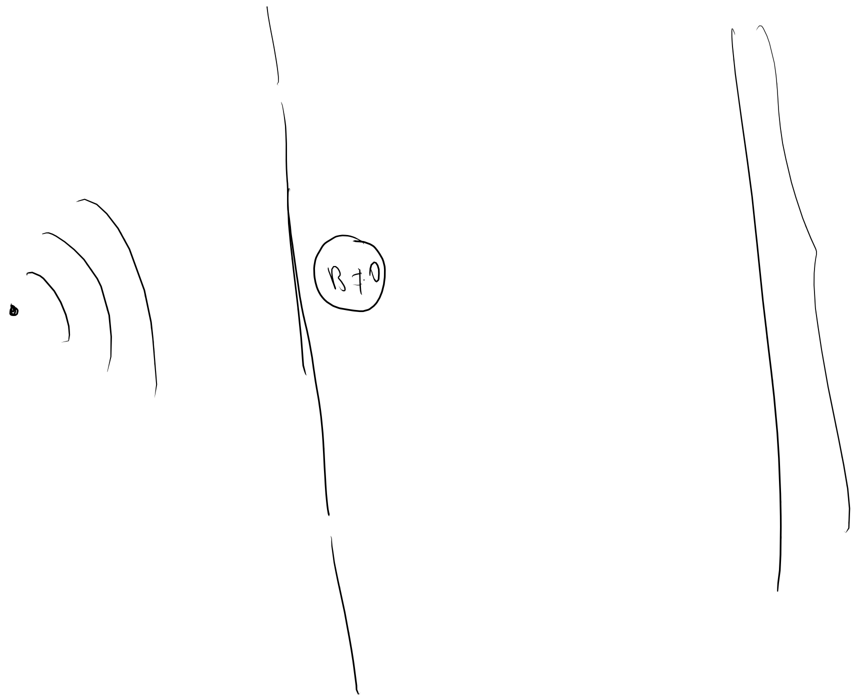
$$\psi = e^{i\lambda\phi} \; ; \; \lambda = 0, \pm 1, \pm 2, \ldots$$

$$\lambda = \beta \pm \sqrt{\beta^2 + \varepsilon} \Longleftarrow$$

$$\beta = \frac{q\phi}{2\pi\hbar}, \quad \varepsilon = \frac{2mb^2E}{\hbar^2} - \beta$$

$$E_n = \frac{\hbar^2}{2mb^2}\left(n - \frac{q\Phi}{2\pi\hbar}\right)^2 \qquad n = 0, \pm 1, \ldots \qquad \boxed{\Phi = B\pi a^2}$$

b: radius of ring.



$\vec{\nabla}\times\vec{A} = 0 \Rightarrow \int_O^P \vec{A}\cdot d\vec{l}$ indep of path as long as they are above the solenoid.

$$\psi = e^{i\frac{q}{\hbar}\int_{O,\ \text{above}}^{P} \vec{A}\cdot d\vec{l}}\,\psi_0 + e^{i\frac{q}{\hbar}\int_{O,\ \text{below}}^{P} \vec{A}\cdot d\vec{l}}\,\psi_0$$

phase differene

$$\frac{q}{\hbar}\left[\int_{O\ \text{below}}^{P} \vec{A}\cdot d\vec{l} - \int_{O\ \text{above}}^{P} \vec{A}\cdot d\vec{l}\right] = \frac{q}{\hbar}\left[\int_{O\ \text{below}}^{P} \vec{A}\cdot d\vec{l} + \int_{P\ \text{above}}^{O} \vec{A}\cdot d\vec{l}\right]$$

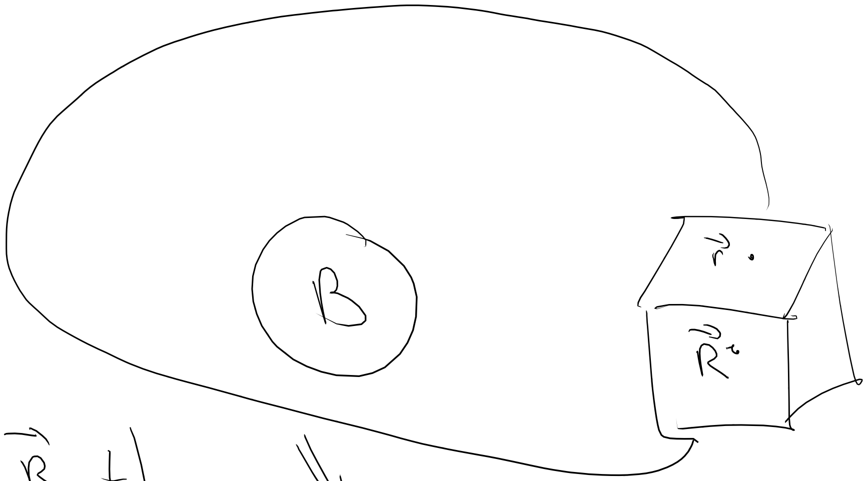
$$= \frac{q}{\hbar}\oint \vec{A}\cdot d\vec{l} = \frac{q}{\hbar}\oint \underbrace{(\vec{\nabla}\times\vec{A})}_{\vec{B}}\cdot d\vec{a}$$

phase difference $= \frac{q}{\hbar}\Phi$

$$\left[\frac{1}{2m}\left[-i\hbar\vec{\nabla}_{\vec{r}} - q\vec{A}\right]^2 + V\right]\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\psi(\vec{r},t) = e^{i\frac{q}{\hbar}\int_0 \vec{A}\cdot d\vec{l}}\,\psi_0$$

$$\boxed{\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi_0 = i\hbar\frac{\partial\psi_0}{\partial t}}$$



$$\psi(\vec{r},\vec{R},t)$$

$$\left\{\frac{1}{2m}\left[-i\hbar\nabla^2 - q\vec{A}\right]^2_{\vec{r}} + V(\vec{r}-\vec{R})\right\}\psi_n(\vec{r},\vec{R}) = E\psi_n(\vec{r},\vec{R})$$

$$\psi_n(\vec{r},\vec{R}) = e^{i\frac{q}{\hbar}\int_{\vec{R}}^{\vec{r}}\vec{A}\cdot d\vec{l}}\,\psi_n^0(\vec{r},\vec{R})$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}-\vec{R})\right]\psi_n^0(\vec{r},\vec{R}) = E_n\psi_n(\vec{r},\vec{R})$$

$$\psi_n^0 = \psi_n^0(\vec{r}-\vec{R})$$

$$\psi_n = e^{i\frac{q}{\hbar}\int_{\vec{R}}^{\vec{r}}\vec{A}\cdot d\vec{l}}\,\psi_n^0(\vec{r}-\vec{R})$$

$$\langle \psi_n | \nabla_{\vec{R}} | \psi_n \rangle$$

$$= \langle \psi_n | \left(-i \frac{q}{\hbar} \vec{A}(\vec{R}) \right) | \psi_n \rangle$$

$$+ \langle \psi_n | e^{i \frac{q}{\hbar} ()} \nabla_R | \psi_n \rangle$$

$$\Rightarrow -i \frac{q}{\hbar} \vec{A}(\vec{R}) + e^{i \frac{q}{\hbar} ()} \underbrace{\langle \psi_n | (-\nabla_r) | \psi_n \rangle}_{=0}$$

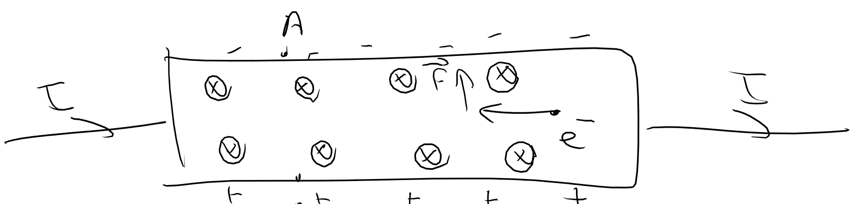
$$\gamma_n(T) = i \oint \langle \psi_n | \vec{\nabla}_R | \psi_n \rangle \cdot d\vec{R}$$

$$= \frac{q}{\hbar} \oint \vec{A}(\vec{R}) \cdot d\vec{R}$$

$$= \frac{q}{\hbar} \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \frac{q}{\hbar} \Phi$$

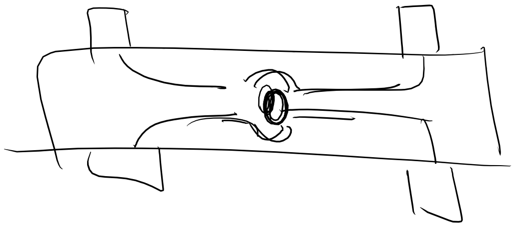
"Quantum Hall Effect"

Hall Effect



$V_{AB} \neq 0$

$$\frac{V_{AB}}{I} = R_H$$



$\frac{h}{e}$ = von Klitzeng constant