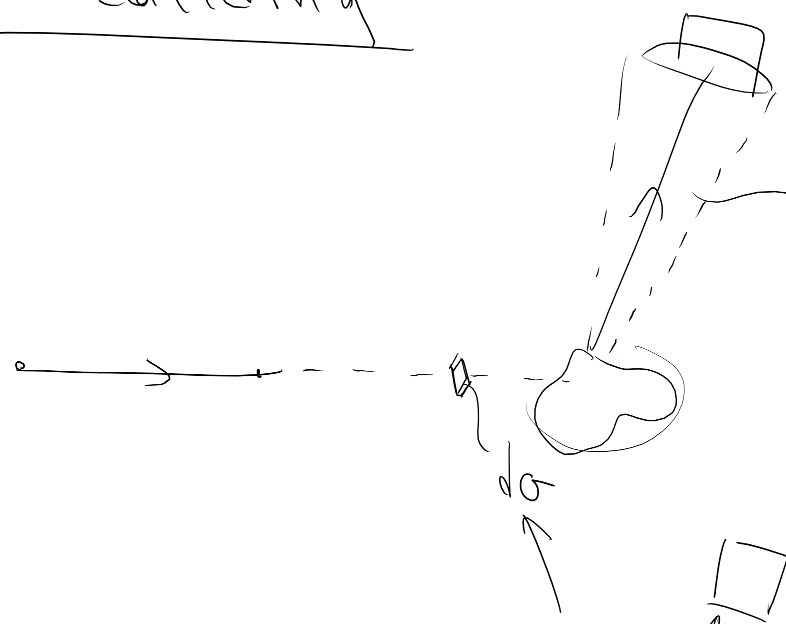
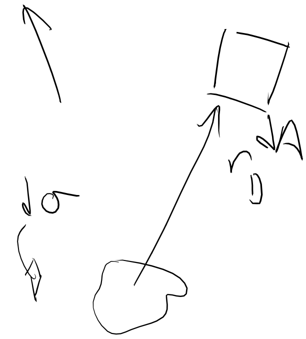
# Scattering

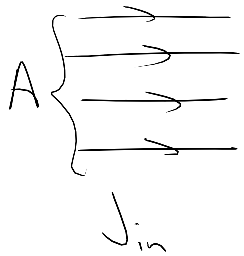



$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega$$




dA : area of the detector

$$\frac{dA}{r_D^2} \equiv d\Omega$$




$J_{in}$ : incident flux of particles : # of particles passing through a unit area per second

$J_{sc}$ : flux of scattered particles.

$dN$ : # of particles measured by the detector per second.

$\frac{d\sigma}{A}$ = fraction of particles that will hit the area $d\sigma$ and scatter into the solid angle $d\Omega$

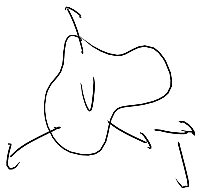
$$\begin{cases} dN = \underbrace{(J_{in} \cdot \cancel{A})}_{} \dfrac{d\sigma}{\cancel{A}} \\ dN = J_{sc}\, r_D^2\, d\Omega \end{cases}$$

$$d\sigma = \left(\frac{J_{sc}\, r_D^2}{J_{in}}\right) d\Omega$$

$$J_{sc} \sim \frac{1}{r_D^2}$$

$(\psi^*\psi)$ : propability density (if normalized to 1)

: number density of particles (if normalized to N)



$$\frac{d}{dt}\int_V (\psi^*\psi)\, d^3r = -\oint_{\partial V} \vec{j}\cdot d\vec{s}$$

$$\int_V \left(\frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t}\right) d^3r =$$

$$= \int_V \left[\underbrace{\left(i\hbar\frac{\partial\psi}{\partial t}\right)^*}_{H\psi}\frac{1}{(-i\hbar)}\psi + \psi^*\underbrace{\left(i\hbar\frac{\partial\psi}{\partial t}\right)}_{H\psi}\frac{1}{i\hbar}\right] d^3r$$

$$= \frac{i}{\hbar}\int_V \left[(H\psi)^*\psi - \psi^* H\psi\right] d^3r$$

$$H = \frac{-\hbar^2}{2m}\nabla^2 + V$$

$$= \frac{i}{\hbar}\int_V \left[\frac{-\hbar^2}{2m}\right]\left[(\nabla^2\psi^*)\psi - \psi^*\nabla^2\psi\right] d^3r$$

$$= -\frac{i\hbar}{2m}\int_V \vec\nabla\cdot\underbrace{\left((\vec\nabla\psi^*)\psi - \psi^*\vec\nabla\psi\right)}_{} d^3r$$

$$= -\frac{i\hbar}{2m} \int_{\partial V} \left[ (\vec{\nabla} \psi^*) \psi - \psi^* \vec{\nabla} \psi \right] \cdot d\vec{S}$$

$$\equiv -\int \vec{j} \cdot d\vec{S}$$

$$\vec{j} = \frac{i\hbar}{2m} \left[ (\vec{\nabla} \psi^*) \psi - \psi^* \vec{\nabla} \psi \right]$$

plane wave with momentum $\vec{p}$: $\psi = e^{\frac{i \vec{p} \cdot \vec{r}}{\hbar}}$

$$\nabla \psi = \frac{i \vec{p}}{\hbar} \psi$$

$$\vec{j} = \frac{i\hbar}{2m} \left( \left( -\frac{i\vec{p}}{\hbar} \right) - \left( \frac{i\vec{p}}{\hbar} \right) \right)$$

$$= \frac{i \cancel{\hbar}}{\cancel{2}m} \left( -\cancel{2} i \frac{\vec{p}}{\cancel{\hbar}} \right) = \frac{\vec{p}}{m} = \vec{v}$$

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) \right] \psi = E \psi$$

far from the potential $V(\vec{r}) \simeq 0$

$$\psi = \frac{u(r)}{r} Y_{lm}$$

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left( -\frac{\hbar^2}{2m} \right) \frac{l(l+1)}{r^2} u = E \cdot u$$

negligible for large r

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} = E u$$

$$\hbar k = \sqrt{2mE}$$

$$u = A e^{ikr} + B e^{-ikr}$$

outgoing wave

ingoing wave

$$\psi(\vec{r}) = A\left(e^{i\vec{k}\cdot\vec{r}} + f(\theta,\phi)\frac{e^{ikr}}{r}\right)$$

$$\vec{J}_{in} = |A|^2 \frac{\hbar \vec{k}}{m}$$

incident wave

scattered wave

$$\vec{J} = \frac{i\hbar}{2m}\left[\left(\vec{\nabla}\psi^*\right)\psi - \psi^*\vec{\nabla}\psi\right]$$

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\Theta(\theta,\phi)\psi$$

$$\lim_{r_0\to\infty} J_{sc}\, r_0^2$$

$$\vec{\nabla}\psi_{sc} = A(ik)\frac{e^{ikr}}{r}\hat{r} + O\left(\frac{1}{r^2}\right)$$

$$\vec{J}_{sc} = \frac{i\hbar}{2m}\left((-ik)|\psi_{sc}|^2\hat{r} - (ik)|\psi_{sc}|^2\hat{r}\right)$$

$$\vec{J}_{sc} = \frac{\hbar k}{m}|A|^2|f(\theta,\phi)|^2\hat{r}\frac{1}{r^2} + O\left(\frac{1}{r^3}\right)$$

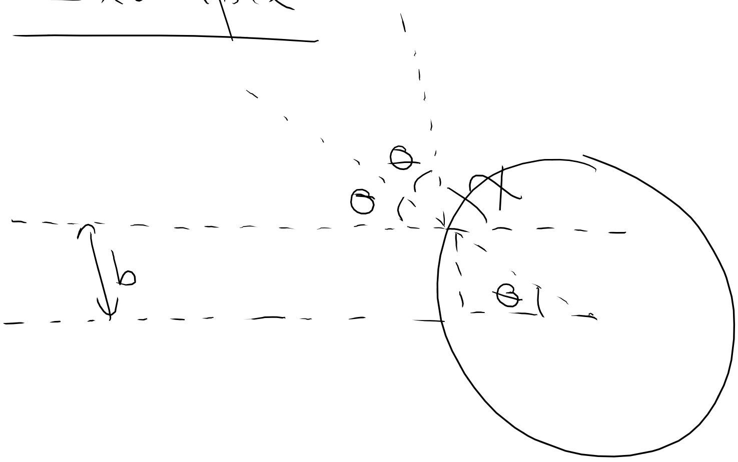
$$J_{sc} = J_{in}|f(\theta,\phi)|^2\frac{1}{r^2}$$

$$\frac{r^2 J_{sc}}{J_{in}} = |f(\theta,\phi)|^2$$

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{} A\left(e^{i\vec{k}\cdot\vec{r}} + f\,\frac{e^{ikr}}{r}\right)$$

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2}$$

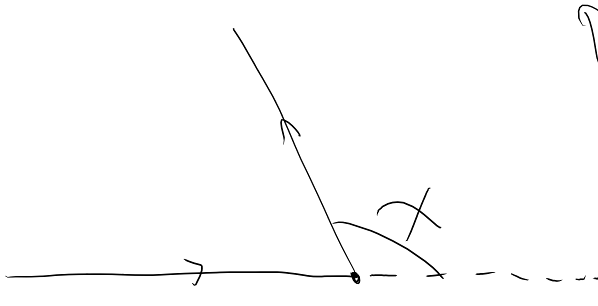
Example




b: impact parameter

χ: scattering angle

hard sphere




$b \to b + db$

$\chi \to \chi + d\chi$

$$\theta = \frac{\pi - \chi}{2} = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\sin\theta = \frac{b}{R} \to b = R\sin\theta = R\sin\left(\frac{\pi}{2} - \frac{\chi}{2}\right)$$

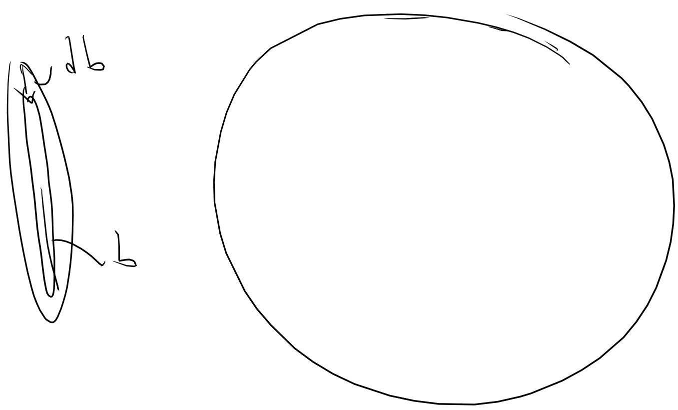
$$\to b = R\cos\left(\frac{\chi}{2}\right)$$

$$db = -\frac{R}{2}\sin\left(\frac{\chi}{2}\right)d\chi$$


$$d\sigma = b\,d\phi\,db$$

$$= R\cos\left(\frac{\chi}{2}\right) d\phi\,\frac{R}{2}\sin\left(\frac{\chi}{2}\right) d\chi$$

$$= \frac{R^2}{4}\underbrace{\sin(\chi)\,d\chi\,d\phi}_{d\Omega}$$




$$\boxed{\frac{d\sigma}{d\Omega} = \frac{R^2}{4}}$$

$$\sigma_T = \int \frac{d\sigma}{d\Omega}\,d\Omega = \frac{R^2}{4}\underbrace{\int d\Omega}_{4\pi} = \pi R^2$$

Example