$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{x}} + f(k;\Omega)\,\frac{e^{ikr}}{r}$$

$k \to iK$ $\quad f(iK)$ has a pole at the location of the bound state.

Potential well in 3D

$$V(\vec{r}) = V_0\,\Theta(a-r) \qquad V_0 < 0$$

Bound States $(l=0)$

$$\psi(\vec{r}) = \frac{u(r)}{r} Y_{lm}(\Omega) = \frac{u(r)}{r}\frac{1}{\sqrt{4\pi}}$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r)\,u = Eu$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V_0\,\Theta(a-r)\,u = Eu$$

$$r = xa$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{a^2dx^2} + V_0\,\Theta(a(1-x))\,u = Eu$$

$$-\frac{d^2u}{dx^2} + \overbrace{\frac{2mV_0a^2}{\hbar^2}}^{v_0}\Theta(1-x)\,u = \overbrace{\frac{2mEa^2}{\hbar^2}}^{\varepsilon}u$$

$$-\frac{d^2u}{dx^2} + V_0\,\Theta(1-x)u = \varepsilon u$$

$x < 1$, $\quad -\frac{d^2u}{dx^2} + V_0 u = \varepsilon u$

$$+\frac{d^2u}{dx^2} = -\underbrace{(\varepsilon - V_0)}_{k^2 = \varepsilon - V_0}u$$

$V_0 < \varepsilon < 0$

$u = A\sin(kx) \qquad u(x=0) = 0$

$x > 1 \qquad \frac{d^2u}{dx^2} = \underbrace{-\varepsilon}_{K^2 = -\varepsilon = |\varepsilon|} u$

$$u = Be^{-K(x-1)}$$

$$u = \begin{cases} A\sin(kx) & x < 1 \\ Be^{-K(x-1)} & x > 1 \end{cases}$$

u is continuous at $x = 1$

$A\sin(k) = B$

$$u = A\begin{cases} \sin(kx) & x < 1 \\ \sin(k)e^{-K(x-1)} & x > 1 \end{cases}$$

$$\boxed{k\cos(k) = -K\sin(k)} \qquad \begin{aligned} k^2 &= \varepsilon - V_0 \\ K^2 &= -\varepsilon \end{aligned}$$

Scattering solns $(\varepsilon > 0)$

$$-\frac{d^2u}{dx^2} + v_0\,\Theta(1-x)\,u = \varepsilon u$$

$x < 1$

$$-\frac{d^2u}{dx^2} + v_0 = \varepsilon u$$

$$-\frac{d^2u}{dx^2} = \underbrace{(\varepsilon - v_0)}_{k_1^2}u$$

$$\Rightarrow \quad u = A\sin(k_1 x)$$

$x > 1$

$$-\frac{d^2u}{dx^2} = \underbrace{\varepsilon}_{k^2=\varepsilon} u \qquad \frac{d^2u}{dx^2} = -k^2 u$$

$$\Rightarrow \quad u = B e^{ik(x-1)} + C e^{-ik(x-1)}$$

$$A\sin(k_1) = B + C$$

$$k_1 A\cos(k_1) = ik(B - C)$$

$$B = \frac{1}{2}\left[A\sin(k_1) + \frac{k_1}{ik}A\cos(k_1)\right]$$

$$C = \frac{1}{2}\left[A\sin(k_1) - \frac{k_1}{ik}A\cos(k_1)\right] = B^*$$

$$u = Be^{ik(x-1)} + Ce^{-ik(x-1)}$$

$$u = Ce^{ik}\left[e^{-ikx} + \frac{Be^{-2ik}}{C} e^{ikx}\right]$$

$$\psi = Ce^{ik}\left[\frac{e^{-i\frac{k}{a}r}}{r} + \underbrace{\frac{Be^{-2ik}}{C}}_{e^{2i\delta_{l=0}}} \frac{e^{i\left(\frac{k}{a}\right)r}}{r}\right]$$

$$\psi = e^{i\vec{k}\cdot\vec{r}} + f\frac{e^{ikr}}{r}$$

$$f = \sum \sqrt{4\pi}\sqrt{2l+1}\,\overbrace{e^{i\delta_l}\sin\delta_l}^{f_l}\, P_l(\cos\Theta)$$

$$\sigma = \sum 4\pi(2l+1)\underbrace{\sin^2\delta_l}$$

$$f_l = e^{i\delta_l}\sin\delta_l = \frac{e^{2i\delta_l}-1}{2i}$$

$$= \frac{Be^{-2ik}/C - 1}{2i}$$

$$f_{l=0} = \frac{Be^{-2ik} - C}{2iC}$$

$k \to iK \qquad k^2 = \varepsilon$

$$C(k) = \frac{1}{2}\left[A\sin(k_1) - \frac{k_1}{ik} A\cos(k_1)\right]$$

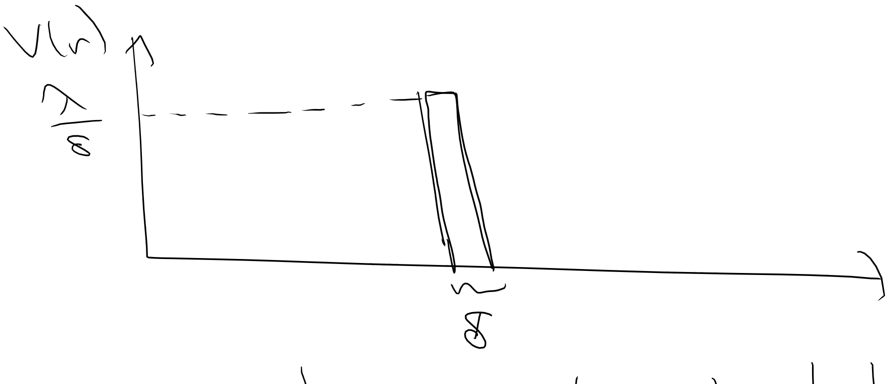
$$C(iK) = \frac{1}{2} A\left[\sin(k_1) + \frac{k_1}{K}\cos(k_1)\right] = 0$$

$$\boxed{\frac{k_1}{K} = -\tan(k_1)}$$

$$k\cos(k) = -K\sin(k)$$

<u>Example</u> Shell potential

$$V(r) = \lambda\delta(r-a)$$



<u>Energy eigenvalues of bound states</u> $(\lambda \to \infty)$

$r < a$

$$-\frac{d^2u}{dx^2} + \tilde{\lambda}\delta(1-x)u = \varepsilon u$$

$$\frac{2m\lambda a}{\hbar^2} = \tilde{\lambda} \quad ; \quad \varepsilon = \frac{2mEa^2}{\hbar^2}$$

$$-\frac{d^2u}{dx^2} = \underbrace{\varepsilon}_{k^2} u \implies u_n(x) = A\sin(k_n x)$$

$$k_n = n\pi \;;\; n = 1, 2, \dots$$

$$\varepsilon_n = k_n^2 = n^2\pi^2$$

Resonance: a long lived state

<u>2^{nd} Midterm</u> 9^{th} of May.

<u>Scattering states</u> $(\lambda < \infty)$

$$-\frac{d^2u}{dx^2} + \tilde{\lambda}\,\delta(1-x)\,u = \varepsilon u$$

$x < 1 \qquad u = A\sin(kx)$

$x > 1 \qquad u = \underbrace{Be^{ik(x-1)}}_{\text{outgoing}} + \underbrace{Ce^{-ik(x-1)}}_{\text{ingoing}}$

$$\boxed{A\sin(k) = B + C}$$

$$\frac{Be^{-2ik}}{C} = e^{2i\delta_{l=0}}$$

$$\int_{1^-}^{1^+} dx\left(-\frac{d^2u}{dx^2} + \tilde{\lambda}\,\delta(1-x)\,u = \varepsilon u\right)$$

$$-\left(\left.\frac{du}{dx}\right|_{x=1^+} - \left.\frac{du}{dx}\right|_{x=1^-}\right) + \tilde{\lambda}\,u(1) = 0$$

$$-\left(ikB - ikC - Ak\cos(k)\right) + \tilde{\lambda}A\sin(k) = 0$$

$$-ik(B-C) + Ak\cos k + \tilde{\lambda}A\sin k = 0$$

$$B - C = \frac{A}{ik}\left(k\cos k + \tilde{\lambda}\sin k\right)$$

$$B - C = \frac{A}{ik}\left(k\cos k + \tilde{\lambda}\sin k\right)$$

$$A\sin(k) = B + C$$

$$B = \left[A\sin(k) + \frac{A}{ik}\left(k\cos k + \tilde{\lambda}\sin k\right)\right]\frac{1}{2}$$

$$C = \left[A\sin(k) - \frac{A}{ik}\left(k\cos k + \tilde{\lambda}\sin k\right)\right]\frac{1}{2}$$

choose $A = 1$

$$B = C^*$$

$$e^{2i\delta_{l=0}} = \frac{Be^{-2ik}}{C} \qquad \frac{e^{2i\delta_{l=0}} - 1}{2i}$$

$$\frac{\frac{Be^{-2ik}}{C} - 1}{2i} = \frac{Be^{-2ik} - C}{2iC} = \frac{\frac{B}{\tilde{\lambda}}e^{-2ik} - \frac{C}{\tilde{\lambda}}}{2i\frac{C}{\tilde{\lambda}}}$$

k, which makes $\frac{C}{\tilde{\lambda}} = 0$,

can be written as

$$k = k^{(0)} + \frac{k^{(1)}}{\tilde{\lambda}} + \frac{k^{(2)}}{\tilde{\lambda}^2} + O\left(\frac{1}{\tilde{\lambda}^3}\right)$$

$$C = \left[A\sin(k) - \frac{A}{ik}\left(k\cos k + \tilde{\lambda}\sin k\right)\right]\frac{1}{2}$$

$$\frac{1}{\tilde\lambda}\sin\left(k^{(0)}+k^{(1)}+k^{(2)}\right)+\frac{i}{\tilde\lambda}\left(\cos\left(k^{(0)}+k^{(1)}+k^{(2)}\right)\right)$$

$$+i\frac{1}{k}\sin\left(k^{(0)}+k^{(1)}+k^{(2)}\right)=0$$

$$O\left(\left(\frac{1}{\tilde\lambda}\right)^0\right):\quad \sin\left(k^{(0)}\right)=0\implies k^{(0)}=n\pi$$

$$O\left(\frac{1}{\tilde\lambda}\right):\quad \frac{\overset{0}{\underset{\|}{\sin}}(\overset{n\pi}{\underset{\|}{k^{0}}})}{\tilde\lambda}+i\frac{\cos(\overset{n\pi}{\underset{\|}{k^{(0)}}})}{\tilde\lambda}+i\sin\left(\overset{n\pi}{\underset{\|}{k^{(0)}}}+k^{(1)}\right)\frac{1}{k^{(0)}+k^{(1)}}$$

$$=0$$

$$\frac{\cancel{i}\,\cancel{(-1)^n}}{\tilde\lambda}+\cancel{i}\,\cancel{(-1)^n}\,k^{(1)}\frac{1}{k^{(0)}}\left(1+\frac{k^{(1)}}{k^{(0)}}\right)^{-1}=0$$

$$\frac{1}{\tilde\lambda}+\frac{k^{(1)}}{n\pi}\left(1-\cancel{\frac{k^{(1)}}{k^{(0)}}}\right)\implies k^{(1)}$$

$$k_n\simeq n\pi-\frac{n\pi}{\tilde\lambda}\simeq n\pi\left(1-\frac{1}{\tilde\lambda}\right)$$

$$O\left(\frac{1}{\tilde\lambda^2}\right):\quad \underline{\frac{1}{\tilde\lambda}}\sin\left(n\pi-\underline{\frac{1}{\tilde\lambda}+k^{(2)}}\right)+\frac{i}{\tilde\lambda}\cos\left(n\pi-\frac{1}{\tilde\lambda}+k^{(2)}\right)$$

$$+i\frac{\sin\left(n\pi-\frac{1}{\tilde\lambda}+k^{(2)}\right)}{n\pi\left(1-\frac{1}{\tilde\lambda}\right)+k^{(2)}}=O\left(\frac{1}{\tilde\lambda^3}\right)$$

$$\cancel{(-1)^n}\frac{1}{\tilde\lambda}\left(-\frac{1}{\tilde\lambda}\right)+i\cancel{(-1)^n}\frac{i}{\tilde\lambda}+i\cancel{(-1)^n}\left(-\frac{1}{\tilde\lambda}+k^{(2)}\right)\frac{1}{n\pi}\left[1-\frac{1}{\tilde\lambda}+\frac{k^{(2)}}{n\pi}\right]^{-1}$$

$$=O\left(\frac{1}{\tilde\lambda^3}\right)$$

$$-\frac{1}{\tilde\lambda^2}-\ldots$$

$$k_n=n\pi\left(1-\frac{1}{\tilde\lambda}\right)+i\left(\frac{n\pi}{\tilde\lambda}\right)^2+\ldots$$

$$\varepsilon_n = k_n^2 \Rightarrow \mathrm{Im}\,\varepsilon_n \propto 2\frac{(n\pi)^3}{\tilde{\lambda}^2} = 2(n\pi)^3\left(\frac{1}{\tilde{\lambda}}\right)^2$$

$$\Uparrow \frac{\Gamma}{2}$$

$$\Gamma = \frac{1}{\tau}$$

τ : life time

$$N \propto |\psi|^2 \longleftarrow \psi(x,t) \simeq \psi(x,t=0)e^{-iEt}$$

$$\underbrace{\frac{1}{N}\frac{dN}{dt}} = -\Gamma \Rightarrow \frac{dN}{dt} = -\Gamma N$$

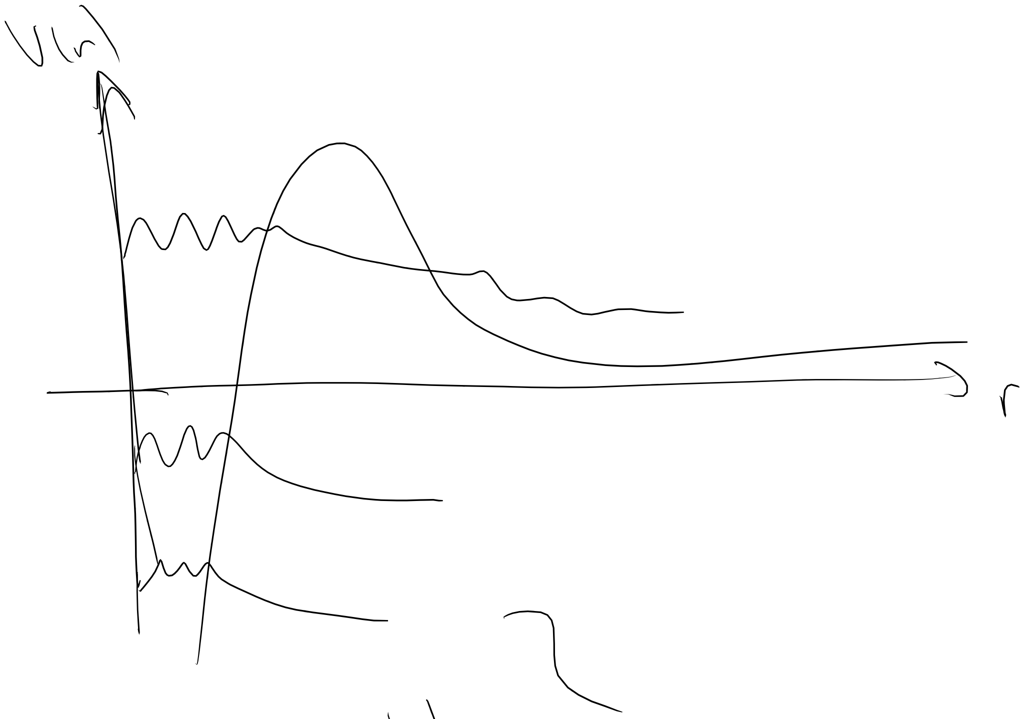
$$N(t) = N(t=0)\underline{e^{-\Gamma t}}$$

$$\psi(x,t) = \psi(x,t=0)e^{-iEt-\frac{\Gamma}{2}t}$$

$$= \psi(x,t=0)e^{-it\underbrace{\left(E-i\frac{\Gamma}{2}\right)}_{E_R}}$$

$\mathrm{Re}\,E_R = E$

$\mathrm{Im}\,E_R = -\frac{\Gamma}{2}$



model