Path Integrals

$$\underbrace{|\psi(t)\rangle} = e^{-\frac{i}{\hbar}Ht}\underbrace{|\psi(t=0)\rangle}$$

$$\psi(x,t) = \langle x|\psi(t)\rangle$$

$$= \langle x|e^{-\frac{i}{\hbar}Ht}\,\mathbb{1}\,|\psi(t=0)\rangle$$

$$\boxed{\int dx'\,|x'\rangle\langle x'| = \mathbb{1}} = \int dx'\,\underbrace{\langle x|e^{-\frac{i}{\hbar}Ht}|x'\rangle}\,\psi(x',t=0)$$

$$G(x,x';t) = \langle x|e^{-\frac{i}{\hbar}Ht}|x'\rangle \leftarrow$$

$$\frac{\partial G}{\partial t} = \langle x|-\frac{i}{\hbar}He^{-\frac{i}{\hbar}Ht}|x'\rangle$$

$$= -\frac{i}{\hbar}H\,G(x,x';t)$$

$$\boxed{i\hbar\frac{\partial}{\partial t}G = HG} \Leftarrow$$

$$G(x,x';t=0) = \langle x|e^{0}|x'\rangle = \delta(x-x')$$

$$G(x,x';t) = \langle x|e^{-\frac{i}{\hbar}Ht}\sum|n\rangle\langle n|x'\rangle$$

$$= \langle x|\sum e^{-\frac{i}{\hbar}E_n t}|n\rangle\langle n|x'\rangle$$

$$\boxed{G(x,x';t) = \sum_n e^{-\frac{i}{\hbar}E_n t}\,\psi_n(x)\,\psi_n^*(x')}$$

$$\int_0^\infty dt\, G(x,x';t)\, e^{\frac{i}{\hbar}Et - \frac{i\varepsilon}{\hbar}t} \equiv G(x,x';E)$$

$$= \sum_n \int_0^\infty dt\, e^{\frac{i}{\hbar}(E - E_n + i\varepsilon)t}\, \psi_n(x)\psi_n^*(x')$$

$$= \sum_n \frac{1}{\frac{i}{\hbar}(E - E_n + i\varepsilon)}\, \psi_n(x)\psi_n^*(x') = G(x,x';E)$$

$$G(x,x';t) = \sum_n e^{-\frac{i}{\hbar}E_n t}\, \psi_n(x)\psi_n^*(x')$$

$$G(x,x';-i\hbar\beta) = \sum_n e^{-E_n\beta}\, \psi_n(x)\psi_n^*(x')$$

$$\int_{-\infty}^{\infty} dx\, G(x,x';-i\hbar\beta) = \sum_n e^{-\beta E_n} \underbrace{\int_{-\infty}^{\infty} dx\, |\psi_n(x)|^2}_{1}$$

$$= \sum_n e^{-\beta E_n} \equiv Z$$

$$\beta = \frac{1}{k_B T}$$

$$\int_{-\infty}^{\infty} dx\, \underbrace{\langle x| e^{-\beta H} |x\rangle}_{\substack{\parallel \\ \mathrm{Tr}\, e^{-\beta H}}} = \int_{-\infty}^{\infty} dx\, G(x,x;-i\hbar\beta)$$

$$G(x,x';t) = \langle x | e^{-\frac{i}{\hbar}Ht} | x' \rangle$$

$$H = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2$$

$$G(x,x';t) = \int dp \langle x | e^{-\frac{i}{\hbar}Ht} | p \rangle \langle p | x' \rangle$$

$$= \int dp \, e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} \langle x | p \rangle \langle p | x' \rangle$$

$$= \int_{-\infty}^{\infty} \frac{dp}{(2\pi\hbar)} e^{-\frac{ip^2}{2m\hbar}t} e^{\frac{i}{\hbar}p(x-x')}$$

$$\langle x | p \rangle = \frac{e^{\frac{i}{\hbar}px}}{(2\pi\hbar)^{1/2}}$$

$$G(x,x';t) = \int_{-\infty}^{\infty} \frac{dp}{(2\pi\hbar)} e^{-i\frac{p^2}{2m\hbar}t} e^{\frac{i}{\hbar}p(x-x')}$$

$$= \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{-i\frac{p^2}{2m\hbar}t(1-i\varepsilon)} e^{\frac{i}{\hbar}p(x-x')}$$

$$= \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{-\underbrace{\frac{p^2 t}{2m\hbar}(i+\varepsilon)}_{\alpha}} e^{\underbrace{\frac{i}{\hbar}p(x-x')}_{\beta}}$$

$$\int_{-\infty}^{\infty} dp \, e^{-\alpha p^2 + \beta p} = \int_{-\infty}^{\infty} dp \, e^{-\alpha\left[p^2 - \frac{\beta}{\alpha}p + \frac{\beta^2}{4\alpha^2} - \frac{\beta^2}{4\alpha^2}\right]}$$

$$= e^{\frac{\beta^2}{4\alpha}} \int_{-\infty}^{\infty} dp \, e^{-\alpha\left(p - \frac{\beta}{2\alpha}\right)^2}$$

$$= e^{\frac{\beta^2}{4\alpha}} \sqrt{\frac{\pi}{\alpha}}$$

$$G(x,x',t) = \frac{1}{2\pi\hbar} \exp\left(-\frac{1}{\hbar^2}(x-x')^2 \frac{1}{4\frac{t}{2m\hbar}(i+\varepsilon)}\right) \sqrt{\frac{\pi}{\frac{t}{2m\hbar}(i+\varepsilon)}}$$

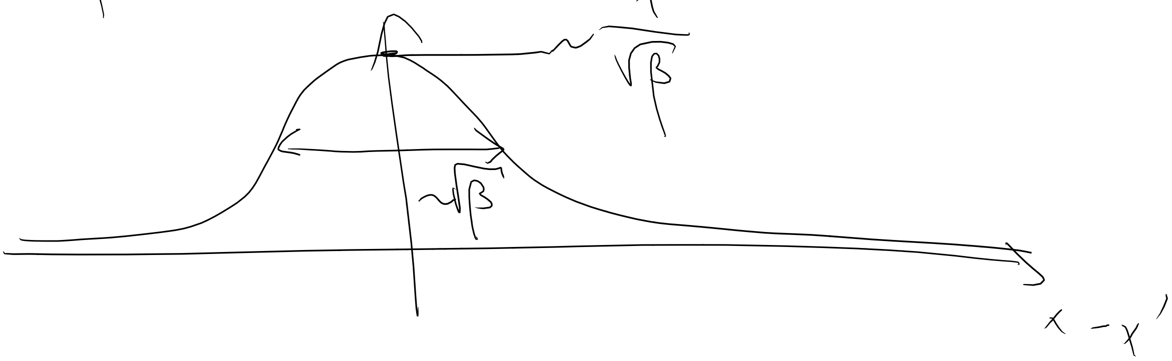
$$= \frac{1}{\sqrt{2\pi\hbar}} \sqrt{\frac{\cancel{2m\hbar\pi}}{t(i+\varepsilon)}} \exp\left\{-\frac{(x-x')^2\, \cancel{2}m\cancel{\hbar}}{2\cancel{4}\hbar^2 t(i+\varepsilon)}\right\}$$

$$\boxed{G(x,x',t) = \sqrt{\frac{m}{2\pi\hbar t(i+\varepsilon)}} \exp\left\{-\frac{m(x-x')^2}{2\hbar t(i+\varepsilon)}\right\}}$$

$t \to -i\hbar\beta$

$$G(x,x',-i\hbar\beta) \overset{\varepsilon\to 0}{=} \sqrt{\frac{m}{2\pi\hbar^2\beta}} \exp\left\{-\frac{m(x-x')^2}{2\hbar^2\beta}\right\}$$

$\beta \to 0$ $\quad G$

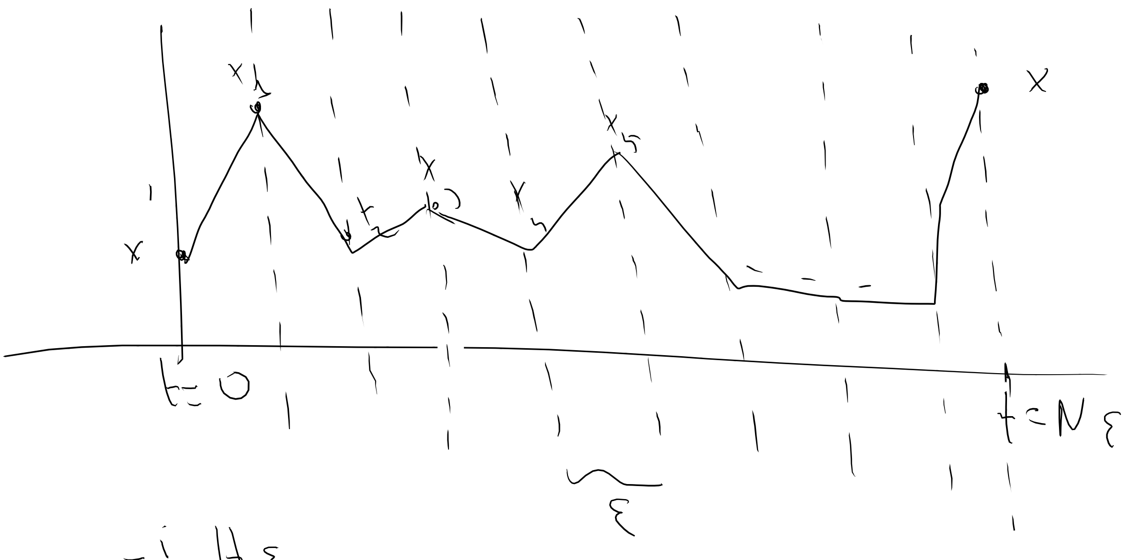


$$G(x,x';t) = \langle x| e^{-\frac{i}{\hbar}Ht} |x'\rangle$$

$$= \underset{\substack{\parallel \\ x_N}}{\langle x|} \underbrace{e^{-\frac{i}{\hbar}H\varepsilon} \cdots e^{-\frac{i}{\hbar}H\varepsilon} \, e^{-\frac{i}{\hbar}H\varepsilon}}_{N} \underset{\substack{\parallel \\ x_0}}{|x'\rangle}$$

$N\varepsilon = t$ $\qquad \int dx_1 |x_1\rangle\langle x_1|$

$$G(x,x';t) = \int dx_1\, dx_2 \dots dx_{N-1} \prod_{i=1}^{N} \langle x_i| e^{-\frac{i}{\hbar}H\varepsilon} |x_{i-1}\rangle$$



$$\langle x_i| e^{-\frac{i}{\hbar}H\varepsilon} |x_{i-1}\rangle$$

$$H = \frac{p^2}{2m} + V(x)$$

$$e^{-\frac{i}{\hbar}H\varepsilon} \neq e^{-\frac{i}{\hbar}\frac{p^2}{2m}\varepsilon}\, e^{-\frac{i}{\hbar}V(x)\varepsilon}$$

$$e^{-\frac{i}{\hbar}H\varepsilon} = e^{-\frac{i}{\hbar}\frac{p^2}{2m}\varepsilon}\, e^{-\frac{i}{\hbar}V(x)\varepsilon} + O(\varepsilon^2)$$

$$e^{-\frac{i}{\hbar} H \varepsilon} = e^{-\frac{i}{\hbar} \frac{V(x)}{2} \varepsilon} e^{-\frac{i}{\hbar} \frac{p^2}{2m} \varepsilon} e^{-\frac{i}{\hbar} \frac{V(x)}{2} \varepsilon} + O(\varepsilon^3)$$