$$\langle x'|e^{-\frac{i}{\hbar}Ht}|x\rangle = G(x',x;t)$$

$$\langle x'|e^{-\frac{i}{\hbar}Ht}|x\rangle = \langle x'|\underbrace{e^{-\frac{i}{\hbar}H\varepsilon}\cdots e^{-\frac{i}{\hbar}H\varepsilon}\,e^{-\frac{i}{\hbar}H\varepsilon}}_{N,\ N\varepsilon=t}|x\rangle$$

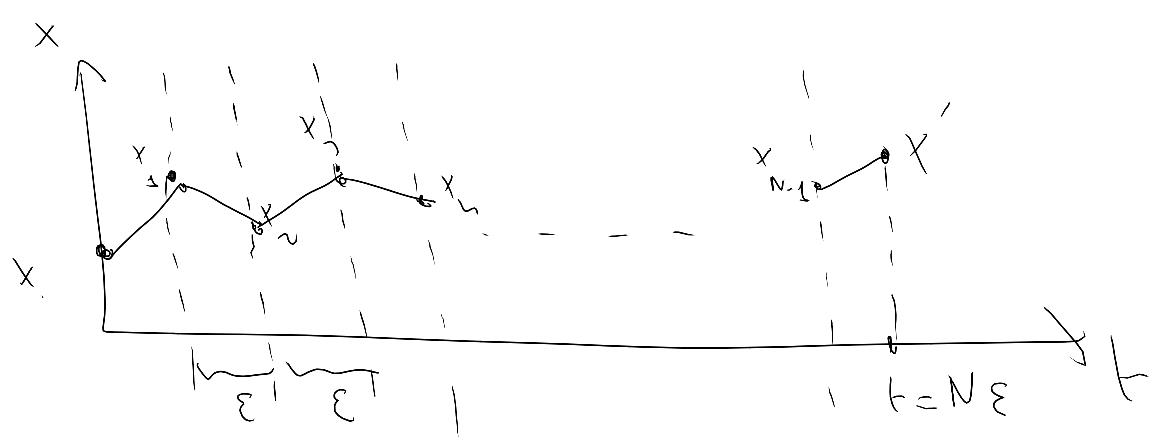
$$\int_{-\infty}^{\infty} dx\,|x\rangle\langle x| = 1$$

$$= \int\left(\prod_{i=1}^{N-1}dx_i\right)\langle x'|e^{-\frac{i}{\hbar}H\varepsilon}|x_{N-1}\rangle\,\langle x_{N-1}|e^{-\frac{i}{\hbar}H\varepsilon}|x_{N-2}\rangle\cdots\langle x_1|e^{-\frac{i}{\hbar}H\varepsilon}|x\rangle$$

$x = x_0$

$x' = x_N$

$$= \int_{-\infty}^{\infty}\prod_{i=1}^{N-1}\left(dx_i\,\overbrace{\langle x_i|e^{-\frac{i}{\hbar}H\varepsilon}|x_{i-1}\rangle}\right)\langle x|e^{-\frac{i}{\hbar}H\varepsilon}|x_{N-1}\rangle$$



$$\langle x_i | e^{-\frac{i}{\hbar}\varepsilon H} | x_{i-1} \rangle$$

$$H = T(\hat{p}) + V(\hat{x})$$

$$e^{-\frac{i}{\hbar}\varepsilon H} = e^{-\frac{i}{\hbar}\varepsilon(T+V)} \neq \underbrace{e^{-\frac{i}{\hbar}\varepsilon T} e^{-\frac{i}{\hbar}\varepsilon V}}_{\left(1-\frac{i}{\hbar}\varepsilon T\right)\left(1-\frac{i}{\hbar}\varepsilon V\right)+O(\varepsilon^2)}$$

$$e^{-\frac{i}{\hbar}\varepsilon(T+V)} = 1 - \frac{i}{\hbar}\varepsilon(T+V) + O(\varepsilon^2)$$

$$\left(1-\frac{i}{\hbar}\varepsilon T\right)\left(1-\frac{i}{\hbar}\varepsilon V\right)+O(\varepsilon^2) = 1 - \frac{i}{\hbar}\varepsilon(T+V) + O(\varepsilon^2)$$

$$e^{-\frac{i}{\hbar}\varepsilon(T+V)} = e^{-\frac{i}{\hbar}\varepsilon T} e^{-\frac{i}{\hbar}\varepsilon V} + O(\varepsilon^2)$$

$$\langle x_i | e^{-\frac{i}{\hbar}H\varepsilon} | x_{i-1} \rangle = \langle x_i | e^{-\frac{i}{\hbar}\varepsilon T} \underset{\int dp\, |p\rangle\langle p|}{\wedge} e^{-\frac{i}{\hbar}\varepsilon V} | x_{i-1} \rangle + O(\varepsilon^2)$$

$$\langle x_i | e^{-\frac{i}{\hbar}H\varepsilon} | x_{i-1} \rangle = \int dp_i \langle x_i | e^{-\frac{i}{\hbar}\varepsilon T} | p_i \rangle \langle p_i | e^{-\frac{i}{\hbar}\varepsilon V} | x_{i-1} \rangle$$

$$= \int dp_i\, e^{-\frac{i}{\hbar}\varepsilon T(p_i)} e^{-\frac{i}{\hbar}\varepsilon V(x_{i-1})} \frac{e^{\frac{i}{\hbar}p_i x_i} e^{-\frac{i}{\hbar}p_i x_{i-1}}}{(2\pi\hbar)}$$

$$\longrightarrow$$

$$\langle x_i | e^{-\frac{i}{\hbar}H\varepsilon} | x_{i-1} \rangle = \int dp_i\, e^{\frac{i}{\hbar}\left[p_i(x_i - x_{i-1}) - \varepsilon H(p_i, x_{i-1})\right]}$$

$$x(t) = x_i \qquad t = \varepsilon i$$

$$p(t) = p_i \qquad t = \varepsilon i$$

$$x_i - x_{i-1} = x(t) - x(t-\varepsilon) = \underbrace{\frac{x(t) - x(t-\varepsilon)}{\varepsilon}}_{\left(\frac{dx}{dt}\right)} \varepsilon \quad \varepsilon$$

$$\langle x_i | e^{-\frac{i}{\hbar} H \varepsilon} | x_{i-1} \rangle = \int dp_i \, e^{\frac{i}{\hbar} \varepsilon (p\dot{x} - H)(t_i)}$$

$$\boxed{p\dot{q} - H \equiv L}$$

$$\int \prod \frac{dx_i \, dp_i}{(2\pi\hbar)} e^{\frac{i}{\hbar} \sum_i \varepsilon (p\dot{x} - H)(t_i)} \Longleftarrow$$

$$= \int \prod dx_i \, dp_i \, e^{\frac{i}{\hbar} \int_0^t dt \, (p\dot{x} - H)}$$

$$\equiv \int \mathcal{D}x \, \mathcal{D}p \, e^{\frac{i}{\hbar} \int_0^t dt \, (p\dot{x} - H)} \Longleftarrow$$

$$\mathcal{D}x = \lim_{\substack{N \to \infty \\ \varepsilon \to 0}} \prod dx_i \; ; \; \mathcal{D}p = \lim_{\substack{N \to \infty \\ \varepsilon \to 0}} \prod dp_i$$

$$T = \frac{p^2}{2m}$$

$$\langle x_i | e^{-\frac{i}{\hbar} H \varepsilon} | x_{i-1} \rangle = \int dp_i \, e^{\frac{i}{\hbar}\left[p_i (x_i - x_{i-1}) - \varepsilon H(p_i, x_{i-1})\right]}$$

$$= \int dp_i \, e^{\frac{i}{\hbar}\left[p_i (x_i - x_{i-1}) - \varepsilon \frac{p_i^2}{2m}\right]} e^{-\frac{i}{\hbar} V(x_{i-1}) \varepsilon}$$

$$= \int dp_i \, e^{-\frac{i\varepsilon}{2m\hbar}\left[p_i^2 - \frac{2m}{\varepsilon}(x_i - x_{i-1}) p_i\right]} e^{-\frac{i}{\hbar}\varepsilon V}$$

$$= \sqrt{\frac{2m\pi\hbar}{i\varepsilon}} \, e^{-\frac{i\varepsilon(-1)}{2m\hbar}\left[+\underbrace{\frac{m}{\varepsilon}(x_i - x_{i-1})}\right]^2} e^{-\frac{i}{\hbar}\varepsilon V}$$

$$= \sqrt{\frac{2m\pi\hbar}{i\varepsilon}} \, e^{\frac{i\varepsilon}{2m\hbar} m^2 \dot{x}(t_i)^2} e^{-\frac{i}{\hbar}\varepsilon V}$$

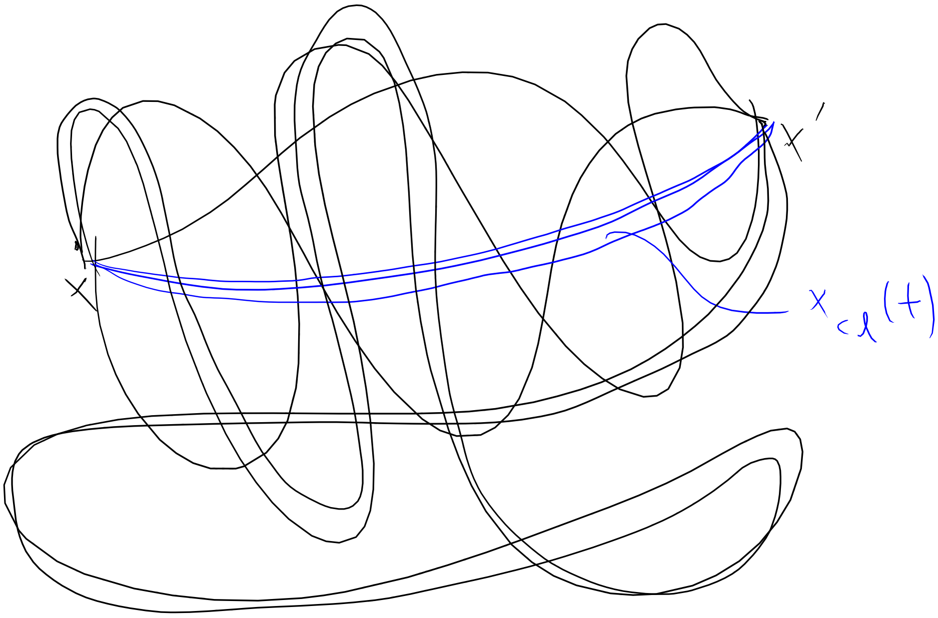
$$= \sqrt{\frac{2m\pi\hbar}{i\varepsilon}} \, e^{i\frac{\varepsilon}{\hbar}\left(\frac{1}{2} m \dot{x}^2\right)} e^{-\frac{i}{\hbar}\varepsilon V} \qquad \frac{1}{(2\pi\hbar)}$$

$$\langle x_i | e^{-\frac{i}{\hbar} H t} | x_{i-1} \rangle$$

$$G(x'; x; t) = \left(\frac{2m\pi\hbar}{i\varepsilon}\right)^{N/2} \int \left(\prod dx_i\right) e^{+i\frac{\varepsilon}{\hbar}\sum\left(\frac{1}{2} m \dot{x}^2 - V(x)\right)}$$

$$\boxed{G(x'; x; t) = \left(\frac{2m\pi\hbar}{i\varepsilon}\right)^{N/2} \int \mathcal{D}x \, e^{\frac{i}{\hbar}\underbrace{\int_0^t dt \, L[x(t)]}_{S}}}$$

in the limit $\hbar \to 0$ only the path that satisfies $\delta S = 0$ contributes



$$G(x,x';t) = \sqrt{\left(\frac{2\cancel{m}\pi\hbar}{i\epsilon}\right)^N} \int \prod_{i=1}^{N-1} dx_i \; e^{+\frac{i}{\hbar}\sum \frac{1}{2}m\left(\frac{x_i-x_{i-1}}{\epsilon}\right)^2}$$

$$\int_{-\infty}^{\infty} dx_1 \, e^{\frac{i}{\hbar}\epsilon\frac{1}{2}m\left[\left(\frac{x_1-x_0}{\epsilon}\right)^2+\left(\frac{x_2-x_1}{\epsilon}\right)^2\right]}$$

$$+\left(\frac{x_1+x_0}{2}\right)^2 - \left(\frac{x_0+x_2}{2}\right)^2$$

$$= \int_{-\infty}^{\infty} dx_1 \, e^{\frac{i}{\hbar}\frac{m}{\cancel{2}\epsilon}\left[\cancel{2}x_1^2 - \cancel{2}x_1(x_2+x_0) + \frac{(x_0^2+x_2^2)}{2}\right]}$$

$$= \sqrt{\frac{\pi\epsilon\hbar}{-im}} \, e^{\frac{i}{\hbar}\frac{m}{\epsilon}\left[\frac{x_0^2+x_2^2}{2} - \frac{(x_2+x_0)^2}{4}\right]}$$

$$= \sqrt{\frac{\pi\epsilon\hbar}{-im}} \, e^{\frac{im}{4\hbar\epsilon}\left[\cancel{2}x_2^2+\cancel{2}x_0^2 - (\cancel{x_2^2}+\cancel{x_0^2}+2x_0x_2)\right]}$$

$$= \sqrt{\frac{\pi\epsilon\hbar}{-im}} \, e^{\frac{im}{4\hbar\epsilon}\left[(x_2-x_0)^2\right]}$$

$$= \sqrt{\frac{\pi \varepsilon \hbar}{-i m}} \; e^{\frac{i m}{2\hbar}\left(\frac{\varepsilon}{2}\right)\left(\frac{x_2 - x_0}{\varepsilon}\right)^2}$$

$$\int_{-\infty}^{\infty} dx_2 \; e^{\frac{i}{\hbar}\frac{\varepsilon m}{2}\left(\frac{x_3 - x_2}{\varepsilon}\right)^2 + \frac{i m}{2\hbar}\left(\frac{\varepsilon}{2}\right)\left(\frac{x_2 - x_0}{\varepsilon}\right)^2}$$

$$= \int_{-\infty}^{\infty} dx_2 \; e^{\frac{i m}{4\hbar\varepsilon}\left[2(x_3 - x_2)^2 + (x_2 - x_0)^2\right]}$$

$$= \int_{-\infty}^{\infty} dx_2 \; e^{\frac{i m}{4\hbar\varepsilon}\left[2(x_3^2 - 2x_3x_2 + x_2^2) + (x_2^2 + x_0^2 - 2x_2x_0)\right]}$$

$$= \int_{-\infty}^{\infty} dx_2 \; e^{\frac{i m}{4\hbar\varepsilon}\left[(2x_3^2 + x_0^2) + 3x_2^2 - 2x_2(2x_3 + x_0)\right]}$$

$$= \int_{-\infty}^{\infty} dx_2 \; e^{\frac{i m 3}{4\hbar\varepsilon}\left[x_2^2 - 2x_2\left(\frac{2x_3 + x_0}{3}\right) + \frac{2x_3^2 + x_0^2}{3}\right]}$$

$$= \sqrt{\frac{\pi 4\hbar\varepsilon}{-i3 \, m}} \; e^{\frac{i 3 m}{\varepsilon 4\hbar}\left[\frac{2x_3^2 + x_0^2}{3} - \left(\frac{2x_3 + x_0}{3}\right)^2\right]}$$

$$= \sqrt{\frac{4\pi\varepsilon\hbar}{-i3 \, m}} \; e^{\frac{i 3 m}{\varepsilon 4\hbar \cdot 9}\left[3(2x_3^2 + x_0^2) - (4x_3^2 + x_0^2 + 4x_0x_3)\right]}$$

$$= \sqrt{\frac{4\pi\varepsilon\hbar}{-i3 \, m}} \; e^{\frac{\cancel{3} i m}{\cancel{2}4\hbar\varepsilon 9 3}\left[\cancel{2}x_3^2 + \cancel{2}x_0^2 - 4x_0x_3\right]}$$

$$= \sqrt{\frac{4\pi\hbar\varepsilon}{-i3 \, m}} \; e^{\frac{i m}{\varepsilon 6\hbar}(x_3 - x_0)^2}$$

$$\int dx_1 \cdots = \sqrt{\frac{\pi \varepsilon \hbar}{-i m}}\; e^{\frac{i m}{2\hbar}\left(\frac{\varepsilon}{2}\right)\left(\frac{x_2 - x_0}{\varepsilon}\right)^2}$$

$$\int dx_2 \int dx_1 \cdots = \sqrt{\left(\frac{\pi \varepsilon \hbar}{-i m}\right)^2 \left(\frac{4}{3}\right)}\; e^{\frac{i m}{2\hbar}\left(\frac{\varepsilon}{3}\right)\left(\frac{x_3 - x_0}{\varepsilon}\right)^2}$$

$$\underline{\qquad\qquad\qquad} \to \frac{''\ \frac{2\cdot 2}{n+1}}{\frac{i}{\hbar}\left(\frac{\varepsilon}{n}\right)\frac{1}{2}m\left(\frac{x_n - x_0}{\varepsilon}\right)^2}$$

$$\int dx_{n-1} \cdots \int dx_1 \cdots = N_{n-1}\, e^{\frac{i}{\hbar}\left(\frac{\varepsilon}{n}\right)\frac{1}{2}m\left(\frac{x_n - x_0}{\varepsilon}\right)^2}$$

$$\int dx_n \int dx_{n-1} \cdots \int dx_1 \cdots = N_{n-1} \int dx_n\, e^{\frac{i}{\hbar}\left[\varepsilon \frac{1}{2} m \left(\frac{x_{n+1} - x_n}{\varepsilon}\right)^2 + \left(\frac{\varepsilon}{n}\right)\frac{1}{2} m \left(\frac{x_n - x_0}{\varepsilon}\right)^2\right]}$$

$$= N_{n-1} \int dx_n \exp\left\{\frac{i}{\hbar}\frac{m}{2}\frac{1}{n\varepsilon}\left[n\left(x_{n+1} - x_n\right)^2 + \left(x_n - x_0\right)^2\right]\right\}$$

$$= N_{n-1} \int dx_n \exp\left\{\frac{i}{\hbar}\frac{m}{2}\frac{1}{n\varepsilon}\left[\left(n x_{n+1}^2 + x_0^2\right) + \underline{(n+1)}\, x_n^2 - 2 x_n\left(n x_{n+1} + x_0\right)\right]\right\}$$

$$= N_{n-1}\, e^{\frac{i}{\hbar}\frac{m(n+1)}{2}\frac{1}{n\varepsilon}\left[\frac{\left(n x_{n+1}^2 + x_0^2\right)}{n+1} - \left(\frac{n x_{n+1} + x_0}{n+1}\right)^2\right]}$$

$$\sqrt{\frac{\pi 2\hbar n \varepsilon}{-i(n+1)m}}$$

$$N_n = \sqrt{\frac{\pi n \varepsilon 2\hbar}{-i m (n+1)}}\; N_{n-1}$$

$$= \mathcal{N}_n \exp\left\{ \frac{i}{\hbar} \frac{m(n+1)}{2n\varepsilon} \frac{1}{(n+1)^2} \left[(n+1)\left(n x_{n+1}^2 + x_0^2\right) - \left(n x_{n+1} + x_0\right)^2 \right] \right\}$$

$$= \mathcal{N}_n \exp\left\{ \frac{i}{\hbar} \frac{m \cancel{(n+1)}}{2\cancel{n}\varepsilon} \frac{1}{(n+1)^{\cancel{2}}} \left[\cancel{n} x_{n+1}^2 + \cancel{n} x_0^2 - 2\cancel{n} x_0 x_{n+1} \right] \right\}$$

$$= \mathcal{N}_n \exp\left\{ \frac{i}{\hbar} \frac{1}{2} m \frac{\varepsilon}{(n+1)} \left(\frac{x_{n+1} - x_0}{\varepsilon} \right)^2 \right\}$$

$$\mathcal{N}_1 = \sqrt{\frac{\pi \varepsilon \hbar}{-i m}}$$

$$\mathcal{N}_n = \sqrt{\frac{\pi n \varepsilon 2\hbar}{-i(n+1)}}\, \mathcal{N}_{n-1} = \sqrt{\frac{\pi \varepsilon \hbar}{-i m}} \left(\frac{2n}{n+1} \right)^{1/2} \mathcal{N}_{n-1}$$

$$\mathcal{N}_{N-1} = \sqrt{\left(\frac{\pi \varepsilon \hbar}{-i m} \right)^{N-1}} \left(\underbrace{\left(\frac{2}{\cancel{2}}\right) \left(\frac{\cancel{4}^{\,2}}{\cancel{3}}\right) \left(\frac{\cancel{6}^{\,2}}{4}\right) \cdots \frac{2(N-1)}{N}}_{N-1} \right)^{1/2}$$

$$\mathcal{N}_{N-1} = \left(\sqrt{\frac{4\pi \varepsilon \hbar}{-i m}} \right)^{N-1} \frac{1}{\sqrt{N}}$$

$$G(x, x'; t) = \sqrt{\frac{2\cancel{m}\pi\hbar}{i\cancel{\varepsilon}}}^{\,N-1} \left(\sqrt{\frac{2\pi \varepsilon \hbar}{\cancel{-im}}} \right)^{N-1} \frac{1}{\sqrt{N}}$$

$$\exp\left\{ \frac{i}{\hbar} \frac{1}{2} m \frac{\varepsilon}{N} \left(\frac{x_N - x_0}{\varepsilon} \right)^2 \right\}$$

$$G(x,x';t) = \cancel{(2\pi\hbar)^{N-1}} \left(\frac{1}{\sqrt{N}}\right) \cancel{\frac{1}{(2\pi\hbar)^{N-1}}}$$

$$\exp\left\{\frac{i}{\hbar}\frac{1}{2}m \frac{1}{N\varepsilon}(x-x')^2\right\}$$

correction
we forgot this
factor before!

$$G(x,x';t) = \sqrt{\left(\frac{2m\pi\hbar}{i\varepsilon}\right)}\left(\frac{1}{\sqrt{N}}\right)\exp\left\{\frac{im}{2\hbar t}(x-x')^2\right\}$$

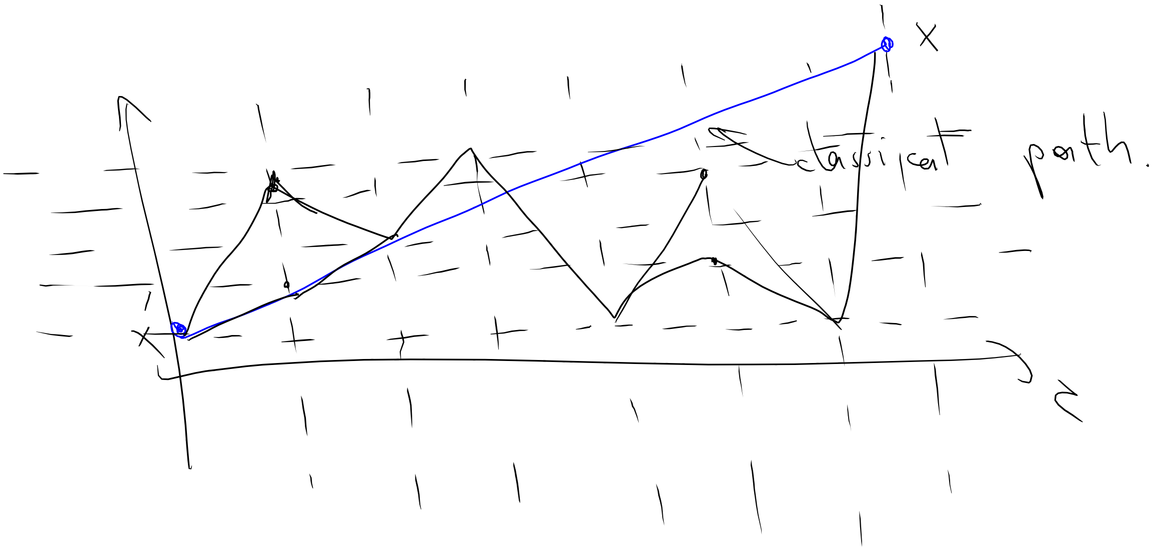
another correction
we have $N-1$ x_i insertions
whereas N p_i insertions.

$$\boxed{G(x,x';t) = \sqrt{\frac{2m\pi\hbar}{it}}\exp\left\{\frac{im}{2\hbar t}(x-x')^2\right\}}$$

$$G(x';x;t) = \left(\frac{2m\pi\hbar}{i\varepsilon}\right)^{N/2}\left(\prod dx_i\right) e^{+\frac{i\varepsilon}{\hbar}\sum\left(\frac{1}{2}m\dot{x}^2 - V(x)\right)}$$

$$e^{-\frac{i}{\hbar}Ht} \xrightarrow{t\to -i\hbar\beta} e^{-\beta H} \quad ; \quad \beta \equiv \frac{1}{k_B T}$$

$$G(x',x,\beta) = \mathcal{N}\int\prod dx_i\, e^{\underbrace{-\frac{1}{\hbar}\int_0^\beta d\lambda\left[\frac{1}{2}m\left(\frac{dx}{d\lambda}\right)^2 + V(x)\right]}_{E[x(\lambda)]}}$$



$$E[x_{trial}] = E_0$$

$$E[x^{(1)}] = E^{(1)}$$

if $E^{(1)} < E_0 \Longrightarrow$ keep the path

if $E^{(1)} > E_0 \Longrightarrow$ keep the path with probability $e^{-\beta(E^{(1)} - E^{(0)})}$

repeat to obtain M different paths.

$$\langle G|H|G\rangle$$

$$\frac{\langle x|He^{-\beta H}|x'\rangle}{\langle x|e^{-\beta H}|x'\rangle} \propto \frac{1}{N}\int \mathcal{D}x\,\mathcal{D}p\, E[x,p]\, e^{-\frac{1}{\hbar}E[x,p]}$$

$$= \sum_{\text{paths}} E(\text{path}) \frac{e^{-\frac{1}{\hbar}E(\text{path})}}{N_{\text{paths}}}$$

"importance sampling"

"Lattice QCD"

$$= \sum_{\substack{\text{paths}\\ \text{in the}\\ \text{ensemble}}} E(\text{path}) \frac{1}{N_{\text{paths in the ensemble}}}$$

$$G(x,x';t) = \int \mathcal{D}x \; e^{\frac{i}{\hbar}\int_0^{t_f} dt \left(\frac{1}{2}m\dot{x}^2 - V(x(t)) - \lambda(t)x(t)\right)}$$

Functional integrals:

$$\frac{\delta f(t)}{\delta f(t')} \equiv \delta(t-t')$$

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \qquad x(t) \equiv x_i \qquad t = \varepsilon i$$

$$\frac{\delta}{\delta \lambda(t')}\int dt\, \lambda(t)x(t) = x(t')$$

$$G(x,x',t) = e^{-\frac{i}{\hbar}\int dt\, V\left(\frac{\hbar}{i}\frac{\delta}{\delta\lambda(t)}\right)} \underbrace{\int \mathcal{D}x \; e^{\frac{i}{\hbar}\int dt \left(\frac{1}{2}m\dot{x}^2 - \lambda x\right)}}$$