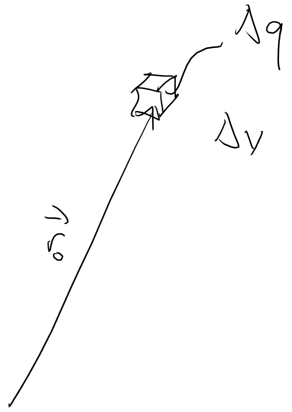
Charge Density



$$\rho(\vec{r}_0) = \lim_{\Delta V \to 1} \frac{\Delta q}{\Delta V}$$

$\Delta q = q$ indep. of ΔV

$$q_{tot} = \int \rho(\vec{r})\, dV$$

$$\rho(\vec{r}) = q\, \delta^{(3)}(\vec{r} - \vec{r}_0)$$

$$\vec{F}_{12} = k \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12}$$

In SI units: $[q] = 1C$

$$k = \frac{1}{4\pi \varepsilon_0}$$

$$[\varepsilon_0] = C^2 / Nm^2$$

$1C = 1As$

Gaussian Units: $\Longleftarrow$

$$k = 1 \qquad |\vec{F}| = \frac{q_1^* q_2^*}{r^2}$$

$$[q] = N^{1/2} m$$

$$q^* \sim \frac{1}{\sqrt{4\pi\varepsilon_0}} q$$

CGS system $\quad [q] = N^{1/2} m$ = stat Coulomb

Natural Units

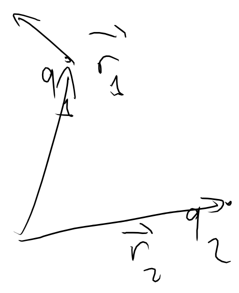
$$k = \frac{1}{4\pi} \qquad c = 1, \; \hbar = 1, \; ..$$

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\Delta E \, \Delta t \gtrsim \frac{\hbar}{2}$$

$$E = \sqrt{\vec{p}^2 + m^2} \qquad (c = 1)$$

$$F \propto \frac{q_1 q_2}{r^2} e^{-m_\gamma r} \qquad (\hbar = 1 = c)$$

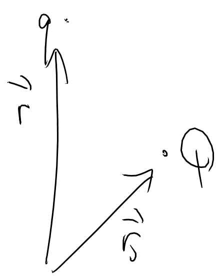


$$\vec{F}_1 = q_1 q_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} = \underline{q_1 \vec{E}(\vec{r}_1)}$$

$$\vec{E}(\vec{r}) = \lim_{q \to 0} \frac{\vec{F}_q}{q} =$$

<u>For a point charge Q</u>

$$\vec{F}_q = \frac{q\,Q\,(\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^3}$$



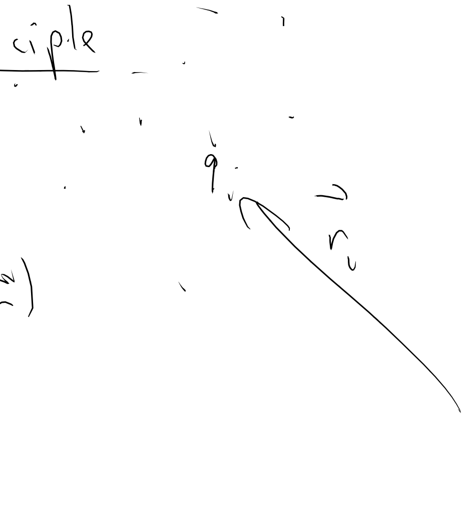
$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}_q}{q}$$

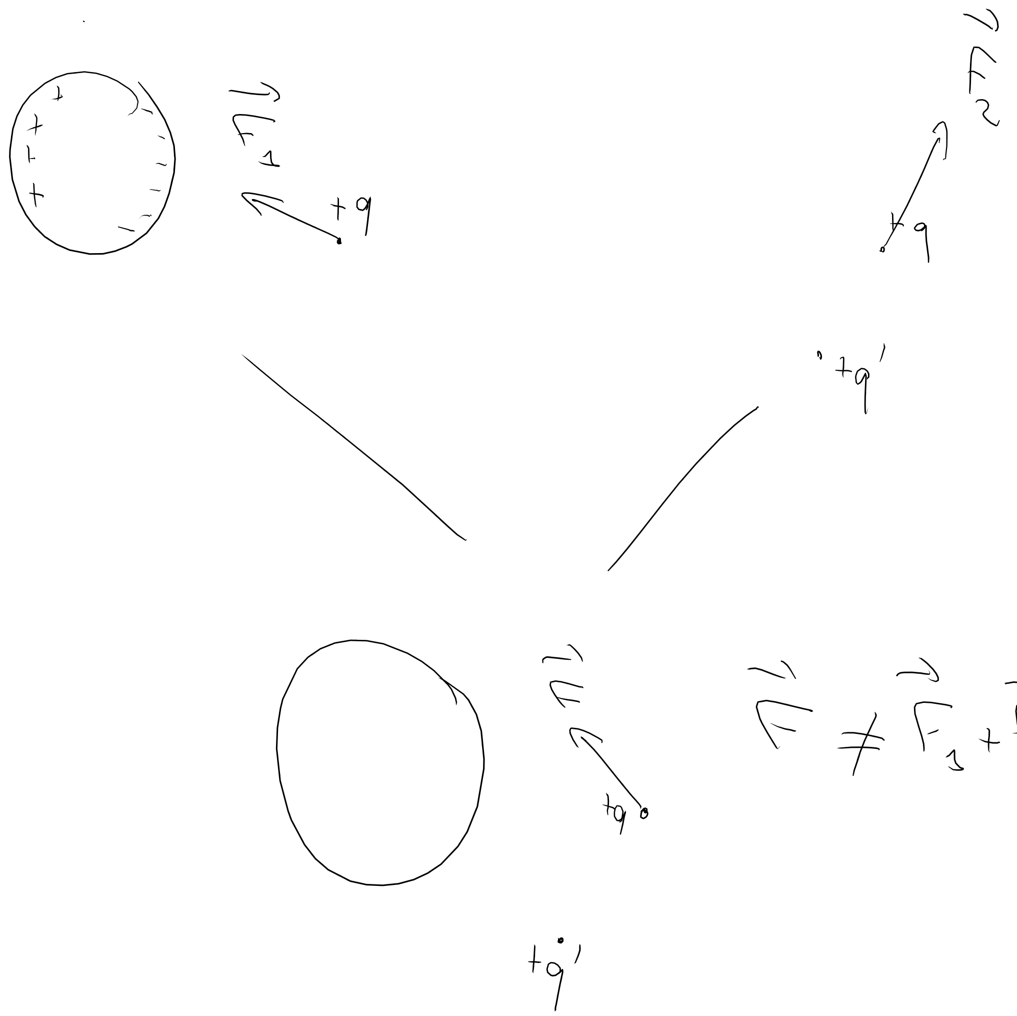
$$\boxed{\vec{E} = Q \frac{(\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^3}}$$

<u>Superposition Principle</u>

$$\vec{E}(\vec{r}) = \sum \vec{E}_i(\vec{r})$$

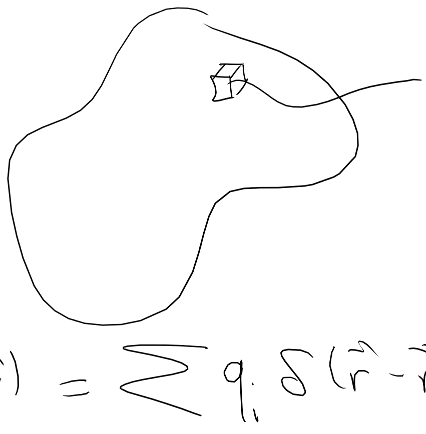
$$\vec{F}(\vec{r}) = \sum q \vec{E}_i(\vec{r})$$





$$\vec{F} \neq \vec{F}_1 + \vec{F}_2$$

$$\vec{E}(\vec{r}) = \sum q_i \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3} = \int \rho(\vec{r}\,') \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} d^3V$$



$$q_i = \rho(\vec{r}_i)\,\Delta V$$

$$\rho(\vec{r}) = \sum q_i\, \delta(\vec{r} - \vec{r}_i) \quad \text{(system of point charges)}$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{r} = -\frac{\vec{r}}{r^3}$$

$$\vec{\nabla}_r \frac{1}{|\vec{r}-\vec{r}\,'|} = -\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$$\vec{E}(\vec{r}) = \int \rho(\vec{r}\,') \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\, dV \quad \Longleftarrow$$

$$= -\int \rho(\vec{r}\,')\, \vec{\nabla}_r \frac{1}{|\vec{r}-\vec{r}\,'|}\, dV$$

$$= -\vec{\nabla}_r \underbrace{\left(\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, dV\right)}_{\phi(\vec{r})\text{: electrostatic potential}}$$

$$\boxed{\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})}$$

$$\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times(\vec{\nabla}\phi) = 0$$

$$\boxed{\vec{\nabla}\times\vec{E} = 0}$$

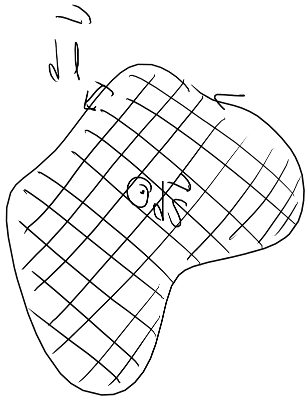
$$\oint \vec{E}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{E})\cdot d\vec{S} = 0$$

$$\Uparrow$$

$$f(a) - f(b) = \int_b^a \frac{df}{dx}\, dx$$

$$\oint \vec{E}\cdot d\vec{S} = \int (\vec{\nabla}\cdot\vec{E})\, dV$$

$$\oint \vec{E} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{S}$$



Helmholtz thm

$$\left.\begin{array}{l} \vec{\nabla} \times \vec{E} \checkmark \\ \vec{\nabla} \cdot \vec{E} \checkmark \end{array} + \begin{array}{l}\text{boundary} \\ \text{conditions}\end{array}\right\} \vec{E} \text{ is uniquely determined!}$$

$$\vec{E}(\vec{r}) = \int \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} dV$$

$$= - \int \rho(\vec{r}') \vec{\nabla}_r \frac{1}{|\vec{r} - \vec{r}'|} dV$$

$$\vec{\nabla}_{\vec{r}} \cdot \vec{E}(\vec{r}) = - \int \rho(\vec{r}') \vec{\nabla}_r \cdot \left(\vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}'|} \right) dV$$

$$= - \int \rho(\vec{r}') \nabla_r^2 \frac{1}{|\vec{r} - \vec{r}'|} dV$$

$$\vec{\nabla} \cdot \vec{E} = - \int \rho(\vec{r}') \left(-4\pi \delta^{(3)}(\vec{r} - \vec{r}') \right) dV$$

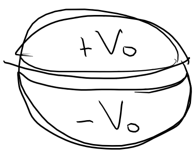
$$\boxed{\vec{\nabla}\cdot\vec{E} = 4\pi\rho(\vec{r})}$$

$$\int_V (\vec{\nabla}\cdot\vec{E})\, dV = \int_V 4\pi\rho(r)\, dV = 4\pi Q_{enc}$$

$$\boxed{\oint_{\partial V} \vec{E}\cdot d\vec{S} = 4\pi Q_{enc}}$$

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}\phi$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\phi) = -4\pi\rho$$

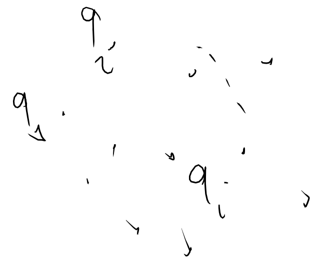


$$\boxed{\nabla^2\phi = -4\pi\rho}$$ Poisson's Eqn.

$$\boxed{\nabla^2\phi = 0}$$ Laplace's Eqn

Energy Stored in the Electric Field

$$\vec{E} = -\vec{\nabla}\phi$$



$$W_1 = 0$$

$$W_2 = -\int q_2 \vec{E}_1 \cdot d\vec{l} = -q_2 \int (-\vec{\nabla}\phi_1) \cdot d\vec{l}$$

$$= q_2 \left(\phi_1(\vec{r}_2) - \underbrace{\phi_1(\infty)}_{=0} \right)$$

$$W_2 = q_2 \phi_1(\vec{r}_2)$$

$$W_3 = -\int q_3 \left(\vec{E}_1 + \vec{E}_2 \right) \cdot d\vec{l}$$

$$= -q_3 \int \left(-\vec{\nabla}\phi_1 - \vec{\nabla}\phi_2 \right) \cdot d\vec{l}$$

$$= q_3 \int \vec{\nabla} (\phi_1 + \phi_2) \cdot d\vec{l}$$

$$W_3 = q_3 \left(\phi_1(\vec{r}_3) + \phi_2(\vec{r}_3) \right)$$

$$W_n = q_n \sum_{i=1}^{n-1} \phi_i(\vec{r}_n)$$

$$W_T = \sum_n W_n = \sum_n q_n \sum_{i<n} \phi_i(\vec{r}_n)$$

$$\boxed{W_T = \sum_{j,\, i<j} q_j \phi_i(\vec{r}_j)}$$

$$W_T = q_2\phi_1 + q_3(\phi_1+\phi_2) + q_4(\phi_1+\phi_2+\phi_3) + \dots$$

$$q_j\phi_i(\vec{r}_j) = \frac{q_i q_j}{|r_i - r_j|} = q_i \underbrace{\left(\frac{q_j}{|r_i - r_j|}\right)}_{\phi_j(\vec{r}_i)} = q_i\phi_j(\vec{r}_i)$$

$$W_T = \sum_{\substack{i,j \\ i<j}} q_j\phi_i(\vec{r}_j) = \sum_{\substack{i,j \\ i<j}} q_i\phi_j(\vec{r}_i)$$

$$\overset{\text{rename indices}}{=} \sum_{\substack{j,i \\ j<i}} q_j\phi_i(\vec{r}_j)$$

$$W_T = \sum_{\substack{i,j \\ i<j}} q_j\phi_i(\vec{r}_j) = \sum_{\substack{i,j \\ i>j}} q_j\phi_i(\vec{r}_j)$$

$$= \frac{1}{2}\left(\sum_{\substack{i,j \\ i<j}} q_j\phi_i(\vec{r}_j) + \sum_{\substack{i,j \\ i>j}} q_j\phi_i(\vec{r}_j)\right)$$

$$W_T = \frac{1}{2}\sum_{\substack{i,j \\ i\neq j}} q_j\phi_i(\vec{r}_j)$$

$$= \frac{1}{2}\sum_j q_j\phi(\vec{r}_j)$$

$$\boxed{W_T = \frac{1}{2}\int \rho(\vec{r})\phi(\vec{r})\, dV}$$

$$W_T = \frac{1}{2}\int\left(-\frac{1}{4\pi}\right)\left(\nabla^2\phi\right)\phi\, dV$$

$$= -\frac{1}{8\pi}\int\left[\vec{\nabla}\cdot\left(\phi\vec{\nabla}\phi\right) - \left(\vec{\nabla}\phi\right)\cdot\left(\vec{\nabla}\phi\right)\right]dV$$

$$= -\frac{1}{8\pi}\int\left(\phi\vec{\nabla}\phi\right)\cdot d\vec{S} + \frac{1}{8\pi}\int\vec{E}^2\, dV$$

$$W_T = \frac{1}{8\pi}\int\vec{E}^2\, dV$$

$$\boxed{u_E = \frac{1}{8\pi}\vec{E}^2}$$