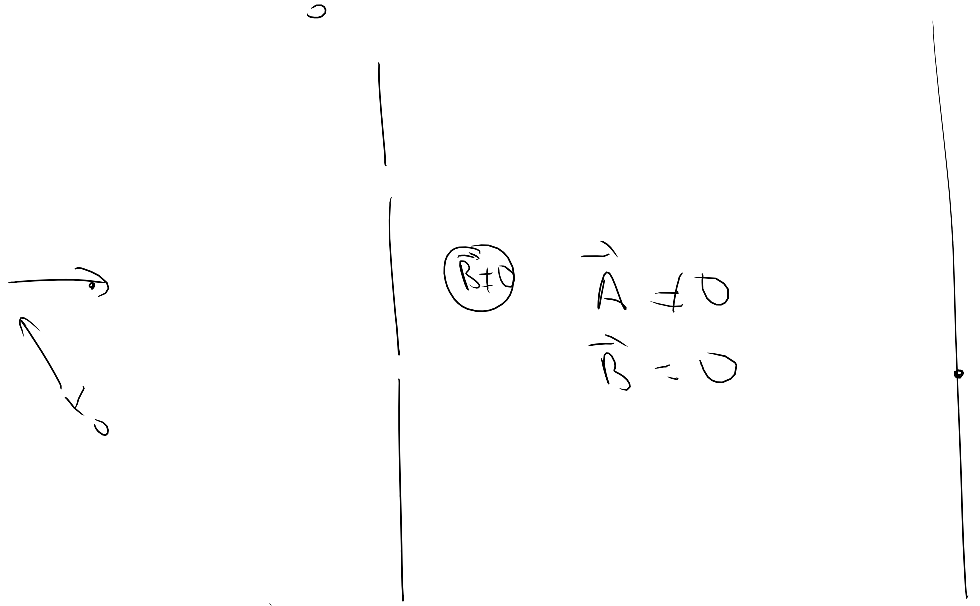
Aharanow-Bohm Effect in Path-Integrals

$$G(x,x';t) \equiv \langle x' | e^{-\frac{i}{\hbar}Ht} | x \rangle$$

$$= \int \mathcal{D}x \, e^{\frac{i}{\hbar}S[x(t)]}$$

$$S[x] = \int_0^t dt' \, \mathcal{L}(\dot{x}, x; t)$$



$$\langle x' | e^{-\frac{i}{\hbar}Ht} | x_0 \rangle$$

$$\mathcal{L} = \mathcal{L}^0 + q\vec{v}\cdot\vec{A}$$

$$\int \mathcal{D}x \, e^{\frac{i}{\hbar}S} = \underbrace{\int \mathcal{D}x \, e^{\frac{i}{\hbar}S}}_{x(t) \text{ goes above}} + \underbrace{\int \mathcal{D}x \, e^{\frac{i}{\hbar}S}}_{x(t) \text{ goes below}}$$

$$\underbrace{S[x(t)] - S[x'(t)]}_{x,\,x' \text{ goes above } B} = \int_0^t dt \, q(\vec{v}\cdot\vec{A} - \vec{v}'\cdot\vec{A})$$

$$\vec{v}\,dt = \frac{d\vec{l}}{dt}\,dt = d\vec{l}$$

$$\int^{t} dt\, q\vec{v}\cdot\vec{A} = q\int_{x_0}^{x'} \vec{A}\cdot d\vec{l}$$

$$\underset{\text{path 1}}{\int_{x_0}^{x'} \vec{A}\cdot d\vec{l}} - \underset{\text{path 2}}{\int_{x_0}^{x'} \vec{A}\cdot d\vec{l}} = \oint \vec{A}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{A})\cdot d\vec{S}$$

$$= \begin{cases} 0 & \text{if both paths go above or below} \\ \phi_B & \text{if one goes above and the other goes below} \end{cases}$$

$$\int \mathcal{D}x\, e^{\frac{i}{\hbar}S} = \underset{x(t)\text{ goes above}}{\int \mathcal{D}x\, e^{\frac{i}{\hbar}S}} + \underset{x(t)\text{ goes below}}{\int \mathcal{D}x\, e^{\frac{i}{\hbar}S}}$$

$$= e^{\frac{qi}{\hbar}\int_{x_0,\ \text{above}}^{x'} \vec{A}\cdot d\vec{l}} \underset{x(t)\text{ goes above}}{\int \mathcal{D}x\, e^{\frac{i}{\hbar}S^0}} + e^{\frac{qi}{\hbar}\int_{x_0,\ \text{below}}^{x'} \vec{A}\cdot d\vec{l}} \underset{x(t)\text{ goes below}}{\int \mathcal{D}x\, e^{\frac{i}{\hbar}S^0}}$$

$$= e^{\frac{qi}{\hbar}\int_{\text{above}} \vec{A}\cdot d\vec{l}} \left[M_1 + e^{\frac{qi}{\hbar}\phi_B} M_2 \right]$$

$$G(x,x';t) = \int \mathcal{D}x \, e^{\frac{i}{\hbar} S[x]}$$

$$\int \mathcal{D}x = \lim_{N \to \infty} \prod_{i=1}^{N-1} dx_i$$

$$\int_{-\infty}^{\infty} dx_i = \int_{-\infty}^{\infty} d(x_i + \delta_i)$$

$$x_i = y_i + \delta_i$$

$$\int_{-\infty}^{\infty} dx_i = \int_{-\infty}^{\infty} dy_i$$

$$\mathcal{D}x = \mathcal{D}y$$

$$\int \mathcal{D}x \, e^{\frac{i}{\hbar} S[x]} = \int \mathcal{D}(y+\delta y) \, e^{\frac{i}{\hbar} S[y+\delta y]}$$

$$= \int \mathcal{D}y \, e^{\frac{i}{\hbar} S[y+\delta y]}$$

$$= \int \mathcal{D}x \, e^{\frac{i}{\hbar} S[\cancel{y} + \delta x]}$$

$$\int \mathcal{D}x \left(e^{\frac{i}{\hbar} S[x]} - e^{\frac{i}{\hbar} S[x+\delta x]} \right) = 0$$

if δx are small and arbitrary

$$\int \mathcal{D}x \, e^{\frac{i}{\hbar} S[x]} \underline{\delta S} = 0 = \langle \delta S \rangle$$

$$G(x,x',t) = \langle x'| e^{-\frac{i}{\hbar}Ht} |x\rangle$$

change of notation

$$e^{-\frac{i}{\hbar}Ht} = e^{-\frac{i}{\hbar}H\varepsilon}\, e^{-\frac{i}{\hbar}H\varepsilon}\, e^{-\frac{i}{\hbar}H\varepsilon} \cdots e^{-\frac{i}{\hbar}H\varepsilon}$$

$$e^{-\frac{i}{\hbar}Ht}\underbrace{|x\rangle}_{t=0} = e^{-\frac{i}{\hbar}H\varepsilon}\, e^{-\frac{i}{\hbar}H\varepsilon}\, \underbrace{e^{-\frac{i}{\hbar}H(t')\varepsilon}}_{t'} \cdots e^{-\frac{i}{\hbar}H\varepsilon} |x\rangle$$

act at the time t'

$$= T\, e^{-\frac{i}{\hbar}\int_0^{t} dt'\, H(t')} |x\rangle$$

$H(t')$ acts on my state at time t'

$$G(x,x',t) = \langle x'| T\, e^{-\frac{i}{\hbar}\int_0^{t} dt'\, H(t')} |x\rangle$$

$x(t') \equiv \hat{x}$ but acts on my state when it evolves until the time t'

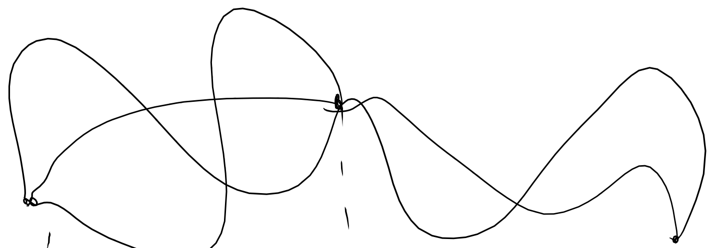
$$\langle x'| e^{-\frac{i}{\hbar}H(t-t')}\, x\, e^{-\frac{i}{\hbar}Ht'} |x\rangle = \langle x'| T\left\{ x(t')\, e^{-\frac{i}{\hbar}\int H\,dt} \right\} |x\rangle$$

$$\langle x'| e^{-\frac{i}{\hbar}H(t-t')} \hat{x} e^{-\frac{i}{\hbar}Ht'} |x\rangle$$

$$= \langle x'| e^{-\frac{i}{\hbar}H(t-t')} \hat{x} \left(\int dx'' |x''\rangle\langle x''|\right) e^{-\frac{i}{\hbar}Ht'} |x\rangle$$

$$= \int dx''\, x'' \langle x'| e^{-\frac{i}{\hbar}H(t-t')} |x''\rangle \langle x''| e^{-\frac{i}{\hbar}Ht'} |x\rangle$$

$$= \int dx''\, x'' \int\limits_{\substack{x(t')=x'' \\ x(t)=x'}} \mathcal{D}x\, e^{\frac{i}{\hbar}\int_{t'}^{t} \mathcal{L}\,dt} \int\limits_{\substack{x(t=0)=x \\ x(t')=x''}} \mathcal{D}x\, e^{\frac{i}{\hbar}\int_{0}^{t'} \mathcal{L}\,dt}$$

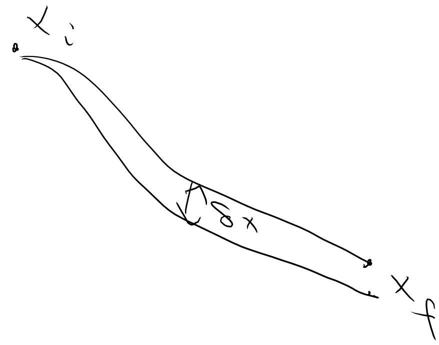


$$= \int\limits_{\substack{x(t=0)=x \\ x(t')=x'}} \mathcal{D}x\; x(t')\, e^{\frac{i}{\hbar}S[x]} \qquad = \langle x'| e\, x(t)\, e |x\rangle$$

$$G(\overset{\text{final}}{x}, x'; t)$$

$$\frac{\partial G}{\partial x} = \int \mathcal{D}x \, \frac{\partial}{\partial x}\left(e^{\frac{i}{\hbar}S}\right)$$

$$\frac{\partial G}{\partial x} = \int \mathcal{D}x \, \frac{i}{\hbar} \underbrace{\left(\frac{\partial S}{\partial x}\right)} e^{\frac{i}{\hbar}S}$$



$$\delta S = \int dt \, \delta \mathcal{L}$$

$$= \int dt \left(\frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}} \delta \dot{x} \right)$$

$$= \int \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \delta x \right) dt$$

$$+ \underbrace{\int dt \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x}_{\delta S \big|_{\delta x(t=0) = \delta x(t_f) = 0}}$$

$$\delta S = \overbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}}}^{p} \delta x \Big|_{t=t_i}^{t_f} = p_f \, \delta x_f - p_i \, \underset{0}{\underset{\parallel}{\delta x_i}}$$

$$\frac{\partial S}{\partial x} = p$$

$$\frac{\partial G}{\partial x} = \int \mathcal{D}x \, \frac{i}{\hbar}\left(\frac{\partial S}{\partial x}\right) e^{\frac{i}{\hbar}S}$$

$$= \int \mathcal{D}x \, \frac{i}{\hbar}\, p \, e^{\frac{i}{\hbar}S}$$

$$\frac{\partial^2 G}{\partial x^2} = \int \mathcal{D}x \left(\frac{i}{\hbar}p\right)^2 e^{\frac{i}{\hbar}S}$$

$$\frac{\partial S}{\partial t} = \frac{\partial}{\partial t}\left(\int_{t_0}^{t} dt\, \mathcal{L}(x,\dot{x},t)\right) = -H$$

$$dL = p\,dq - H\,dt$$

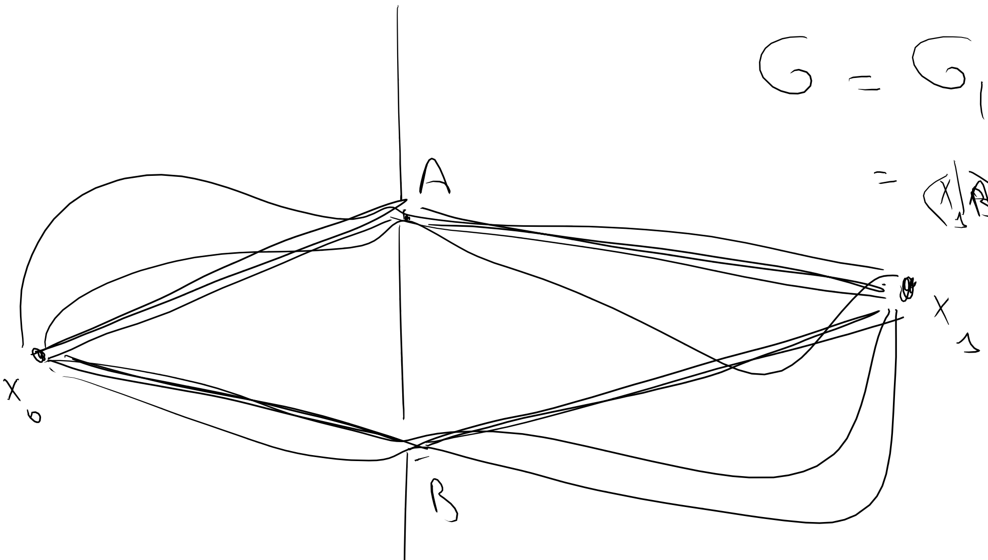
$$\frac{\partial G}{\partial t} = \int \mathcal{D}x \left(\frac{i}{\hbar}\frac{\partial S}{\partial t}\right) e^{\frac{i}{\hbar}S}$$

$$= \int \mathcal{D}x \left(-\frac{i}{\hbar}H\right) e^{\frac{i}{\hbar}S}$$

$$= \int \mathcal{D}x \left(-\frac{i}{\hbar}\right)\left(\frac{p^2}{2m}+V(x)\right) e^{\frac{i}{\hbar}S}$$

$$= -\left(\frac{i}{\hbar}\right)\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\frac{1}{2m}+V(x)\right] G$$

$$\psi(x,t) = \int d^3x' \; G(x, x', t)\, \psi(x'; t=0)$$



$$G = G_{lower} + G_{higher}$$
$$= \langle x_1 | B \rangle \langle B | x_0 \rangle + \langle x_1 | A \rangle \langle A | x_0 \rangle$$

$\langle x | B \rangle$
$\langle x | A \rangle$

$$S^{free}_{cl} = \int dt \; \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m v^2 t$$
$$= \frac{1}{2} m \frac{(x_f - x_i)^2}{t}$$
$$= \frac{1}{2} m \, \Delta x \, v$$

$$\sim e^{\frac{i}{\hbar} \Delta L}$$

SHO

$$\propto \det A^{1/2}$$

$$G(x_i, x_f; t) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega(t_f - t_i)}}$$

$$\exp\Big\{ \underbrace{\frac{i}{\hbar}\frac{1}{2}m\omega \frac{(x_i^2 + x_f^2)\cos\omega(t_f - t_i) - 2x_i x_i}{\sin\omega(t_f - t_i)}}_{\frac{i}{\hbar} S[x_{cl}]} \Big\}$$

$$= \langle x_f | e^{-\frac{i}{\hbar}Ht} | x_i \rangle \qquad t \equiv t_f - t_i$$

$$\frac{i}{\hbar} t = \beta \Rightarrow t = -i\beta\hbar$$

$$\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i} = \frac{e^{\beta\hbar\omega} - e^{-\beta\hbar\omega}}{2i} = -i\sinh\beta\hbar\omega$$

$$\cos(\omega t) = \frac{e^{\beta\hbar\omega} + e^{-\beta\hbar\omega}}{2} = \cosh(\beta\hbar\omega)$$

$$G(x_i, x_f; t) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega(t_f - t_i)}}$$

$$\exp\Big\{ \frac{i}{\hbar}\frac{1}{2}m\omega \frac{(x_i^2 + x_f^2)\cos\omega(t_f - t_i) - 2x_i x_f}{\sin\omega(t_f - t_i)} \Big\}$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar \sinh(\hbar\omega\beta)}} \exp\Big\{ -\frac{m\omega}{2\hbar} \frac{(x_i^2 + x_f^2)\cosh(\omega\hbar\beta) - 2x_i x_f}{\sinh(\hbar\omega\beta)} \Big\}$$

$$Z = \int dx \, \langle x | e^{-\beta H} | x \rangle \qquad \int_{-\infty}^{\infty} dx \, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

$$= \int dx \sqrt{\frac{m\omega}{2\pi\hbar \sinh(\hbar\omega\beta)}} \exp\left\{ -\frac{m\omega}{\cancel{2}\hbar} \cancel{2} x^2 \frac{(\cosh \omega\hbar\beta - 1)}{\sinh(\hbar\omega\beta)} \right\}$$

$$= \sqrt{\frac{\cancel{m\omega}}{\cancel{2\pi\hbar \sinh(\hbar\omega\beta)}}} \sqrt{\frac{\cancel{\pi\hbar\sinh(\hbar\omega\beta)}}{\cancel{m\omega}(\cosh \hbar\omega\beta - 1)}}$$

$$Z = \sqrt{\frac{1}{2}} \frac{1}{\sqrt{\cosh(\hbar\omega\beta) - 1}}$$

$$\cosh(\Theta) = \frac{e^{\Theta} + e^{-\Theta}}{2} = \frac{\left(e^{\Theta/2} + e^{-\Theta/2}\right)^2}{2} \pm 1$$

$$\cosh(\Theta) = 2\left(\sinh \frac{\Theta}{2}\right)^2 + 1$$

$$Z = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2 \sinh\left(\frac{\hbar\omega\beta}{2}\right)^2}} = \frac{1}{2} \frac{1}{\sinh\left(\frac{\hbar\omega\beta}{2}\right)}$$

$$Z = \frac{1}{e^{\frac{\hbar\omega\beta}{2}} - e^{-\frac{\hbar\omega\beta}{2}}} = \frac{e^{-\frac{\hbar\omega\beta}{2}}}{1 - e^{-\hbar\omega\beta}}$$

$$= e^{-\frac{\hbar\omega\beta}{2}} \sum_n e^{-\hbar\omega\beta n}$$

$$= \sum_n e^{-\underbrace{\hbar\omega\left(n + \frac{1}{2}\right)}_{E_n}\beta}$$

$$\langle x_f | e^{-\beta H} | x_i \rangle = \sum e^{-\beta E_n} \psi_n(x_f) \psi_n(x_i)$$

$$\xrightarrow[\beta \to \infty]{} e^{-\beta E_0} \psi_0(x_f) \psi_n(x_i)$$

$$\sinh(\hbar\omega\beta) \xrightarrow[\beta \to \infty]{} \frac{e^{\hbar\omega\beta}}{2}$$

$$\cosh(\hbar\omega\beta) \xrightarrow[\beta \to \infty]{} \frac{e^{\hbar\omega\beta}}{2}$$

$$G = \sqrt{\frac{m\omega}{2\pi\hbar \sinh(\hbar\omega\beta)}} \exp\left\{ -\frac{m\omega}{2\hbar} \frac{(x_i^2 + x_f^2)\cosh(\omega\hbar\beta) - 2x_i x_f}{\sinh(\hbar\omega\beta)} \right\}$$

$$\overset{\beta \to \infty}{\Rightarrow} \sqrt{\frac{m\omega}{2\pi\hbar\, e^{\hbar\omega\beta}}} \exp\left\{ -\frac{m\omega}{2\hbar} (x_i^2 + x_f^2) \right\}$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar}} \exp\left\{ -\frac{m\omega}{2\hbar} x_i^2 \right\} \exp\left\{ -\frac{m\omega}{2\hbar} x_f^2 \right\} e^{-\frac{\hbar\omega\beta}{2}}$$

$$E_0 = \frac{\hbar\omega}{2} \; ; \; \psi_0(x) \propto e^{-\frac{m\omega}{2\hbar} x^2}$$

$$G = \langle x_f | e^{-H(\beta - \beta')} e^{-\beta' H} | x_i \rangle$$

$$\lim_{\beta' \to \infty} e^{-\beta' H} | x_i \rangle = e^{-\beta' E_0} | 0 \rangle \psi_0(x_i)$$

$$\lim_{\beta' \to \infty} \hat{x}^n e^{-\beta' H} |x_i\rangle \propto |1\rangle$$

$$\lim_{\substack{\beta' \to \infty \\ \beta - \beta' \to \infty}} \underbrace{\langle x_f | e^{-(\beta-\beta')H}}\ \underbrace{x^n H x^n}\ \underbrace{e^{-\beta' H} |x_i\rangle} \propto \underbrace{E_1 e^{-\beta E_0}}$$

$$K[J] = \int \mathcal{D}x\, e^{\frac{i}{\hbar} \int dt \left(\mathcal{L}^0 + \mathcal{L}^1(x) + J(t) x(t) \right)}$$

$$\Rightarrow \int \mathcal{D}x\, e^{\frac{i}{\hbar} \int dt \left(\mathcal{L}^0 + \mathcal{L}^1 \right)}$$

$$= e^{\frac{i}{\hbar} \int dt\, \mathcal{L}^1 \left(\frac{\hbar}{i} \frac{\delta}{\delta J(t)} \right)} \int \mathcal{D}x\, e^{\frac{i}{\hbar} \int dt \left(\mathcal{L}^0 + J(t) x(t) \right)}$$

$$K = e^{\frac{i}{\hbar} \int dt\, \mathcal{L}^1 \left(\frac{\hbar}{i} \frac{\delta}{\delta J(t)} \right)} K^0 e^{\int dt_1 dt_2\, J(t_1) G(t_1, t_2) J(t_2)} \Big|_{J=0}$$

$$E_G = \lim_{\beta \to \infty} \frac{1}{K} \frac{\partial}{\partial \beta} K$$