$$E = \frac{p^2}{2m} + V \longleftarrow$$ only non relativistic limit

$$p = -i\hbar \frac{\partial}{\partial x}$$

$$E^2 = p^2c^2 + m^2c^4 \quad ; \quad c = 1, \hbar = 1$$

$$\boxed{E^2 = p^2 + m^2}$$

$$E = \sqrt{p^2 + m^2}$$

$$i\hbar \frac{\partial \psi}{\partial t} = \sqrt{-\hbar^2 \nabla^2 + m^2}\, \psi$$

$$\left(-\frac{\partial^2}{\partial t^2} = -\nabla^2 + m^2\right)\psi$$

$$\left(\underbrace{\frac{\partial^2}{\partial t^2} - \nabla^2}_{} + m^2\right)\psi = 0$$

$$(\Box + m^2)\,\psi = 0 \Longleftarrow$$ Klein-Gordon Eqn.

$m = 0$ case: $\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\psi = 0$

$$\psi(x,t) = e^{+\frac{i}{\hbar}mt}\, \varphi(x,t)$$

$$\frac{\partial \psi}{\partial t} = \left(\frac{i}{\hbar} m\, \varphi + \frac{\partial \varphi}{\partial t}\right) e^{\frac{i}{\hbar}mt}$$

$$\frac{\partial^2 \psi}{\partial t^2} = \left( \frac{i}{\hbar} m \frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial t^2} \right) e^{\frac{i}{\hbar} m t}$$

$$+ \frac{i}{\hbar} m \left( \frac{i}{\hbar} m \varphi + \frac{\partial \varphi}{\partial t} \right) e^{\frac{i}{\hbar} m t}$$

$$+ \hbar^2 \frac{\partial^2 \psi}{\partial t^2} = \left[ + \hbar^2 \frac{\partial^2 \varphi}{\partial t^2} - 2 i \hbar m \frac{\partial \varphi}{\partial t} - m^2 \varphi \right] e^{\frac{i}{\hbar} m t}$$

$$0 = \underline{\frac{\partial^2}{\partial t^2}} \psi - \nabla^2 \psi + m^2 \psi$$

$$2 i m \frac{\partial \varphi}{\partial t} + \hbar^2 \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = 0$$

$$i \frac{\partial \varphi}{\partial t} + \left( \frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial t^2} \right) - \frac{\nabla^2}{2m} \varphi = 0$$

$$-i \frac{\partial \varphi}{\partial t} = - \frac{\nabla^2}{2m} \varphi + \underbrace{\frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial t^2}}_{\text{negligable}}$$

---

$$\psi(\vec{r}, t) = e^{i \vec{k} \cdot \vec{r} - i E t}$$

$$(\Box + m^2) \psi = (-E^2 + p^2 + m^2) \psi = 0$$

$$E^2 = p^2 + m^2 \Rightarrow E = \mp \sqrt{p^2 + m^2}$$

$m = 0$ case

$$E = p = \sqrt{\vec{p}^2}$$

$$\left( E + \vec{\alpha} \cdot \vec{p} \right) \psi = 0$$

$$(E - p)(E + p)\psi = 0 \Rightarrow \quad p = \pm E$$

$$(E^2 - p^2)\psi = 0$$

$$\left( E - \vec{\alpha} \cdot \vec{p} \right) \left( E + \vec{\alpha} \cdot \vec{p} \right) \psi = 0$$

$$\left[ E^2 + \cancelto{0}{E(\vec{\alpha} \cdot \vec{p}) - (\vec{\alpha} \cdot \vec{p}) E} - (\vec{\alpha} \cdot \vec{p})(\vec{\alpha} \cdot \vec{p}) \right] \psi = 0$$

$$\left[ E^2 - \left( \alpha_x p_x + \alpha_y p_y + \alpha_z p_z \right)^2 \right] \psi = 0$$

$$\Big[ \left( E^2 - \alpha_x^2 p_x^2 - \alpha_y^2 p_y^2 - \alpha_z^2 p_z^2 \right)$$
$$- \left( \alpha_x \alpha_y + \alpha_y \alpha_x \right) p_x p_y$$
$$- \left( \alpha_x \alpha_z + \alpha_z \alpha_x \right) p_x p_z$$
$$- \left( \alpha_y \alpha_z + \alpha_z \alpha_y \right) p_z p_y \Big] \psi = 0$$

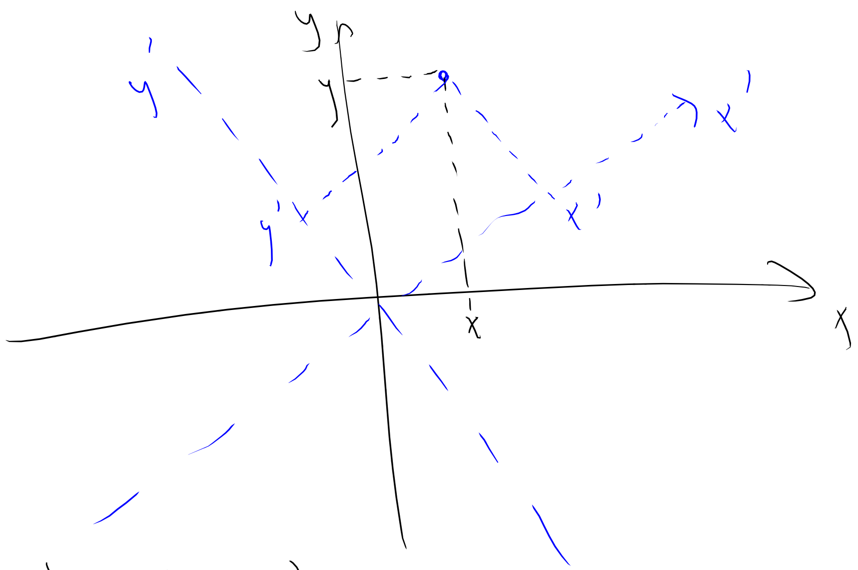
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \qquad \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$$

$$\alpha_i \equiv I\sigma_i$$

$$(E - \vec{\sigma}\cdot\vec{p})\psi_+ = 0$$

$$(E + \vec{\sigma}\cdot\vec{p})\psi_- = 0$$




scalar $(s=0)$ $\qquad \Lambda \equiv e^{i\vec{\Theta}\cdot\vec{S}} \qquad R = e^{i\vec{\Theta}\cdot\vec{L}}$

$$\psi'(x', y'; t) = R\psi(x, y, ; t) \qquad \vec{S} = 0$$

vectors $(s=1)$ $\qquad = \Lambda R \psi(x', y'; t)$

$$A'_j(x', y', t) = \Lambda_j{}^i A_i(x, y, t)$$

Rotation around z $\qquad \begin{pmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \Lambda \equiv e^{i\Theta S_z}$

$$S_z \overset{\text{in diagonal form}}{=} \begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & -1 \end{pmatrix}$$

$$\Rightarrow (E \mp \vec{\sigma}\cdot\vec{p})\,\psi_\pm = 0 \qquad \Leftarrow \text{ massless Dirac Eqn.}$$

consider a rotation

$$\Rightarrow \quad \psi'_\pm = \Lambda \psi_\pm$$

$$\Rightarrow (E \mp \vec{\sigma}\cdot\vec{p}\,')\,\psi'_\pm = 0$$

$$\Lambda = ?$$

$$\Rightarrow \Lambda = e^{i\vec{\theta}\cdot\left(\frac{\vec{\sigma}}{2}\right)}$$

$$\Rightarrow \vec{S} = \frac{\vec{\sigma}}{2} \qquad S^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)$$

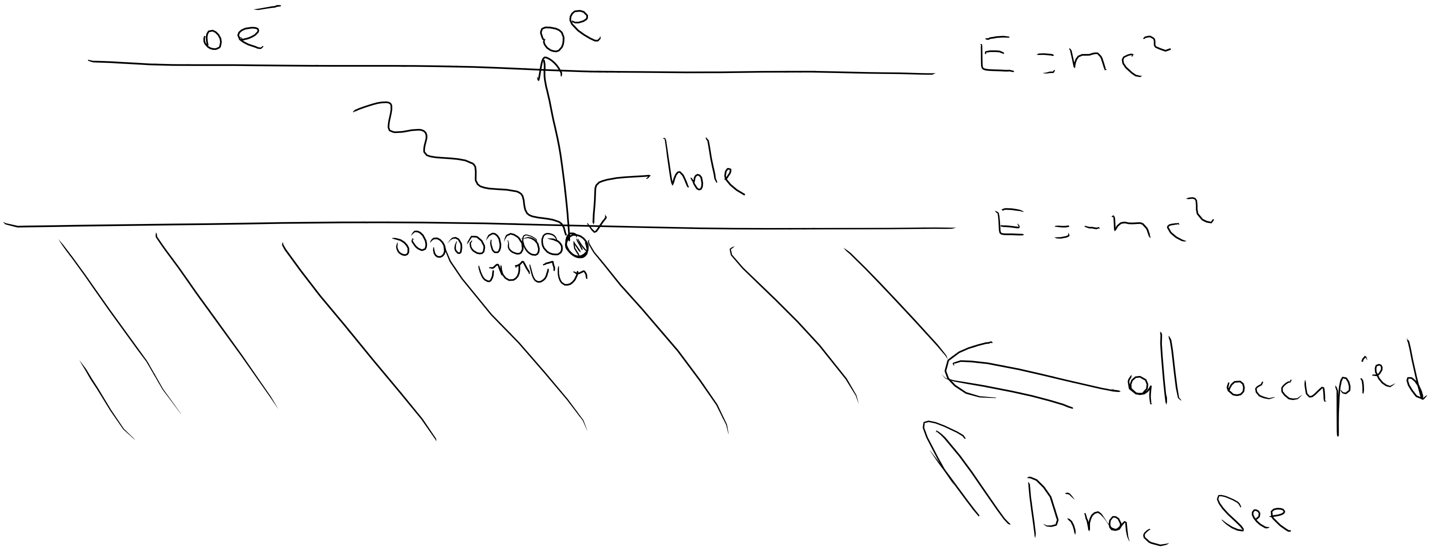
$$(E\mathbb{1} \mp \vec{\sigma}\cdot\vec{p})\,\psi = 0$$

$$\det(E\mathbb{1} \mp \vec{\sigma}\cdot\vec{p}) = 0$$

$$\det\begin{pmatrix} E \mp p_z & \pm(p_x + i p_y) \\ \pm(p_x - i p_y) & E \pm p_z \end{pmatrix} = E^2 - p_z^2 - (p_x^2 + p_y^2) = 0$$

$$\Rightarrow E^2 = \vec{p}^{\,2}$$

$$E = \pm|\vec{p}|$$



$$\psi = e^{-iEt} a + e^{iEt} b^{+}$$

wave function

$\Downarrow$ "second quantization"

$\psi$ field operator

$a; b$: annihilation operators.

$a$: annihitates electron lowering the energy by $E$

$b^{+}$: creates positron increasing the energy by $E$

$a$ & $b^{+}$ increase the charge by $+1$

$$(E \mp \vec{\sigma} \cdot \vec{p})\psi_{\pm} = 0$$

$$E\psi_{\pm} = \pm\vec{\sigma} \cdot \vec{p}\,\psi_{\pm}$$

$E = |\vec{p}|$ for positive energy solutions (particles)

$E = -|\vec{p}|$ for negative energy solutions (anti particles.

for particles

$$\psi_{\pm} = \pm\vec{\sigma} \cdot \hat{p}\,\psi_{\pm} \Rightarrow \vec{\sigma} \cdot \hat{p}\,\psi_{\pm} = \pm\psi_{\pm} \Leftarrow$$

$\vec{\sigma} \cdot \hat{p}$ : spin projection on the direction of momentum.

$\vec{\sigma} \cdot \hat{p}$ = helicity operator for anti particles

$\psi_\pm$ : spinors
$\sigma$ : Pauli spin matrices.

$$\vec{\sigma}\cdot\hat{p}\,\psi_\pm = \mp\psi_\pm$$

$$(E + \vec{\alpha}\cdot\vec{p} + \beta m)\psi = 0$$

$$(E - \vec{\alpha}\cdot\vec{p} - \beta m)$$

$$\left[E^2 - (\vec{\alpha}\cdot\vec{p} + \beta m)^2\right]\psi = 0$$

$$i \neq j$$

$$\Big[E^2 - \alpha_i^2 p_i^2 - \beta^2 m^2 - (\alpha_i\alpha_j + \alpha_j\alpha_i)p_i p_j - (\alpha_i\beta + \beta\alpha_i)p_i m\Big]\psi = 0$$

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$$

$$\beta^2 = 1$$

$$(\alpha_i\beta + \beta\alpha_i) = 0 \qquad \mathrm{Tr}\,AB = \mathrm{Tr}\,BA$$

$$\mathrm{Tr}\,\alpha_i = \mathrm{Tr}\,\beta\beta\alpha_i = \mathrm{Tr}\,\beta\alpha_i\beta$$

$$= -\mathrm{Tr}\,\underbrace{\beta\beta}_{1}\alpha_i$$

$$= -\mathrm{Tr}\,\alpha_i$$

$$\Rightarrow \mathrm{Tr}\,\alpha_i = 0$$

$$\beta \equiv \gamma^0 \qquad \{\gamma^0, \vec{\gamma}\} : \text{Dirac Matrices.}$$

$$\beta\vec{\alpha} \equiv \vec{\gamma}$$