Ex

$$\vec{E}(\vec{r}) = \begin{cases} \frac{q}{r^2}\hat{r} & r > R \\ 0 & r < R \end{cases} = \left(\frac{q}{r^2}\hat{r}\right)\Theta(r-R)$$

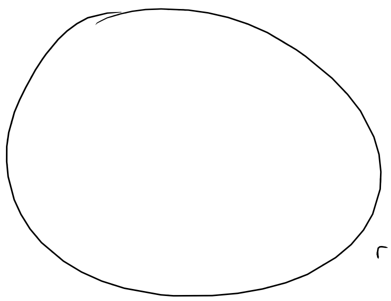
$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho(\vec{r})$$

$$\vec{\nabla}\left(\frac{\hat{r}}{r^2}\right) = 0 \qquad r > R$$

$$\nexists\ \vec{\nabla}\cdot\vec{E} = 0 \qquad r = R$$

$$\Delta(E^{\perp}) = 4\pi\sigma \Leftarrow$$

$$\Delta(\vec{E}^{\parallel}) = 0$$



$r = R$

$$\vec{E}^{\parallel} = 0$$

$$E^{\perp} = \frac{q}{r^2}\Theta(r-R)$$

$$\Delta E^{\perp} = \frac{q}{R^2} = 4\pi\sigma$$

$$\boxed{\sigma = \frac{q}{4\pi R^2}}$$

Ex

$$\vec{E}(\vec{r}) = \begin{cases} \left(\frac{1}{2}\right) q \frac{r}{R^3}\hat{r} & r < R \\ \frac{q}{r^2}\hat{r} & r > R \end{cases} \qquad \Delta\vec{E} = 0$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} + \dots \begin{cases} \left(\frac{1}{2}\right)\frac{3q}{R^3} & r < R \\ 0 & r > R \end{cases} = 4\pi\rho(\vec{r})$$

$$\rho(\vec r) = \left(\frac{1}{2}\right)\frac{3q}{4\pi R^3}\,\Theta(R-r)$$

$$\boxed{\rho(\vec r) = \frac{(q/2)}{\frac{4\pi}{3}R^3}\,\Theta(R-r)} + (\text{surface charge})$$

surface charge density $\sigma = \dfrac{q/2}{4\pi R^2}$ (check)

$$\rho = \sigma\,\delta(r-R)$$

Ex $\rho(\vec r) = q\,\delta^{(3)}(\vec r - \vec R)$

$$= \frac{q}{r^2}\,\delta(r)\quad \delta\left(\theta-\frac{\pi}{2}\right)\delta(\phi-\phi_0)$$

$$\int_0^\infty dr\,\delta(r) \equiv 1$$

in general

$$\int_a^{b\,(>a)} dr\,\delta(x-a) \qquad\qquad \int_{a-\varepsilon}^{a+\varepsilon} dx\,\delta(x-a)$$

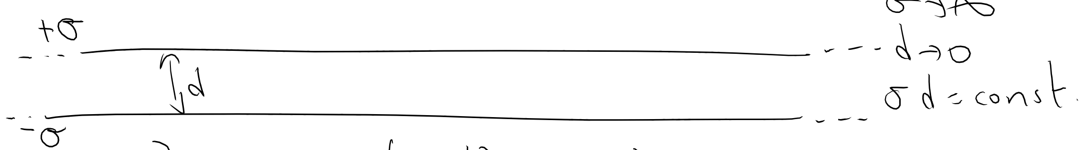
$$\delta(x-a) = \begin{cases} \\ \end{cases} \qquad\qquad \int_{a+\varepsilon}^{b} dx\,\delta(x-a)$$

Question

$\cdot\,\vec r_0$

$\vec E = 0$

$\lim_{\substack{\sigma\to\infty \\ d\to 0 \\ \sigma d = \text{const.}}}$



$\vec E, V \propto (\sigma d)$ $\quad\cdot\,\vec r_1$ $\quad\vec E = 0$

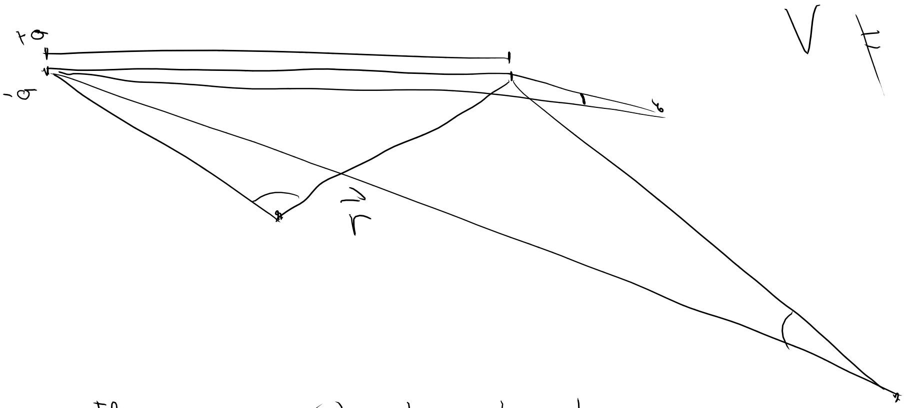
$$V(\vec{r}) = \int D \, d\Omega$$

$$D = \lim_{\substack{d \to 0 \\ \sigma \to \infty}} (\sigma d)$$

$$\to V(\vec{r}_0) - V(\vec{r}_1) = 4\pi D$$

$$\vec{E}(\vec{r}_0) = -\vec{\nabla}_{r_0} \left[V(\vec{r}_0) - V(\vec{r}_1) \right] = 0$$

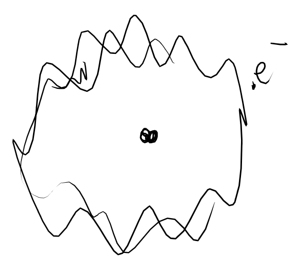
For a finite plate

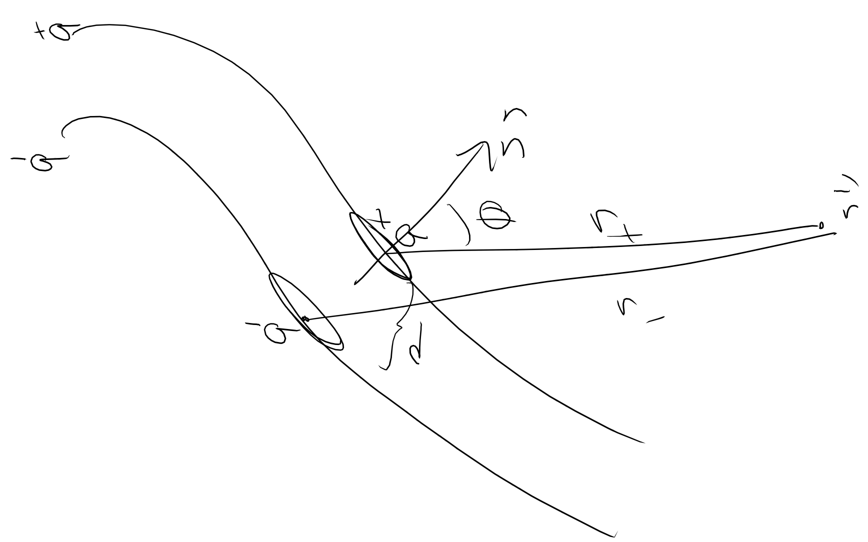


$$V \propto \frac{\vec{p} \cdot \hat{r}}{r^3}$$

$\lim_{\substack{q \to \infty \\ d \to 0 \\ qd = \text{constant}}}$ $+q$, $-q$, d : ideal dipole

Bare Coulomb Potential vs. dressed / screened Coul. Pot.





$$dV(\vec{r}) = \frac{\sigma\, dA}{r_+} + \frac{(-\sigma\, dA)}{r_-} \leftarrow$$

$$\boxed{\vec{r}_- = \vec{r}_+ - \hat{n}\, d}$$

$r_- \to r_+$

$d \ll r_-, r_+$

$$r_-^2 = r_+^2 + d^2 - 2 r_+ d \cos\theta$$

$$r_- = r_+ \left[1 + \underbrace{\left(\frac{d}{r_+}\right)^2 - 2\frac{d}{r_+}\cos\theta}_{\ll 1}\right]^{1/2}$$

$(1+x)^n \simeq 1 + nx + O(x^2)$

$$r_- \simeq r_+ \left[1 - \frac{d}{r_+}\cos\theta + O\left(\left(\frac{d}{r_+}\right)^2\right)\right]$$

$$r_- \simeq r_+ - d\cos\theta$$

$$dV = \sigma\, dA \left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$= \sigma\, dA \left[r_+^{-1} - (r_+ - d\cos\theta)^{-1}\right]$$

$$= \frac{\sigma\, dA}{r_+}\left[1 - \left(1 - \frac{d}{r_+}\cos\theta\right)^{-1}\right]$$

$$dV \approx \frac{\sigma\, dA}{r_+}\left[\cancel{1} - \left(\cancel{1} + \frac{d}{r_+}\cos\theta\right)\right]$$

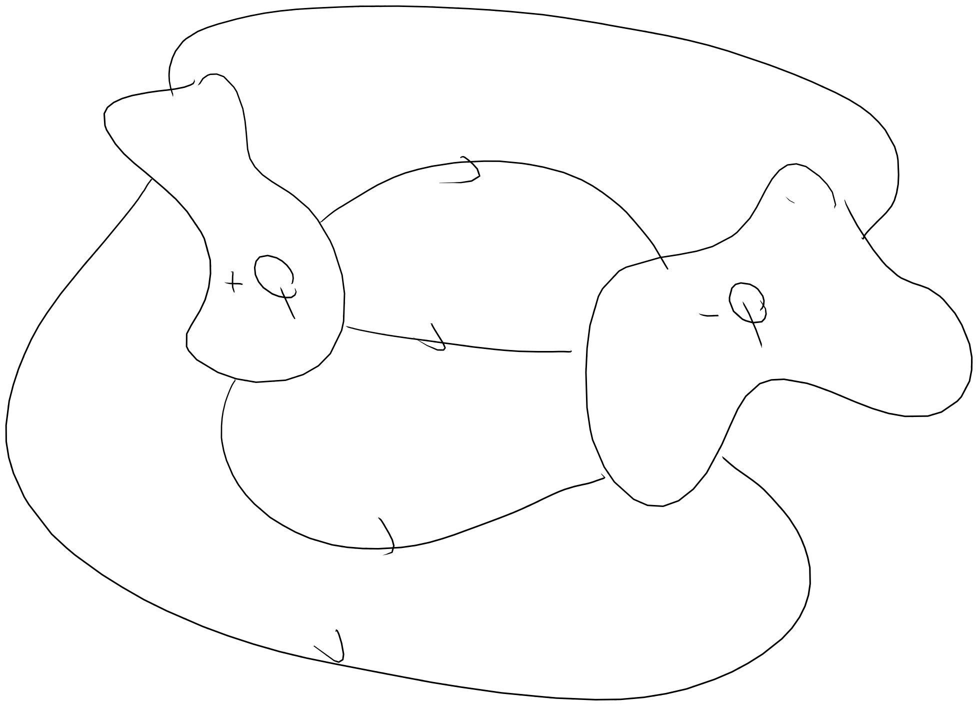
$$dV \simeq \frac{\sigma\, dA}{r_+}\left(-\frac{d}{r_+}\cos\theta\right)$$

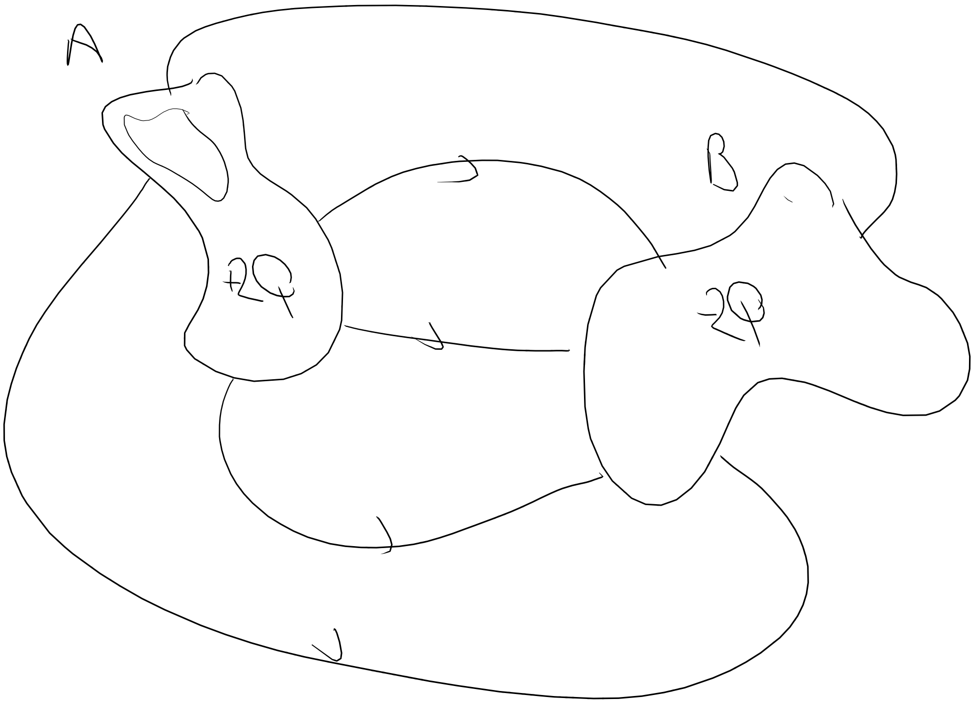
$$= -\frac{\sigma\, dA}{r_+^2}\underbrace{\left(d\, r_+ \cos\theta\right)}_{d\,\hat{n}\cdot\hat{r}}$$

$$\boxed{dV = -\frac{(\sigma d)(dA)}{r_+^2}\,\hat{n}\cdot\hat{r}} \qquad \sigma d \equiv D$$

$$dV = -D\,\frac{(dA)}{r_+^2}\,\hat{n}\cdot\hat{r}$$

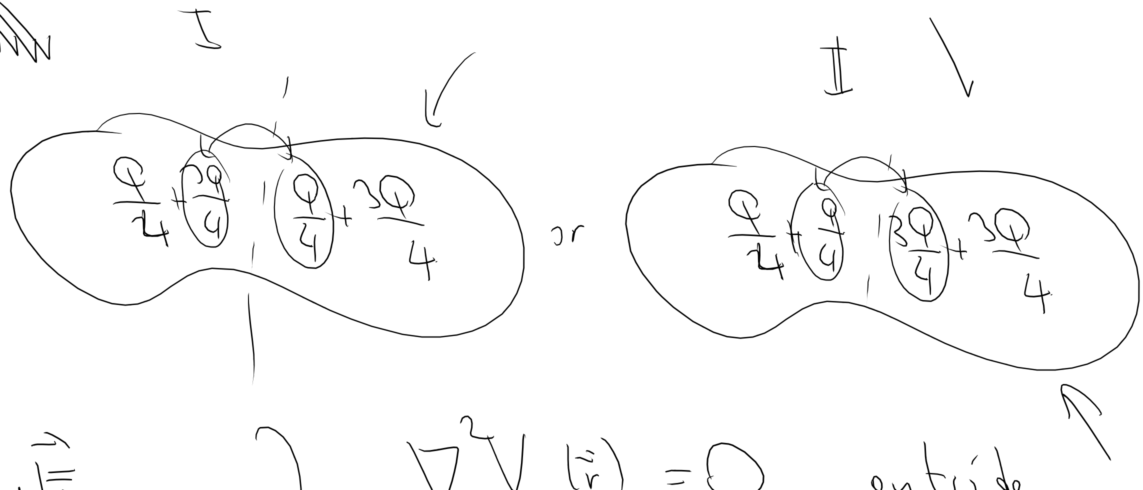
Capacitance





$$\rho_1(\vec{r}) = Q\,\tilde{\rho}(\vec{r}) \qquad \mp 1 = \int \tilde{\rho}(\vec{r})\, dV = \int \tilde{\rho}'(\vec{r}')\, dV$$

$$\rho_2(\vec{r}) = 2Q\,\tilde{\rho}'(\vec{r}')$$



$$\left.\begin{array}{l} \vec{\nabla}\cdot\vec{E}_Q \\ \vec{\nabla}\times\vec{E}_Q = 0 \end{array}\right\} \quad \nabla^2 V_Q(\vec{r}) = 0$$ outside the conductors.

$\Delta V_Q = 0$ on the surfaces

$\vec{\nabla} V_Q$ is perpendicular to the surfaces. and is zero inside the conductor

$V_Q \to 0$ at infinity

$$\oint (\vec{\nabla} V_Q)\cdot d\vec{S} = -4\pi Q$$

$$\begin{cases} \nabla^2 V_{2Q}(\vec{r}) = 0 & \text{outside conductors} \\ \Delta V_{2Q} = 0 & \text{on the surfaces} \\ \vec{\nabla} V_{2Q} \text{ is perpendicular to the surfaces and is zero inside the conductor} \\ \oint (\vec{\nabla} V_{2Q}) \cdot d\vec{S} = -4\pi (2Q) \\ V_{2Q} \to 0 \text{ at infinity} \end{cases}$$

$$\boxed{V_{2Q} = 2 V_Q}$$

$$V_Q(\vec{r}) = Q\, f(\vec{r})$$

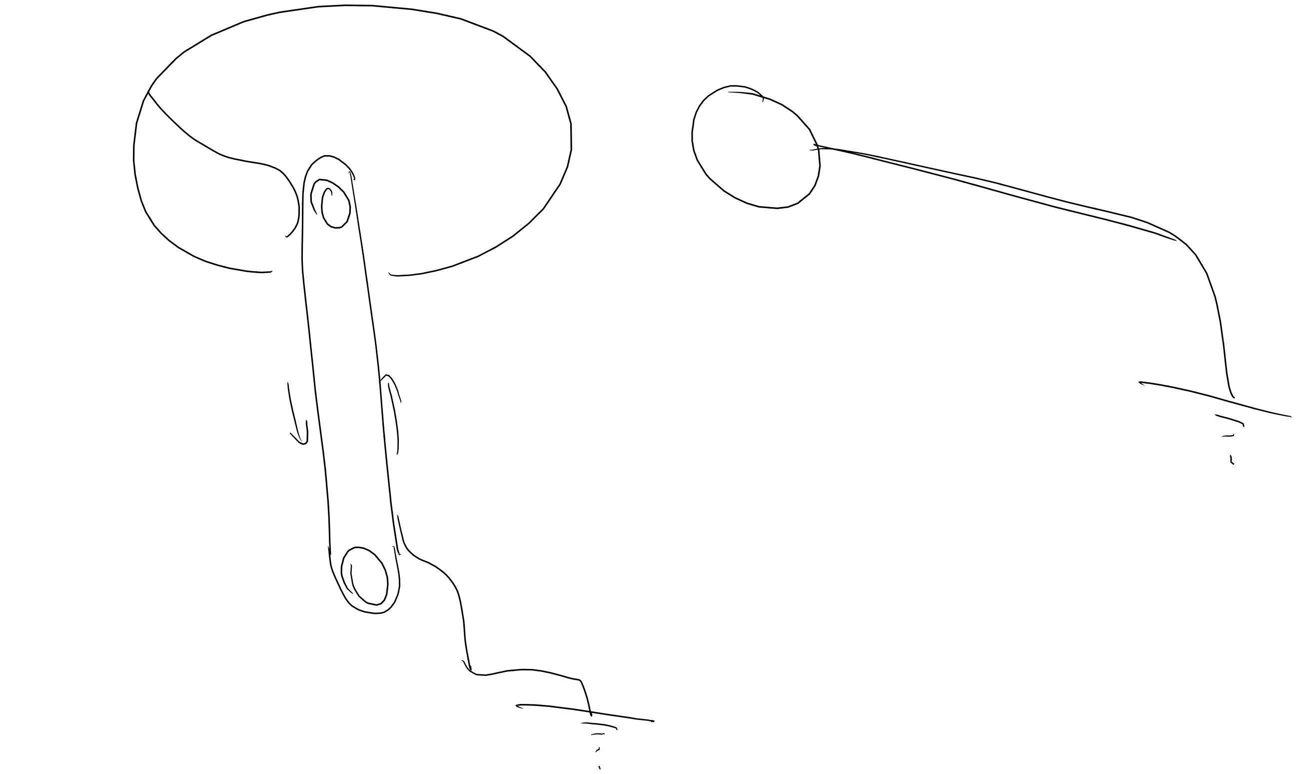
$f(\vec{r})$ depends <u>only</u> on the geometry of the problem.

$$\boxed{\Delta V = \frac{Q}{C}} \Longleftarrow \qquad C: \boxed{\text{capacitance}}$$

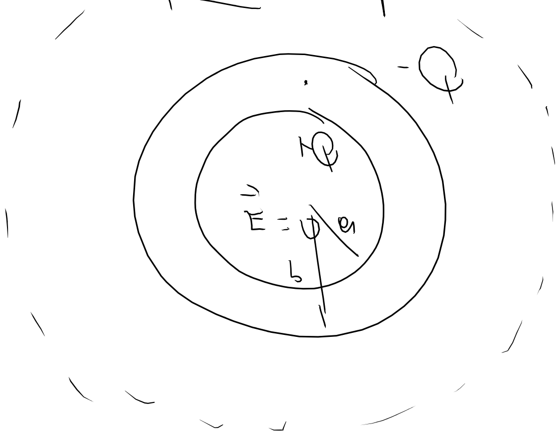
$$\frac{1}{C} = \Delta f(\vec{r}) \equiv f(\vec{r}_+) - f(\vec{r}_-)$$

$\vec{r}_\pm$ arbitrary points on the $\pm$ conductor.

Van de Graaf Generators



Example — Spherical Capacitor



$$\vec{E}(\vec{r}) = \frac{Q}{r^2}\hat{r}$$

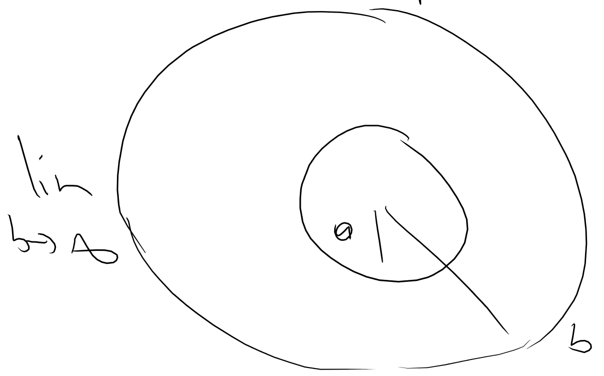
$$\vec{E} = 0 \qquad V(r) = \frac{Q}{r}$$

$$\Delta V = \frac{Q}{a} - \frac{Q}{b} \equiv \frac{Q}{C}$$

$$\frac{1}{C} = \frac{1}{a} - \frac{1}{b}$$

$$\boxed{C = \frac{ab}{b-a}}$$

Ex Capacitance of a conducting sphere



$$C = \lim_{b\to\infty} \frac{ab}{b-a} = a$$

$$\Delta V = \frac{Q}{C}$$

$[Q] \neq C$

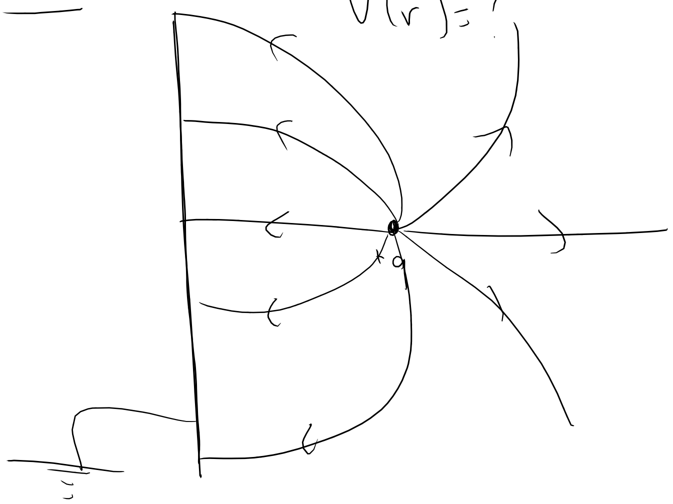
$[Q] = \text{stat}C = \sqrt{Nm^2}$

$$[\Delta V\, Q] = \left[\frac{Q^2}{C}\right]$$

$$Nm = \frac{Nm^2}{[C]} \Rightarrow [C] = \text{length}$$

Method of Images

Ex



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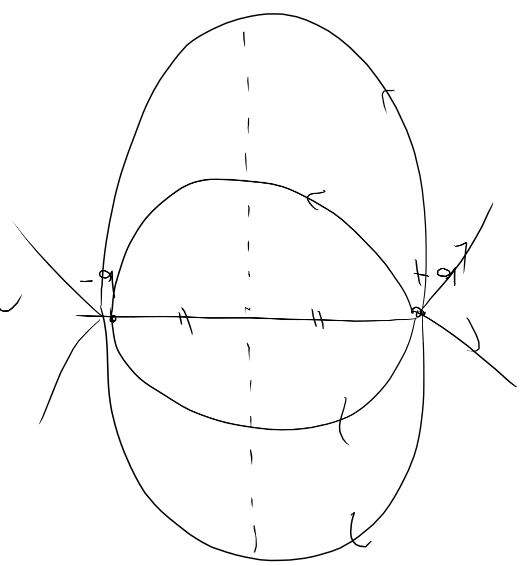

$$V_{tr} = \frac{q}{r_+} - \frac{q}{r_-}$$

$r_+ = |\vec{r} - \vec{r}_+|$
$r_- = |\vec{r} - \vec{r}_-|$

$$\nabla^2 V = -4\pi \rho(\vec{r}) = -4\pi \delta^{(3)}(\vec{r} - \vec{r}_+)$$

$$\nabla^2 V_{tr} = -4\pi \delta^{(3)}(\vec{r} - \vec{r}_+) - 4\pi \delta^{(3)}(\vec{r} - \vec{r}_-)$$

in our volume $\vec{r} \neq \vec{r}_- \Rightarrow \delta^{(3)}(\vec{r} - \vec{r}_-) = 0$

V_{tr}