$$\vec{V} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$$

$$a\hat{x} + b\hat{y} + c\hat{z} = 0 \Rightarrow a = b = c = 0$$

$$\vec{V} \cdot \vec{V}' = V V' \cos\theta$$

$$= V_x V'_x + V_y V'_y + V_z V'_z$$

$$\{ f_n(x) \, ; \, x \in D \} \, , \quad \langle f, g \rangle = \int dx \, f^*(x) g(x) w(x)$$

Ex polynomials: $\{x^n\}$ : orthonormal basis.

$$f(x) = \sum_n a_n x^n$$

$$\langle f(x), g(x) \rangle = \sum_n a_n^* b_n$$

$$f(x) = \sum_n a_n x^n \, ; \quad g(x) = \sum_n b_n x^n$$

Ex Polynomials defined in the interval $-1 \leq x \leq 1$

Define the inner product by $\langle f, g \rangle = \int_{-1}^{1} dx \, f(x) g(x)$

$$\boxed{f_0(x) = \frac{1}{\sqrt{2}}} \qquad \langle f_0, f_0 \rangle = \int_{-1}^{1} dx \, \frac{1}{2} = 1$$

$$f_1(x) = a x$$

$$\langle f_0, f_1 \rangle = \int_{-1}^{1} dx \, \frac{1}{\sqrt{2}} a x = 0$$

$$\langle f_1, f_1 \rangle = \int_{-1}^{1} dx \, a^2 x^2 = \frac{2a^2}{3} = 1$$

$$a = \sqrt{\frac{3}{2}}$$

$$\boxed{f_1(x) = \sqrt{\frac{3}{2}}\, x}$$

$$f_2(x) = ax^2 + b \qquad \langle f_2, f_0\rangle = \int_{-1}^{1} dx\,(ax^2)\,\frac{1}{\sqrt{2}} \neq 0$$

$$\langle f_2, f_1\rangle = \int_{-1}^{1} dx\,(ax^2+b)\sqrt{\frac{3}{2}}\,x = 0$$

$$\int_{-1}^{1} (ax^2)\,\frac{1}{\sqrt{2}}\,dx = \frac{1}{\sqrt{2}}\,a\,\frac{2}{3}$$

$$f_2(x) = ax^2 - \frac{\sqrt{2}}{3}\,a\,\frac{1}{\sqrt{2}} = a\left(x^2 - \frac{1}{3}\right)$$

$$\langle f_2, f_0\rangle = a\int_{-1}^{1} dx\left(x^2 - \frac{1}{3}\right)\frac{1}{\sqrt{2}} = \frac{a}{\sqrt{2}}\left(\frac{x^3}{3} - \frac{1}{3}x\right)\Bigg|_{x=-1}^{1}$$

$$= \frac{a}{\sqrt{2}}\left[\left(\frac{1}{3} - \frac{1}{3}\right) - \left(-\frac{1}{3} + \frac{1}{3}\right)\right] = 0$$

$$\langle f_2, f_2\rangle = a^2 \int_{-1}^{1} dx\left(x^2 - \frac{1}{3}\right)^2$$

$$= a^2 \int_{-1}^{1} dx\left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)$$

$$= 2a^2\left(\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\right)$$

$$= 2a^2\,\frac{9-5}{45} = \frac{8}{45}a^2 = 1 \Rightarrow a = \sqrt{\frac{45}{8}} = \frac{3\sqrt{5}}{2\sqrt{2}}$$

$$\boxed{f_2(x) = \frac{3}{2}\sqrt{\frac{5}{2}}\left(x^2 - \frac{1}{3}\right)}$$

$$f_3(x) = a x^3 - \underset{=0}{\langle a x^3, f_2 \rangle} f_2 - \langle a x^3, f_1 \rangle f_1 - \underset{=0}{\langle a x^3, f_0 \rangle} f_0$$

$$f_3(x) = a x^3 - \langle a x^3, f_1 \rangle f_1$$

$$\langle f_3, f_3 \rangle = 1 \Rightarrow a = \ldots$$

Gramm- Schmidt orthogonalization

$$\underset{-1 \leq x \leq 1}{f(x)} = \sum_{n=0}^{\infty} a_n f_n(x)$$

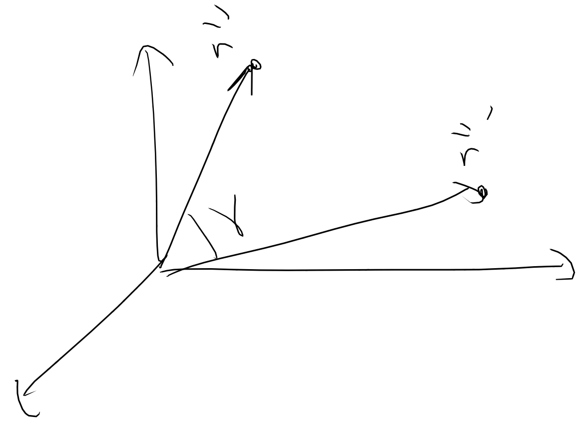
$$\langle f_m, f \rangle = \sum_{n=0}^{\infty} a_n \underbrace{\langle f_m, f_n \rangle}_{\delta_{nm}} = a_m$$

<u>Ex</u> Periodic functions with period $L$

$$\langle f, g \rangle = \int_0^L dx\, f(x)\, g(x)$$

$$\left\{ \sin\left(\frac{2n\pi x}{L}\right), \cos\left(\frac{2n\pi x}{L}\right) \right\}$$

$$\phi(\vec r) = \int \rho(\vec r')\,\frac{1}{|\vec r - \vec r'|}\,d^3 r'$$




$$|\vec r - \vec r'| = \sqrt{(\vec r - \vec r')^2}$$

$$= \sqrt{r^2 + r'^2 - 2rr'\cos\gamma}$$

$$r_> = \max(r, r')$$

$$r_< = \min(r, r')$$

$$\frac{1}{|\vec r - \vec r'|} = \frac{1}{\sqrt{r_>^2 + r_<^2 - 2r_< r_> \cos\gamma}}$$

$$= \frac{1}{r_>}\left(1 + x^2 - 2x\cos\gamma\right)^{-1/2}$$

$$x = \frac{r_<}{r_>} \leq 1 \quad ; \quad z \equiv \cos\gamma$$

$$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \dots$$

$$\frac{1}{|\vec r - \vec r'|} = \frac{1}{r_>}\Big(1 + \underbrace{x^2 - 2xz}_{<1}\Big)^{-1/2}$$

$$= \frac{1}{r_>}\left[1 + \left(-\tfrac{1}{2}\right)(x^2 - 2xz) + \left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\frac{(x^2 - 2xz)^2}{2!} + O(x^3)\right]$$

$$= \frac{1}{r_>}\left[1 - \frac{x^2}{2} + xz + \frac{3}{4}\,\frac{(-2zx)^2}{2} + O(x^3)\right]$$

$$= \frac{1}{r_>}\left[1 + xz + x^2\left(-\tfrac{1}{2} + \tfrac{3}{2}z^2\right) + O(x^3)\right]$$

$$\frac{1}{|\vec r - \vec r'|} = \frac{1}{r_>}\left[1 + xz + x^2\,\tfrac{3}{2}\left(z^2 - \tfrac{1}{3}\right) + O(x^3)\right]$$

$$f_0(x) = \frac{1}{\sqrt{2}} x$$

$$P_0(z) \equiv 1$$

$$P_1(z) \equiv z$$

$$f_1(x) = \sqrt{\frac{3}{2}}\, x$$

$$P_2(z) \equiv \frac{3}{2}\left(z^2 - \frac{1}{3}\right)$$

$$P_n(z) \equiv \dots$$

$$f_2(x) = \frac{3}{2}\sqrt{\frac{5}{2}}\left(x^2 - \frac{1}{3}\right)$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r_>} \sum_{n=0}^{\infty} x^n P_n(z) = \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+1}} P_n(\cos\gamma)$$

$$\vec{r} \cdot \vec{r}' = r r' \cos\gamma$$

$$= rr' \big[\cos\theta \cos\theta' + \sin\theta \cos\phi \sin\theta' \cos\phi' + \sin\theta \sin\theta' \sin\phi \sin\phi'\big]$$

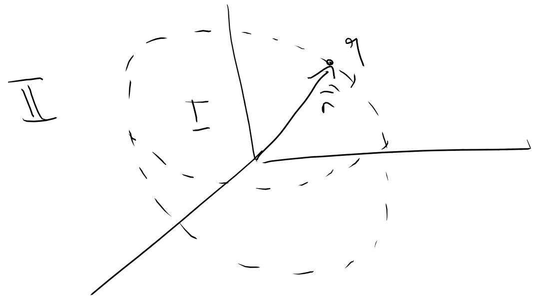
$$= rr' \big[\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi')\big]$$

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi')$$

$$\frac{1}{\sqrt{1 + x^2 - 2xz}} \equiv \sum_{n=0}^{\infty} x^n P_n(z)$$

Exercise Calculate the potential on a point charge using separation of variables in spherical coordinates.

$$\nabla^2 \phi = -4\pi q\, \delta^{(3)}(\vec{r} - \vec{r}\,')$$




if $r < r'$ or $r > r'$

$$\nabla^2 \phi = 0$$

$$\sigma = \frac{q}{R^2}\, \delta(\phi - \phi')\, \delta(\cos\theta - \cos\theta')$$

$$q = \int \sigma \, dS = \int \frac{q}{\cancel{R^2}}\, \delta(\phi - \phi')\, \delta(\cos\theta - \cos\theta')\, \cancel{R^2}\, d\phi\, d(\cos\theta)$$

$$= q \checkmark$$

$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2}$$

$$z = \cos\theta \qquad -1 \leq z \leq 1$$

$$\frac{\partial}{\partial \theta} = -\sin\theta \frac{\partial}{\partial z}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) = \frac{1}{\cancel{\sin\theta}}\left(+\cancel{\sin\theta}\frac{\partial}{\partial z}\right)\left(\sin\theta(+\sin\theta)\frac{\partial \Phi}{\partial z}\right)$$

$$= \frac{\partial}{\partial z}\left[(1 - z^2)\frac{\partial \Phi}{\partial z}\right]$$

$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial z}\left[(1-z^2)\frac{\partial \Phi}{\partial z}\right]$$

$$+ \frac{1}{r^2(1-z^2)}\frac{\partial^2 \Phi}{\partial \phi^2}$$

$$\Phi = R(r)\,\varphi(\phi)\,Z(z) \Leftarrow$$

$$\frac{\nabla^2 \Phi}{\Phi} = \frac{1}{R}\frac{1}{r^2}\frac{d}{dr}\left[r^2 R'\right] + \frac{1}{r^2}\frac{1}{Z}\frac{d}{dz}\left[(1-z^2)\,Z'\right]$$

$$+ \frac{1}{r^2}\frac{1}{(1-z^2)}\underbrace{\frac{\varphi''}{\varphi}} = 0$$

$$\frac{\varphi''}{\varphi} = -m^2 \Rightarrow \boxed{\varphi = e^{im\phi}} \quad m = 0, \mp 1, \mp 2, \ldots$$

$$\frac{1}{R}\frac{1}{r^2}\frac{d}{dr}\left[r^2 R'\right] + \frac{1}{r^2}\underbrace{\left\{\frac{1}{Z}\frac{d}{dz}\left[(1-z^2)Z'\right] - \frac{m^2}{(1-z^2)}\right\}}_{\text{depends only on } z} = 0$$

$$\frac{1}{Z}\frac{d}{dz}\left[(1-z^2)Z'\right] - \frac{m^2}{(1-z^2)} = -l(l+1)$$

$$\frac{d}{dz}\left[(1-z^2)Z'\right] + \left[l(l+1) - \frac{m^2}{(1-z^2)}\right]Z = 0$$

Consider $m=0$ case.

$$\frac{d}{dz}\left[(1-z^2)Z'\right]+l(l+1)Z=0$$

if $l=0$ $\quad \frac{d}{dz}\left((1-z^2)Z'\right)=0$

$$(1-z^2)Z'=C$$

$$Z'=\frac{C_1}{1-z^2}=\frac{C_1}{2}\left(\frac{1}{1-z}+\frac{1}{1+z}\right)$$

$$Z_{l=0}=\frac{C_1}{2}\ln\left(\frac{1+z}{1-z}\right)+C_2$$

$Z(z=\mp1)$ should be finite $\Rightarrow C_1=0$

$$Z_{l=0}=\text{const}\propto P_0(z)$$

$l=1$

$$\frac{d}{dz}\left[(1-z^2)Z'\right]+2Z=0$$

$$Z\overset{?}{=}az\propto P_1(z)$$

$$\frac{d}{dz}\left[(1-z^2)a\right]+2Z\overset{?}{=}0$$

$$=-2za+2az=0\quad\checkmark$$

$l=2$

$$\frac{d}{dz}\left[(1-z^2)Z'\right]+6Z=0$$

$$Z=a\left(z^2-\frac{1}{3}\right)\propto P_2(z)$$

$$\frac{d}{dz}\left[(1-z^2)\,2az\right] + 6a\left(z^2-\frac{1}{3}\right)$$

$$= 2a\,\frac{d}{dz}\left[z-z^3\right] + 6a\left(z^2-\frac{1}{3}\right)$$

$$= 2a\left(1-3z^2\right) + 6a\left(z^2-\frac{1}{3}\right) = 0 \checkmark$$

## General Case

$$\frac{d}{dz}\left[(1-z^2)Z'\right] + \left[l(l+1) - \frac{m^2}{(1-z^2)}\right]Z = 0$$

$$(1-z^2)Z'' - 2zZ' + \left[l(l+1) - \frac{m^2}{(1-z^2)}\right]Z = 0$$

$$Z(z \sim \mp 1) = (z \pm 1)^{\nu}$$

$$(1-z^2)\,\nu(\nu-1)\,(z\pm 1)^{\nu-2} - 2z\nu\,(z\pm 1)^{\nu-1}$$

$$+ \left[l(l+1) - \frac{m^2}{(1-z^2)}\right](z\pm 1)^{\nu} = 0$$

$$1-z^2 = (1-z)(1+z) = 2(1\mp z) = \pm 2(z\pm 1)$$

$$\pm 2\nu(\nu-1)\,(z\pm 1)^{\nu-1} \pm 2\nu\,(z\pm 1)^{\nu-1} + \underbrace{\left[l(l+1) \pm \frac{m^2}{2(z\pm 1)}\right]}(z\pm 1)^{\nu} = 0$$

$$\pm 2\nu(\nu-1) \pm 2\nu \pm \frac{m^2}{2} = 0$$

$$2\nu^2 \pm \frac{m^2}{2} = 0$$

For $m = 0$ $\quad \nu = 0 \longrightarrow$ $Z(z) = \text{const}$

$$Z(z) = \ln\frac{1+z}{1-z} + \text{const}$$

$$z = \pm 1$$

$$Z(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$Z(z) = P_\ell^m(z) \qquad -\ell \leq m \leq \ell$$

$$\Phi = R(r)\, P_\ell^m(z)\, e^{im\phi}$$

$$Y_{\ell m} = (\ \ )\, P_\ell^m(z)\, e^{im\phi}$$

$$\int Y_{\ell m}^*(\theta,\varphi)\, Y_{\ell' m'}(\theta,\phi)\, d(\cos\theta)\, d\phi = \delta_{\ell\ell'}\delta_{mm'}$$

$$\frac{1}{R}\,\frac{1}{r^2}\frac{d}{dr}\left[r^2 R'\right] + \frac{1}{r^2}(-\ell)(\ell+1) = 0$$

$$\left[2rR' + r^2R''\right] - \ell(\ell+1)R = 0$$

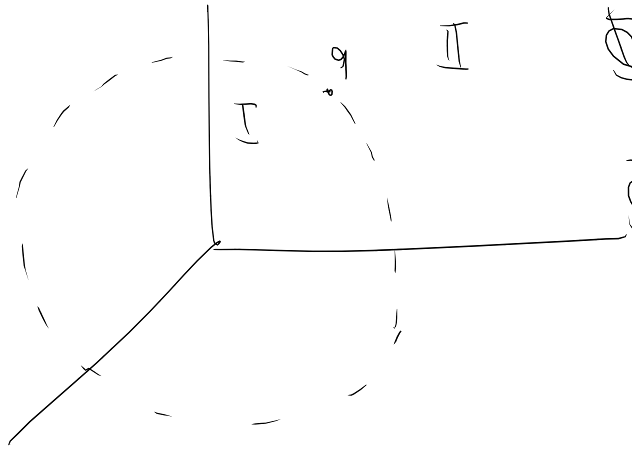
$$r^2 R'' + 2rR' - \ell(\ell+1)R = 0$$

$$R = r^n$$

$$n(n-1)\cancel{r^n} + 2n\cancel{r^n} - \ell(\ell+1)\cancel{r^n} = 0 \Rightarrow n(n+1) - \ell(\ell+1) = 0$$

$$n = \ell \qquad n = -(\ell+1)$$

$$\boxed{\Phi(\vec{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[a_{lm} r^{l} + b_{lm} r^{-(l+1)}\right] Y_{lm}(\Omega)}$$




$$\Phi^{I}(\vec{r}) = \sum_{lm} a_{lm} r^{l} Y_{lm}(\Omega)$$

$$\Phi^{II}(\vec{r}) = \sum_{lm} b_{lm} \frac{1}{r^{l+1}} Y_{lm}(\Omega)$$

on the boundary $\Phi^{I} \overset{r=R}{=} \Phi^{II}$

$$\sum a_{lm} R^{l} Y_{lm}(\Omega) = \sum b_{lm} \frac{1}{R^{l+1}} Y_{lm}(\Omega)$$

$$\boxed{a_{lm} R^{l} = b_{lm} \frac{1}{R^{l+1}}}$$

on the boundary $\Delta\vec{E} = 4\pi\sigma\,\hat{n}$

$$E_r = -\frac{\partial\phi}{\partial r}$$

$$E_r^{I} \underset{r=R}{=} -\sum_{lm} a_{lm}\, l\, R^{l-1}\, Y_{lm}(\Omega)$$

$$E_r^{II} \underset{r=R}{=} \sum_{lm} b_{lm} \frac{(l+1)}{R^{l+2}} Y_{lm}(\Omega)$$

$$\hat{r}\cdot\Delta\vec{E} = E_r^{II} - E_r^{I} = \sum_{lm}\left(\frac{b_{lm}(l+1)}{R^{l+2}} + a_{lm}\, l\, R^{l-1}\right) Y_{lm}(\Omega)$$

$$= 4\pi \frac{q}{R^2}\delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta')$$

multiply both sites by $Y^*_{l'm'}(\Omega)$ and integrate over $\Omega$

$$\int Y^*_{l'm'}(\Omega) Y_{lm}(\Omega)\, d\Omega = \delta_{ll'}\delta_{mm'}$$

$$\Rightarrow \frac{b_{l'm'}(l'+1)}{R^{l+2}} + a_{l'm'}\, l' R^{l'-1} = 4\pi \frac{q}{R^2} Y^*_{l'm'}(\Omega')$$

$$b_{lm}\frac{l+1}{R^{l+1}} + a_{lm}\, l R^{l} = \frac{4\pi}{R} q Y^*_{lm}(\Omega')$$

$$\Rightarrow a_{lm} R^{l} = b_{lm}\frac{1}{R^{l+1}}$$

$$b_{lm}\frac{l+1}{R^{l+1}} + \frac{b_{lm}}{R^{l+1}} l = \frac{4\pi}{R} q Y^*_{lm}(\Omega')$$

$$\boxed{b_{lm} = 4\pi q R^{l} \frac{Y^*_{lm}(\Omega')}{2l+1}}$$

$$a_{lm} = \frac{1}{R^{2l+1}} b_{lm} = 4\pi q \frac{1}{R^{l+1}} \frac{Y^*_{lm}(\Omega')}{2l+1}$$

$$\Phi^{I}(\vec{r}) = \sum_{lm} a_{lm} r^{l} Y_{lm}(\Omega) = q\sum_{lm}\frac{4\pi}{2l+1}\frac{r^{l}}{R^{l+1}} Y^*_{lm}(\Omega') Y_{lm}(\Omega)$$

$$\Phi^{II}(\vec{r}) = \sum_{lm} b_{lm}\frac{1}{r^{l+1}} Y_{lm}(\Omega) = q\sum_{lm}\frac{4\pi}{2l+1}\frac{R^{l}}{r^{l+1}} Y^*_{lm}(\Omega') Y_{lm}(\Omega)$$

$R = r'$

region I $\quad r < R = r' \quad r_< = r \, ; \, r_> = r' = R$

region I $\quad r > R = r' \quad r_< = R \, ; \, r_> = r$

$$\Phi(\vec{r}) = q \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^{l}}{r_>^{l+1}} Y^*_{lm}(\Omega') Y_{lm}(\Omega) = q \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\hookrightarrow \frac{1}{|\vec{r} - \vec{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^{l}}{r_>^{l+1}} Y^*_{lm}(\Omega') Y_{lm}(\Omega)$$

$$= \sum_{l} \frac{r_<^{l}}{r_>^{l+1}} P_l(\cos\gamma)$$

$$\boxed{P_l(\cos\gamma) = \sum_{m} \frac{4\pi}{2l+1} Y^*_{lm}(\Omega') Y_{lm}(\Omega)}$$

$$\Phi(\vec{r}) = \sum_{lm} \frac{1}{r^{l}} Y_{lm}(\Omega) Q_{lm}$$