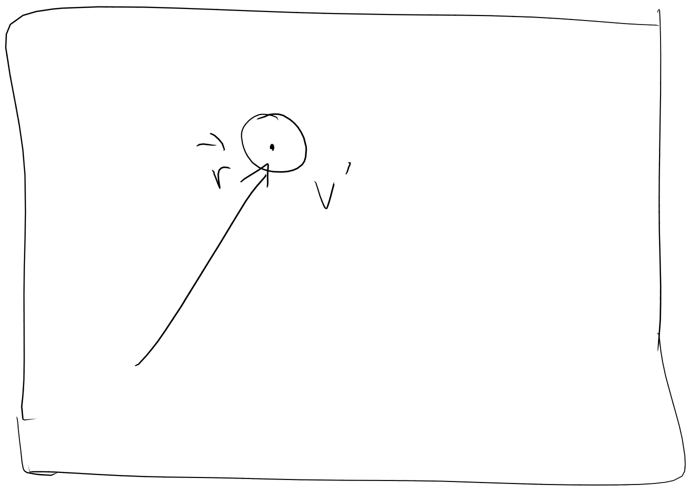


$$\langle \vec E(\vec r)\rangle = \frac{1}{V'}\int dV\, \vec E(\vec r)$$

$$\vec E(\vec r) = \vec E_{in}(\vec r) + \vec E_{out}(\vec r)$$

$$\langle \vec E(\vec r)\rangle = \langle \vec E_{in}(\vec r)\rangle + \langle \vec E_{out}(\vec r)\rangle$$

$$\langle \vec E_{in}(\vec r)\rangle = -\frac{4\pi}{3}\vec P(\vec r)$$

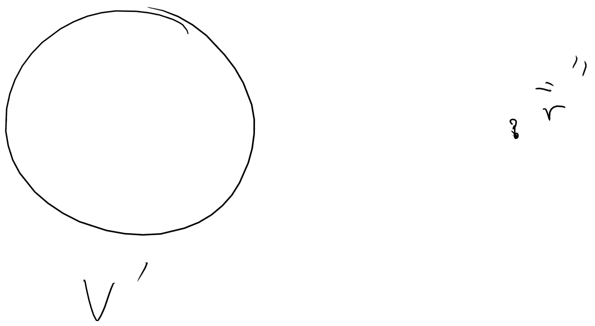
$$\vec P(\vec r) = \frac{1}{V'}\int_{V'} d^3r'\, \rho(\vec r+\vec r')\,\vec r'$$

$$\langle \vec E_{out}(\vec r)\rangle = \frac{1}{V'}\int d^3r'\, \vec E_{out}(\vec r+\vec r')$$

$$= \frac{1}{V'}\int_{V'} d^3r' \int_{outside} d^3r''\, \rho(\vec r+\vec r'')\frac{(\vec r+\vec r')-(\vec r+\vec r'')}{|\qquad|^3}$$

$$= \frac{1}{V'}\int_{outside} d^3r''\, \rho(\vec r+\vec r'')\int_{V'} d^3r'\, \frac{\vec r'-\vec r''}{|\vec r'-\vec r''|^3}$$

$$\underset{V'}{\int} d^3r' \, \frac{\vec r' - \vec r''}{|\vec r' - \vec r''|^3} = - \underbrace{\underset{V'}{\int} d^3r' \, \frac{\vec r'' - \vec r'}{|\vec r'' - \vec r'|}}_{} \quad (\rho = 1)$$

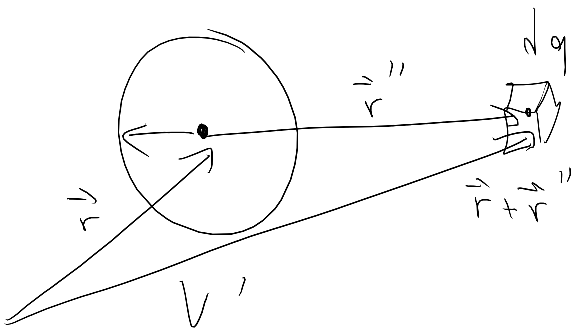


$$\underset{V'}{\int} d^3r' \, \frac{\vec r' - \vec r''}{|\vec r' - \vec r''|^3} = - V' \frac{1}{r''^2} \hat r''$$

$$\langle \vec E_{out}(\vec r) \rangle = \frac{1}{V'} \int_{outside} d^3r'' \, \rho(\vec r + \vec r'') \underset{V'}{\int} d^3r' \frac{\vec r' - \vec r''}{|\vec r' - \vec r''|^3}$$

$$= \frac{1}{\cancel{V'}} \int_{outside} d^3r'' \, \rho(\vec r + \vec r'') \, (-\cancel{V'}) \frac{\vec r''}{r''^3}$$

$$\langle \vec E_{out}(\vec r) \rangle = - \int_{outside} d^3r'' \, \rho(\vec r + \vec r'') \frac{\vec r''}{r''^3} \quad \Longleftarrow$$



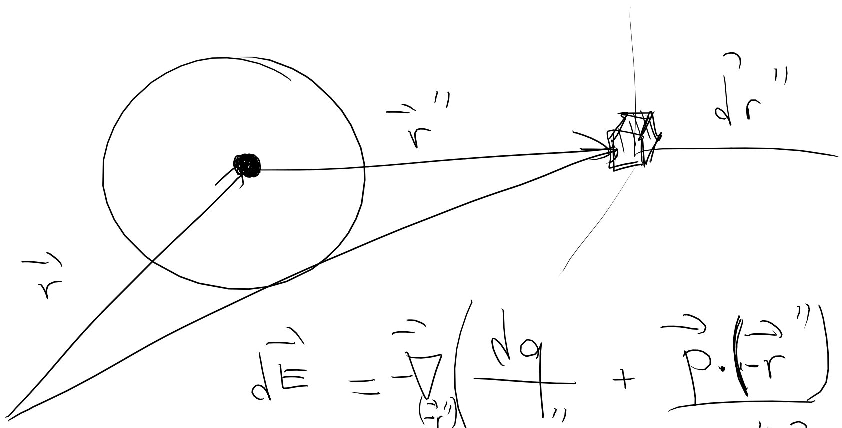
$$dq = \rho(\vec r + \vec r'') \, d^3r$$

$$d\vec E = \rho(\vec r + \vec r'') d^3r \frac{(-\hat r'')}{r''^2}$$

$$d\vec E = -\rho(\vec r + \vec r'') \, d^3r \frac{\vec r''}{r''^3}$$

$$\langle \vec{E}_{out}(\vec{r})\rangle = -\int_{outside} d\vec{r}''\, \rho(\vec{r}+\vec{r}'')\,\frac{\vec{r}''}{r''^3}$$

$$\frac{\vec{p}\cdot\vec{r}}{r^3}$$



$$\frac{d\vec{p}}{d\vec{r}''} = \vec{P}(\vec{r}+\vec{r}'')$$

$$d\vec{E} = -\vec{\nabla}_{(-\vec{r}'')}\underbrace{\left(\frac{dq}{r''} + \frac{\vec{p}\cdot(-\vec{r}'')}{r''^3}\right)}$$

$$\langle \vec{E}_{out}(\vec{r})\rangle = \vec{E}_{out}(\vec{r}) = \int_{outside} d\vec{r}''\left(-\vec{\nabla}_{(-\vec{r}'')}\right)\left[\frac{\rho(\vec{r}+\vec{r}'')}{r''} + \frac{\vec{P}(\vec{r}+\vec{r}'')\cdot(-\vec{r}'')}{r''^3}\right]$$

$$\vec{\delta}'' = \vec{r}+\vec{r}'' \Rightarrow -\vec{r}'' = \vec{r}-\vec{\delta}''$$

$$d\vec{r}'' = d\vec{\delta}''$$

$$\langle \vec{E}_{out}(\vec{r})\rangle = \int_{outside} d\vec{\delta}''\left(-\vec{\nabla}_{\vec{r}-\vec{\delta}''}\right)\left[\frac{\rho(\vec{\delta}'')}{|\vec{\delta}''-\vec{r}|} + \frac{\vec{P}(\vec{\delta}'')\cdot(\vec{r}-\vec{\delta}'')}{|\vec{r}-\vec{\delta}''|^3}\right]$$

$$\vec{\nabla}_{\vec{r}-\vec{\delta}''} = \vec{\nabla}_{\vec{r}} = -\vec{\nabla}_{\vec{\delta}''}$$

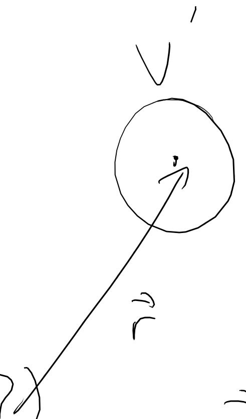
$$= -\vec{\nabla}_{\vec{r}}\underbrace{\int_{outside} d\vec{\delta}''\left[\frac{\rho(\vec{\delta}'')}{|\vec{\delta}''-\vec{r}|} + \frac{\vec{P}(\vec{\delta}'')\cdot(\vec{r}-\vec{\delta}'')}{|\vec{r}-\vec{\delta}''|^3}\right]}$$

$$\langle \vec{E}(\vec{r})\rangle = -\frac{4}{3}\pi\vec{P}(\vec{r}) - \vec{\nabla}_{\vec{r}}\int_{outside} d\vec{\delta}''\left[\frac{\rho(\vec{\delta}'')}{|\vec{\delta}''-\vec{r}|} + \frac{\vec{P}(\vec{\delta}'')\cdot(\vec{r}-\vec{\delta}'')}{|\vec{r}-\vec{\delta}''|^3}\right]$$

$$-\vec{\nabla}\left(\int_{V'} d^3\vec{r}\,''\,\frac{\rho(\vec{r}\,'')}{|\vec{r}\,''-\vec{r}|}\right)$$

$$=\int_{V'} d^3\vec{r}\,''\,\rho(\vec{r}\,'')\frac{(\vec{r}-\vec{r}\,'')}{|\vec{r}\,''-\vec{r}|^3}$$

$$\simeq \rho(\vec{r})\int_{V'} d^3\vec{r}\,''\,\frac{\vec{r}-\vec{r}\,''}{|\vec{r}-\vec{r}\,''|^3}=0$$

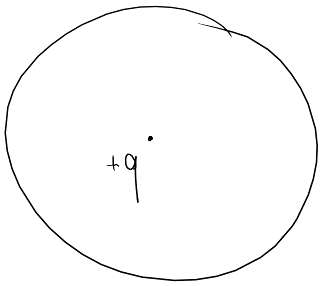


$$-\vec{\nabla}_{\vec{r}}\left(\int_{V'} d^3\vec{r}\,''\,\frac{\vec{P}(\vec{r}\,'')\cdot(\vec{r}-\vec{r}\,'')}{|\vec{r}-\vec{r}\,''|^3}\right) \overset{(?)}{=} -\frac{4\pi}{3}\vec{P}(\vec{r})$$

$$\langle\vec{E}(\vec{r})\rangle = -\frac{4}{3}\pi\vec{P}(\vec{r}) - \vec{\nabla}_{\vec{r}}\int_{\text{outside}} d^3\vec{r}\,''\left[\frac{\rho(\vec{r}\,'')}{|\vec{r}\,''-\vec{r}|}+\frac{\vec{P}(\vec{r}\,'')\cdot(\vec{r}-\vec{r}\,'')}{|\vec{r}-\vec{r}\,''|^3}\right]$$

$$\boxed{\langle\vec{E}(\vec{r})\rangle = -\vec{\nabla}_{\vec{r}}\left(\int d^3\vec{r}\,''\left[\frac{\rho(\vec{r}\,'')}{|\vec{r}-\vec{r}\,''|}+\frac{\vec{P}(\vec{r}\,'')\cdot(\vec{r}-\vec{r}\,'')}{|\vec{r}-\vec{r}\,''|^3}\right]\right)}$$

$$V(\vec{r}) = \int d^3\vec{r}\,''\left[\frac{\rho(\vec{r}\,'')}{|\vec{r}-\vec{r}\,''|}+\frac{\vec{P}(\vec{r}\,'')\cdot(\vec{r}-\vec{r}\,'')}{|\vec{r}-\vec{r}\,''|^3}\right]$$



$$V_P = \int d^3r' \, \frac{\vec{P}(\vec{r}'') \cdot (\vec{r} - \vec{r}'')}{|\vec{r} - \vec{r}''|^3}$$

$$= \int d^3r' \, \vec{P}(\vec{r}'') \, \vec{\nabla}_{r''} \frac{1}{|\vec{r} - \vec{r}''|}$$

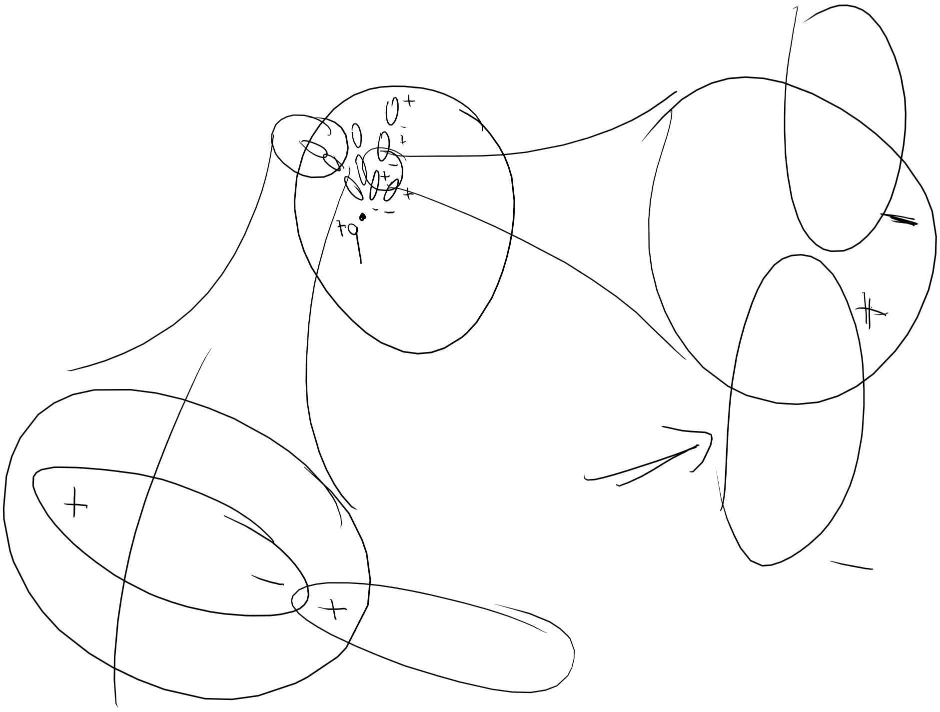
$$= \int d^3r' \left\{ \vec{\nabla}_{r''} \cdot \left[\frac{\vec{P}(\vec{r}'')}{|\vec{r} - \vec{r}''|} \right] - \frac{1}{|\vec{r} - \vec{r}''|} (\vec{\nabla} \cdot \vec{P}) \right\}$$

$$= \oint_{\partial V} d\vec{S} \cdot \frac{\vec{P}(\vec{r}'')}{|\vec{r} - \vec{r}''|} - \int_V d^3r'' \, \frac{1}{|\vec{r} - \vec{r}''|} (\vec{\nabla} \cdot \vec{P})$$

$$= \int dS'' \, \frac{\sigma_b(\vec{r}'')}{|\vec{r} - \vec{r}''|} + \int d^3r'' \, \frac{\rho_b(\vec{r}'')}{|\vec{r} - \vec{r}''|}$$

$$\sigma_b(\vec{r}'') \equiv \vec{P}(\vec{r}'') \cdot \hat{n}$$

$$\rho_b(\vec{r}'') = -\vec{\nabla} \cdot \vec{P}$$



$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \left(\rho + \underbrace{\rho_b + \sigma_b \delta(\ )}\right)$$

$$\vec{\nabla} \cdot \vec{P} = -\left(\rho_b + \sigma_b \delta(\ )\right)$$

$$\vec{\nabla} \cdot \left(\underbrace{\vec{E} + 4\pi \vec{P}}_{\vec{D}(\vec{r})}\right) = 4\pi\rho$$

$$\boxed{\vec{\nabla} \cdot \vec{D}(\vec{r}) = 4\pi \rho_f}$$

$$\boxed{\vec{\nabla} \times \vec{E} = 0} \Rightarrow \vec{E} = -\vec{\nabla}\phi$$

$$\vec{D}(\vec{E})$$

$$D_x = \varepsilon_{xx} E_x + \varepsilon_{xy} E_y + \varepsilon_{xz} E_z + \mathcal{O}(E_i^2)$$

$$D_y = \varepsilon_{yx} E_x + \varepsilon_{yy} E_y + \varepsilon_{yz} E_z + \mathcal{O}(E_i^2)$$

$$D_z = \ldots$$

isotropic dielectric : $\varepsilon_{ij} = \delta_{ij}\,\varepsilon$

$$\boxed{\vec{D} = \varepsilon \vec{E}}$$ (linear isotropic dielectric)

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho$$

$$\vec{\nabla}\cdot(\varepsilon\vec{E}) = 4\pi\rho$$

if the material is homogeneous

$$\varepsilon\,\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

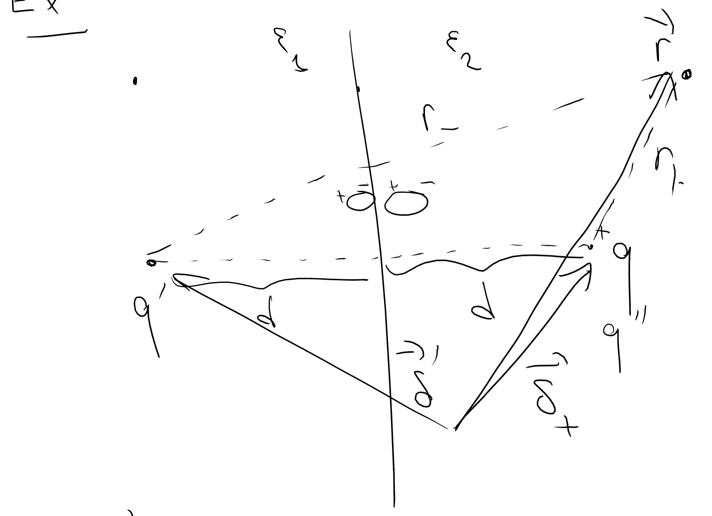
$$\vec{\nabla}\cdot\vec{E} = \frac{4\pi}{\varepsilon}\rho$$

ε: permitivity

<u>Ex</u> point charge in an infinite uniform isotropic dielectric

$$\vec{E} = \frac{1}{\varepsilon}\frac{q}{r^2}\hat{r}$$

Ex



$$V_{II}(\vec{r}) = \frac{1}{\varepsilon_2}\frac{q}{r_+} + \frac{1}{\varepsilon_2}\frac{q'}{r'}$$

$$\nabla^2 V = -\frac{4\pi}{\varepsilon}\rho$$

in region II

$$\nabla^2 V_{II}(\vec{r}) = \frac{1}{\varepsilon_2} q \nabla^2\left(\frac{1}{r_+}\right) + \frac{1}{\varepsilon_2} q' \overbrace{\nabla^2\left(\frac{1}{r'}\right)}^{0}$$

$$\nabla^2 V_{II}(\vec{r}) = -\frac{4\pi}{\varepsilon_2} q \,\delta^{(3)}(\vec{r} - \vec{\delta}_+)$$

in region I

$$V_I = \frac{1}{\varepsilon_1}\frac{q''}{r_+}$$

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \Delta E_{\parallel} = 0 \Rightarrow V \text{ is continuous}$$

$$\frac{1}{\varepsilon_1}\frac{q''}{\cancel{r_+}} = \frac{1}{\varepsilon_2}\frac{q}{\cancel{r_+}} + \frac{1}{\varepsilon_2}\frac{q'}{\cancel{r_+}}$$

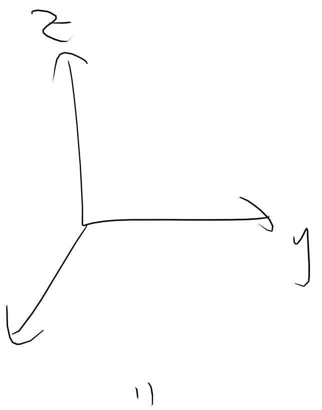
$$\boxed{\varepsilon_2 q'' = \varepsilon_1 (q + q')}$$

$\vec{\nabla} \cdot \vec{D} = 0$ on the boundary

$\Delta D_\perp = 0$

$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$

$\Rightarrow \vec{E}^{\,\text{outside}} - \vec{E}^{\,\text{inside}} = 4\pi\sigma\hat{n}$



$$V_I = \frac{1}{\varepsilon_1}\frac{q''}{r_+} = \frac{1}{\varepsilon_1}\frac{q''}{\sqrt{x^2+z^2+(y-d)^2}}$$

$$E^\perp = -\frac{\partial V_I}{\partial y} = \frac{1}{\varepsilon_1}\, q''\, \frac{(y-d)}{\left(\sqrt{x^2+z^2+(y-d)^2}\right)^3}$$

$$\left(D_I^\perp\right)^{\text{boundary}} = \varepsilon_1 \left(E_I^\perp\right)^{\text{boundary}} = q''\frac{(-d)}{\left(\sqrt{x^2+z^2+d^2}\right)^3}$$

$$V_{II} = \frac{1}{\varepsilon_2}\frac{q}{r_+} + \frac{1}{\varepsilon_2}\frac{q'}{r'}$$

$$V_{II} = \frac{1}{\varepsilon_2}\frac{q}{\sqrt{x^2+z^2+(y-d)^2}} + \frac{1}{\varepsilon_2}\frac{q'}{\sqrt{x^2+z^2+(y+d)^2}}$$

$$\left(D_{II}^\perp\right)^{\text{boundary}} = \frac{q\,(-d)}{\left(\sqrt{x^2+z^2+d^2}\right)^3} + \frac{q'\,d}{\left(\sqrt{}\right)^3}$$

$$q'' = q - q'$$

$$\varepsilon_2 q'' = \varepsilon_1 (q + q')$$

$$\varepsilon_2 (q - q') = \varepsilon_1 (q + q')$$

$$(\varepsilon_1 + \varepsilon_2) q' = (-\varepsilon_1 + \varepsilon_2) q$$

$$\boxed{q' = \frac{-\varepsilon_1 + \varepsilon_2}{\varepsilon_1 + \varepsilon_2} q}$$

$$q'' = q - q' = q - \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} q$$

$$\boxed{q'' = \frac{2\varepsilon_1}{\varepsilon_2 + \varepsilon_1} q}$$