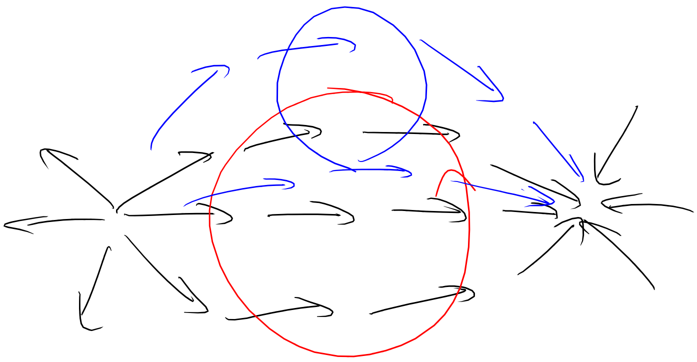
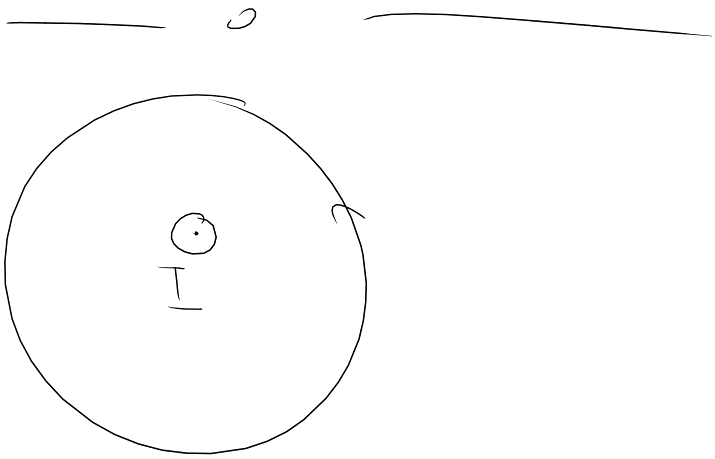
$\vec{A}$



$$\int_{A} (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oint_{\partial A} \vec{A} \cdot d\vec{l}$$



Helmholtz Thm

$\vec{V}$

$\vec{\nabla} \times \vec{V}$ ✓

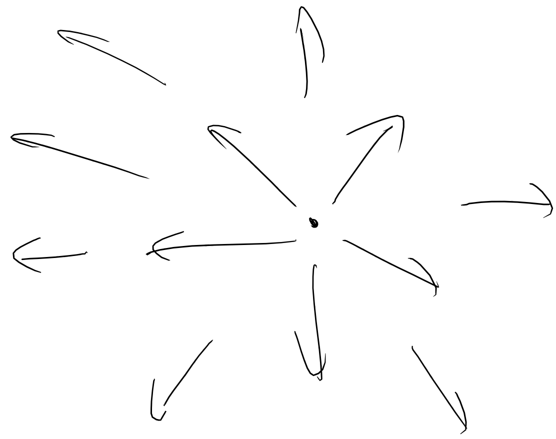
$\vec{\nabla} \cdot \vec{V}$ ✓ + Boundary conditions ✓ $\Big\}$ $\vec{V}$ is uniquely determined

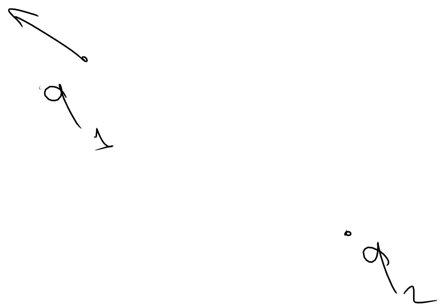
Electric Force & Electric Field

Example

$$\vec{\nabla}\cdot\vec{E} \neq 0$$

$$\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = 4\pi\delta^{(3)}(\vec{r}) \leftarrow$$





$$\vec{F}_1 = k\, q_1 q_2 \frac{(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3}$$

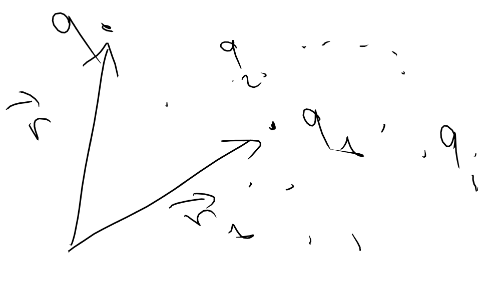
$\searrow$ SI : $[q] = 1\,C$ $\quad k = \frac{1}{4\pi\varepsilon_0}$

Gaussian Units $\quad [k] = 1 \quad [q] = $ stat Coulomb $\simeq (N\,m^2)^{1/2}$

Natural units / Heaviside units $\quad [k] = \frac{1}{4\pi} \quad \hbar = c = 1$

Superposition Principle

$$\vec{F}_q = \frac{1}{4\pi\varepsilon_0}\, q\, q_1 \frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|^3} + \frac{1}{4\pi\varepsilon_0}\, q\, q_2 \frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|^3} + \dots$$



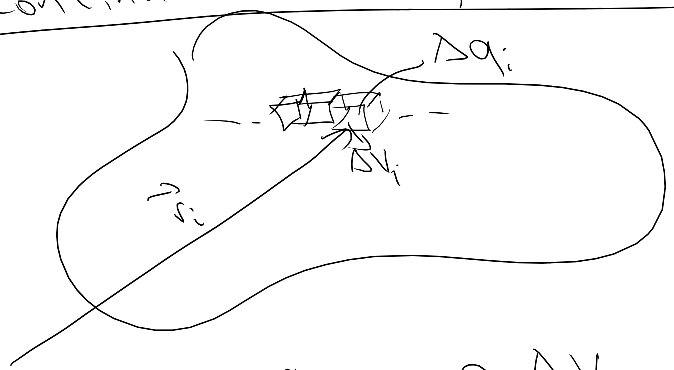
$$\boxed{\vec{F}_q = q\vec{E}}$$

electric field

$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}_q}{q}$$

$$\boxed{\vec{E}(\vec{r}) = \sum_i \frac{1}{4\pi\varepsilon_0} \frac{q_i}{|\vec{r}-\vec{r}_i|^3} (\vec{r}-\vec{r}_i)}$$

Continuous Charge Distribution



$\Delta V_i \to 0$

$$\rho(\vec{r}_i) = \lim_{\Delta V_i \to 0} \frac{\Delta q_i}{\Delta V_i}$$

↳ charge density

$$\Delta q_i = \rho \, \Delta V_i$$

$$\vec{E}(\vec{r}) = \lim_{\Delta V \to 0} \sum_i \frac{1}{4\pi\varepsilon_0} \frac{\Delta q_i}{|\vec{r}-\vec{r}_i|^3} (\vec{r}-\vec{r}_i)$$

$$= \lim_{\Delta V \to 0} \sum_{\vec{r}_i} \frac{1}{4\pi\varepsilon_0} \rho(\vec{r}_i) \, \Delta V_i \, \frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}$$

$$\boxed{\vec{E}(\vec{r}) = \int dV \, \frac{1}{4\pi\varepsilon_0} \rho(\vec{r}\,') \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}}$$

$$\vec\nabla\cdot\frac{\vec r}{|\vec r|^3} = 4\pi\,\delta^{(3)}(\vec r)$$

$$\vec\nabla_r\cdot\frac{\vec r-\vec r\,'}{|\vec r-\vec r\,'|^3} = \vec\nabla_\delta\cdot\frac{\vec\delta}{|\vec\delta|^3} \qquad \vec\delta = \vec r-\vec r\,'$$

$$\boxed{\vec\nabla_r\cdot\frac{\vec r-\vec r\,'}{|\vec r-\vec r\,'|^3} = 4\pi\,\delta^{(3)}(\vec r-\vec r\,')}$$

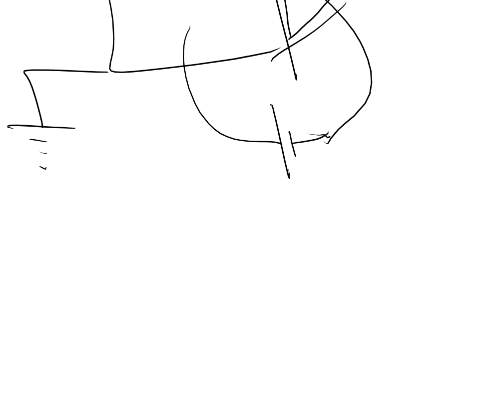
$$\vec E(\vec r) = \int dV\,\rho(\vec r\,')\,\frac{(\vec r-\vec r\,')}{|\vec r-\vec r\,'|^3}\,\frac{1}{4\pi\varepsilon_0}$$

$$\vec\nabla\cdot\vec E(\vec r) = \int dV\,\frac{1}{4\pi\varepsilon_0}\,\rho(\vec r\,')\,\underbrace{\vec\nabla_r\cdot\left(\frac{(\vec r-\vec r\,')}{|\vec r-\vec r\,'|^3}\right)}_{4\pi\delta^{(3)}(\vec r-\vec r\,')}$$

$$\boxed{\vec\nabla\cdot\vec E(\vec r) = \frac{\rho(\vec r)}{\varepsilon_0}}$$ Gauss' Law

$$\int_V d^3r'\,\delta^{(3)}(\vec r-\vec r\,')\,f(\vec r\,') = \begin{cases} f(\vec r) & \text{if } \vec r\in V \\ 0 & \text{if } \vec r\notin V\end{cases}$$

Example



$\rho(\vec r) = 0$ inside

$\Rightarrow \vec\nabla\cdot\vec E = 0$ inside

$\Rightarrow \vec E$ is perpendicular to the surface everywhere on the surface

$\vec\nabla\times\vec E = 0$

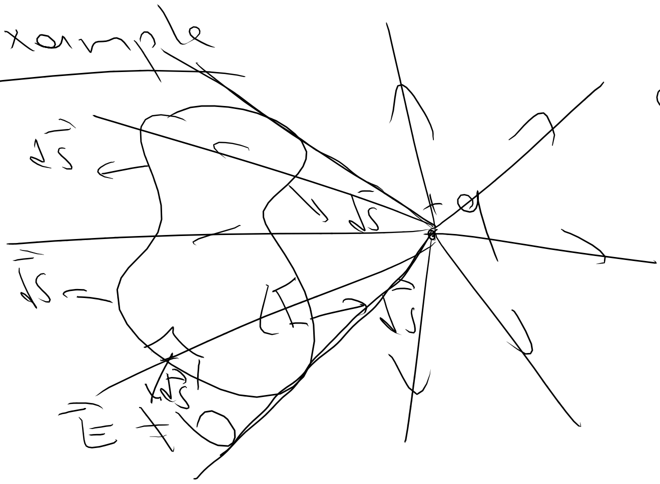
$$\int_V (\vec{\nabla}\cdot\vec{E})\,dV = \oint_{\partial V} \vec{E}\cdot d\vec{S}$$

$d\vec{S}$: perpendicular to the surface, pointing outside

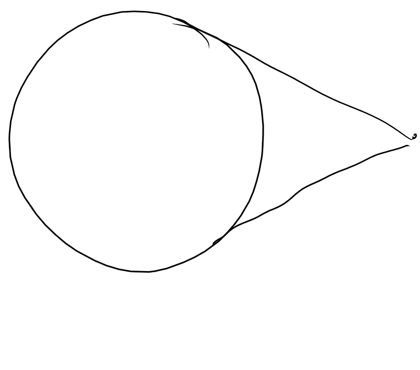
$$\oint_{\partial V} \vec{E}\cdot d\vec{S} = \int_V \frac{\rho(\vec{r})}{\varepsilon_0}\,dV = \frac{Q_{enc}}{\varepsilon_0}$$

$$\boxed{\oint \vec{E}\cdot d\vec{S} = \frac{Q_{enc}}{\varepsilon_0}}$$

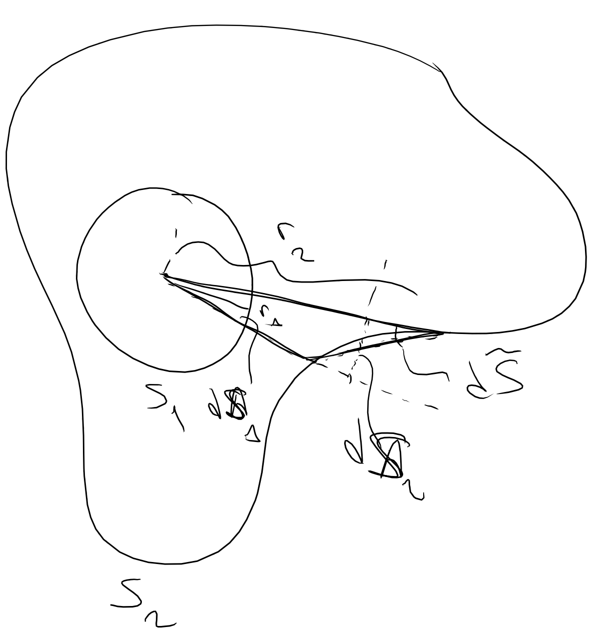
Example



$$\oint \vec{E}\cdot d\vec{S} = 0$$



$$\int_{S_1} \vec{E}\cdot d\vec{S} = \int_{S_2} \vec{E}\cdot d\vec{S}$$



$$dS_1 \propto r_1^2$$

$$dS_2 \propto r_2^2$$

$$\frac{dS_1}{dS_2} = \frac{r_1^2}{r_2^2} = \frac{\frac{1}{r_2^2}}{\frac{1}{r_1^2}}$$

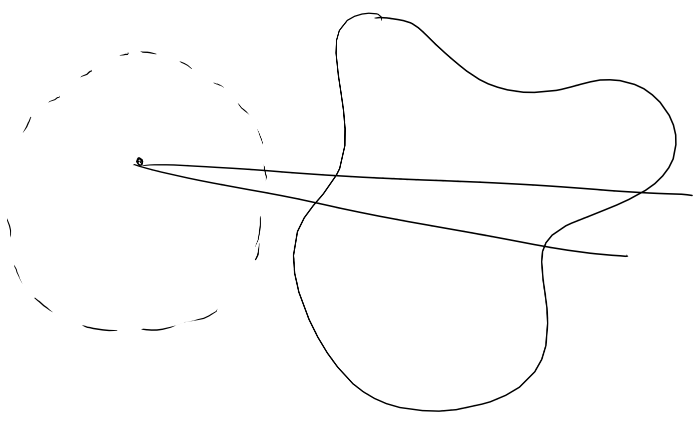
$$dS_2 = d\tilde{S}\cos\theta$$

$$E_2 dS_2 = E_2 d\tilde{S}\cos\theta$$

$$= \vec{E}_2 \cdot d\vec{\tilde{S}}$$

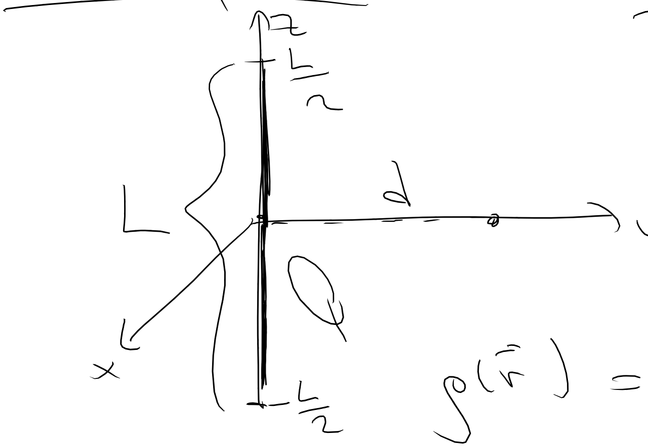
$$\frac{dS_1}{dS_2} = \frac{E_2}{E_1}$$

$$E_1 dS_1 = E_2 dS_2$$



Examples

Example 1



$$\vec{E}(\vec{r}) = \int dV\, \rho(\vec{r}\,') \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$\vec{r} = d\hat{y}$$

$$\rho(\vec{r}) = A\,\delta(x)\,\delta(y)\,\Theta\left(\frac{L^2}{4}-z^2\right)$$

$$\Theta(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x>0 \end{cases}$$

$$\rho(\vec{r}) = \lim_{dz\to 0} dz\, \cancel{z}\, A = 0$$

$$\Delta q = \rho(\vec{r})\,\Delta V$$

$$Q = \int \rho(\vec{r})\,dx\,dy\,dz = A\int_{-\infty}^{\infty} dx\,\delta(x)\int_{-\infty}^{\infty} dy\,\delta(y)\int_{-\infty}^{\infty} dz\,\Theta\left(\frac{L^2}{4}-z^2\right)$$

$$Q = A\int_{-\infty}^{\infty} dx\,\delta(x)\int_{-\infty}^{\infty} dy\,\delta(y)\int_{-L/2}^{L/2} dz$$

$$= LA$$

$$A = \frac{Q}{L} = \lambda : \text{linear charge density}$$

$$\boxed{\rho(\vec{r}) = \frac{Q}{L}\,\delta(x)\,\delta(y)\,\Theta\left(\frac{L^2}{4}-z^2\right)}$$

$$\vec{E} = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \, \frac{Q}{L}\delta(x')\delta(y')\Theta\left(\frac{L^2}{4}-z'^2\right)$$

$$\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$\vec{r}\,' = x'\hat{x} + y'\hat{y} + z'\hat{z}$$

$$\vec{r} = d\hat{y}$$

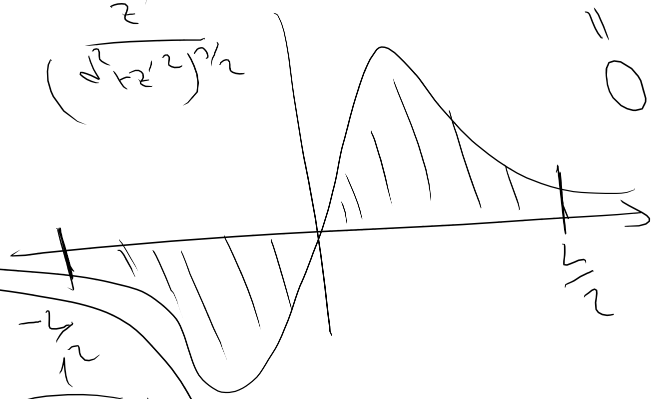
$$\vec{r}-\vec{r}\,' = -x'\hat{x} + (d-y')\hat{y} - z'\hat{z} \leftarrow$$

$$|\vec{r}-\vec{r}\,'| = \left[x'^2 + (d-y')^2 + z'^2\right]^{1/2}$$

$$\vec{E} = \frac{Q}{L}\int_{-L/2}^{L/2} dz' \, \frac{(d\hat{y} - z'\hat{z})}{(d^2+z'^2)^{3/2}}$$

$$= \frac{Qd\hat{y}}{L}\int_{-L/2}^{L/2} dz' \, \frac{1}{(d^2+z'^2)^{3/2}} - \frac{Q}{L}\hat{z}\int_{-L/2}^{L/2} dz' \, \frac{z'}{(d^2+z'^2)^{3/2}}$$

$$\frac{z'}{(d^2+z'^2)^{3/2}}$$



$$\boxed{\vec{E} = \frac{Qd}{L}\hat{y}\int_{-L/2}^{L/2} dz' \, \frac{1}{(d^2+z'^2)^{3/2}}}$$