$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$dU = (\vec{\nabla} U)\cdot d\vec{\ell}$$

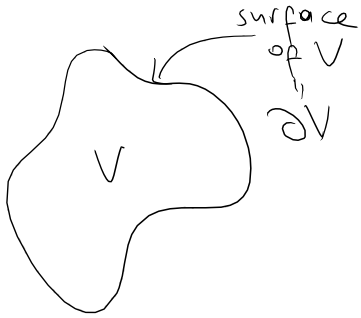
$$\text{curl}\,\vec{A} = \vec{\nabla}\times\vec{A}$$

$$\text{div}\,\vec{A} = \vec{\nabla}\cdot\vec{A}$$

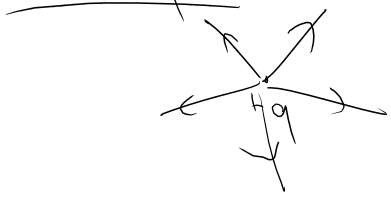
$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

$$\int_{P_0}^{P_1} (\vec{\nabla} U)\cdot d\vec{\ell} = U(P_1) - U(P_0)$$

$$\int_V (\vec{\nabla}\cdot\vec{A})\,dV = \int_{\partial V} \vec{A}\cdot d\vec{S}$$




Example



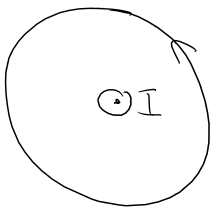

$$\oint \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0} q_{enc}$$

$$\int (\vec{\nabla}\cdot\vec{E})\,dV$$

$$\oint \vec{E}\cdot d\vec{\ell} = 0$$

$$\Rightarrow \vec{\nabla}\times\vec{E} = 0 \quad \text{(electrostatics)}$$

Example




$$\oint \vec{B}\cdot d\vec{S} = 0$$

$$\int (\vec{\nabla}\cdot\vec{B})\,dV = 0 \Rightarrow \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = 0$$

$$\int_S (\nabla \times \vec{A}) \cdot d\vec{S} = \oint_{\partial S} \vec{A} \cdot d\vec{l}$$

$$\oint \vec{B} \cdot d\vec{l} = \frac{\mu_0}{4\pi} I_{enc}$$

$$\int (\nabla \times \vec{B}) \, d\vec{S}$$

i) $\vec{\nabla} \cdot (\vec{\nabla} \phi) = \nabla^2 \phi$

ii) $\vec{\nabla} \times (\vec{\nabla} \phi) = 0$

iii) $\vec{\nabla} (\vec{\nabla} \cdot \vec{A})$

iv) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

v) $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

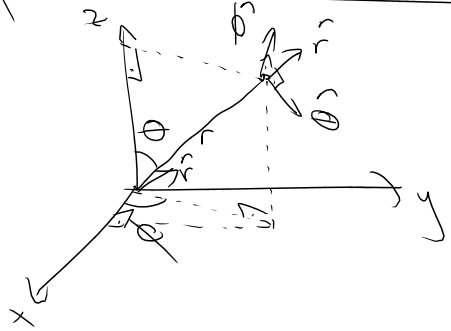
i) $\vec{\nabla} \cdot (\vec{\nabla} \phi) = \vec{\nabla} \cdot \left( \hat{x} \frac{\partial \phi}{\partial x} + \hat{y} \frac{\partial \phi}{\partial y} + \hat{z} \frac{\partial \phi}{\partial z} \right)$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad : \text{Laplacian.}$$

ii) $\left[ \vec{\nabla} \times (\vec{\nabla} \phi) \right]_x = \frac{\partial}{\partial y} \frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z} \frac{\partial \phi}{\partial y} = 0$

## Spherical Coordinates




$(r, \theta, \phi)$

$r \geq 0$

$0 \leq \theta \leq \pi$

$0 \leq \phi \leq 2\pi$

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

$\vec{r} = r\hat{r}(\theta,\phi)$ $\qquad \hat{r}\cdot\hat{r} = 1$

$\hat{r} = \frac{1}{r}(x\hat{x} + y\hat{y} + z\hat{z})$

$0 = \frac{\partial \hat{r}^2}{\partial \theta} = 2\hat{r}\cdot\frac{\partial \hat{r}}{\partial \theta}$

$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$

$0 = \frac{\partial \hat{r}^2}{\partial \phi} = 2\hat{r}\cdot\frac{\partial \hat{r}}{\partial \phi}$

$\frac{\partial \hat{r}}{\partial \phi} = \sin\theta\,\hat{\phi}$

$\hat{\theta}\cdot\hat{\phi} = 0$

$dU = (\vec{\nabla} U)\cdot d\vec{r}$ $\Leftarrow$

$dU = U(r+dr,\theta+d\theta,\phi+d\phi) - U(r,\theta,\phi)$

$d\vec{r} = d\vec{r} = d(r\hat{r})$

$= \frac{\partial}{\partial r}(r\hat{r})\,dr + \frac{\partial(r\hat{r})}{\partial \theta}\,d\theta + \frac{\partial(r\hat{r})}{\partial \phi}\,d\phi$

$d\vec{r} = dr\,\hat{r} + r\,\hat{\theta}\,d\theta + r\sin\theta\,d\phi\,\hat{\phi}$

case i: $d\theta = d\phi = 0$ ; $dr \to 0$

$U(r+dr,\theta,\phi) - U(r,\theta,\phi) = (\vec{\nabla} U)\cdot(dr\,\hat{r})$

$= (\vec{\nabla} U)\cdot\hat{r} = \frac{U(r+dr,\theta,\phi) - U(r,\theta,\phi)}{dr} = \frac{\partial U}{\partial r}$

$(\vec{\nabla} U)_r = \frac{\partial U}{\partial r}$

case ii: $d\phi = 0$ ; $dr = 0$ ; $d\theta \to 0$

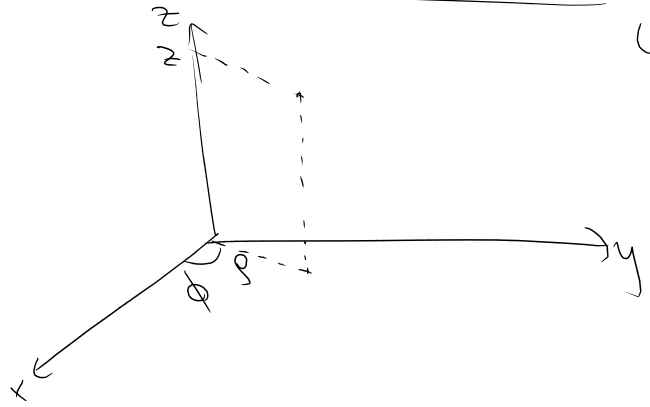
$(\vec{\nabla} U)_\theta = \frac{1}{r}\frac{\partial U}{\partial \theta}$

case iii $\quad d\theta = 0 \; ; \; dr = 0 \; ; \; d\phi \neq 0$

$$(\vec{\nabla} u)_\phi = \frac{1}{r \sin\theta} \frac{\partial u}{\partial \phi}$$

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}$$

## Cylindrical Coordinates




$(z, \rho, \phi)$

$z = z$

$x = \rho \cos\phi$

$y = \rho \sin\phi$

## Dirac Delta Function

### Example

$$\vec{\nabla}\left(\frac{1}{r}\right) = \hat{r} \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = \hat{r}\left(\frac{-1}{r^2}\right) = -\frac{\hat{r}}{r^2}$$

$$\nabla^2\left(\frac{1}{r}\right) = \vec{\nabla} \cdot \left(\vec{\nabla} \frac{1}{r}\right) = \vec{\nabla} \cdot \left(\frac{-\hat{r}}{r^2}\right) = \vec{\nabla} \cdot \left(\frac{-\vec{r}}{r^3}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{-y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{-z}{r^3}\right)$$

$$= -\frac{1}{r^3} - x \frac{\partial}{\partial x} \frac{1}{r^3} \quad \leftarrow \quad r \neq 0$$

$$-\frac{1}{r^3} - y \frac{\partial}{\partial y} \frac{1}{r^3}$$

$$-\frac{1}{r^3} - z \frac{\partial}{\partial z}\left(\frac{1}{r^3}\right)$$

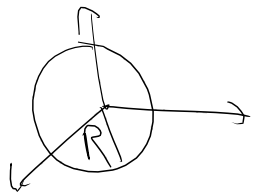
$$= -\frac{3}{r^3} - x\left(\frac{-3}{r^4}\right)\frac{x}{r} - y\left(\frac{-3}{r^4}\right)\left(\frac{y}{r}\right) - z\left(\frac{-3}{r^4}\right)\left(\frac{z}{r}\right)$$

$$= -\frac{3}{r^3} + \frac{3}{r^5} r^2 = 0$$

$$\nabla^2\left(\frac{1}{r}\right) = 0$$

**Caution**

$$0 = \int \nabla^2\left(\frac{1}{r}\right) dV = \int_V \vec{\nabla}\cdot\left(\vec{\nabla}\frac{1}{r}\right) dV$$




$$d\vec{S} = dS\,\hat{r}$$

$$= \oint_{\partial V} \vec{\nabla}\left(\frac{1}{r}\right)\cdot d\vec{S}$$

$$= \oint \left(\vec{\nabla}\frac{1}{r}\right)\cdot \hat{r}\, dS$$

$$= \oint \frac{\partial}{\partial r}\left(\frac{1}{r}\right) dS$$

$$= \oint \left(-\frac{1}{r^2}\right) dS = -\frac{1}{R^2}\oint dS = -\frac{1}{R^2}\,4\pi R^2$$

$$\oint \nabla^2\left(\frac{1}{r}\right) dV = -4\pi \neq 0$$

**Example**

$$\Theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$\frac{d\Theta}{dx} = \begin{cases} 0 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$a < 0 < b$

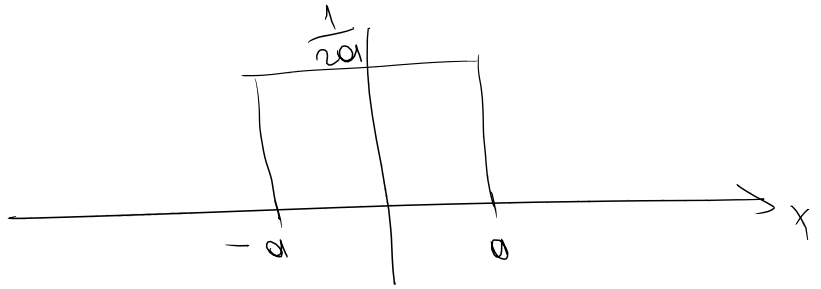
$$\int_a^b \frac{d\Theta}{dx}\, dx = \Theta(b) - \Theta(a) = 1$$

$$\delta(x-x_0) = \begin{cases} 0 & \text{if } x \neq x_0 \\ \infty & \text{if } x = x_0 \end{cases}$$

$$\int_{x_0^-}^{x_0^+} \delta(x-x_0)\,dx = 1$$

Example

$$\tilde{\delta}_a(x) = \begin{cases} 0 & \text{if } |x| > a \\ \frac{1}{2a} & \text{if } |x| < a \end{cases}$$

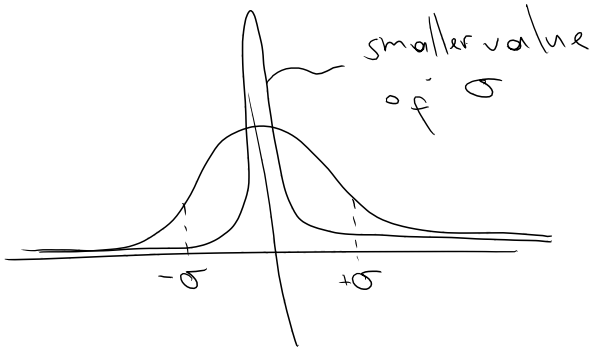



$$\int_{-\infty}^{\infty} \tilde{\delta}_a(x)\,dx = 1 \qquad \delta(x) = \lim_{a \to 0} \tilde{\delta}_a(x)$$

Example

$$\Delta_a(x) = \frac{1}{\sqrt{\pi}\,a} e^{-x^2/a^2}$$

$$\int_{-\infty}^{\infty} \Delta_a(x)\,dx = 1$$




$$\boxed{\delta(x) = \lim_{a \to 0} \Delta_a(x)}$$

$$\delta^{(3)}(\vec{r}-\vec{r}_0) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$$

$$\int \delta^{(3)}(\vec{r}-\vec{r}_0)\,dV = \int \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\,dx\,dy\,dz$$

$$= 1$$

$$\int_a^b dx\,\delta(x-x_0)f(x) = \begin{cases} 0 & \text{if } x_0 < a \\ 0 & \text{if } x_0 > b \\ f(x_0) & \text{if } a < x_0 < b \end{cases}$$

$$\int_a^b dx\,\delta(x-x_0)f(x) = \int_a^{x_0^-} dx\,f(x)\delta(x-x_0) + \int_{x_0^-}^{x_0^+} dx\,f(x)\delta(x-x_0) + \int_{x_0^+}^{b} dx\,f(x)\delta(x-x_0)$$

$$= \int_{x_0^-}^{x_0^+} dx\,f(x)\delta(x-x_0)$$

$$= f(x_0)\int_{x_0^-}^{x_0^+} dx\,\delta(x-x_0)$$

$$= 1 \quad \text{if } a < x_0 < b$$

$$\int_a^b dx\,\delta(x-x_0)f(x) = f(x_0)$$

## 3D Dirac Delta in Spherical Coordinates

$$\delta^{(3)}(\vec{r}-\vec{r}_0) = A\,\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)$$

$$1 = \int \delta^{(3)}(\vec{r}-\vec{r}_0)\,dV = \int \overbrace{r^2\sin\theta\,d\theta\,d\phi\,dr}^{dV} A\,\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)$$

$$= r_0^2\sin\theta_0\,A$$

$$A = \frac{1}{r_0^2\sin\theta_0}$$

$$\int_V \delta(\vec{r}_0 - \vec{r}) f(\vec{r}) \, dV = \begin{cases} f(\vec{r}_0) & \text{if } \vec{r}_0 \in V \\ 0 & \text{if } \vec{r}_0 \notin V \end{cases}$$