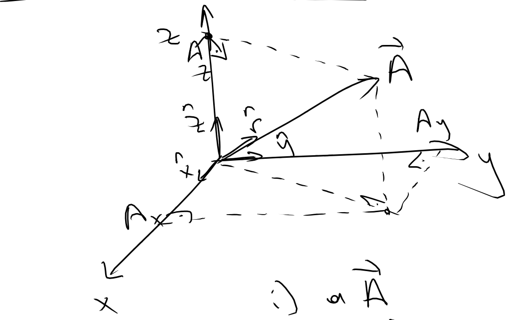
Vector Calculus



$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

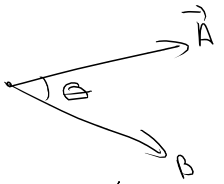
$$\vec{A} = A_r\hat{r}$$

i) $a\vec{A}$

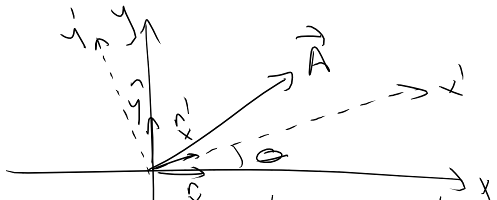
ii) $\vec{A} + \vec{B}$

$\vec{A} - \vec{B} = \vec{A} + (-1)\vec{B}$

iii) $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$



$$\vec{A} = (\vec{A}\cdot\hat{x})\hat{x} + (\vec{A}\cdot\hat{y})\hat{y} + (\vec{A}\cdot\hat{z})\hat{z}$$



$$\vec{A} = A_x\hat{x} + A_y\hat{y}$$

$$\vec{A} = A'_x\hat{x}' + A'_y\hat{y}'$$

$$A'_x = (\vec{A}\cdot\hat{x}') = (A_x\hat{x} + A_y\hat{y})\cdot\hat{x}'$$

$$= A_x\,\hat{x}\cdot\hat{x}' + A_y\,\hat{y}\cdot\hat{x}'$$

$\hat{x}\cdot\hat{x}' = \cos\theta$

$\hat{y}\cdot\hat{x}' = \cos(\frac{\pi}{2}-\theta) = \sin\theta$

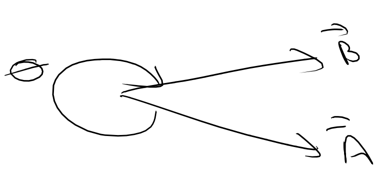
$$A'_x = A_x\cos\theta + A_y\sin\theta$$

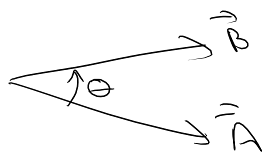
$$A'_y = -A_x\sin\theta + A_y\cos\theta$$

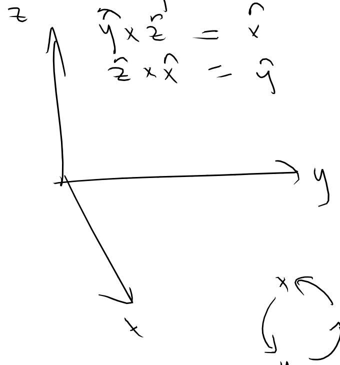
$$\begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \qquad A'_i = R_{ij}A_j$$

iv) $\vec{A}\times\vec{B}$

$$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$$







$\hat{x}\times\hat{y} = \hat{z}$
$\hat{y}\times\hat{z} = \hat{x}$
$\hat{z}\times\hat{x} = \hat{y}$
(RH coordinate system)

$\hat{x}\times\hat{y} = -\hat{z}$ (LH coordinate system)

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$
$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

$$(\vec{A}\times\vec{B})\cdot\vec{C} = (\vec{B}\times\vec{C})\cdot\vec{A} = (\vec{C}\times\vec{A})\cdot\vec{B}$$
$$(\vec{A}\times\vec{B})\times\vec{C} = \vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{A}\cdot(\vec{B}\cdot\vec{C})$$

$$\left[(\vec{A}\times\vec{B})\times\vec{C}\right]_i$$

$\varepsilon_{123} \equiv \varepsilon_{xyz} = 1 = \varepsilon_{231} = \varepsilon_{312}$
$\varepsilon_{ijk} = -\varepsilon_{jik} \cdots$

$$(\vec{A}\times\vec{B})_i = \varepsilon_{ijk}A_jB_k$$

$$\begin{aligned}
\left[(\vec{A}\times\vec{B})\times\vec{C}\right]_i &= \varepsilon_{ijk}\,(\vec{A}\times\vec{B})_j\,C_k \\
&= \varepsilon_{ijk}\,\varepsilon_{jlm}A_lB_mC_k \\
&= -\varepsilon_{jik}\,\varepsilon_{jlm}A_lB_mC_k \\
&= -(\delta_{il}\delta_{km}-\delta_{im}\delta_{kl})A_lB_mC_k \\
&= -A_i(\vec{B}\cdot\vec{C}) + B_i(\vec{A}\cdot\vec{C})
\end{aligned}$$

$$\left[(\vec{A}\times\vec{B})\times\vec{C}\right]_i = \left[\vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{A}\,(\vec{B}\cdot\vec{C})\right]_i$$

$$\Rightarrow \boxed{(\vec{A}\times\vec{B})\times\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{A}\,(\vec{B}\cdot\vec{C})}$$

$$A'_x = A_x \cos\theta + A_y \sin\theta \qquad R_{11} \equiv \cos\theta$$
$$A'_y = -A_x \sin\theta + A_y \cos\theta \qquad R_{12} \equiv \sin\theta$$
$$\qquad R_{21} \equiv -\sin\theta$$
$$\begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \qquad R_{22} \equiv \cos\theta$$

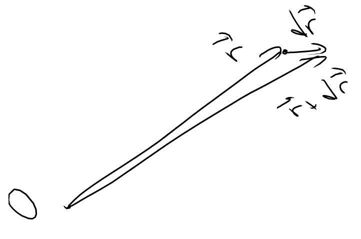
$$A'_x = R_{11} A_x + R_{12} A_y \Rightarrow A'_1 = R_{11} A_1 + R_{12} A_2 \Rightarrow A'_1 = \sum_j R_{1j} A_j$$
$$A'_y = R_{21} A_x + R_{22} A_y \Rightarrow A'_2 = R_{21} A_1 + R_{22} A_2 \Rightarrow A'_2 = \sum_j R_{2j} A_j$$

$$A'_i = \sum_j R_{ij} A_j$$
$$A'_i = R_{ij} A_j \quad \text{(Einsten's summation convention)}$$

Derivatives and Integrals

$$df = f(x+dx) - f(x) = \frac{df}{dx} dx$$

$$d\phi = \phi(\vec{r} + d\vec{r}) - \phi(\vec{r}) = \phi(x+dx, y+dy, z+dz) - \phi(x,y,z)$$
$$= \left(\phi(x,y,z) + \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz\right) - \phi(x,y,z)$$



$$= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$
$$= \underbrace{\left(\frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y} + \frac{\partial \phi}{\partial z} \hat{z} \right)}\cdot \underbrace{(dx\hat{x} + dy\hat{y} + dz\hat{z})}_{d\vec{r}}$$

$$d\phi = (\vec{\nabla}\phi) \cdot (d\vec{r})$$
$$\vec{\nabla}\phi \equiv \frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y} + \frac{\partial \phi}{\partial z} \hat{z}$$

$\vec{\nabla}\phi$: gradient of ϕ

$$d\phi \equiv (\vec{\nabla}\phi) \cdot d\vec{r} \Leftarrow \text{definition of } \vec{\nabla}\phi$$

in cartesian coordinates $d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

choose $dy = dz = 0$

$$d\phi = \phi(x+dx, y, z) - \phi(x,y,z) = (\vec{\nabla}\phi) \cdot (dx\,\hat{x})$$
$$= dx\,(\vec{\nabla}\phi)_x$$

$$(\vec{\nabla}\phi)_x = \lim_{dx\to 0} \frac{\phi(x+dx,y,z)-\phi(x,y,z)}{dx} = \frac{\partial\phi}{\partial x}$$

$$\vec{\nabla}\phi = \left(\hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}\right)$$

$$= \underbrace{\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)}\phi$$

$$= \vec{\nabla}\phi$$

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \Longleftarrow$$

$$d\phi = (\vec{\nabla}\phi)\cdot d\vec{r} = |\vec{\nabla}\phi||d\vec{r}|\cos\theta$$

Example 3D surface

$$z = f(x,y)$$

$$\vec{\nabla}f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}$$

$$df = (\vec{\nabla}f)\cdot d\vec{r} = |\vec{\nabla}f||d\vec{r}|\cos\theta$$

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right)$$

$$= \left(\frac{\partial}{\partial x}A_x\right) + \left(\frac{\partial}{\partial y}A_y\right) + \left(\frac{\partial}{\partial z}A_z\right)$$: divergence of a vector field.

curl of $\vec{A}$

$\Downarrow$

$$\vec{\nabla}\times\vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\times\left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{y}\left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$(\vec{\nabla}\times\vec{A})_i = \varepsilon_{ijk}\frac{\partial A_k}{\partial x_j}$$